CHAPTER TWO

4.8 Indefinite Integrals

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

1. $\int dx = x + c$
2. $\int kf(x)\,dx = k\int f(x)\,dx$
3. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
4. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ if n≠-1
5. $\int (f(x))^n f(x)'\,dx = \dfrac{(f(x))^{n+1}}{n+1} + c$ if n≠-1

Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\,dx = x^2 + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int (2x + \cos x)\,dx = x^2 + \sin x + C.$$

EXAMPLE 7 Indefinite Integration Done Term-by-Term and Rewriting the Constant of Integration

Evaluate

$$\int (x^2 - 2x + 5)\, dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\, dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \int x^2\, dx - 2\int x\, dx + 5\int 1\, dx \\ &= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

$$\begin{aligned}\int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Example: Evaluate

$$\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \frac{x^{3/2}}{3/2} + c$$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

1- $\int \cos u\,du = \sin u + c$

2- $\int \sin u\,du = -\cos u + c$

3- $\int \sec^2 u\,du = \tan u + c$

4- $\int \csc^2 u\,du = -\cot u + c$

5- $\int \sec u \tan u\,du = \sec u + c$

6- $\int \csc u \cot u\,du = -\csc u + c$

Example: Evaluate

$\int \tan x \sec^2 x\,dx$

Solution

$\frac{\tan^2 x}{2} + c$

EXERCISES 4.8

Finding Indefinite Integrals In Exercises 17–54, find the most general indefinite integral. Check your answers by differentiation.

17. $\int (x + 1)\,dx$

18. $\int (5 - 6x)\,dx$

19. $\int \left(3t^2 + \frac{t}{2}\right) dt$

20. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$

21. $\int (2x^3 - 5x + 7)\,dx$

22. $\int (1 - x^2 - 3x^5)\,dx$

23. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$

24. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

25. $\int x^{-1/3}\,dx$

26. $\int x^{-5/4}\,dx$

27. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

29. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

30. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

31. $\int 2x(1 - x^{-3})\, dx$

32. $\int x^{-3}(x + 1)\, dx$

33. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$

34. $\int \frac{4 + \sqrt{t}}{t^3}\, dt$

35. $\int (-2 \cos t)\, dt$

36. $\int (-5 \sin t)\, dt$

37. $\int 7 \sin \frac{\theta}{3}\, d\theta$

38. $\int 3 \cos 5\theta\, d\theta$

39. $\int (-3 \csc^2 x)\, dx$

40. $\int \left(-\frac{\sec^2 x}{3}\right) dx$

41. $\int \frac{\csc \theta \cot \theta}{2}\, d\theta$

42. $\int \frac{2}{5} \sec \theta \tan \theta\, d\theta$

43. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$

44. $\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$

45. $\int (\sin 2x - \csc^2 x)\, dx$

46. $\int (2 \cos 2x - 3 \sin 3x)\, dx$

47. $\int \frac{1 + \cos 4t}{2}\, dt$

48. $\int \frac{1 - \cos 6t}{2}\, dt$

49. $\int (1 + \tan^2 \theta)\, d\theta$

(*Hint:* $1 + \tan^2 \theta = \sec^2 \theta$)

50. $\int (2 + \tan^2 \theta)\, d\theta$

51. $\int \cot^2 x\, dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

52. $\int (1 - \cot^2 x)\, dx$

53. $\int \cos \theta (\tan \theta + \sec \theta)\, d\theta$

54. $\int \frac{\csc \theta}{\csc \theta - \sin \theta}\, d\theta$

Solution

17. $\int (x+1)\,dx = \frac{x^2}{2} + x + C$

18. $\int (5-6x)\,dx = 5x - 3x^2 + C$

19. $\int \left(3t^2 + \frac{t}{2}\right) dt = t^3 + \frac{t^2}{4} + C$

20. $\int \left(\frac{t^2}{2} + 4t^3\right) dt = \frac{t^3}{6} + t^4 + C$

25. $\int x^{-1/3}\,dx = \frac{x^{2/3}}{\frac{2}{3}} + C = \frac{3}{2}x^{2/3} + C$

26. $\int x^{-5/4}\,dx = \frac{x^{-1/4}}{-\frac{1}{4}} + C = \frac{-4}{\sqrt[4]{x}} + C$

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx = \int \left(\frac{1}{2}x^{1/2} + 2x^{-1/2}\right) dx = \frac{1}{2}\left(\frac{x^{3/2}}{\frac{3}{2}}\right) + 2\left(\frac{x^{1/2}}{\frac{1}{2}}\right) + C = \frac{1}{3}x^{3/2} + 4x^{1/2} + C$

33. $\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt = \int \left(\frac{t^{3/2}}{t^2} + \frac{t^{1/2}}{t^2}\right) dt = \int \left(t^{-1/2} + t^{-3/2}\right) dt = \frac{t^{1/2}}{\frac{1}{2}} + \left(\frac{t^{-1/2}}{-\frac{1}{2}}\right) + C = 2\sqrt{t} - \frac{2}{\sqrt{t}} + C$

35. $\int -2\cos t\,dt = -2\sin t + C$

36. $\int -5\sin t\,dt = 5\cos t + C$

37. $\int 7\sin\frac{\theta}{3}\,d\theta = -21\cos\frac{\theta}{3} + C$

38. $\int 3\cos 5\theta\,d\theta = \frac{3}{5}\sin 5\theta + C$

43. $\int (4\sec x\tan x - 2\sec^2 x)\,dx = 4\sec x - 2\tan x + C$

47. $\int \frac{1+\cos 4t}{2}\,dt = \int \left(\frac{1}{2} + \frac{1}{2}\cos 4t\right) dt = \frac{1}{2}t + \frac{1}{2}\left(\frac{\sin 4t}{4}\right) + C = \frac{t}{2} + \frac{\sin 4t}{8} + C$

Integrations of the form $\int f(ax+b)\,dx$

For the integral $\int f(ax+b)\,dx$, make the substitution $u = ax + b$.

Example 7.5 Find $\int \sin(3x+2)\,dx$.

Solution Substitute $u = 3x + 2$. Then $du/dx = 3 \Rightarrow du = 3\,dx \Rightarrow dx = du/3$. Then the integral becomes

$$\int \sin(u)\frac{du}{3} = -\frac{\cos(u)}{3} + C$$

Re-substitute $u = 3x + 2$ to give

$$\int \sin(3x+2)\,dx = -\frac{\cos(3x+2)}{3} + C.$$

Check:

$$\begin{aligned}\frac{d}{dx}\left(-\frac{\cos(3x+2)}{3} + C\right) &= \frac{\sin(3x+2)}{3}\frac{d}{dx}(3x+2)\\ &= \frac{3\sin(3x+2)}{3} = \sin(3x+2).\end{aligned}$$

Example 7.6 Integrate

$$\frac{1}{\sqrt{1-(3-x)^2}}$$

with respect to x.

Solution Notice that this is very similar to the expression which integrates to $\sin^{-1}(x)$ or $\cos^{-1}(x)$. We substitute for the expression in the bracket $u = 3 - x$ giving $du/dx = -1 \Rightarrow dx = -du$. The integral

becomes

$$\int \frac{1}{\sqrt{1-(u)^2}}(-du) = \int \frac{(-du)}{\sqrt{1-(u)^2}}$$

From Table 7.1, this integrates to give

$$\cos^{-1}(u) + C$$

Re-substituting $u = 3 - x$ gives

$$\int \frac{1}{\sqrt{1-(3-x)^2}}\,dx = \cos^{-1}(3-x) + C.$$

Check:

$$\begin{aligned}\frac{d}{dx}(\cos^{-1}(3-x) + C) &= -\frac{1}{\sqrt{1-(3-x)^2}}\frac{d}{dx}(3-x)\\ &= \frac{1}{\sqrt{1-(3-x)^2}}.\end{aligned}$$

Integrals of the form $\int f(u)(\mathrm{d}u/\mathrm{d}x)\,\mathrm{d}x$

Example 7.7 Find $\int x\sin(x^2)\,\mathrm{d}x$.

Solution Substitute $u = x^2 \Rightarrow \mathrm{d}u/\mathrm{d}x = 2x \Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x \Rightarrow \mathrm{d}x = \mathrm{d}u/2x$ to give

$$\int x\sin(x^2)\,\mathrm{d}x = \int x\sin(u)\frac{\mathrm{d}u}{2x} = \int \frac{1}{2}\sin(u)\,\mathrm{d}u$$
$$= -\frac{1}{2}\cos(u) + C.$$

As $u = x^2$, we have

$$\int x\sin(x^2)\,\mathrm{d}x = -\frac{1}{2}\cos(x^2) + C.$$

Check:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{2}\cos(x^2) + C\right) = \frac{1}{2}\sin(x^2)\frac{\mathrm{d}}{\mathrm{d}x}(x^2)$$
$$= \frac{1}{2}\sin(x^2)(2x) = x\sin(x^2).$$

Example 7.8 Find

$$\int \frac{3x}{(x^2+3)^4}\,\mathrm{d}x.$$

,,,

Example 7.9 Find $\int \cos^2(x)\sin(x)\,\mathrm{d}x$.

,, ,

Example 7.10 Find

$$\int \frac{x^2}{(x^2+1)^2}\,\mathrm{d}x.$$

Integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$$
$$\Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}(uv) - \frac{\mathrm{d}u}{\mathrm{d}x}v = u\frac{\mathrm{d}v}{\mathrm{d}x}$$

(subtracting $(\mathrm{d}u/\mathrm{d}x)\ v$ from both sides)

$$\Leftrightarrow u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - \frac{\mathrm{d}u}{\mathrm{d}x}v$$
$$\Rightarrow \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \quad \text{(integrating both sides)}$$

$$\boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.}$$

Example 7.11 Find $\int x \sin x \, dx$

Solution Use $u = x$; $dv = \sin(x)\, dx$. Then

$$\frac{du}{dx} = 1 \quad \text{and} \quad v = \int \sin x \, dx = -\cos(x).$$

Substitute in $\int u \, dv = uv - \int v \, du$ to give

$$\begin{aligned}\int x \sin x \, dx &= -x\cos(x) - \int -\cos(x) 1 \, dx \\ &= -x\cos(x) + \sin(x) + C.\end{aligned}$$

Check:

$$\begin{aligned}\frac{d}{dx}(-x\cos(x) + \sin(x) + C) &= -\cos(x) + x\sin(x) + \cos(x) \\ &= x\sin(x).\end{aligned}$$

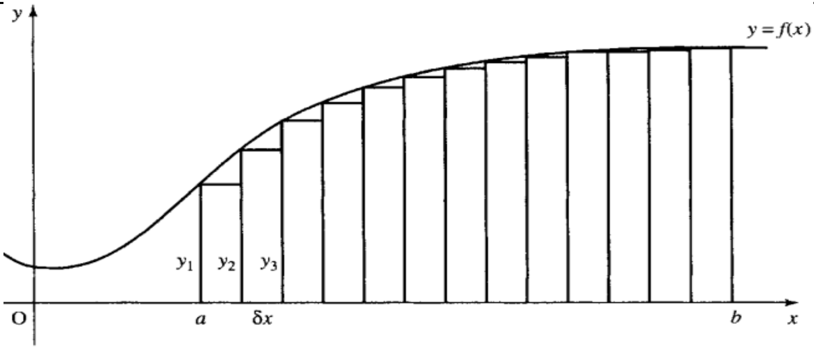
Example 7.12 Find

$$\int \frac{x^2}{(x^2+1)^2} \, dx.$$

<u>The Definite Integral</u>

- ❖ The definite integral from **(x=a to x = b)** is defined as the area under the curve between those two points.
- ❖ In the graph in Figure below, the area under the graph has been approximated by dividing it into rectangles.
- ❖ The height of each is the value of **y** and if each rectangle is the same width then the area of the rectangle is **(yδx)**.
- ❖ If the rectangle is very thin, then **y** will not vary very much over its width and the area can logically be approximated as the sum of all of these rectangles.

$$A = y_1\delta x + y_2\delta x + y_3\delta x + y_4\delta x + \cdots = \sum_{x=a}^{x=b-\delta x} y\delta x.$$



When δx=0.1, the approximate calculation gives

1×0.1+1.1×0.1+1.2×0.1+1.3×0.1+1.4×0.1+1.5×0.1+1.6×0.1+1.7×0.1+1.8×0.1+1.9×0.1=1.45

When δx =0.01, the calculation gives

1×0.01+1.01×0.01+1.02×0.01+ ⋯+1.98×0.01+1.99×0.01=1.495

When δx=0.001, the calculation gives

1×0.001+1.001×0.001+1.002×0.001+ ⋯+1.998×0.001+1.999×0.001=1.4995

- ✓ ***The area under the curve, y=f(x) between (x=a and x= b) is found as:***

$$\int_a^b y\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{x=b-\delta x} y\,\delta x$$

- ✓ The definite integral of **y** from **x = a to x = b** equals the limit as δx tends to 0 of the sum of y times δx for all x from x = a to x = b−δx.
- ✓ This is the definition of the definite integral which gives a number asits result, not a function.

Example 7.17 Find $\int_2^3 2t\,dt$.

This is the area under the graph from $t = 2$ to $t = 3$. As $\int 2t\,dt = t^2 + C$, the area up to 2 is $(2)^2 + C = 4 + C$ and the area up to 3 is $(3)^2 + C = 9 + C$. The difference in the areas is $9 + C - (4 + C) = 9 - 4 = 5$. Therefore, $\int_2^3 2t\,dt = 5$.

The working of a definite integral is usually laid out as follows

$$\int_2^3 2t\,dt = \left[t^2\right]_2^3 = (3)^2 - (2)^2 = 5.$$

Example 7.18 Find

$$\int_{-1}^{1} 3x^2 + 2x - 1\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_{-1}^{1} 3x^2 + 2x - 1\,\mathrm{d}x &= \left[x^3 + x^2 - x\right]_{-1}^{1}\\ &= (1^3 + 1^2 - 1) - ((-1)^3 + (-1)^2 - (-1))\\ &= 1 - (-1 + 1 + 1) = 1 - 1 = 0.\end{aligned}$$

Example 7.19 Find

$$\int_0^{\pi/6} \sin(3x + 2)\,\mathrm{d}x.$$

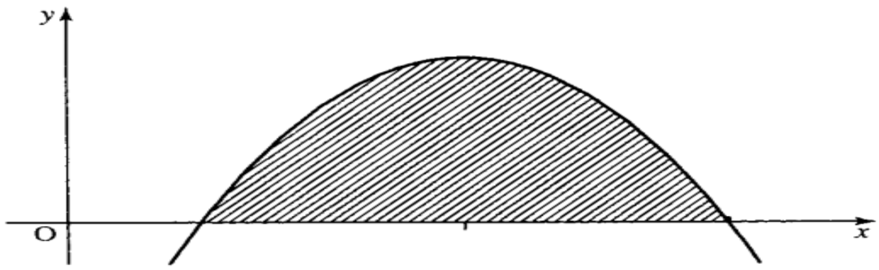
Solution

$$\begin{aligned}\int_0^{\pi/6} \sin(3x+2)\,\mathrm{d}x &= \left[-\frac{1}{3}\cos(3x+2)\right]_0^{\pi/6}\\ &= \frac{1}{3}\cos\left(3\frac{\pi}{6}+2\right) - \left(-\frac{1}{3}\cos(2)\right)\\ &= \frac{1}{3}\cos\left(\frac{\pi}{2}+2\right) + \frac{1}{3}\cos(2) \approx 0.1644.\end{aligned}$$

Example 7.20 Find the shaded area in Figure 7.6, where $y = -x^2 + 6x - 5$.

Solution First, we find where the curve crosses the x-axis, that is, when $y = 0$

$$\begin{aligned}0 = -x^2 + 6x - 5 \quad &\Leftrightarrow \quad x^2 - 6x + 5 = 0\\ \Leftrightarrow \quad (x-5)(x-1) = 0 \quad &\Leftrightarrow \quad x = 5 \vee x = 1\end{aligned}$$



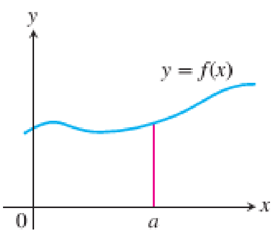
This has given the limits of the integration. Now we integrate:

$$\begin{aligned}\int_1^5 -x^2 + 6x - 5\,\mathrm{d}x &= \left[-\frac{x^3}{3} + \frac{6x^2}{2} - 5x\right]_1^5\\ &= -\frac{(5)^3}{3} + \frac{6(5)^2}{2} - 5(5)\\ &\quad - \left(-\frac{(1)^3}{3} + \frac{6(1)^2}{2} - 5(1)\right)\\ &= -\frac{125}{3} + 75 - 25 + \frac{1}{3} - 3 + 5 = \frac{32}{3} = 10\tfrac{2}{3}\end{aligned}$$

Therefore, the shaded area is $10\frac{2}{3}$ units^2.

TABLE 5.3 Rules satisfied by definite integrals

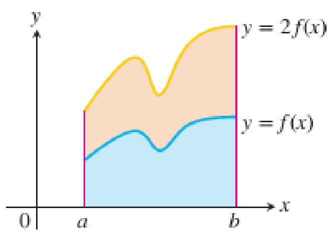
1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

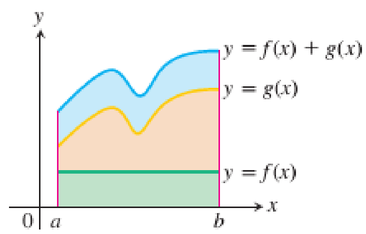
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

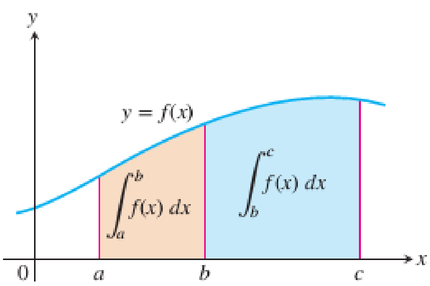
(Shown for $k = 2$.)



(c) *Sum:*

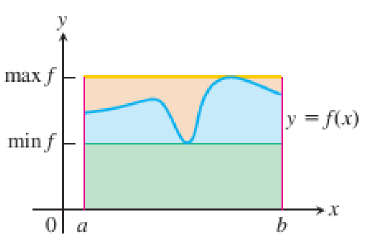
$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)



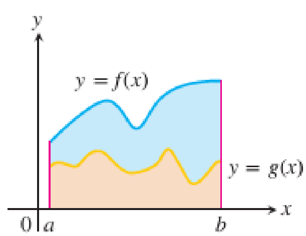
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$

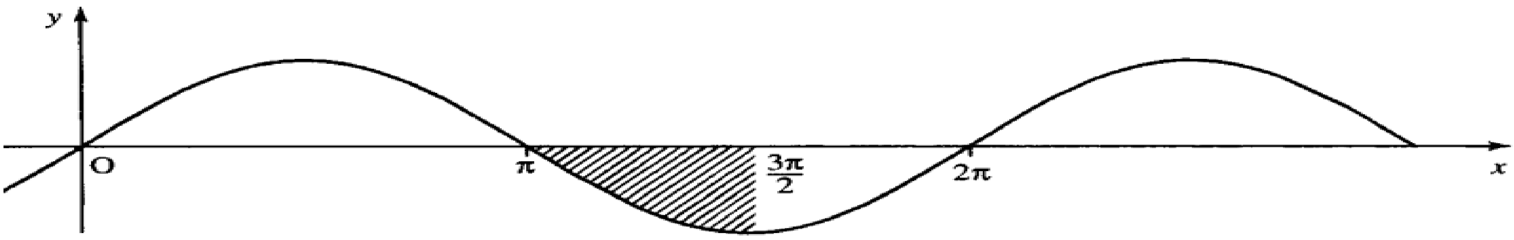


(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

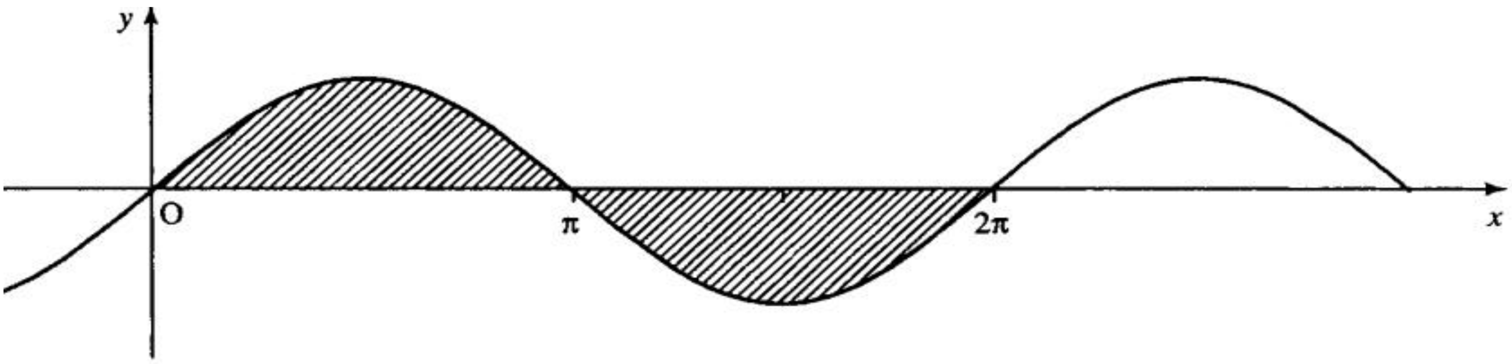
FIGURE 5.11

Example: Find the negative area. y =sin(x) from x =π to x= 3π/2 .



$$\int_{\pi}^{3\pi/2} \sin(x)\,dx = [-\cos(x)]_{\pi}^{3\pi/2} = -\cos\left(\frac{3\pi}{2}\right) + \cos(\pi) = -1$$

Example: Find the negative area. y =sin(x) from x =0 to 2π



The area under the graph $y = \sin(x)$ *from* $x = 0$ *to* 2π.

$$\int_{0}^{2\pi} \sin(x)\,dx = [-\cos(x)]_{0}^{2\pi} = -\cos(2\pi) - (-\cos(0))$$
$$= -1 - (-1) = 0$$

- ❖ To prevent cancellation of the positive and negative parts of the integration.
- ❖ we find the total shaded area in two stages.

$$\int_{0}^{\pi} \sin(x)\,dx = [-\cos(x)]_{0}^{\pi} = -\cos(\pi) - (-\cos(0)) = 2$$

and

$$\int_{\pi}^{2\pi} \sin(x)\,dx = [-\cos(x)]_{\pi}^{2\pi} = -\cos(2\pi) - (-\cos(\pi)) = -2$$

So, the total area is $2 + |-2| = 4$.

Example:

Find the area bounded by the curve $y=x^2-x$ and the x-axis and the lines $x=-1$ and $x=1$.

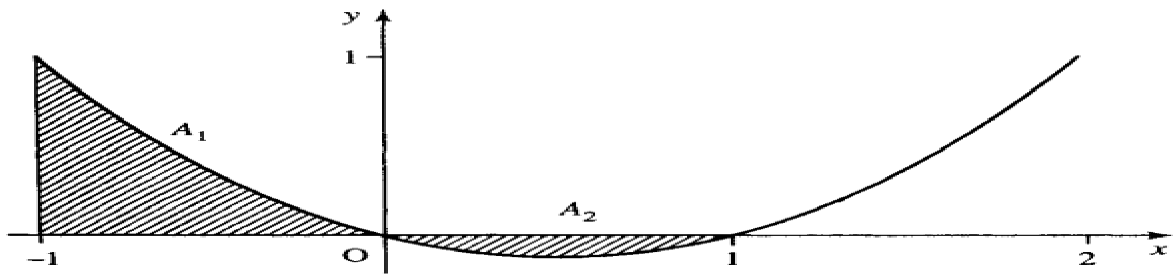
Solution First, we find if the curve crosses thex-axis.$x^2 - x = 0$
$x(x-1) = 0 \Leftrightarrow x= 0$ or $x =1$.

The sketch of the graph with the required area shaded is given in Figure below. Therefore, the area is the sum ofA1and A2.

❖ We find A1 by integrating from −1 to 0.

$$\int_{-1}^{0}(x^2-x)\,dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{0} = 0-\left(\frac{(-1)^3}{3}-\frac{(-1)^2}{2}\right)$$
$$= \frac{1}{3}+\frac{1}{2}=\frac{5}{6}$$

$A_1 = \frac{5}{6}$.



Find A_2 by integrating from 0 to 1 and taking the modulus

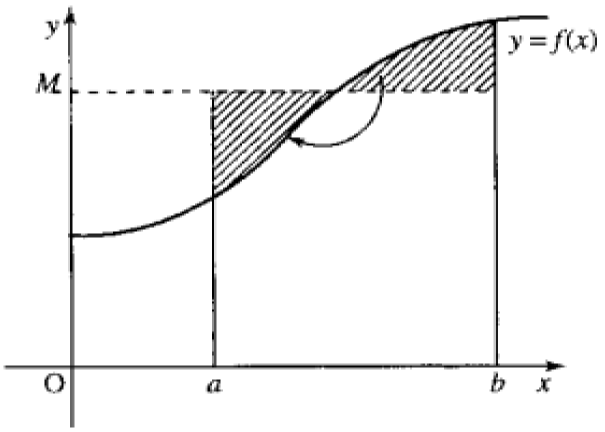
$$\int_{0}^{1}(x^2-x)\,dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{3}-\frac{1}{2}=-\frac{1}{6}$$

Therefore, $A_2 = \frac{1}{6}$.
Then, the total area is $A_1 + A_2 = \frac{5}{6}+\frac{1}{6} = 1$.

THE MEAN VALUE AND R.M.S. VALUE

The mean value of a function is the value it would have take if it were constant over the range but with the same area under the graph, that is, with the same integral as shown.

The formula for the mean value is



$$M = \frac{1}{b-a}\int_a^b y\,\mathrm{d}x.$$

Example 7.22 Find the mean value of $i(t) = 20 + 2\sin(\pi t)$ for $t = 0$ to 0.5.

Solution Using the formula $a = 0, b = 0.5$ gives

$$M = \frac{1}{0.5-0}\int_0^{0.5} 20 + 2\sin(\pi t)\,\mathrm{d}t$$

$$2\left[20t - \frac{2}{\pi}\cos(\pi t)\right]_0^{0.5} = 2(10 - 0 - \left(0 - \frac{2}{\pi}(1)\right) \approx 21.27$$

THE ROOT MEAN SQUARED (R.M.S) VALUE

The (r.m.s. value) means the square root of the mean value of the square of y. The formula for the r.m.s. value of y between x=a and x=b is.

$$\text{r.m.s.}(y) = \sqrt{\frac{1}{b-a}\int_a^b y^2\,\mathrm{d}x}$$

- ❖ The advantage of the r.m.s. value is that as all the values for y are squared, they are positive, so the r.m.s. value will not give 0 unless we are considering the zero function.

Example 7.23 Find the r.m.s. value of $y = x^2 - 3$ between $x = 1$ and $x = 3$.

$$(\text{r.m.s.}(y))^2 = \frac{1}{3-1}\int_1^3 (x^2-3)^2\,dx = \frac{1}{2}\int_1^3 (x^4 - 6x^2 + 9)\,dx$$
$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{6x^3}{3} + 9x\right]_1^3$$
$$= \frac{1}{2}\left(\left(\frac{243}{5} - 54 + 27\right) - \left(\frac{1}{5} - 2 + 9\right)\right) = 7.2$$

Therefore, the r.m.s value is $\sqrt{7.2} \approx 2.683$.

EXERCISES 5.3

Using Area to Evaluate Definite Integrals

In Exercises 15–22, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^{4}\left(\frac{x}{2}+3\right)dx$
16. $\int_{1/2}^{3/2}(-2x+4)\,dx$
17. $\int_{-3}^{3}\sqrt{9-x^2}\,dx$
18. $\int_{-4}^{0}\sqrt{16-x^2}\,dx$
19. $\int_{-2}^{1}|x|\,dx$
20. $\int_{-1}^{1}(1-|x|)\,dx$
21. $\int_{-1}^{1}(2-|x|)\,dx$
22. $\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)dx$

Use areas to evaluate the integrals in Exercises 23–26.

23. $\int_0^b \frac{x}{2}\,dx, \quad b > 0$
24. $\int_0^b 4x\,dx, \quad b > 0$
25. $\int_a^b 2s\,ds, \quad 0 < a < b$
26. $\int_a^b 3t\,dt, \quad 0 < a < b$

Evaluations

Use the results of Equations (1) and (3) to evaluate the integrals in Exercises 27–38.

27. $\int_1^{\sqrt{2}} x\,dx$
28. $\int_{0.5}^{2.5} x\,dx$
29. $\int_{\pi}^{2\pi} \theta\,d\theta$
30. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$
31. $\int_0^{\sqrt[3]{7}} x^2\,dx$
32. $\int_0^{0.3} s^2\,ds$
33. $\int_0^{1/2} t^2\,dt$
34. $\int_0^{\pi/2} \theta^2\,d\theta$
35. $\int_a^{2a} x\,dx$
36. $\int_a^{\sqrt{3}a} x\,dx$
37. $\int_0^{\sqrt[3]{b}} x^2\,dx$
38. $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.3 and Equations (1)–(3) to evaluate the integrals in Exercises 39–50.

39. $\int_3^1 7\,dx$
40. $\int_0^{-2} \sqrt{2}\,dx$
41. $\int_0^2 5x\,dx$
42. $\int_3^5 \frac{x}{8}\,dx$
43. $\int_0^2 (2t-3)\,dt$
44. $\int_0^{\sqrt{2}} (t-\sqrt{2})\,dt$
45. $\int_2^1 \left(1+\frac{z}{2}\right)dz$
46. $\int_3^0 (2z-3)\,dz$
47. $\int_1^2 3u^2\,du$
48. $\int_{1/2}^1 24u^2\,du$
49. $\int_0^2 (3x^2+x-5)\,dx$
50. $\int_1^0 (3x^2+x-5)\,dx$

20.10 $\int_0^{\pi/4} \cos x\, dx$. ,,,, **20.11** $\int_0^{\pi/3} \sec^2 x\, dx$. ,,,, **20.22** $\int_0^{\pi/4} \tan x \sec^2 x\, dx$.

20.24 $\int_0^{\pi/2} \sqrt{\sin x + 1}\, \cos x\, dx$.

,,,,

20.32 Find the average value of $f(x) = \sqrt[3]{x}$ on $[0, 1]$.

20.33 Compute the average value of $f(x) = \sec^2 x$ on $[0, \pi/4]$.

20.60 The region above the x-axis and under the curve $y = \sin x$, between $x = 0$ and $x = \pi$, is divided into two parts by the line $x = c$. If the area of the left part is one-third the area of the right part, find c.

20.61 Find the value(s) of k for which $\int_0^2 x^k\, dx = \int_0^2 (2 - x)^k\, dx$.