VECTOR ANALYSIS (4 hour $= 6$ hour)

1.1 INTRODUCTION

Electromagnetics (EM) may be regarded as the study of the interactions between electric charges at rest and in motion. It entails the analysis, synthesis, physical interpretation, and application of electric and magnetic fields.

Electromagnetics (EM): is a branch of physics or electrical engineering in which electric and magnetic phenomena are studied.

EM devices include transformers, electric relays, radio/TV, telephone, electric motors, transmission lines, waveguides, antennas, optical fibers, radars, and lasers. The design of these devices requires thorough knowledge of the laws and principles of EM.[2]

1.2 SCALARS AND VECTORS

The scalar refers to a quantity that has only magnitude. The x, y and z in basic algebra, distance l, time t, temperature T, mass, density, pressure, volume, volume resistivity and voltage are all scalars quantities.

A vector is a quantity that has both a magnitude and a direction. Force, velocity, acceleration, and a straight line from the positive to the negative terminal of a storage battery are vectors quantities. Each quantity is characterized by both a magnitude and a direction.

Note: The vector are denoted by boldface symbols or large letters with arrow over them.[1 and 2]

1.3 THE CARTESIAN COORDINATE SYSTEM

In order to describe a vector accurately, some specific lengths, directions, angles, projections, or components must be given. There are three simple methods of doing this, and about eight or ten other methods which are useful in very special cases. We are going to use only the three simple methods, and the simplest of these is the cartesian, or rectangular, coordinate system.

In the cartesian coordinate system we set up three coordinate axes mutually at right angles to each other (x, y and z axes). A rotation of the x axis (through the smaller angle) into the y axis would cause a right-handed screw to progress in the direction of the z axis. If the right hand is used, then the thumb, forefinger, and middle finger may be identified, respectively, as the x, y and z axes. Figure 1.1 shows a right-handed cartesian coordinate system.

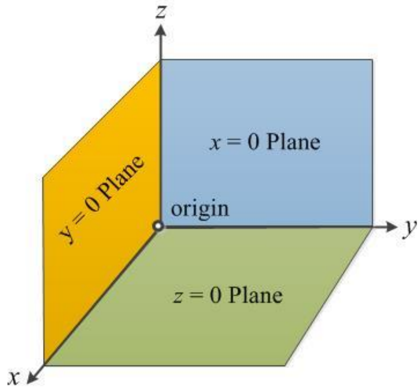


Fig. 1.1 a right-handed cartesian coordinate system

A point is located by giving its x, y and z coordinates. These are, respectively, the distances from the origin to the intersection of a perpendicular dropped from the point to the x, y and z axes. An alternative method of interpreting coordinate values, that which must be used in all other coordinate systems, is to consider the point as being at the common intersection of three surfaces, the planes $x = \text{const.}$, $y = \text{cons.}$, and $z = \text{const.}$, the constants being the coordinate values of the point.[1]

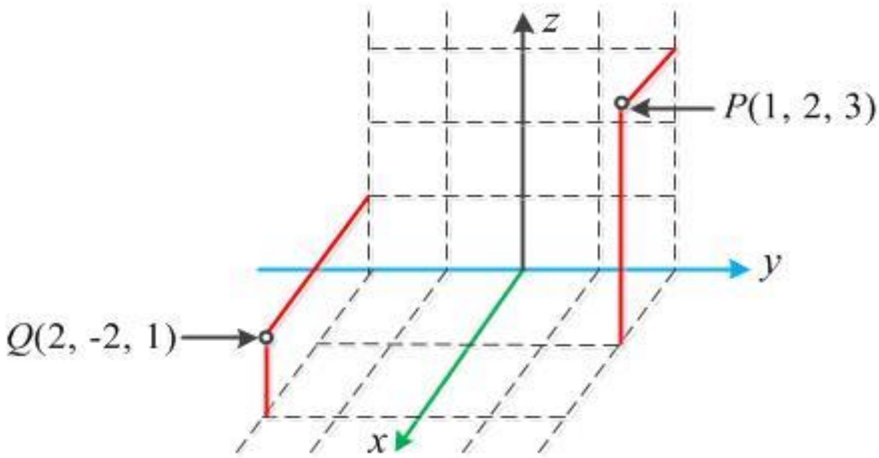


Fig. 1.2 points P and Q whose coordinates are $(1, 2, 3)$ and $(2, -2, 1)$, respectively.

- Point P is located at the common point of intersection of the planes $x = 1$, $y = 2$, and $z = 3$,
- Point Q is located at the common point of intersection of the planes $x = 2$, $y = -2$, and $z = 1$.

1.4 VECTOR COMPONENTS AND UNIT VECTORS

If the component vectors of the vector $\mathbf{r}$ are x, y and z, then $\mathbf{r} = x + \mathrm{y} + z$. The component vectors are shown in Fig. 1.3.

The component vectors have magnitudes which depend on the given vector (such as $\mathbf{r}$ above), but they each have a known and constant direction.

A unit vectors $\mathbf{a}_{\mathrm{A}}$ in the direction of the vector $\mathbf{A}$ is defined as a vector whose magnitude is unity (i.e. 1) and its direction is along $\mathbf{A}$.

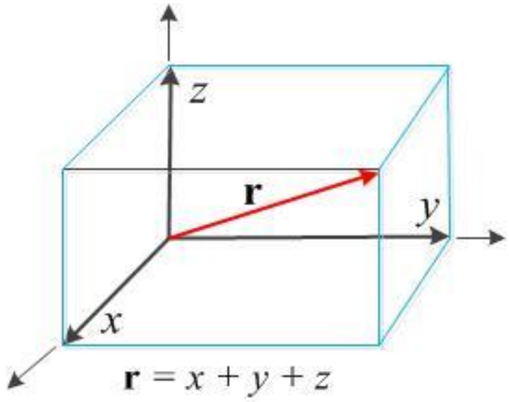


Fig 1.3 the component vectors x, y and z of vector $\mathbf{r}$

Any vector $\mathbf{B}$ may be described by:

$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z.$$

The magnitude of the vector $\mathbf{B}$ written $|\mathbf{B}|$ or simply B, is given by:

$$|\mathbf{B}| = B = \sqrt{{B_x}^2 + {B_y}^2 + {B_z}^2}$$

The component vector $\mathbf{B}$ are:

- $B_x\mathbf{a}_x$
- $B_y\mathbf{a}_y$ and
- $B_z\mathbf{a}_z$.

Thus $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ are the unit vectors in the x, y and z axes in the cartesian coordinate system, as shown in Fig. 1.4.

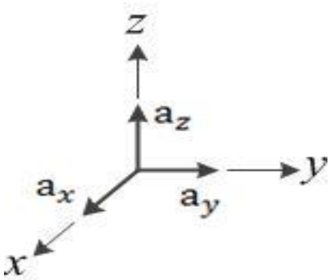


Fig. 1.4 the unit vectors of the cartesian coordinate system

A unit vector in a given direction is merely a vector in that direction divided by its magnitude.

A unit vector in the direction of the vector B is:

$$\mathbf{a}_B = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

1.5 POSITION AND DISTANCE VECTORS

The position vector $\mathbf{r}_P$ (or radius vector) of point P is as the directed distance from the origin O to P:

A vector $\mathbf{r}_\mathrm{P}$ pointing from the origin to the point $P(1,2,3)$ is written as:

$$\mathbf{r}_P = \mathbf{r}_{OP} = (1-0)\mathbf{a}_x + (2-0)\mathbf{a}_y + (3-0)\mathbf{a}_z = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z.$$

The distance vector is the displacement from one point to another.[2]

The vector from P to Q may be obtained by applying the rule of vector addition. This rule shows that the vector from the origin to P plus the vector from P to Q is equal to the vector from the origin to Q. The desired vector from $\mathrm{P}(1,2,3)$ to $\mathrm{Q}(2,-2,1)$ is therefore

$$\mathbf{R}_{\mathrm{PQ}} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z = \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

The vectors $\mathbf{r}_\mathrm{P}$, $\mathbf{r}_\mathrm{Q}$ and $\mathbf{R}_{\mathrm{PQ}}$ are shown in Fig. 1.5.

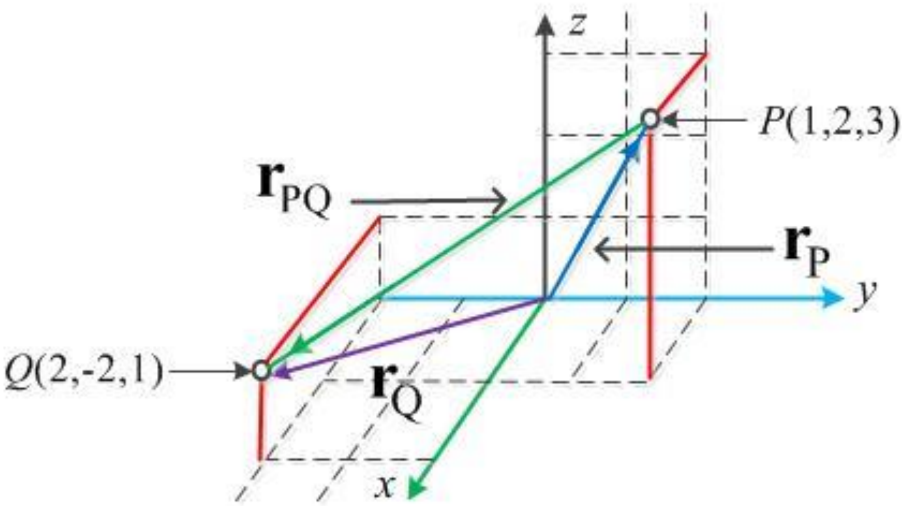


Fig. 1.5 the vector $\mathbf{R}_{\mathrm{PQ}}$ is equal to the vector difference $\mathbf{r}_\mathrm{Q} - \mathbf{r}_\mathrm{P}$

If we are discussing a force vector $\mathbf{F}$. The component scalars are F_x, F_y and F_z. We may then write $\mathbf{F} = F_x\mathbf{a}_x + F_y\mathbf{a}_y + F_z\mathbf{a}_z$. The component vectors are $F_x\mathbf{a}_x$, $F_y\mathbf{a}_y$ and $F_z\mathbf{a}_z$.

where F_x, F_y and F_z as a function of x, y and z in the cartesian coordinate system.[1]

1.6 THE VECTOR FIELD

A vector field is defined as a vector function of a position vector. In general, the magnitude and the direction of the function will change as we move throughout the region, and the value of the vector function must be determined by using the coordinate values of the point. Since we have considered the cartesian coordinate system, we should expect the vector to be a function of the variables x, y and z.

If we represent the position vector as $\mathbf{r}$, then a vector field $\mathbf{G}$ can be expressed in functional notation as $\mathbf{G}(\mathbf{r})$; a scalar field T is written as $T(\mathbf{r})$.

The velocity vector may be written as:

$$\mathbf{V} = v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z; \qquad \text{or} \qquad \mathbf{V}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z;$$

Each of components v_x, v_y and v_z may be a function of the variables x, y and z.[1]

1.7 VECTOR ALGEBRA

With the definitions of vectors and vector fields now accomplished, we may proceed to define the rules of vector arithmetic, vector algebra, and (later) of vector calculus. Some of the rules will be similar to those of scalar algebra, some will differ slightly, and some will be entirely new and strange. Vector algebra has set of rules.

1.7.1 ADDITION OF VECTORS

Vectorial addition follows the parallelogram law, and this is easily, if inaccurately, accomplished graphically. Fig. 1.6 shows the sum of two vectors, if the two vectors:

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$$

$$\mathbf{A} + \mathbf{B} = \left(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z\right) + \left(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z\right).$$

$$\mathbf{A} + \mathbf{B} = [(A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z].$$

- Vector addition obeys the commutative law; $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
- Vector addition obeys the associative law; $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}.$

Fig. 1.6 two vectors may be added graphically

Coplanar vectors: are vectors lying in a common plane, such as shown in Fig. 1.6, which both lie in the plane of the paper, may also be added by expressing each vector in terms of "horizontal" and "vertical" components and adding the corresponding components. $\mathbf{A}+\mathbf{B}$

1.7.2 SUBTRACTION OF VECTORS

The rule for the subtraction of vectors follows easily from that for addition, for we may always express $\mathbf{A}-\mathbf{B}$ as $\mathbf{A}+(-\mathbf{B})$; the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition.

$$\mathbf{A}-\mathbf{B}=\left(A_x\mathbf{a}_x+A_y\mathbf{a}_y+A_z\mathbf{a}_z\right)-\left(B_x\mathbf{a}_x+B_y\mathbf{a}_y+B_z\mathbf{a}_z\right).$$

$$\mathbf{A}-\mathbf{B}=[(A_x-B_x)\mathbf{a}_x+(A_y-B_y)\mathbf{a}_y+(A_z-B_z)\mathbf{a}_z].$$

Multiplication of a vector by a scalar

Vectors may be multiplied by scalars. The magnitude of the vector changes, but its direction does not when the scalar is positive, although it reverses direction when multiplied by a negative scalar.

Multiplication of a vector by a scalar obeys

- the associative and
- distributive laws.

$$k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B}$$

$$(r+s)(\mathbf{A}+\mathbf{B})=r(\mathbf{A}+\mathbf{B})+s(\mathbf{A}+\mathbf{B})=r\mathbf{A}+r\mathbf{B}+s\mathbf{A}+s\mathbf{B}$$

Division of a vector by a scalar

Division of a vector by a scalar is merely multiplication by the reciprocal of that scalar.

Two vectors are said to be equal if their difference is zero, or $\mathbf{A}=\mathbf{B}\quad if\ \ \mathbf{A}-\mathbf{B}=0.$

1.7.3 THE MULTIPLICATION OF VECTORS

1.7.3.1 THE DOT PRODUCT

The dot product, or scalar product of two vectors **A** and **B**, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle (θ) between them, or

$$\mathbf{A}\,.\,\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$$

- the dot, or scalar, product is a scalar, $\mathbf{A}\,.\,\mathbf{B}$ is read "**A** dot **B**".
- it obeys the commutative law, $\mathbf{A}\,.\,\mathbf{B} = \mathbf{B}\,.\,\mathbf{A}$
- it also obeys the distributive law, $\mathbf{A}\,.\,(\mathbf{B} + \mathbf{C}) = \mathbf{A}\,.\,\mathbf{B} + \mathbf{A}\,.\,\mathbf{C}$

Since the angle between two different unit vectors of the cartesian coordinate system is 90^{o}, we then have

- $\mathbf{a}_x.\mathbf{a}_y = 0$
- $\mathbf{a}_y.\mathbf{a}_z = 0$
- $\mathbf{a}_z.\mathbf{a}_x = 0$
- $\mathbf{a}_x.\mathbf{a}_x = 1$
- $\mathbf{a}_y.\mathbf{a}_y = 1$
- $\mathbf{a}_z.\mathbf{a}_z = 1$

giving finally, an expression involving no angles:

$$\mathbf{A}\,.\,\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

A vector dotted with itself yields the magnitude squared, or $\mathbf{A}\,.\,\mathbf{A} = A^2 = |\mathbf{A}|^2$

and any unit vector dotted with itself is unity, $\mathbf{a}_A.\mathbf{a}_A = 1$

One of the most important applications of the dot product is that of finding the component of a vector in a given direction. Referring to Fig. 1.7(a), we can obtain the component (scalar) of **B** in the direction specified by the unit vector **a** as

$$\mathbf{B}.\,\mathbf{a} = |\mathbf{B}||\mathbf{a}|\cos\theta = |\mathbf{B}|\cos\theta$$

The sign of the component is positive if $0 \leq \theta \leq 90^{\mathrm{o}}$ and negative whenever $90^{\mathrm{o}} \leq \theta \leq 180^{\mathrm{o}}$.

In order to obtain the component vector of **B** in the direction of **a**, we simply multiply the component (scalar) by **a**, as illustrated by Fig. 1.7(b).

For example, the component of vector $\mathbf{B}$ in the direction of $\mathbf{a}_x$ is $\mathbf{B} \cdot \mathbf{a}_x = B_x$ and the component vector is $B_x\ \mathbf{a}_x$, or $(\mathbf{B} \cdot \mathbf{a}_x)\mathbf{a}_x$.

The geometrical term projection is also used with the dot product. Thus, $\mathbf{B} \cdot \mathbf{a}_x$ is the projection of **B** in the a direction.[1]

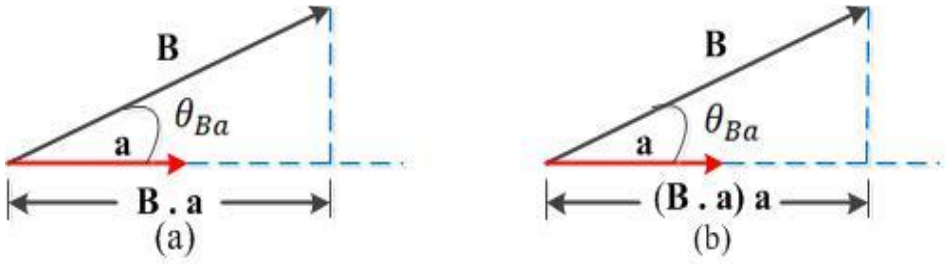


Fig. 1.7 (a) the scalar component of **B** in the direction of the unit vector **a** is $\mathbf{B} \cdot \mathbf{a}$
(b) the vector component of **B** in the direction of the unit vector **a** is $(\mathbf{B} \cdot \mathbf{a})\mathbf{a}$.

1.7.3.2 THE CROSS PRODUCT

The cross product, or vector product, of two vectors **A** and **B**, written with a cross between the two vectors as $(\mathbf{A} \times \mathbf{B})$.

- the cross product $\mathbf{A} \times \mathbf{B}$ is a vector; $(\mathbf{A} \times \mathbf{B})$ read “**A** cross **B**”.
- it is not commutative; $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$ or $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$
- it is not associative; $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$
- it is distributive; $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
- the magnitude of $\mathbf{A} \times \mathbf{B}$ is equal to the product of the magnitude of **A**, the magnitude of **B** and the sine of the smaller angle (θ) between **A** and **B**;

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin \theta$$

- the direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane containing **A** and **B** and is along that one of the two possible perpendiculars which is in the direction of advance of a right-handed screw as **A** is turned into **B**, as shown in Fig. 1.8.

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_{\mathrm{N}}|\mathbf{A}||\mathbf{B}| \sin \theta$$

where $\mathbf{a}_N$ is a unit vector normal to the plane determined by **A** and **B** when they are drawn from a common point.

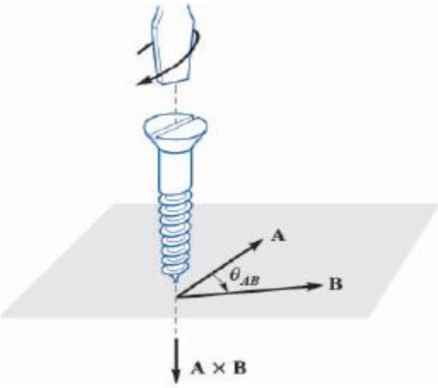


Fig. 1.8 the direction of $\mathbf{A} \times \mathbf{B}$ is in the direction of advance of a right-handed screw as $\mathbf{A}$ is turned into $\mathbf{B}$.

If the definition of the cross product is applied to the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, we find.

- $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$
- $\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$
- $\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$
- $\mathbf{a}_y \times \mathbf{a}_x = -\mathbf{a}_z$
- $\mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$
- $\mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$
- $\mathbf{a}_x \times \mathbf{a}_x = 0$
- $\mathbf{a}_y \times \mathbf{a}_y = 0$
- $\mathbf{a}_z \times \mathbf{a}_z = 0$

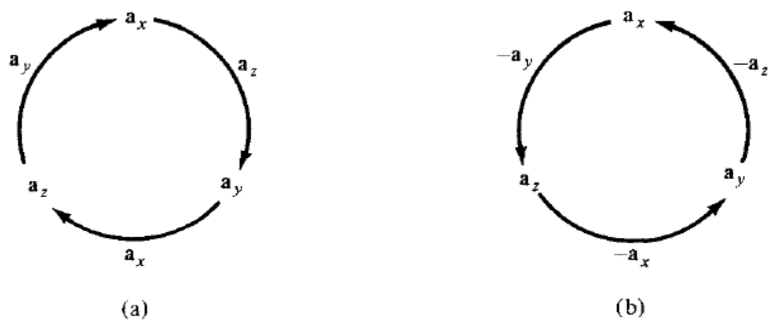


Fig. 1.9 cross product using cyclic permutation: (a) moving clockwise leads to positive results: (b) moving counterclockwise leads to negative results.

The evaluation of a cross product by means of its definition. This work may be avoided by using cartesian components for the two vectors $\mathbf{A}$ and $\mathbf{B}$ and expanding the cross product as a sum of nine simpler cross products, each involving two unit vectors,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} = {} & A_x B_x(\mathbf{a}_x \times \mathbf{a}_x) + A_x B_y(\mathbf{a}_x \times \mathbf{a}_y) + A_x B_z(\mathbf{a}_x \times \mathbf{a}_z) \\ & + A_y B_x(\mathbf{a}_y \times \mathbf{a}_x) + A_y B_y(\mathbf{a}_y \times \mathbf{a}_y) + A_y B_z(\mathbf{a}_y \times \mathbf{a}_z) \\ & + A_z B_x(\mathbf{a}_z \times \mathbf{a}_x) + A_z B_y(\mathbf{a}_z \times \mathbf{a}_y) + A_z B_z(\mathbf{a}_z \times \mathbf{a}_z)\end{aligned}$$

These results may be combined to give

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z$$

or written as a determinant in a more easily remembered form,[1]

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

1.7.3.3 Scalar Triple Product

Given three vectors A, B, and C, we define the scalar triple product as

$$\mathbf{A}\,.\,(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,.\,(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\,.\,(\mathbf{A}\times\mathbf{B})$$

If $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$, $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$, and $\mathbf{C} = C_x\mathbf{a}_x + C_y\mathbf{a}_y + C_z\mathbf{a}_z$, then $\mathbf{A}\,.\,(\mathbf{B}\times\mathbf{C})$ is the volume of a parallelepiped having **A**, **B**, and **C** as edges and is easily obtained by finding the determinant of the 3×3 matrix formed by **A**, **B**, and **C**; that is,

$$\mathbf{A}\,.\,(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Since the result of this vector multiplication is scalar, equations above are called the scalar triple product.[2]

1.7.3.4 Vector Triple Product

For vectors **A**, **B**, and **C**, we define the vector triple product as

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\,.\,\mathbf{C}) - \mathbf{C}(\mathbf{A}\,.\,\mathbf{B})$$

obtained using the "bac-cab" rule. It should be noted that

$$(\mathbf{A}\,.\,\mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B}\,.\,\mathbf{C})$$

But

$$(\mathbf{A}\,.\,\mathbf{B})\mathbf{C} = \mathbf{C}(\mathbf{A}\,.\,\mathbf{B})$$

Example 1.1:[1]

Specify the unit vector extending from the origin toward the point $G(2,-2,-1)$.

Solution:

We first construct the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

the magnitude of vector **G** is $\quad |\mathbf{G}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

The unit vector

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \frac{2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z}{3} = \frac{2}{3}\mathbf{a}_x - \frac{2}{3}\mathbf{a}_y - \frac{1}{3}\mathbf{a}_z$$

$$\mathbf{a}_G = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.334\mathbf{a}_z$$

Example 1.2:[2]

If $\mathbf{A} = 10\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y$, find: (a) the component of **A** along $\mathbf{a}_y$, (b) the magnitude of $3\mathbf{A} - \mathbf{B}$, (c) a unit vector along $\mathbf{A} + 2\mathbf{B}$ (d) the component of **B** along $\mathbf{a}_z$.

Solution: (a) The component of **A** along $\mathbf{a}_y$ is $A_y = -4$.

(b) $3\mathbf{A} - \mathbf{B} = \left[3\left(10\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z\right) - \left(2\mathbf{a}_x + \mathbf{a}_y\right)\right]$

$3\mathbf{A} - \mathbf{B} = \left(30\mathbf{a}_x - 12\mathbf{a}_y + 18\mathbf{a}_z\right) - \left(2\mathbf{a}_x + \mathbf{a}_y\right) = 28\mathbf{a}_x - 13\mathbf{a}_y + 18\mathbf{a}_z$

the magnitude of $3\mathbf{A} - \mathbf{B} = \sqrt{28^2 + (-13)^2 + 18^2} = \sqrt{1277} = 35.74$

(c) Let $\mathbf{C} = \mathbf{A} + 2\mathbf{B} = (10\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z) + 2(2\mathbf{a}_x + \mathbf{a}_y) = 14\mathbf{a}_x - 2\mathbf{a}_y + 6\mathbf{a}_z$.

A unit vector along **C** is:

$$\mathbf{a}_C = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{14\mathbf{a}_x - 2\mathbf{a}_y + 6\mathbf{a}_z}{\sqrt{14^2 + (-2)^2 + 6^2}} = 0.9113\mathbf{a}_x - 0.1302\mathbf{a}_y + 0.3906\mathbf{a}_z$$

(d) The component of **B** along $\mathbf{a}_z$ is $B_z = 0$.

Example 1.3:[2]

Points P and Q are located at $(0,2,4)$ and $(-3,1,5)$. Calculate

(a) The position vector P (b) The distance vector from P to Q (c) The distance between P and Q (d) A vector parallel to PQ with magntude of 10.

Solution: (a) $\mathbf{r}_P = \mathbf{r}_{OP} = (0-0)\mathbf{a}_x + (2-0)\mathbf{a}_y + (4-0)\mathbf{a}_z = 2\mathbf{a}_y + 4\mathbf{a}_z$

(b) $\mathbf{r}_Q = \mathbf{r}_{OQ} = (-3-0)\mathbf{a}_x + (1-0)\mathbf{a}_y + (5-0)\mathbf{a}_z = -3\mathbf{a}_x + \mathbf{a}_y + 5\mathbf{a}_z$

$\mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (-3\mathbf{a}_x + \mathbf{a}_y + 5\mathbf{a}_z) - (2\mathbf{a}_y + 4\mathbf{a}_z) = -3\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z$

(c) $d = |\mathbf{r}_{PQ}| = \sqrt{(-3)^2 + (-1)^2 + (1)^2} = \sqrt{11} = 3.31$

(d) Let the required vector be $\mathbf{A}$, then $\mathbf{A} = A\,.\,\mathbf{a}_A$

where $A = 10$ is the magnitude of $\mathbf{A}$. Since $\mathbf{A}$ is parallel to PQ, it must have the same unit vector as $\mathbf{r}_{PQ}$ or $\mathbf{r}_{QP}$. Hence,

$$\mathbf{a}_A = \mp \frac{\mathbf{r}_{PQ}}{|\mathbf{r}_{PQ}|} = \mp \frac{-3\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z}{3.31}$$

$$\mathbf{A} = \mp \frac{10(-3\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)}{3.31} = \mp(-9.04\mathbf{a}_x - 3.015\mathbf{a}_y + 3.015\mathbf{a}_z)$$

Example 1.4:[1]

The vector field $\mathbf{G} = y\mathbf{a}_x - 2.5x\mathbf{a}_y + 3\mathbf{a}_z$, and the point $Q(4,5,2)$. find: (i) $\mathbf{G}$ at Q; (ii) the scalar component of $\mathbf{G}$ at Q in the direction of $\mathbf{a}_N = \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$; (iii) the vector component of $\mathbf{G}$ at Q in the direction of $\mathbf{a}_N$ and finally, (iv) the angle θ_{Ga} between $\mathbf{G}_{(\mathbf{r}Q)}$ and $\mathbf{a}_N$.

Solution:

(i) Substituting the coordinates of point Q into the expression for $\mathbf{G}$, we have

$\mathbf{G}_{(\mathbf{r}Q)} = 5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z$

(ii) We find the scalar component. Using the dot product, we have

$$\mathbf{G}.\mathbf{a}_N = (5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z).\left(\frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)\right) = \frac{1}{3}(10 - 10 - 6) = -2$$

(iii) The vector component is obtained by multiplying the scalar component by the unit vector in the direction of $\mathbf{a}_N$:

$$(\mathbf{G}.\mathbf{a}_N).\mathbf{a}_N = -2\left(\frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)\right) = -1.333\mathbf{a}_x - 0.667\mathbf{a}_y + 1.333\mathbf{a}_z$$

(iv) The angle between $\mathbf{G}_{(\mathbf{r}Q)}$ and $\mathbf{a}_N$, is found from

$\mathbf{G}.\mathbf{a}_N = |\mathbf{B}| \cos\theta_{Ga}$

$$-2 = \sqrt{25 + 100 + 9}\cos\theta_{Ga} \quad \Rightarrow \quad \theta_{Ga} = \cos^{-1}\frac{-2}{\sqrt{134}} = 99.9^o$$

Example 1.5:[1]

If $\mathbf{A} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$; and $\mathbf{B} = -4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$; find (i) $\mathbf{A}\,.\,\mathbf{B}$ (ii) $\mathbf{A} \times \mathbf{B}$ and (iii) the angle between **A** and **B**.

Solution:

(i)
$$\mathbf{A}\,.\,\mathbf{B} = \left(2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z\right).\left(-4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z\right)$$
$$\mathbf{A}\,.\,\mathbf{B} = (2)(-4) + (-3)(-2) + (1)(5) = -8 + 6 + 5 = 3$$

(ii)
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ -4 & -2 & 5 \end{vmatrix}$$
$$= [(-3)(5) - (1)(-2)]\mathbf{a}_x - [(2)(5) - (1)(-4)]\mathbf{a}_y + [(2)(-2) - (-3(-4)]\mathbf{a}_z$$
$$\mathbf{A} \times \mathbf{B} = -13\mathbf{a}_x - 14\mathbf{a}_y - 16\mathbf{a}_z$$

(iii)
$$\mathbf{A}\,.\,\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$$
$$\mathbf{A}\,.\,\mathbf{B} = 3$$
$$|\mathbf{A}| = \sqrt{(2)^2 + (-3)^2 + (1)^2} = 3.74 \quad \text{and} \quad |\mathbf{B}| = \sqrt{(-4)^2 + (-2)^2 + (5)^2} = 6.7$$
$$3 = (3.74)(6.7)\cos\theta$$
$$\cos\theta = \left(\frac{3}{(3.74)(6.7)}\right) = 0.119 \quad \Rightarrow \quad \theta = \cos^{-1} 0.119 = 83.16^o$$

or
$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$$
$$|\mathbf{A} \times \mathbf{B}| = \sqrt{(-13)^2 + (-14)^2 + (16)^2} = 24.9$$
$$24.9 = (3.74)(6.7)\sin\theta$$
$$\sin\theta = \frac{24.9}{(3.74)(6.7)} = 0.993 \quad \Rightarrow \quad \theta = \sin^{-1}(0.993) = 83.2^o$$

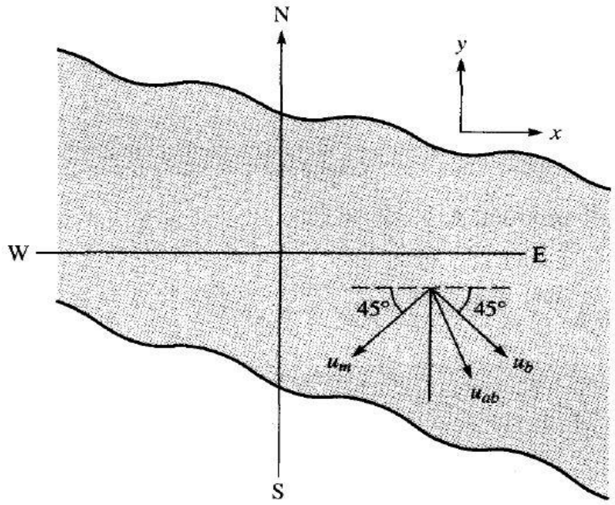
Example 1.6:[2]

A river flows southeast at 10 km/hr and a boat flows upon it with its bow pointed in the direction of travel. A man walks upon the deck at 2 km/hr in a direction to the right and perpendicular to the direction of the boat's movement. Find the velocity of the man with respect to the earth.

Solution:

Consider Figure bellow as illustrating the example. The velocity of the boat is

$$\mathbf{u}_b = 10(\cos 45\,\mathbf{a}_x - \sin 45\,\mathbf{a}_y) = 7.071\mathbf{a}_x - 7.071\mathbf{a}_y \text{ km/hr}$$



The velocity of the man with respect to the boat (relative velocity) is

$$\mathbf{u}_m = 2(-\cos 45\,\mathbf{a}_x - \sin 45\,\mathbf{a}_y) = -1.414\mathbf{a}_x - 1.414\mathbf{a}_y \text{ km/hr}$$

Thus the absolute velocity of the man is

$$\mathbf{u}_{ab} = \mathbf{u}_m + \mathbf{u}_b = 5.657\mathbf{a}_x - 8.485\mathbf{a}_y$$

$$|\mathbf{u}_{ab}| = 10.2\angle{-56.3^{\circ}}$$

that is, 10.2 km/hr at 56.3° south of east.

Example 1.7:[2]

Three field quantities are given by $\mathbf{P} = 2\mathbf{a}_x - \mathbf{a}_z$, $\mathbf{Q} = 2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$ and $\mathbf{R} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$

Determine

(a) $(\mathbf{P}+\mathbf{Q}) \times (\mathbf{P}-\mathbf{Q})$ (b) $\mathbf{Q}\,.\,\mathbf{R} \times \mathbf{P}$ (c) $\mathbf{P}.\,\mathbf{Q} \times \mathbf{R}$ (d) $\sin\theta_{QR}$ (e) $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$

(f) A unit vector perpendicular to both $\mathbf{Q}$ and $\mathbf{R}$ (g) The component of $\mathbf{P}$ along $\mathbf{Q}$

Solution:

(a)
$$(\mathbf{P}+\mathbf{Q}) \times (\mathbf{P}-\mathbf{Q}) = \mathbf{P} \times (\mathbf{P}-\mathbf{Q}) + \mathbf{Q} \times (\mathbf{P}-\mathbf{Q})$$

$$= \mathbf{P}\times\mathbf{P} - \mathbf{P}\times\mathbf{Q} + \mathbf{Q}\times\mathbf{P} - \mathbf{Q}\times\mathbf{Q} = 0 + \mathbf{Q}\times\mathbf{P} + \mathbf{Q}\times\mathbf{P} - 0$$

$$= 2\mathbf{Q}\times\mathbf{P} = 2\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 2\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z$$

(b) The only way $\mathbf{Q}\,.\,(\mathbf{R}\times\mathbf{P})$ makes sense is

$$\mathbf{Q}\,.\,(\mathbf{R}\times\mathbf{P}) = (2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z)\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$\mathbf{Q}\,.\,(\mathbf{R}\times\mathbf{P}) = \left(2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right).\left(3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z\right) = 6 - 4 + 12 = 14$$

Alternatively: $\mathbf{Q}\,.\,(\mathbf{R}\times\mathbf{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 6 + 0 - 2 + 12 - 0 - 2 = 14$

(c) $$\mathbf{P}.\,(\mathbf{Q}\times\mathbf{R}) = \mathbf{Q}\,.\,(\mathbf{R}\times\mathbf{P}) = 14$$

(d)

$$\sin\theta_{QR} = \frac{|\mathbf{Q}\times\mathbf{R}|}{|\mathbf{Q}||\mathbf{R}|}$$

$$\mathbf{Q}\times\mathbf{R} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 5\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z$$

$$|\mathbf{Q}\times\mathbf{R}| = \sqrt{(5)^2 + (2)^2 + (-4)^2} = 6.7$$

$$|\mathbf{Q}| = \sqrt{(2)^2 + (-1)^2 + (2)^2} = 3 \qquad \text{and} \qquad |\mathbf{R}| = \sqrt{(2)^2 + (-3)^2 + (1)^2} = 3.74$$

$$\sin\theta_{QR} = \frac{|\mathbf{Q}\times\mathbf{R}|}{|\mathbf{Q}||\mathbf{R}|} = \frac{6.7}{(3)(3.74)} = 0.597$$

(e) $$(\mathbf{Q}\times\mathbf{R}) = \left(5\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z\right)$$

$$\mathbf{P}\times(\mathbf{Q}\times\mathbf{R}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 0 & -1 \\ 5 & 2 & -4 \end{vmatrix} = 2\mathrm{a}_x + 3\mathrm{a}_y + 4\mathrm{a}_z$$

(f) A unit vector perpendicular to both **Q** and **R** is given by

$$\mathbf{a} = \mp\frac{\mathbf{Q}\times\mathbf{R}}{|\mathbf{Q}\times\mathbf{R}|} = \frac{5\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z}{6.7} = \mp\left(0.745\mathbf{a}_x + 0.298\mathrm{a}_y - 0.596\mathrm{a}_z\right)$$

(g) The component of **P** along **Q** is

$$\mathbf{P}_Q = |\mathbf{P}|\cos\theta_{PQ}\,\mathbf{a}_Q = \left(\mathbf{P}.\mathbf{a}_Q\right)\mathbf{a}_Q = \frac{(\mathbf{P}.Q)Q}{|Q|^2}$$

$$\mathbf{P}_Q = \frac{\left[(2\mathbf{a}_x - \mathbf{a}_z).\left(2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right)\right]\left(2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right)}{\left(\sqrt{(2)^2 + (-1)^2 + (2)^2}\right)^2}$$

$$\mathbf{P}_Q = \frac{2}{9}\left(2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right) = 0.444\mathbf{a}_x - 0.222\mathbf{a}_y + 0.444\mathbf{a}_z$$

1.8 OTHER COORDINATE SYSTEM

1.8.1 Circular cylindrical coordinate system (ρ, ϕ, z)

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry. A point P in cylindrical coordinates is represented as (ρ, ϕ, z) shown in Fig. 1.10. Observe Fig. 1.10 define each space variable:

- ρ is the radius of the cylinder passing through P or the radial distance from the $z-$axis;
- ϕ called the azimuthal angle is measured from the $x-$axis in the $xy-$plane;
- z is the same as in the Cartesian system.

According to these definitions, the ranges of the variables are:

- $0 \leq \rho < \infty$
- $0 \leq \phi < 2\pi$
- $-\infty < z < \infty$

A vector **A** in cylindrical coordinates can be written as

$$\mathbf{A} = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$$

where $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ are unit vectors in the ρ, ϕ, and $z-$directions as illustrated in Fig. 1.10. Note that $\mathbf{a}_\phi$ is not in degrees; it assumes the unit vector of **A**.

For example if a force of 10 N acts on a particle in a circular motion, the force may be represented as $\mathbf{F} = 10\ \mathbf{a}_\phi\ N$. In this case, $\mathbf{a}_\phi$ is in newtons.

The magnitude of **A** is:

$$|\mathbf{A}| = \sqrt{(A_\rho)^2 + (A_\phi)^2 + (A_z)^2}$$

Notice that the unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ are mutually perpendicular because our coordinate system is orthogonal; $\mathbf{a}_\rho$ points in the direction of increasing ρ; $\mathbf{a}_\phi$ in the direction of increasing ϕ; and $\mathbf{a}_z$ in the positive $z-direction$. Thus,

- $\mathbf{a}_\rho . \mathbf{a}_\phi = 0$
- $\mathbf{a}_\rho . \mathbf{a}_\rho = 1$
- $\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$
- $\mathbf{a}_\phi . \mathbf{a}_z = 0$
- $\mathbf{a}_\phi . \mathbf{a}_\phi = 1$
- $\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho$
- $\mathbf{a}_z . \mathbf{a}_\rho = 0$
- $\mathbf{a}_z . \mathbf{a}_z = 1$
- $\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$

where equations above are obtained in cyclic permutation (see Fig. 1.9).

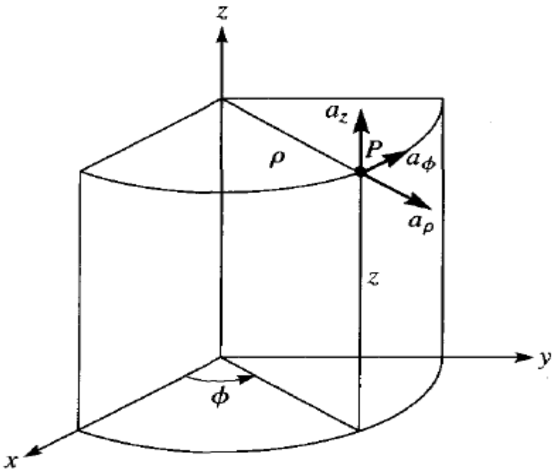


Fig. 1.10 point P and unit vectors in the cylindrical coordinate system.[2]

1.8.2 THE SPHERICAL COORDINATE SYSTEM (r, θ, ϕ)

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry. A point P can be represented as (r, θ, ϕ) and is illustrated in Fig. 1.11. From Fig. 1.11, we notice that:

- r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P;
- θ called the colatitude is the angle between the $z-$ axis and the position vector of P; and
- ϕ is measured from the $x-$ axis (the same as in cylindrical coordinate system).

According to these definitions, the ranges of the variables are

- $0 \le r < \infty$
- $0 \le \theta \le \pi$
- $0 \le \phi < 2\pi$

A vector **A** in spherical coordinates may be written as

$$\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi$$

where $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are unit vectors along the r, θ, and $\phi - directions$.

The magnitude of **A** is

$$|\mathbf{A}| = \sqrt{(A_r)^2 + (A_\theta)^2 + (A_\phi)^2}$$

The unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are mutually orthogonal; $\mathbf{a}_r$ being directed along the radius or in the direction of increasing r, $\mathbf{a}_\theta$ in the direction of increasing θ and $\mathbf{a}_\phi$ in the direction of increasing ϕ. Thus

- $\mathbf{a}_r \, . \, \mathbf{a}_\theta = 0$
- $\mathbf{a}_\theta \, . \, \mathbf{a}_\phi = 0$
- $\mathbf{a}_\phi \, . \, \mathbf{a}_r = 0$
- $\mathbf{a}_r \, . \, \mathbf{a}_r = 1$
- $\mathbf{a}_\theta \, . \, \mathbf{a}_\theta = 1$
- $\mathbf{a}_\phi \, . \, \mathbf{a}_\phi = 1$
- $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$
- $\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r$
- $\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$

where equations above are obtained in cyclic permutation (see Fig. 1.9).[2]

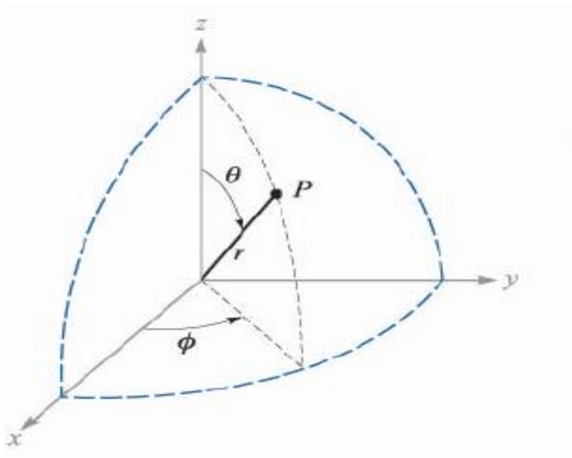


Fig. 1.11 the three spherical coordinates.

1.9 THE TRANSFORMATION BETWEEN COORDINATES

1.9.1 between Cartesian and cylindrical coordinate systems

The variables of the rectangular and cylindrical coordinate systems are easily related to each other. With reference to Fig. 1.12, we see that

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

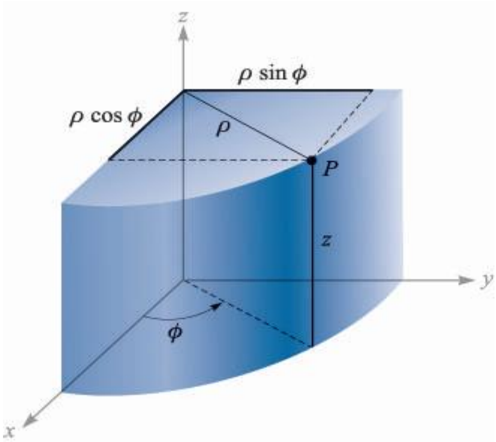


Fig. 1.12 the relationship between the cartesian variables x, y, z and the cylindrical coordinate variables ρ, ϕ, z.

We may express the cylindrical variables in terms of $x, y,$ and z:

$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$

$\phi = tan^{-1} \frac{y}{x}$

$z = z$

A vector function in one coordinate system, however, requires two steps in order to transform it to another coordinate system, because a different set of component vectors is generally required. That is, we may be given a cartesian vector

$\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$

where each component is given as a function of $x, y,$ and z, and we need a vector in cylindrical coordinates

$\mathbf{A} = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$

where each component is given as a function of ρ, ϕ, and z.

To find any desired component of a vector, we recall from the discussion of the dot product that a component in a desired direction may be obtained by taking the dot product of the vector and a unit vector in the desired direction. Hence,

$$A_\rho = \mathbf{A} \, . \, \mathbf{a}_\rho$$

$$A_\phi = \mathbf{A} \, . \, \mathbf{a}_\phi$$

Expanding these dot products, we have

$$A_\rho = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z).\mathbf{a}_\rho = A_x\mathbf{a}_x.\mathbf{a}_\rho + A_y\mathbf{a}_y.\mathbf{a}_\rho$$

$$A_\phi = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z).\mathbf{a}_\phi = A_x\mathbf{a}_x.\mathbf{a}_\phi + A_y\mathbf{a}_y.\mathbf{a}_\phi$$

$$A_z = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z).\mathbf{a}_z = A_z\mathbf{a}_z.\mathbf{a}_z = A_z$$

In order to complete the transformation of the components, it is necessary to know the dot products $\mathbf{a}_x.\mathbf{a}_\rho$, $\mathbf{a}_y.\mathbf{a}_\rho$, $\mathbf{a}_x.\mathbf{a}_\phi$, and $\mathbf{a}_y.\mathbf{a}_\phi$. Applying the definition of the dot product, we see that since we are concerned with unit vectors, the result is merely the cosine of the angle between the two unit vectors in question. Referring to Fig. 1.12 and thinking mightily, we identify the angle between $\mathbf{a}_x$ and $\mathbf{a}_\rho$ as ϕ, and thus $\mathbf{a}_x.\mathbf{a}_\rho = \cos\phi$, but the angle between $\mathbf{a}_y$ and $\mathbf{a}_\rho$ is $90^\circ - \phi$, and $\mathbf{a}_y.\mathbf{a}_\rho = \cos(90^\circ - \phi) = \sin\phi$. The remaining dot products of the unit vectors are found in a similar manner, and the results are tabulated as functions of ϕ in Table 1.1.[1]

Table 1.1.

Dot product	a_ρ	a_ϕ	a_z
a_x	$\cos\phi$	$-\sin\phi$	0
a_y	$\sin\phi$	$\cos\phi$	0
a_z	0	0	1

1.9.1 between Cartesian and spherical coordinate system

The transformation of scalars from the cartesian to the spherical coordinate system is easily made by using Fig. 1.11 to relate the two sets of variables:

$x = rsin\theta cos\phi$

$y = rsin\theta sin\phi\phi$

$z = rcos\theta$

Table 1.2.

Dot product	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	$\cos\theta$	$-\sin\theta$	0

The transformation in the reverse direction is achieved by

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \qquad (0^o \leq \theta \leq 180)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

The transformation of vectors requires the determination of the products of the unit vectors in cartesian and spherical coordinates. We work out these products from Fig. 1.11 and a pinch of trigonometry. Since the dot product of any spherical unit vector with any cartesian unit vector is the component of the spherical vector in the direction of the cartesian vector, the dot products with $\mathbf{a}_z$ are found to be

$\mathbf{a}_z \,.\, \mathbf{a}_r = \cos\theta$

$\mathbf{a}_z \,.\, \mathbf{a}_\theta = -\sin\theta$

$\mathbf{a}_z \,.\, \mathbf{a}_\phi = 0$

The dot products involving $\mathbf{a}_x$ and $\mathbf{a}_y$ require first the projection of the spherical unit vector on the xy $-$plane and then the projection onto the desired axis. For example $\mathbf{a}_r \,.\, \mathbf{a}_x$ is obtained by projecting $\mathbf{a}_r$ onto the $xy-$plane, giving $sin\theta$, and then projecting $sin\theta$ on the $x-$axis, which yields $sin\theta\ cos\phi$. The other dot products are found in a like manner, all are shown in Table 1.2.

1.10 Differential, volume, surface, and line elements

1.10.1 In the cartesian coordinate system

If we visualize three planes intersecting at the general point P, whose coordinates are x, y and z, we may increase each coordinate value by a differential amount and obtain three slightly displaced planes intersecting at point P' whose coordinates are $x + dx$, $y + dy$, and $z + dz$. The six planes define a rectangular parallelepiped whose

(1) Differential displacement is given by

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

(2) Differential normal area (the surfaces areas $d\mathbf{S}$) is given by

$$d\mathbf{S} = dy\,dz\,\mathbf{a}_x;$$

$$d\mathbf{S} = dx\,dz\,\mathbf{a}_y;$$

$$d\mathbf{S} = dx\,dy\,\mathbf{a}_z;$$

(3) Differential volume is given by

$$dv = dx\,dy\,dz;$$

The volume element is shown in Fig. 1.13; point P' is indicated, but point P is located at the only invisible corner.

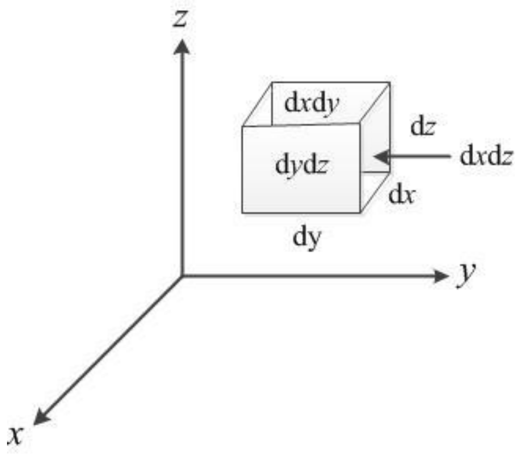


Fig. 1.13 the differential volume element in Cartesian coordinates; dx,dy, and dz: are, in general, independent differentials.

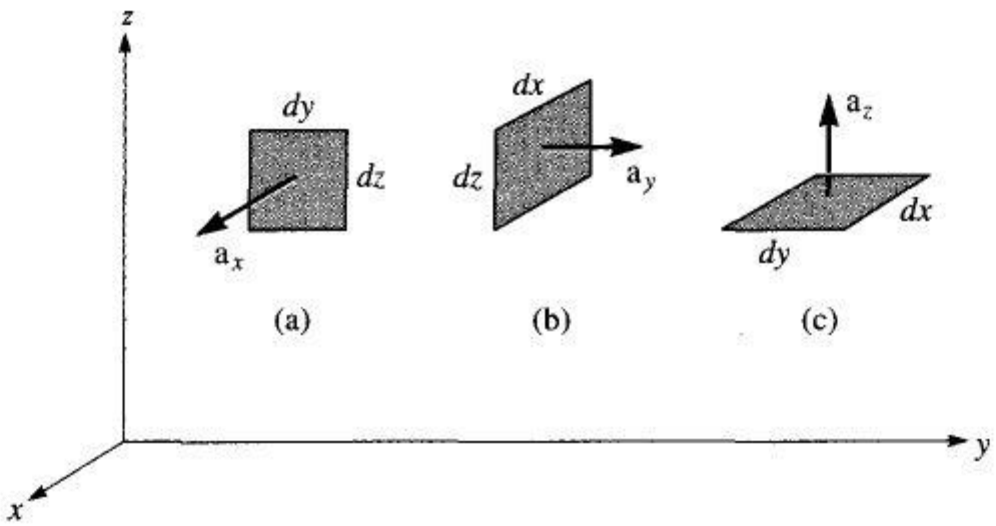


Fig. 1.14 Differential normal areas in Cartesian coordinates

1.10.2 In the cylindrical coordinates system

A differential volume element in cylindrical coordinates may be obtained by increasing ρ, ϕ, and z by the differential increments $d\rho$, $d\phi$ and dz. The two cylinders of radius ρ and $\rho + d\rho$, the two radial planes at angles ϕ and $\phi + d\phi$ and the two "horizontal" planes at "elevations" z and $z + dz$ now enclose a small volume, as shown in Fig. 1.15, having the shape of a truncated wedge. As the volume element becomes very small, its shape approaches that of a rectangular parallelepiped having sides of length $d\rho$, $\rho d\phi$ and dz. Note that $d\rho$ and dz are dimensionally lengths, but $d\phi$ is not; $\rho d\phi$ is the length.

(1) Differential displacement is given by

$$d\mathbf{L} = d\rho\, \mathbf{a}_\rho + \rho d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z$$

(2) Differential normal area is given by

$$d\mathbf{S} = \rho\, d\phi\, dz\, \mathbf{a}_\rho;$$

$$d\mathbf{S} = d\rho\, dz\, \mathbf{a}_\phi;$$

$$d\mathbf{S} = \rho\, d\rho\, d\phi\, \mathbf{a}_z;$$

(3) Differential volume is given by

$$dv = \rho\, d\rho\, d\phi\, dz.$$

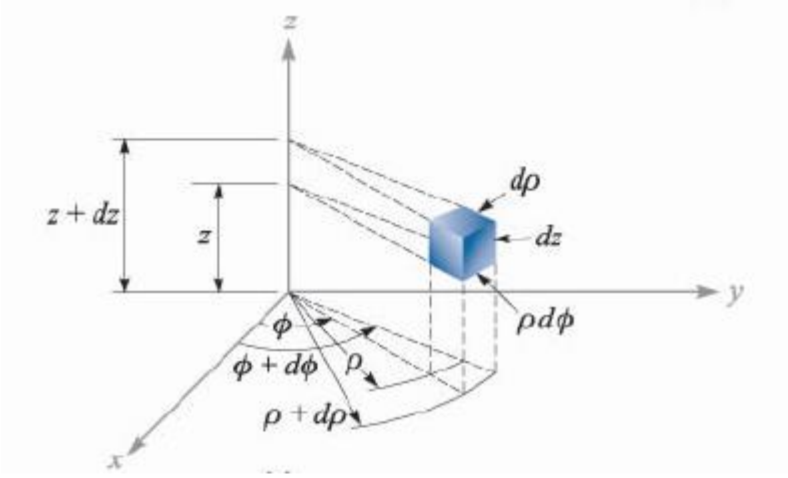


Fig. 1.15 the differential volume unit in the circular cylindrical coordinate system; $d\rho$, $\rho d\phi$, and dz: are all elements of length.

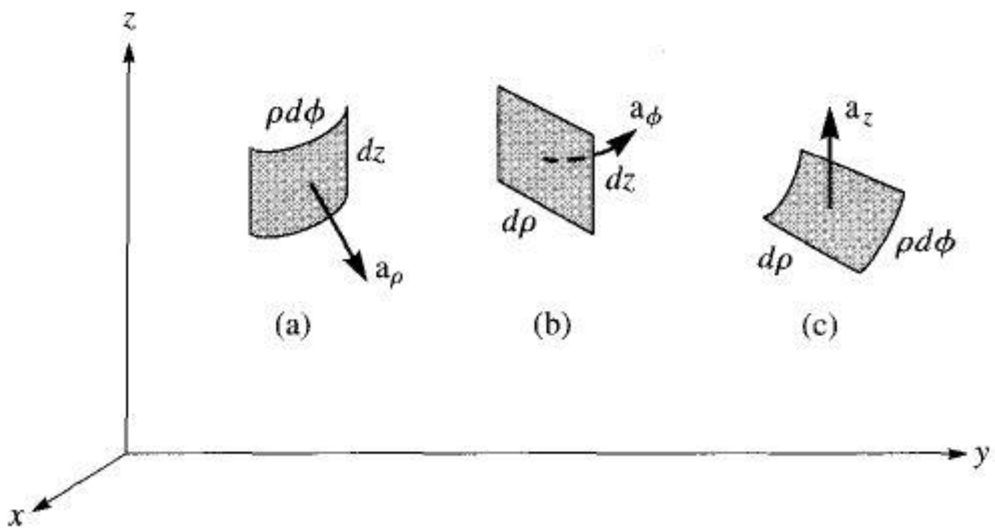


Fig. 1.16 Differential normal areas in cylindrical coordinates

1.10.3 In the spherical coordinate system

A differential volume element may be constructed in spherical coordinates by increasing r, θ, and ϕ by dr, $d\theta$, and $d\phi$, as shown in Fig. 1.17. The distance between the two spherical surfaces of radius r and $r + dr$ is dr; the distance between the two cones having generating angles of θ and $\theta + d\theta$ is $rd\theta$; and the distance between the two radial planes at angles ϕ and $\phi + d\phi$ is found to be $r \sin\theta \; d\phi$.

(1) The differential displacement is

$$d\mathbf{L} = dr\, \mathbf{a}_r + rd\theta\, \mathbf{a}_\theta + r\sin\theta\, d\phi\, \mathbf{a}_\phi$$

(2) The differential normal area is

$$d\mathbf{S} = r^2 \, sin\theta \, d\theta \, d\phi \, \mathbf{a}_r;$$

$$d\mathbf{S} = r \, sin\theta \, dr \, d\phi \, \mathbf{a}_\theta;$$

$$d\mathbf{S} = rdrd\theta \, \mathbf{a}_\phi;$$

(3) The differential volume is

$$dv = r^2 \, sin \, \theta \, dr \, d\theta \, d\phi$$

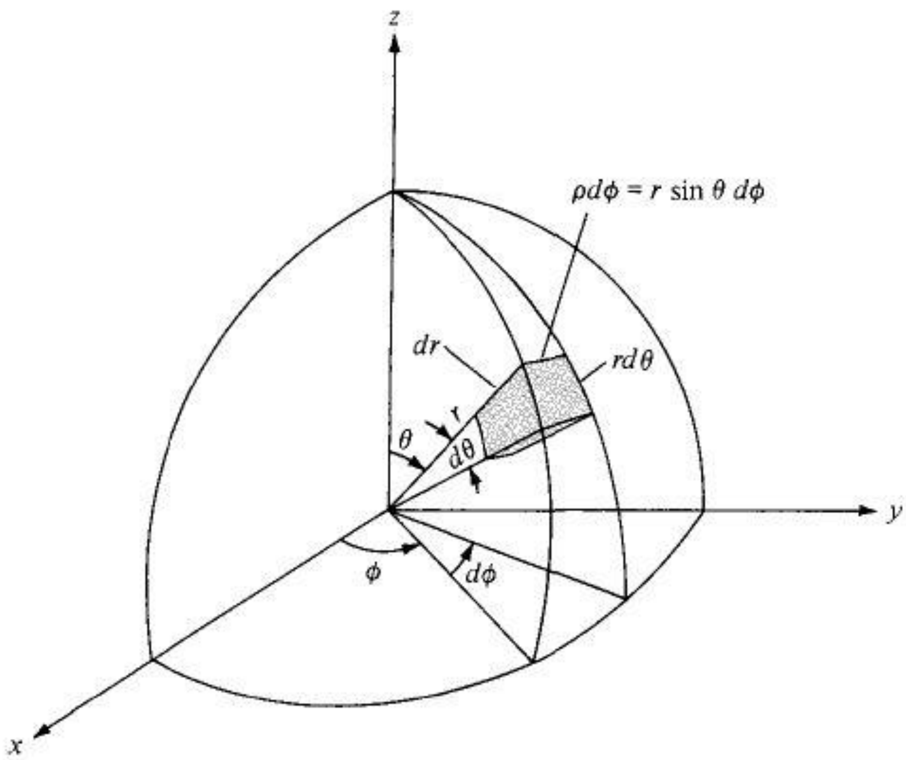


Fig. 1.17 the differential volume element in the spherical coordinate system.

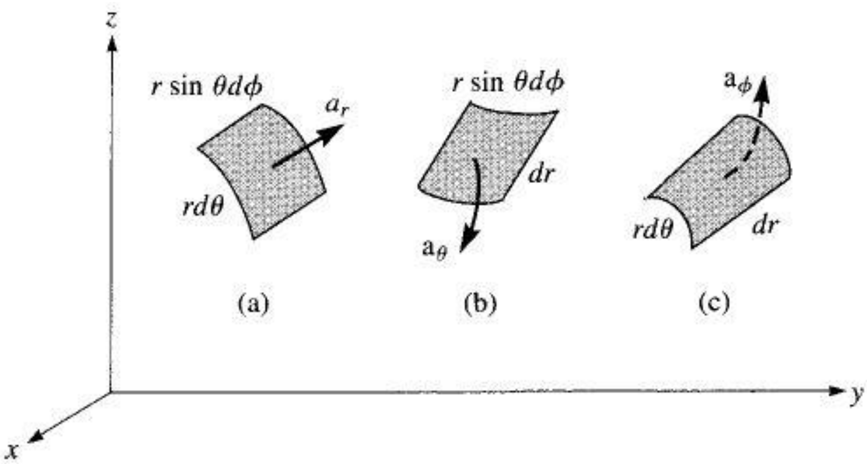


Fig. 1.18 Differential normal areas in spherical coordinates

Example 1.8:[1]

Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

Solution. The new components are

$$B_\rho = \mathbf{B}.\mathbf{a}_\rho = (y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z).\mathbf{a}_\rho = y(\mathbf{a}_x.\mathbf{a}_\rho) - x(\mathbf{a}_y.\mathbf{a}_\rho)$$

$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$$B_\phi = \mathbf{B}.\mathbf{a}_\phi = (y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z).\mathbf{a}_\phi = y(\mathbf{a}_x.\mathbf{a}_\phi) - x(\mathbf{a}_y.\mathbf{a}_\phi)$$

$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

$$\therefore \qquad \mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$$

Example 1.9:[1]

Transform the vector $\mathbf{G} = (xz/y)\mathbf{a}_x$ into spherical components and variables.

Solution.

$$G_r = \mathbf{G}.\mathbf{a}_r = \left(\frac{xz}{y}\mathbf{a}_x\right).\mathbf{a}_r = \frac{xz}{y}(\mathbf{a}_x.\mathbf{a}_r) = \frac{xz}{y}\sin\theta\cos\phi$$

$$G_r = r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi} = r\cos\theta\cos\phi\sin\theta\cot\phi$$

$$G_\theta = \mathbf{G}.\mathbf{a}_\theta = \left(\frac{xz}{y}\mathbf{a}_x\right).\mathbf{a}_\theta = \frac{xz}{y}(\mathbf{a}_x.\mathbf{a}_\theta) = \frac{xz}{y}\cos\theta\cos\phi$$

$$G_\theta = r\cos^2\theta\frac{\cos^2\phi}{\sin\phi} = r\cos^2\theta\cos\phi\cot\phi$$

$$G_\phi = \mathbf{G}.\mathbf{a}_\phi = \left(\frac{xz}{y}\mathbf{a}_x\right).\mathbf{a}_\phi = \frac{xz}{y}(\mathbf{a}_x.\mathbf{a}_\phi) = \frac{xz}{y}(-\sin\phi)$$

$$G_\phi = -r\cos\theta\cos\phi$$

$$\therefore \qquad \mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$

Example 1.10:[2]

Given point $P(-2, 6, 3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x+z)\mathbf{a}_y$, express P and $\mathbf{A}$ in cylindrical and spherical coordinates. Evaluate $\mathbf{A}$ at P in the Cartesian, cylindrical, and spherical systems.

Solution:

At point P: $x = -2$, $y = 6$, $z = 3$. Hence, $z = 3$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (6)^2} = 6.3 \;:\; \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = -70.56^o$$

$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-2)^2 + (6)^2 + (3)^2} = 7$

$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{3}{\sqrt{(-2)^2+(6)^2+(3)^2}}\right) = 64.62^{\text{o}}$

Thus,

$P(-2, 6, 3)$ in Cartesian coordinate

$P(6.32, -70.56^{\text{o}}, 3)$ in cylindrical coordinate

$P(7, 64.62^{\text{o}}, -70.56^{\text{o}})$ in spherical coordinate

In the Cartesian coordinate system, **A** at $P(-2, 6, 3)$ is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

Vector **A** in the cylindrical coordinate system

$$A_\rho = \mathbf{A}.\mathbf{a}_\rho = [y\mathbf{a}_x + (x+z)\mathbf{a}_y].\mathbf{a}_\rho = y\cos\phi + (x+z)\sin\phi$$

$$A_\phi = \mathbf{A}.\mathbf{a}_\phi = [y\mathbf{a}_x + (x+z)\mathbf{a}_y].\mathbf{a}_\phi = -y\sin\phi + (x+z)\cos\phi$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, and substituting in equations above, we get

$$A_\rho = \rho\sin\phi\cos\phi + (\rho\cos\phi + z)\sin\phi$$

$$A_\phi = -\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi$$

$$\therefore \quad \mathbf{A} = [\rho\sin\phi\cos\phi + (\rho\cos\phi + z)\sin\phi]\mathbf{a}_\rho + [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\mathbf{a}_\phi$$

In the cylindrical coordinate system, **A** at $P(6.32, -70.56^{\text{o}}, 3)$ is

$$\mathbf{A} = -0.9487\mathbf{a}_\rho - 6.008\mathbf{a}_\phi$$

Vector **A** in the spherical coordinate system

$$A_r = \mathbf{A}.\mathbf{a}_r = [y\mathbf{a}_x + (x+z)\mathbf{a}_y].\mathbf{a}_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$$

$$A_\theta = \mathbf{A}.\mathbf{a}_\theta = [y\mathbf{a}_x + (x+z)\mathbf{a}_y].\mathbf{a}_\theta = y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$$

$$A_\phi = \mathbf{A}.\mathbf{a}_\phi = [y\mathbf{a}_x + (x+z)\mathbf{a}_y].\mathbf{a}_\phi = -y\sin\phi + (x+z)\cos\phi$$

But $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. Substituting in equations above, we get

$$A_r = r\sin^2\theta\sin\phi\cos\phi + (r\sin\theta\cos\phi + r\cos\theta)\sin\theta\sin\phi$$

$$A_\theta = r\sin\theta\cos\theta\sin\phi\cos\phi + (r\sin\theta\cos\phi + r\cos\theta)\cos\theta\sin\phi$$

$$A_\phi = -r\sin\theta\sin^2\phi + (r\sin\theta\cos\phi + r\cos\theta)\cos\phi$$

$$\therefore \qquad \mathbf{A} = r[\sin^2\theta \sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\mathbf{a}_r$$
$$+ r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\mathbf{a}_\theta$$
$$+ r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\mathbf{a}_\phi$$

In the spherical coordinate system, $\mathbf{A}$ at $P(7, 64.62^o, -70.56^o)$ is

$$\mathbf{A} = -0.8571\mathbf{a}_r - 0.4066\mathbf{a}_\theta - 6.008\mathbf{a}_\phi$$

Example 1.11:[2]

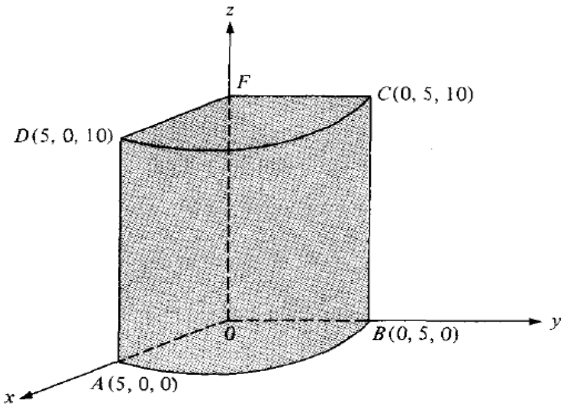
Consider the object shown in Figure below. Calculate (a) The distance BC (b) The distance CD (c) The surface area $ABCD$ (d) The surface area ABO (e) The surface area $AOFD$ (f) The volume $ABDCFO$.

Solution:

The points are transformed from Cartesian to cylindrical coordinates as follows:

$A(5,0,0)$ to $A(5, 0^o, 0)$; $B(0,5,0)$ to $B(5, \pi/2, 0)$; $C(0,5,10)$ to $C(5, \pi/2, 10)$

$D(5,0,10)$ to $D(5, 0^o, 10)$



(a) Along BC, $dl = dz$; hence, $BC = \int dl = \int_0^{10} dz = 10$

(b) Along CD, $dl = \rho d\phi$ and $\rho = 5$, so, $CD = \int dl = \int_0^{\pi/2} \rho d\phi = 2.5\pi$

(c) For $ABCD$, $dS = \rho d\phi dz$, $\rho = 5$, so, $ABCD = \int dS = \int_{\phi=0}^{\pi/2}\int_{z=0}^{10} \rho dz d\phi = 25\pi$

(d) For ABO, $dS = \rho d\phi d\rho$, $z = 0$, so, $ABO = \int dS = \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{5} \rho d\rho d\phi = 6.25\pi$

(e) For $AOFD$, $dS = d\rho dz$, $\phi = 0^o$, so, $AOFD = \int dS = \int_{\rho=0}^{5}\int_{z=0}^{10} d\rho dz = 50$

(f) For volume $ABDCFO$, $dv = \rho d\rho d\phi dz$. Hence,

$$v = \int dv = \int_{\rho=0}^{5}\int_{\phi=0}^{\pi/2}\int_{z=0}^{10} \rho d\rho d\phi dz = 62.5\pi$$