2014-2015

ELECTROMAGNETIC FIELDS

"CHAPTER THREE: ELECTRIC FLUX DENSITY, GAUSS'S LAW, AND DIVERGENCE "

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ELECTRIC FLUX AND GAUSS'S LAW

3.1 NET CHARGE IN A REGION

With charge density defined as in chapter 2, it is possible to obtain the net charge contained in a specified volume by integration. From

$$dQ = \rho_v dv \quad \text{in (C)}$$

It follows that

$$Q = \int_v \rho_v dv \quad \text{in (C)} \tag{3.1}$$

ρ_v will not be constant throughout the volume v.

Example 2.1: Find the charge in the volume $1 \leq r \leq 2$ m in the spherical coordinate system, if $\rho_v = \frac{5\cos^2\phi}{r^4}$ $(^{\mathrm{C}}/_{\mathrm{m}^3})$

Solution:

$$Q = \int_v \rho_v dv$$

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_1^{2}\left(\frac{5\cos^2\phi}{r^4}\right) r^2 \sin\theta \, dr d\theta d\phi = 5\pi \text{ C}$$

3.2 ELECTRIC FLUX AND FLUX DENSITY

The electric flux due to the electric field **E** originates (emanates) from positive charge and terminates on negative charge. In the absence of negative charge, the flux ψ terminates at infinity. Also by definition, one coulomb of electric charge gives rise to one coulomb of electric flux. Hence

$$\psi = Q \quad \text{in (C)} \tag{3.2}$$

In Fig. 3.1 the flux lines leave $+Q$ and terminate on $-Q$. This assumes that the two charges are of equal magnitude. The case of positive charge with no negative charge in the region is illustrated in Fig. 3.2. Here the flux lines are equally spaced throughout the solid angle, and reach out toward infinity.

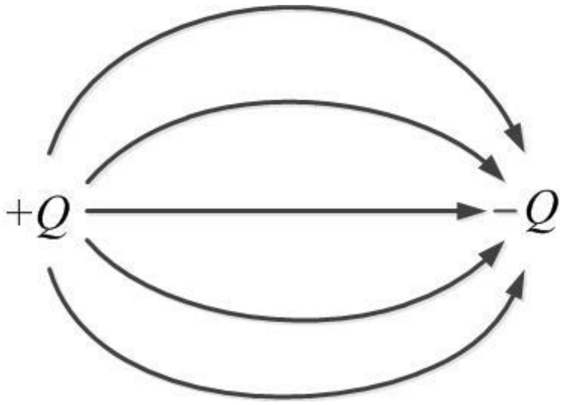

Fig. (3.1)

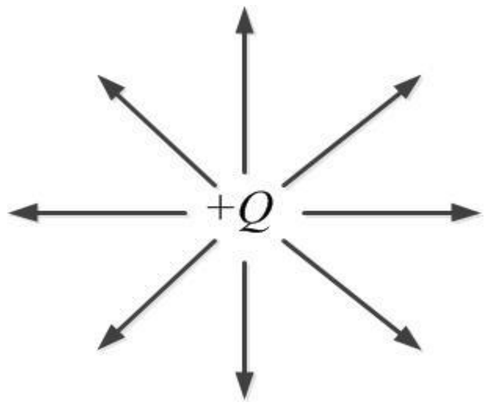

Fig. (3.2)

The electric flux ψ is a scalar quantity, while the electric flux density, **D**, is a vector field which takes its direction from the lines of flux. If in the neighborhood of point P the lines of flux have the direction of the unit vector **a** Fig. 3.3 and if an amount of flux $d\psi$ crosses the differential area, dS, which is normal to **a**. then the electric flux density at P is:

$$\mathbf{D} = \frac{d\psi}{dS}\mathbf{a} \quad \text{in } {}^{\mathrm{C}}/_{\mathrm{m}^2} \tag{3.3}$$

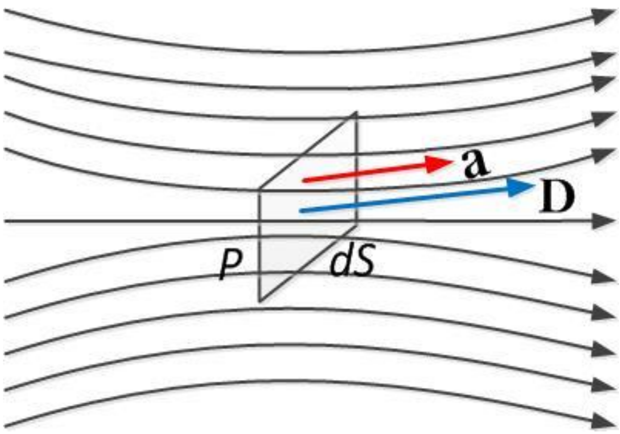

Fig. 3.3

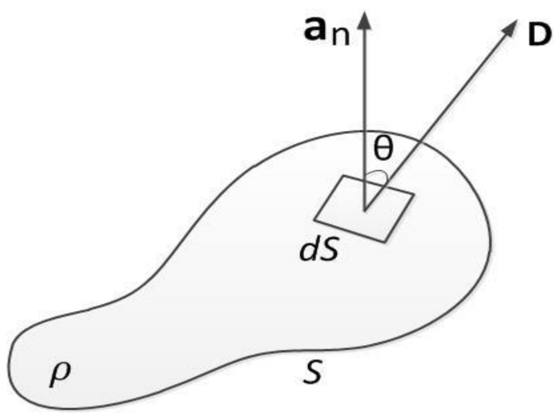

Fig. 3.4

A volume charge distribution of density ρ_v $(\mathrm{C/m^3})$ is shown enclosed by surface S in Fig. 3.4. Since each coulomb of charge Q has, by definition, one coulomb of flux, ψ. It follows that the net flux crossing the closed surface S is an exact measure of the net charge enclosed. However, the density **D** may vary in magnitude and direction from point to point of S; in general, **D** will not be along the normal to S. If at the surface element, dS, **D** makes an angle θ with the normal, then the differential flux crossing dS is given by

$$d\psi = D\, dS\, \cos\theta \tag{3.3a}$$

$$d\psi = \mathbf{D}.\, dS\, \mathbf{a}_n \tag{3.3b}$$

$$d\psi = \mathbf{D}.\, dS \tag{3.3c}$$

Where, dS is the vector surface element of magnitude dS and direction $\mathbf{a}_n$. The unit vector $\mathbf{a}_n$ always taken to point out of S, so that $d\psi$ is the amount of flux passing from the interior of S to the exterior of S through dS.

3.3 GAUSS'S LAW- MAXWELL'S EQUATION

Gauss's law constitutes one of the fundamental laws of electromagnetism.

Gauss's law states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface. Thus

$$\psi = Q_{enc} \tag{3.4}$$

Integration of the above expression for $d\psi$ over the closed surface S gives, since

$$\psi = \oint d\psi = \oint_s \mathbf{D}\,.\, dS = \text{Total charge enclosed } Q = \int_v \rho_v \,.\, dv \tag{3.5}$$

$$Q = \oint_s \mathbf{D}\,.\, dS = \int_v \rho_v \,.\, dv \tag{3.6}$$

It will be seen that a great deal of valuable information can be obtained the application of Gauss's law without actually carrying out the integration.

3.4. RELATION BETWEEN ELECTRIC FLUX DENSITY AND ELECTRIC FIELD INTENSITY

Consider a point charge Q (assumed positive, for simplicity) at the origin Fig. 3.5. If this is enclosed by a spherical surface of radius, r, then, by symmetry, $\mathbf{D}$ due to Q is of constant magnitude over the surface and is everywhere normal to the surface. Gauss's law then gives

$$Q = \oint_s \mathbf{D}\,.\, dS = D \oint_s dS = D(4\pi r^2) \tag{3.7}$$

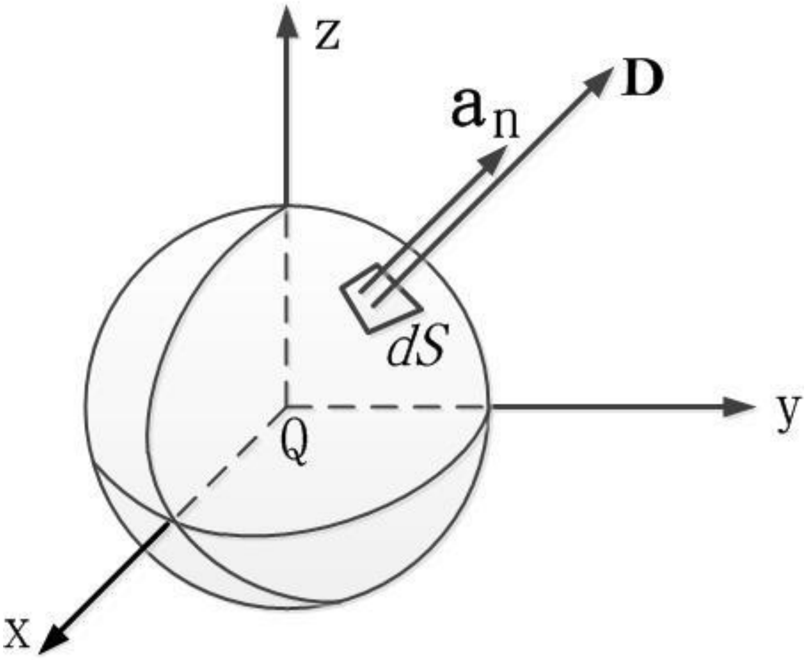

Fig. 3.5

From which

$$D = \frac{Q}{4\pi r^2} \tag{3.8}$$

Therefore

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_n = \frac{Q}{4\pi r^2}\mathbf{a}_r \tag{3.9}$$

But, the electric field intensity due to Q is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r \tag{3.10}$$

It follows that

$$\mathbf{D} = \epsilon_o \mathbf{E} \tag{3.11}$$

More generally, for any electric field in an isotropic medium of permittivity, ϵ.

$$\mathbf{D} = \epsilon\ \mathbf{E} \tag{3.12}$$

Thus, **D** and **E** fields will have exactly the same form, since they differ only by a factor which is a constant of the medium. While the electric field **E** due to a charge configuration is a function of permittivity, ϵ, the electric flux density **D** is not.[3]

3.5 SPECIAL GAUSSIAN SURFACES

The spherical surface used in the derivation of Section 3.4 was a special Gaussian surface in that satisfied the following defining conditions:

1. the surface is closed
2. at each point of the surface **D** is either normal or tangential to the surface
3. **D** has the some value at all points of the surface where **D** is normal

Example 3.2: Use special Gaussian surface to find **D** due to a uniform line change, ρ_l, (C/m).

Solution: Take the line charge as the z-axis of cylindrical coordinates Fig. 3.6. By cylindrical symmetry, **D** can only have an r component, and this component can only depend on r. Thus, the special Gaussian surface for this problem is a closed right circular cylinder whose axis is the z-axis Fig. 3.7. Applying Gauss's law.

$$Q = \oint_1 \mathbf{D} \,.\, dS + \oint_2 \mathbf{D} \,.\, dS + \oint_3 \mathbf{D} \,.\, dS$$

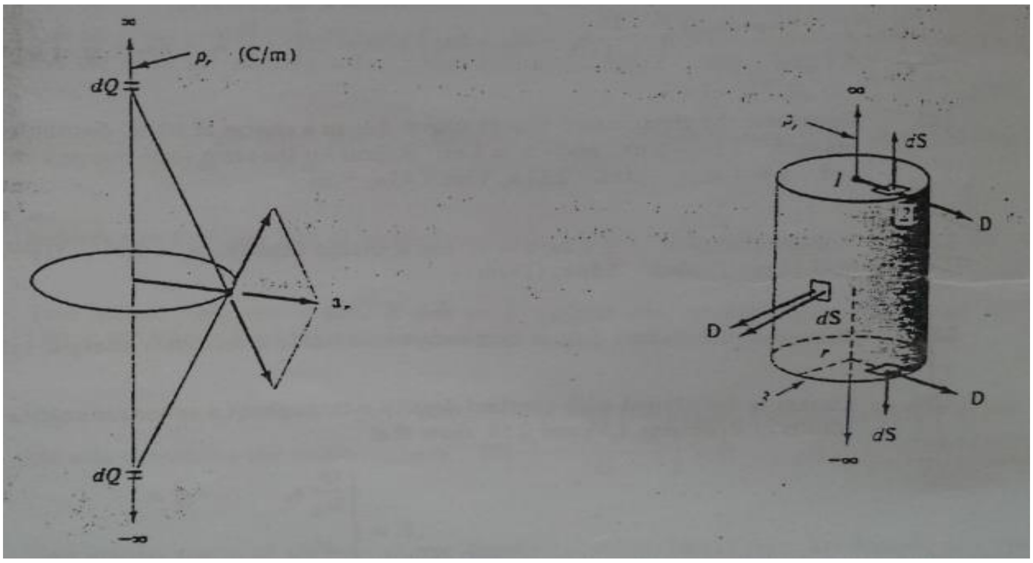

Fig. 3.6 Fig. 3.7

Over surfaces 1and 3, **D** and dS are orthogonal and so the integrals vanish. Over surface 2, **D** and dS are parallel or antiparallel if ρ_l is negative and **D** is constant because r is constant Thus,

$$Q = D \oint_{2} dS = D\,(2\pi r l)$$

Where, l is the length of the cylinder. But the enclosed charge is, $Q = \rho_l l$. Hence,[3]

$$D = \frac{\rho_l}{2\pi r}$$

and

$$\mathbf{D} = \frac{\rho_l}{2\pi r}\mathbf{a}_r$$

3.6 APPLICATIONS OF GAUSS'S LAW

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a Gaussian surface).

The surface is chosen such that **D** is normal or tangential to the Gaussian surface.

- When, **D** is normal to the surface, $\mathbf{D}.dS = D\,dS$ because **D** is constant on the surface.
- When **D** is tangential to the surface, $\mathbf{D}.dS = 0$.

Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution. We shall now apply these basic ideas to the following cases.

A. Point Charge

Suppose a point charge Q is located at the origin. To determine **D** at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a spherical surface centered at the origin is the Gaussian surface in this case and is shown in Fig. 3.8.

Since **D** is everywhere normal to the Gaussian surface, that is, $\mathbf{D} = D_r\mathbf{a}_r$

By applying Gauss's law gives,

$$\psi = Q_{enclosed}$$

$$Q = \oint_{s} \mathbf{D}\,.\,dS = D_r \oint_{s} dS = D_r\,(4\pi r^2) \qquad (3.13)$$

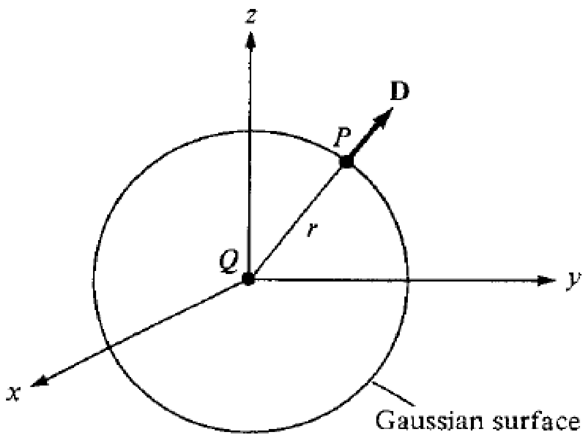

Fig. 3.8 Gaussian surface about a point charge.

$$\oint_s dS = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta d\phi = 4\pi r^2$$

$(4\pi r^2)$ is the surface area of the Gaussian surface. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r \qquad (3.14)$$

B. Infinite Line Charge

Suppose the infinite line of uniform charge ρ_l (C/m) lies along the z-axis. To determine **D** at a point P, we choose a cylindrical surface containing P to satisfy symmetry condition as shown in Fig. 3.9. **D** is constant on and normal to the cylindrical Gaussian surface; that is, $\mathbf{D} = D_\rho \mathbf{a}_\rho$.

Apply Gauss's law to an arbitrary length l of the line.

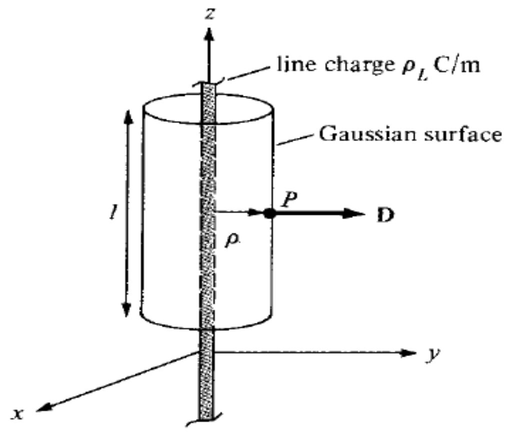

Fig. 3.9. Gaussian surface about an infinite line charge.

$$Q = \rho_l\, l = \oint_S \mathbf{D} \,.\, dS = D_\rho \oint_S dS = D_\rho\,(2\pi\rho l) \tag{3.15}$$

Where,

$$S = \oint_S dS = \int_{\phi=0}^{2\pi} \int_{z=0}^{l} \rho\, d\phi dz = 2\pi\rho l$$

$(2\pi\rho l)$ is the surface area of the Gaussian surface. Note that $\int \mathbf{D} \,.\, dS$ evaluated on the top and bottom surfaces of the cylinder is zero since **D** has no z-component; that means that **D** is tangential to those surfaces. Thus

$$\mathbf{D} = \frac{\rho_l}{2\pi\rho}\mathbf{a}_\rho \tag{3.16}$$

C. Infinite Sheet of Charge

Consider the infinite sheet of uniform charge ρ_s $(\mathrm{C/m^2})$ lying on the $z = 0$ plane (in the xy-plane). To determine **D** at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in Fig. 3.10. As **D** is normal to the sheet, $\mathbf{D} = D_z\mathbf{a}_z$.

Applying Gauss's law gives

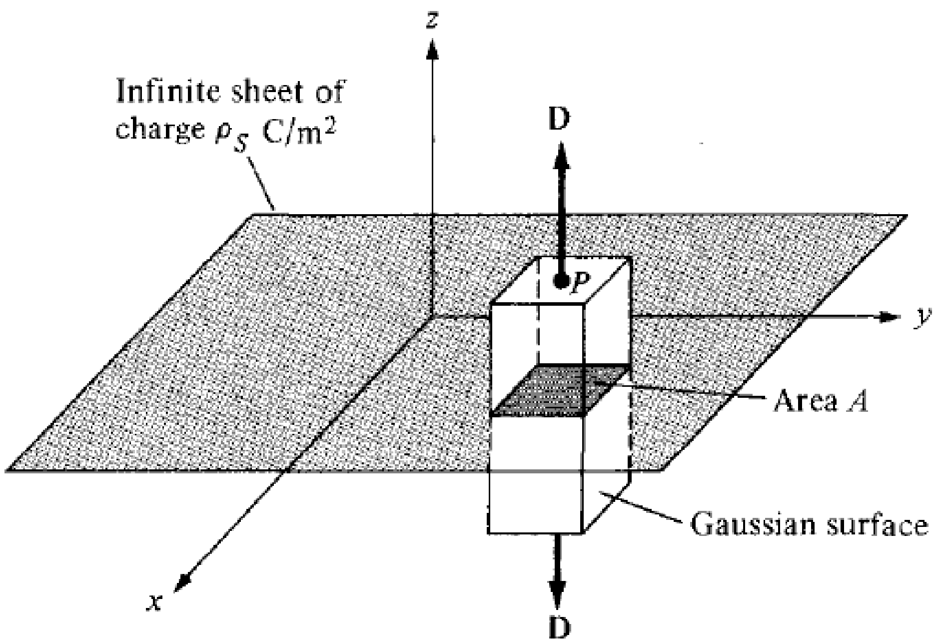

Fig. 3.10. Gaussian surface about an infinite line sheet of charge.

$$Q = \int \rho_s \, dS = \oint_s \mathbf{D} \, . \, dS = D_z \oint_s dS = D_z \left[\int_{top} dS + \int_{bottom} dS \right] \tag{3.17}$$

Note that $\mathbf{D} \, . \, dS$ evaluated on the sides of the box is zero because $\mathbf{D}$ has no components along $\mathbf{a}_x$ and $\mathbf{a}_y$. If the top and bottom area of the box each has area A, equation above becomes

$$\rho_s A = D_z (A + A) \tag{3.18}$$

And thus

$$\mathbf{D} = \frac{\rho_s}{2} \mathbf{a}_z$$

or

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_o} = \frac{\rho_s}{2\epsilon_o} \mathbf{a}_z \tag{3.19}$$

D. Uniformly Charged Sphere

Consider a sphere of radius a with a uniform charge ρ_v $(\mathrm{C/m^3})$. To determine $\mathbf{D}$ everywhere, we construct Gaussian surfaces for eases $r \le a$ and $r \ge a$ separately. Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

For $r \le a$, the total charge enclosed by the spherical surface of radius r, as shown in Figure 3.11(a), is

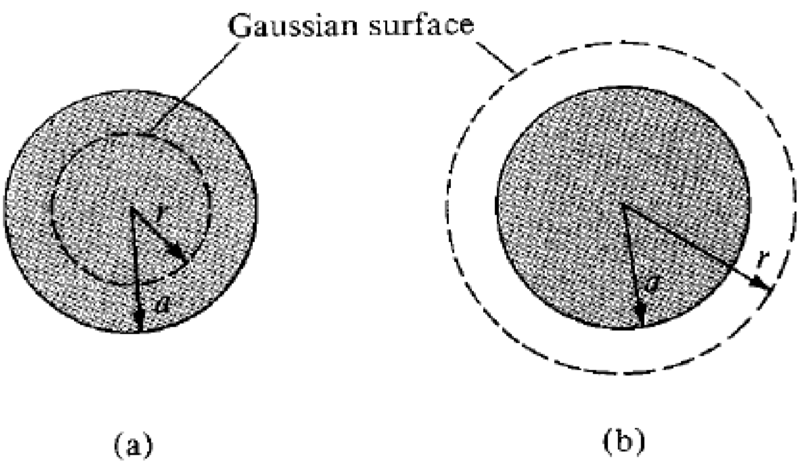

Fig. 3.11 Gaussian surface for a uniformly charged sphere: (a) $r \le a$ and (b) $r \ge a$.

$$Q_{enclosed} = \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{enclosed} = \rho_v \frac{4}{3}\pi r^3 \tag{3.20}$$

and

$$\psi = \oint d\psi = \oint_s \mathbf{D} \, . \, dS = D_r \oint_s dS$$

$$\psi = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta d\phi = D_r 4\pi r^2 \tag{3.21}$$

Hence,

$\psi = Q_{enclosed}$ gives,

$$D_r 4\pi r^2 = \frac{4\pi r^3}{3} \rho_v$$

$$D_r = \frac{r}{3}\rho_v \tag{3.22}$$

or

$$\mathbf{D} = \frac{r}{3}\rho_v \mathbf{a}_r \qquad 0 < r \le a \tag{3.23}$$

For $r \ge a$, the Gaussian surface is shown in Figure 3.11(b). The charge enclosed by the surface is the entire charge in this case, that is,

$$Q_{enclosed} = \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{enclosed} = \rho_v \frac{4}{3}\pi a^3 \tag{3.24}$$

While,

$$\psi = \oint d\psi = \oint_s \mathbf{D} \, . \, dS = D_r 4\pi r^2 \tag{3.25}$$

Hence,

$\psi = Q_{enclosed}$ gives,

$$D_r 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_v$$

or

$$\mathbf{D} = \frac{a^3}{3r^2}\rho_v \mathbf{a}_r \quad r \geq a \tag{3.26}$$

Thus from equations (3.23) and (3.26), **D** everywhere is given by

$$D = \begin{cases} \frac{r}{3}\rho_v \mathbf{a}_r & 0 < r \leq a \\ \\ \frac{a^3}{3r^2}\rho_v \mathbf{a}_r & r \geq a \end{cases} \tag{3.27}$$

and |**D**| is as sketched in Figure 3.12.

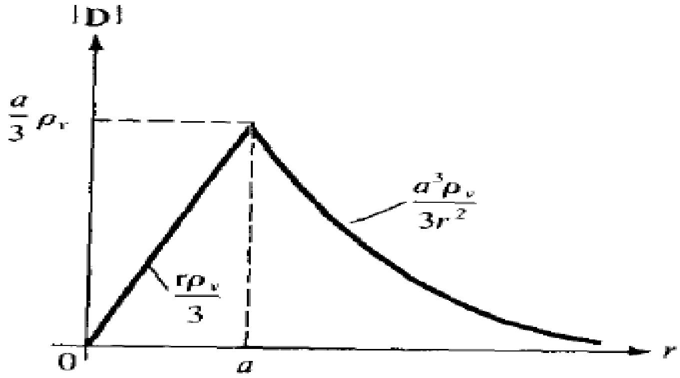

Figure 3.12 Sketch of |**D**| against r for a uniformly charged sphere.

Notice from equations (3.23), (3.15), (3.17) and (3.21) that the ability to take **D** out of the integral sign is the key to finding **D** using Gauss's law. In other words, **D** must be constant on the Gaussian surface.

3.7 DIVERGENCE OF A VECTOR AND DIVERGENCE THEOREM

We have noticed that the net outflow of the flux of a vector field **A** from a closed surface S is obtained from the integral

$$\oint_S \mathbf{A}\,.\,dS$$

We now define the divergence of **A** as the net outward flow of flux per unit volume over a closed incremental surface.

The divergence of a vector **A** at a given point P is the net outward flux per unit volume as the volume shrinks about P. Hence,

$$div\ A = \nabla.\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\,.\,dS}{\Delta v} \tag{3.28}$$

Where, Δv is the volume enclosed by the closed surface S in which P is located.

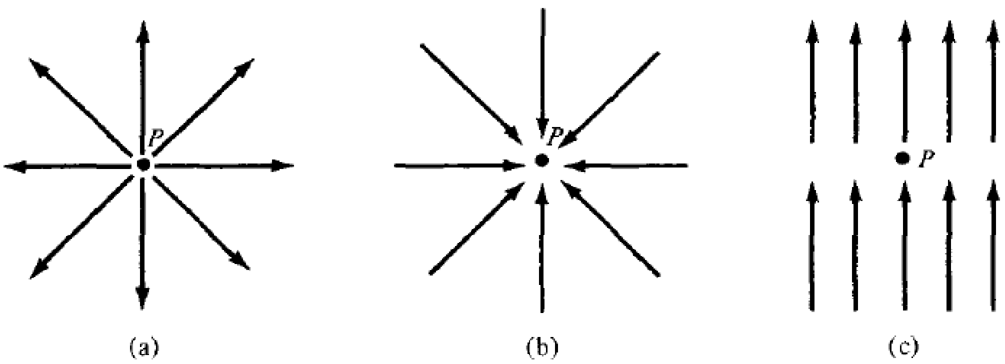

Fig. 3.13 illustration of the divergence of a vector field at P; (a) positive divergence, (b) negative divergence, (c) zero divergence.

The divergence of a vector field the limit of the field's source strength per unit volume (or source density); it is positive at a source point in the field, and negative at a sink point, or zero where there is neither sink nor source.

3.7.1 Divergence In Cartesian Coordinates

We can obtain an expression for Divergence ($\nabla.\mathbf{A}$) in Cartesian coordinates from the definition in equation (3.28). Suppose we wish to evaluate the divergence of a vector field **A** at point, $P(x_o, y_o, z_o)$; we let the point be enclosed by a differential volume as in Fig. 3.14. The surface integral in equation (3.28) is obtained from

$$\oint_s \mathbf{A} \,.\, dS = \left(\int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom} \right) \mathbf{A} \,.\, d\mathbf{S} \tag{3.29}$$

A three-dimensional Taylor series expansion of A_x about P is:

$$A_x(x,y,z) = A_x(x_o,y_o,z_o) + (x - x_o)\left.\frac{\partial A_x}{\partial x}\right|_P + (y - y_o)\left.\frac{\partial A_x}{\partial y}\right|_P + (z - z_o)\left.\frac{\partial A_x}{\partial z}\right|_P$$
$$+ \text{higher} - \text{order terms} \tag{3.30}$$

For the front side, $x = x_o + dx/2$ and $dS = dydz\, \mathbf{a}_x$. Then,

$$\int_{front} \mathbf{A} \,.\, dS = dydz \left[A_x(x_o,y_o,z_o) + \frac{dx}{2} \left.\frac{\partial A_x}{\partial x}\right|_P \right] + \text{higher} - \text{order terms}$$

For the back side, $x = x_o - dx/2$ and $dS = dydz(-\mathbf{a}_x)$. Then,

$$\int_{back} \mathbf{A} \,.\, dS = -dydz \left[A_x(x_o,y_o,z_o) - \frac{dx}{2} \left.\frac{\partial A_x}{\partial x}\right|_P \right] + \text{higher} - \text{order terms}$$

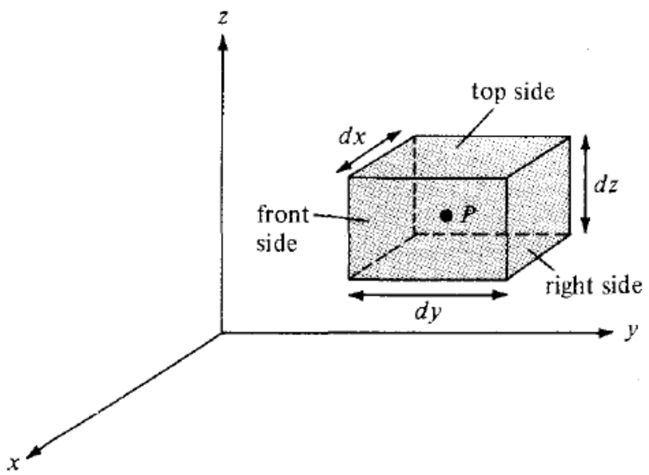

Figure 3.14 Evaluation of $\nabla . \mathbf{A}$ at point $P(x_o, y_o, z_o)$.

$$\int_{front} \mathbf{A} \,.\, d\mathbf{S} + \int_{back} \mathbf{A} \,.\, d\mathbf{S} = dxdydz \left.\frac{\partial A_x}{\partial x}\right|_P + \text{higher} - \text{order terms} \tag{3.31}$$

By taking similar steps, we obtain

$$\int_{left} \mathbf{A} \,.\, dS + \int_{right} \mathbf{A} \,.\, dS = dxdydz \left.\frac{\partial A_y}{\partial y}\right|_P + \text{higher} - \text{order terms} \tag{3.32}$$

and

$$\int_{top} \mathbf{A} \,.\, dS + \int_{bottom} \mathbf{A} \,.\, dS = dxdydz \frac{\partial A_z}{\partial z}\bigg|_P + \text{higher} - \text{order terms} \qquad (3.33)$$

Substituting equations (3.31) to (3.33) into eq. (3.29) and noting that $\Delta v = dx\, dy\, dz$, we get

$$\lim_{\Delta v \to 0} \frac{\oint_s \mathbf{A} \,.\, dS}{\Delta v} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\bigg| \qquad (3.34)$$

Because the higher-order terms will vanish as $\Delta v \to 0$. Thus, the divergence of $\mathbf{A}$ at point $P(x_o, y_o, z_o)$ in a Cartesian system is given by

$$\mathbf{\nabla}.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{(in a Cartesian coordinate system)} \qquad (3.35)$$

3.7.2 Divergence In Other Coordinates System

Similar expressions for $\mathbf{\nabla}.\mathbf{A}$ in other coordinate systems can be obtained directly from eq. (3.28) or by transforming eq. (3.35) into the appropriate coordinate system.

1. In Cylindrical Coordinate System:

$$\mathbf{\nabla}.\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\Phi}{\partial \Phi} + \frac{\partial A_z}{\partial z} \qquad \text{(in cylindrical coordinate system)} \qquad (3.36)$$

2. In Spherical Coordinate System:

$$\mathbf{\nabla}.\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta} A_\theta \sin\theta + \frac{1}{r\sin\theta}\frac{\partial A_\Phi}{\partial \Phi} \qquad \text{(in spherical)} \qquad (3.37)$$

Note the following properties of the divergence of a vector field:

1. It produces a scalar field (because scalar product is involved).
2. The divergence of a scalar V, $div\ V$, makes no sense.
3. $\mathbf{\nabla}.(\mathbf{A} + \mathbf{B}) = \mathbf{\nabla}.\mathbf{A} + \mathbf{\nabla}.\mathbf{B}$
4. $\mathbf{\nabla}.(\mathrm{V}\mathbf{A}) = \mathrm{V}\,\mathbf{\nabla}.\mathbf{A} + \mathbf{A}\,.\mathbf{\nabla}\mathrm{V}$

Example 3.3: If $\mathbf{A} = 5x^2(\sin\frac{\pi x}{2})\mathbf{a}_x$ find $div\ \mathbf{A}$ at $x = 1$

Solution:

$$div\ \mathbf{A} = \nabla.\mathbf{A} = \frac{\partial}{\partial x}\left(5x^2 \sin\frac{\pi x}{2}\right) = 5x^2\left(\cos\frac{\pi x}{2}\right)\frac{\pi}{2} + 10x\left(\sin\frac{\pi x}{2}\right)$$

$$= \frac{5\pi}{2}x^2\left(\cos\frac{\pi x}{2}\right) + 10x\left(\sin\frac{\pi x}{2}\right)$$

$$div\ \mathbf{A}|_{x=1} = 10$$

Example 3.4: If the vector field by cylindrical coordinates

$\mathbf{A} = \rho\sin\phi\,\mathbf{a}_r + \rho^2\cos\phi\,\mathbf{a}_\phi + 2\rho e^{-5z}\mathbf{a}_z$, find $div\ \mathbf{A}$ at $({}^1/_2\,,{}^\pi/_2\,,0)$

Solution:

$$div\ \mathbf{A} = \nabla.\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\Phi}(\rho^2\cos\phi) + \frac{\partial}{\partial z}(2\rho e^{-5z})$$

$$div\ \mathbf{A} = \frac{1}{\rho}(2\rho\sin\phi) - \frac{1}{\rho}(\rho^2\sin\phi) - 10\rho e^{-5z} = 2\sin\phi - \rho\sin\phi - 10\rho e^{-5z}$$

$$div\ \mathbf{A}\Big|_{({}^1/_2\,,{}^\pi/_2,0)} = 2\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{2} - 10\frac{1}{2}e^0 = -\frac{7}{2}$$

H.w 3.1: If the field vector in spherical coordinate

$\mathbf{A} = \frac{5}{r^2}\sin\theta\,a_r + r\cot\theta\,a_\theta + r\sin\theta\cos\phi\,a_\Phi$, find $div\ \mathbf{A}$

3.8 DIVERGENCE OF D (first Maxwell equation for electrostatic field)

From Gauss's law

$$\frac{\oint_s \mathbf{D}\,.\,dS}{\Delta v} = \frac{Q_{enclosed}}{\Delta v} \tag{3.38}$$

In the limit,

$$\lim_{\Delta v\to 0}\frac{\oint_s \mathbf{D}\,.\,dS}{\Delta v} = div\ \mathbf{D} = \lim_{\Delta v\to 0}\frac{Q_{enclosed}}{\Delta v} = \rho \tag{3.39}$$

This important result is one of Maxwell's equations for static fields:

$$div\ \mathbf{D} = \rho_v \qquad \text{and} \qquad div\ \mathbf{E} = \frac{\rho_v}{\epsilon_0} \tag{3.40}$$

Example 3.5:

In spherical coordinates the region $r \leq a$ contains a uniform charge density ρ, while for $r > a$ the charge density is zero. Since $\mathbf{E} = E_r \mathbf{a}_r$ where $(E_r = \frac{\rho r}{3\epsilon_0})$ for $r \leq a$ and $E_r = \frac{\rho a^3}{3\epsilon_0 r^2}$ for $r > a$.

Solution: Then, for $r \leq a$.

$$div\ \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\rho r}{3\epsilon_0}\right) = \frac{1}{r^2}\left(3r^2 \frac{\rho}{3\epsilon_0}\right) = \frac{\rho}{\epsilon_0}$$

and, for $r > a$.

$$div\ \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\rho a^3}{3\epsilon_0 r^2}\right) = 0$$

3.9 THE DEL OPERATOR

We define the del operator ∇ as a vector operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \tag{3.41}$$

$$\nabla . \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right).\left(D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z\right) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$div\ \mathbf{D} = \nabla . \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{3.42}$$

The del operator is defined only in Cartesian coordinate system. When $\nabla.\mathbf{D}$ is writting as the divergence of $\mathbf{D}$ in other coordinate systems, it does not mean that a del operator defined for these systems. For example, the divergence in cylindrical coordinate written as

$$\nabla . \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\Phi}{\partial \Phi} + \frac{\partial A_z}{\partial z}$$

3.10 THE DIVERGANCE THEOREM

From the definition of the divergence of $\mathbf{D}$ in eq. (3.28), the divergence of $\mathbf{D}$

$$\oint_s \mathbf{D} . dS = \oint_v \rho_v\, dv = Q_{enc}$$

But, $\rho_v = \nabla . \mathbf{D}$

$$\oint_S \mathbf{D} \,.\, dS = \oint_v (\nabla . \mathbf{D})\, dv \qquad \text{(divergence theorem for Gauss's law)} \qquad (3.43)$$

For any vector field

$$\oint_S \mathbf{A} \,.\, dS = \oint_v (\nabla . \mathbf{A})\, dv \qquad \text{(divergence theorem)}$$

This is called the divergence theorem, otherwise known as the Gauss-Ostrogradsky theorem.

The divergence theorem: states that the total outward flux of a vector field **A** through the closed surface S is the same as the volume integral of the divergence of **A**.

Example 3.6: The region $r \leq a$ in spherical coordinate system has an electric field intensity $\mathbf{E} = \frac{\rho r}{3\epsilon_0}\mathbf{a}_r$

Examine both sides of the divergence theorem, for this vector field. For S_1 choose the spherical surface $r = b \leq a$

$$\oint_S \mathbf{E} \,.\, dS = \oint_v (\nabla . \mathbf{E})\, dv \qquad \text{(divergence theorem)}$$

Solution: For the left side

$$\oint_S \mathbf{E} \,.\, dS = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\frac{\rho r}{3\epsilon_0} a_r\right) b^2 \sin\theta \, d\theta d\phi a_r = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\rho b^3}{3\epsilon_0} \sin\theta \, d\theta d\phi$$

$$\oint_S \mathbf{E} \,.\, dS = \frac{4\pi\rho b^3}{3\epsilon_0}$$

For the right side

$$\nabla . \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\rho r}{3\epsilon_0}\right) = \frac{\rho}{\epsilon_0}$$

$$\oint_v (\nabla . \mathbf{E})\, dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{b} \frac{\rho}{\epsilon_0} r^2 \sin\theta \, dr d\theta d\phi = \frac{4\pi\rho b^3}{3\epsilon_0}$$

The right side = the left side

Example 3.7: The finite sheet $0 \le x \le 1$, $0 \le y \le 1$ on the $z = 0$ plane has a charge density $\rho_s = xy(x^2 + y^2 + 25)^{3/2}\ nC/m^2$. Find

(a) The total charge on the sheet (b) The electric field at $(0, 0, 5)$

(c) The force experienced by a $-1\ mC$ charge located at $(0, 0, 5)$

Solution: $Q = \int \rho_s ds = \int_0^1 \int_0^1 xy(x^2 + y^2 + 25)^{3/2}\, dx\, dy$

Since, $xdx = 1/2\ d(x^2)$, we now integrate with respect to x^2 (or change variables: $x^2 = u$ so that $x\,dx = du/2$).

$$Q = \frac{1}{2}\int_0^1\int_0^1 y(x^2+y^2+25)^{\frac{3}{2}}\, d(x^2)dy = \frac{1}{2}\int_0^1 y\frac{2}{5}(x^2+y^2+25)^{\frac{5}{2}}\Big|_0^1\, dy$$

$$= \frac{1}{5}\int_0^1 \frac{1}{2}(y^2+26)^{\frac{5}{2}} - (y^2+25)^{\frac{5}{2}} d(y^2)$$

$$= \frac{1}{10}\cdot\frac{2}{7}\left[(y^2+26)^{(7/2)} - (y^2+25)^{(7/2)}\right]_0^1$$

$$= \frac{1}{35}\left[(27)^{\frac{7}{2}} + (25)^{\frac{7}{2}} - 2(25)^{\frac{7}{2}}\right] = 33.15\ \text{nc}$$

$$\mathbf{E} = \int \frac{\rho_s ds}{4\pi\epsilon_o R^2}\mathbf{a}_R = \int \frac{\rho_s ds}{4\pi\epsilon_o}\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = (0, 0, 5) - (x, y, 0) = (-x, -y, 5)$. Hence

$$\mathbf{E} = \int_0^1\int_0^1 \frac{10^{-9}xy(x^2+y^2+25)^{\frac{3}{2}}(-x\mathbf{a}_x - y\mathbf{a}_y - 5\mathbf{a}_z)dxdy}{4\pi\frac{10^{-9}}{36\pi}(x^2+y^2+25)^{\frac{3}{2}}}$$

$$\mathbf{E} = 9\left[-\int_0^1 x^2\,dx\int_0^1 y^2\,dy\mathbf{a}_x - \int_0^1 y^2\,dy\mathbf{a}_y + 5\int_0^1 x\,dx + \int_0^1 y\,dy\mathbf{a}_z\right]$$

$$\mathbf{E} = 9\left[-\frac{1}{6}, -\frac{1}{6}, \frac{5}{4}\right]$$

$$\mathbf{E} = 9\left(-\frac{1}{6}\mathbf{a}_x - \frac{1}{6}\mathbf{a}_y + \frac{5}{4}\mathbf{a}_z\right) = -1.5\mathbf{a}_x - 1.5\mathbf{a}_y + 11.25\mathbf{a}_z\ \text{V/m}$$

$$\mathbf{F} = q\mathbf{E} = -1(-1.5\mathbf{a}_x - 1.5\mathbf{a}_y + 11.25\mathbf{a}_z) = 1.5\mathbf{a}_x + 1.5\mathbf{a}_y - 11.25\mathbf{a}_z\ N$$