2014-2015

ELECTROMAGNETIC FIELDS

"CHAPTER FOUR: ENERGY AND POTENTIAL"

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4. ENERGY AND POTENTIAL

4.1 ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD

The electric field intensity was defined as the force on a unit test charge at that point. To move a charge Q a distance $d\mathbf{L}$ in an electric field $\mathbf{E}$. The force on Q due to the electric field $\mathbf{F}_E$ is

$$\mathbf{F}_E = Q\mathbf{E} \tag{4.1}$$

The component of this force in the direction $d\mathbf{L}$ which we must overcome is

$$F_{EL} = \mathbf{F}.\mathbf{a}_L = Q\mathbf{E}\,.\,\mathbf{a}_L$$

Where, $\mathbf{a}_L$ is the unit vector in the direction of $d\mathbf{L}$.

The force which must apply is equal and opposite to the force due to the field,

$$F_{appl} = -Q\mathbf{E}\,.\,\mathbf{a}_L$$

Expenditure of energy is the product of the force and distance. That is, Differential work done by external source moving charge, Q:

$$dW = -Q\mathbf{E}\,.\,\mathbf{a}_L dL$$

or

$$dW = -Q\mathbf{E}\,.\,d\mathbf{L} \tag{4.2}$$

If replaced $\mathbf{a}_L dL$ by the $d\mathbf{L}$.

This differential amount of work required may be zero under several conditions from (4.2). There are for which $\mathbf{E}$, Q, or $d\mathbf{L}$ is zero, and a much more important case in which $\mathbf{E}$ and $d\mathbf{L}$ are perpendicular. Here the charge is moved always in a direction at right angles to the electric field.

The work required to move the charge a finite distance in electric field must be determined by:

$$W = \int_{init}^{final} \mathbf{E}\,.\,d\mathbf{L} \tag{4.3}$$

Where the path must be specified before the integral can be evaluated. The charge is assumed to be at rest at both its initial and final positions.

Example. 4.1:

Given the electric field, $\mathbf{E} = \frac{1}{z^2}(8xyz\mathbf{a}_x + 4x^2z\mathbf{a}_y - 4x^2y\mathbf{a}_z)\text{V/m}$, find the differential amount of work done in moving a 6 nC charge a distance of 2 μm, starting at $P(2,-2,3)$ and proceeding in the direction $\mathbf{a}_L$ (a) $\left(-\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z\right)$ (b) $\left(\frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z\right)$ (c) $\left(\frac{3}{7}\mathbf{a}_x + \frac{6}{7}\mathbf{a}_y\right)$

Solution:

$$dW = -Q\mathbf{E} \,.\, \mathbf{a}_L dL$$

$$E_{(2,-2,3)} = (-\frac{96}{9}\mathbf{a}_x + \frac{48}{9}\mathbf{a}_y + \frac{32}{9}\mathbf{a}_z)$$

(a) where $\mathbf{a}_L = \left(-\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z\right)$

$$dW = -6 \times 10^{-9} \times 2 \times 10^{-6}\left(-\frac{96}{9}\mathbf{a}_x + \frac{48}{9}\mathbf{a}_y + \frac{32}{9}\mathbf{a}_z\right)\left(-\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z\right)$$

$$dW = -12 \times 10^{-9} \times 10^{-6}\left(\frac{(-96) \times (-6)}{9 \times 7} + \frac{48 \times 3}{9 \times 7} + \frac{32 \times 2}{9 \times 7}\right) = -149.3 \text{ J}$$

(b) where $\mathbf{a}_L = \left(\frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z\right)$

$$dW = -6 \times 10^{-9} \times 2 \times 10^{-6}\left(-\frac{96}{9}\mathbf{a}_x + \frac{48}{9}\mathbf{a}_y + \frac{32}{9}\mathbf{a}_z\right)\left(\frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z\right)$$

$$dW = -12 \times 10^{-9} \times 10^{-6}\left(-\frac{96 \times 6}{9 \times 7} - \frac{48 \times 3}{9 \times 7} - \frac{32 \times 2}{9 \times 7}\right) = 149.3 \text{ J}$$

(c) where $\mathbf{a}_L = \left(\frac{3}{7}\mathbf{a}_x + \frac{6}{7}\mathbf{a}_y\right)$

$$dW = -6 \times 10^{-9} \times 2 \times 10^{-6}\left(-\frac{96}{9}\mathbf{a}_x + \frac{48}{9}\mathbf{a}_y + \frac{32}{9}\mathbf{a}_z\right)\left(\frac{3}{7}\mathbf{a}_x + \frac{6}{7}\mathbf{a}_y\right)$$

$$dW = -12 \times 10^{-9} \times 10^{-6}\left(-\frac{96 \times 3}{9 \times 7} + \frac{48 \times 6}{9 \times 7}\right) = 0 \text{ J}$$

4.2 THE LINE INTEGRAL

The integral expression for the work done in moving a point charge Q from one position to another, equation (4.3), is an example of a line integral.

Without using vector analysis we should have to write:

$$W = -Q\int_{init}^{final} E_L \,.\, dL$$

Where E_L is a component of **E** along $d\mathbf{L}$.

The integral is obtained exactly only when the number of segments becomes infinite. This procedure is indicated in Fig. 4.1.

- Path of integral (from an initial position B to a final position A).
- The path is divided into six segments, $\Delta\mathrm{L}_1$, $\Delta\mathrm{L}_2$, ... , $\Delta\mathrm{L}_6$.
- The components of **E** along each segment denoted by $\mathrm{E}_{\mathrm{L}1}$, $\mathrm{E}_{\mathrm{L}2}$,. . . , $\mathrm{E}_{\mathrm{L}6}$.

The work involved in moving a charge Q from B to A is:

$$W = -Q(E_{\mathrm{L}1}\Delta\mathrm{L}_1 + E_{\mathrm{L}2}\Delta\mathrm{L}_2 + \ldots + E_{\mathrm{L}6}\Delta\mathrm{L}_6)$$

By using vector notation,

$$W = -Q(\mathbf{E}_1\Delta\mathrm{L}_1 + \mathbf{E}_2\Delta\mathrm{L}_2 + \ldots + \mathbf{E}_6\Delta\mathrm{L}_6)$$

For a uniform field, $\mathbf{E}_1 = \mathbf{E}_2 = \cdots = \mathbf{E}_6 = \mathbf{E}$

$$W = -Q\mathbf{E}(\Delta\mathrm{L}_1 + \Delta\mathrm{L}_2 + \ldots + \Delta\mathrm{L}_6)$$

Where,

$$\Delta\mathrm{L}_1 + \Delta\mathrm{L}_2 + \ldots + \Delta\mathrm{L}_6 = \mathbf{L}_{BA}$$

Therefore, if uniform field **E**

$$W = -Q\mathbf{E}\,.\,\mathbf{L}_{BA} \qquad (4.4)$$

Therefore,

$$W = -Q\int_B^A \mathbf{E}\,.\,d\mathbf{L}$$

As applied to a uniform field

$$W = -Q\mathbf{E}\,.\int_B^A d\mathbf{L} = -Q\mathbf{E}\,.\,\mathbf{L}_{BA}$$

The expressions for $d\mathbf{L}$ in coordinate systems (cartesian, cylindrical, and spherical):

$d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$ (cartesian coordinate)

$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z$ (cylindrical coordinate)

$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta\, d\phi\mathbf{a}_\phi$ (spherical coordinate)

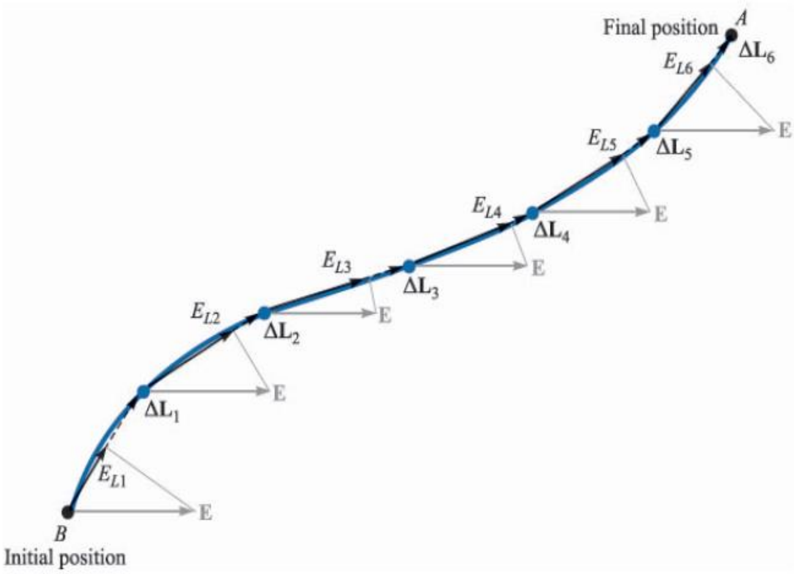

Fig. 4.1. The line integral of **E** between points *B* and *A*

Example 4.2:

We are given the nonuniform field $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$.

Determine the work expended in carrying 2 C from $B(1,0,1)$ to $A(0.8,0.6,1)$ along the shorter arc of the circle, $x^2 + y^2 = 1$, $z = 1$.

Solution: The differential path is, $d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$, where, **E** is not constant.

$$\mathrm{W} = -\mathrm{Q}\int_{\mathrm{B}}^{\mathrm{A}} \mathbf{E} \,. d\mathbf{L}$$

$$\mathrm{W} = -2\int_{\mathrm{B}}^{\mathrm{A}} (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z).\,(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

$$\mathrm{W} = -2\int_{1}^{0.8} y dx - 2\int_{0}^{0.6} x dy - 4\int_{1}^{1} dz$$

Using the equation of the circular path

$$\mathrm{W} = -2\int_{1}^{0.8} \sqrt{1 - x^2} dx - 2\int_{0}^{0.6} \sqrt{1 - y^2} dy - 0$$

$$\mathrm{W} = -\left[x\sqrt{1 - x^2} + \sin^{-1} x\right] - \left[y\sqrt{1 - y^2} + \sin^{-1} y\right]$$

$$\mathrm{W} = -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0) = -0.96 \text{ J}$$

Example 4.3:

Find the work required to carry 2 C from $B(1,0,1)$ to $A(0.8,0.6,1)$ along the straight-line path from B to A in the field, $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$.

Solution:

The differential path $d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$,

We start by determining the equations of the straight line.

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

$$x - x_B = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

From the first and second equation above we have

$$y = -3(x-1)$$

$$z = 1$$

$$W = -Q\int_B^A \mathbf{E}\,.\,d\mathbf{L} = -2\int_1^{0.8} ydx - 2\int_0^{0.6} xdy - 4\int_1^1 dz$$

$$W = 6\int_1^{0.8}(x-1)dx - 2\int_0^{0.6}\left(1 - \frac{y}{3}\right)dy = -0.96\ \text{J}$$

Example 4.4:

Determine W in cylindrical coordinates, and the circular path Fig. 4.2a.

The work done is:

$$\text{W} = -Q\int_B^A \mathbf{E}\,.\,d\mathbf{L}$$

$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\epsilon_o\rho}\mathbf{a}_\rho$$

$$d\mathbf{L} = \rho d\phi\ \mathbf{a}_\phi$$

$$W = -Q\int_{initial}^{final}\frac{\rho_L}{2\pi\epsilon_o\rho}\mathbf{a}_\rho\,.\,\rho d\phi\ \mathbf{a}_\phi = -\,Q\int_{initial}^{final}\frac{\rho_L}{2\pi\epsilon_o}d\phi\ (\mathbf{a}_\rho\,.\,\mathbf{a}_\phi) = 0\ \text{J}$$

Example 4.5:

Determine W if carry a charge from $\rho = a$ to $\rho = b$ along a radial path Fig. 4.2b.

$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\epsilon_o \rho}\mathbf{a}_\rho$$

$$d\mathbf{L} = d\rho \mathbf{a}_\rho$$

$$\mathrm{W} = -Q\int_{initial}^{final} \frac{\rho_L}{2\pi\epsilon_o\rho}\mathbf{a}_\rho \,.\, d\rho\mathbf{a}_\rho = -\,Q\int_{initial}^{final} \frac{\rho_L}{2\pi\epsilon_o\rho} d\rho(\mathbf{a}_\rho \,.\, \mathbf{a}_\rho\,)$$

$$W = -Q\int_a^b \frac{\rho_L}{2\pi\epsilon_o}\frac{d\rho}{\rho} = -\frac{Q\,\rho_L}{2\pi\epsilon_o}\,Ln\frac{b}{a}$$

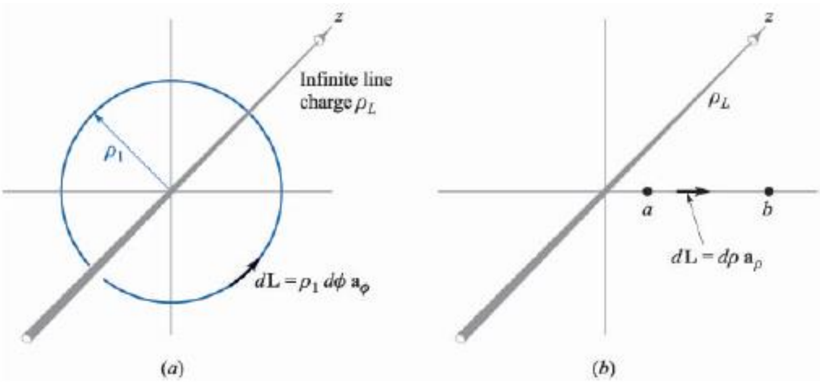

Fig. 4.2 (a) A circular path and (b) a radial path along which a charge of Q is carried in the field of an infinite line charge.

4.3 DEFINITION OF POTENTIAL DIFFERENCE AND POTENTIAL

Potential difference (V): define as the work done (by an external source) in moving a unit positive charge from one point to another in an electric field,

$$\text{Potential difference} = \mathrm{V} = \frac{W}{Q} = -\int_{initial}^{final} \mathbf{E} \, . \, d\mathbf{L}$$

Potential difference is measured in joules per coulomb; (volt) is common unit. Hence the potential difference between points A and B is

$$V_{AB} = V_A - V_B = -\int_{\mathrm{B}}^{\mathrm{A}} \mathbf{E} \, . \, d\mathbf{L} \qquad (\mathrm{V})$$

The potential difference between points at $\rho = a$ to $\rho = b$ is

$$V_{AB} = \frac{W}{Q} = \frac{\rho_L}{2\pi\epsilon_o} Ln\frac{b}{a}$$

Example 4.6:

Find the potential difference between points A and B at radial distances $r_A$and r_B from a point charge Q.

Solution: Choosing an origin at Q.

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon_o r^2} \mathbf{a}_r$$

$$d\mathbf{L} = dr \ \mathbf{a}_r$$

$$V_{AB} = -\int_{\mathrm{B}}^{\mathrm{A}} \mathbf{E} \, . \, d\mathbf{L} = -\int_{\mathrm{B}}^{\mathrm{A}} \frac{Q}{4\pi\epsilon_o r^2} \mathbf{a}_r \, . \, dr \ \mathbf{a}_r = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_o r^2} dr$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_o} (\frac{1}{r_A} - \frac{1}{r_B})$$

If the potential at point A is V_A and that at B is V_B, then

$$V_{AB} = V_A - V_B$$

V_A and V_B shall have the same zero reference point.

Example 4.7: An electric field is expressed in Cartesian coordinates system by $\mathbf{E} = 6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z$ V/m. Find:

(a) V_{MN} if points M and N are specified by $M(2, 6, -1)$ and $N(-3, -3, 2)$;

(b) V_M if $V = 0$ at $Q(4, -2, -35)$; (c) V_N if $V = 2$ at $P(1, 2, -4)$.

Solution: (a) Where $d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$ in Cartesian coordinates

$$V_{MN} = -\int_{N}^{M} \mathbf{E} \, . \, d\mathbf{L}$$

$$V_{MN} = -\int_{N}^{M} \mathbf{E} \, . \, d\mathbf{L} = -\int_{N}^{M} (6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z) \, . \, (dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

$$V_{MN} = -\int_{N}^{M} (6x^2dx + 6ydy + 4dz) = -[\int_{-3}^{2} 6x^2dx + \int_{-3}^{6} 6ydy + \int_{2}^{-1} 4dz]$$

$$V_{MN} = -\left\{ \left[\frac{6x^3}{3}\right]_{-3}^{2} + \left[\frac{6y^2}{2}\right]_{-3}^{6} + [4z]_{2}^{-1} \right\}$$

$$V_{MN} = -\left\{ \left[\frac{6(2)^3}{3} - \frac{6(-3)^3}{3}\right] + \left[\frac{6(6)^2}{2} - \frac{6(-3)^2}{2}\right] + [4(-1) - 4(2)] \right\} = -139 \text{ V}$$

(b) $$V_{MQ} = -\int_{Q}^{M} \mathbf{E} \, . \, d\mathbf{L} = -\int_{Q}^{M} (6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z) . (dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

$$V_{MQ} = -\int_{Q}^{M} (6x^2dx + 6ydy + 4dz) = -[\int_{4}^{2} 6x^2dx + \int_{-2}^{6} 6ydy + \int_{-35}^{-1} 4dz]$$

$$V_{MQ} = -\left\{ \left[\frac{6x^3}{3}\right]_{4}^{2} + \left[\frac{6y^2}{2}\right]_{-2}^{6} + [4z]_{-35}^{-1} \right\}$$

$$V_{MQ} = -\left\{ \left[\frac{6(2)^3}{3} - \frac{6(4)^3}{3}\right] + \left[\frac{6(6)^2}{2} - \frac{6(-2)^2}{2}\right] + 4[(-1) - (-35)] \right\} = -120 \, V$$

$$V_{MQ} = V_M - V_Q \quad \Rightarrow \quad -120 = V_M - 0 \quad \Rightarrow \quad V_M = -120 \text{ V}$$

(c) $$V_{NP} = -\int_{P}^{N} \mathbf{E} \, . \, d\mathbf{L} = -\int_{P}^{N} (6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z) \, . \, (dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$$

$$V_{NP} = -\int_{P}^{N} (6x^2dx + 6ydy + 4zdz) = -[\int_{1}^{-3} 6x^2dx + \int_{2}^{-3} 6ydy + \int_{-4}^{2} 4dz]$$

$$V_{NP} = -\left\{\left[\frac{6x^3}{3}\right]_1^{-3} + \left[\frac{6y^2}{2}\right]_2^{-3} + [4z]_{-4}^{2}\right\}$$

$$V_{NP} = -\left\{\left[\frac{6(-3)^3}{3} - \frac{6(1)^3}{3}\right] + \left[\frac{6(-3)^2}{2} - \frac{6(2)^2}{2}\right] + [4(2) - 4(-4)]\right\} = 17 \text{ V}$$

$$V_{NP} = V_N - V_P \quad \Rightarrow \quad 17 = V_N - 2 \quad \Rightarrow \quad V_N = 19 \text{ V}$$

4.4 THE POTENTIAL FIELD OF A POINT CHARGE

The potential difference between two points located at r_A and r_B in the field of a point charge Q placed at the origin, on the same radial line or had the same θ and ϕ coordinate values.

$$V_{AB} = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B$$

We now should take different θ and ϕ coordinate values for the initial and final position. The points A and B in Fig. 4.3 at radial distances of r_A and r_B.

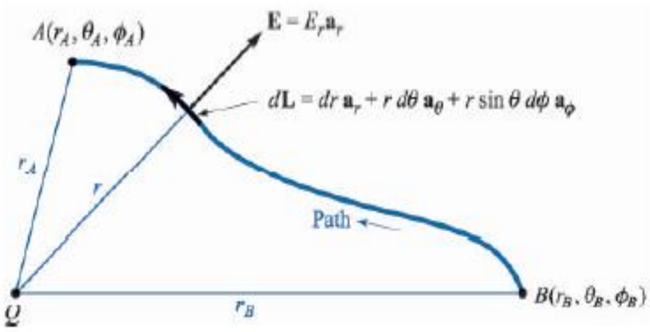

Fig. 4.3

The differential path length $d\mathbf{L}$ is:

$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta\, d\phi\mathbf{a}_\phi$$

The $\mathbf{E}$ has only a radial component.

$$\mathbf{E} = E_r\mathbf{a}_r = \frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r$$

$$V_{AB} = -\int_{r_B}^{r_A} \mathbf{E}\,.\,d\mathbf{L} = -\int_{B}^{A} \frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r\,.\,(dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta\, d\phi\mathbf{a}_\phi)$$

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_o r^2} dr = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

The potential difference between two points in the field of a point charge

- Depends only on the distance of each point from the charge.
- Does not depend on the particular path used to carry our unit charge from one point to the other.[1]

If $V = 0$ at infinity ($r_B = \infty$) i.e. $\left(\frac{1}{r_B} = 0\right)$. The potential at r_A becomes:

$$V_A = \frac{Q}{4\pi\epsilon_o r_A} \quad \text{(V)}$$

Example 4.8: A 15 nC point charge is at the origin in free space. Calculate V_P if point P is located at $P(-2, 3, -1)$ and: (a) $V = 0$ at point $Q(6, 5, 4)$; (b) $V = 0$ at infinity; (c) $V = 5$ V at point $M(2, 0, 4)$.

Solution: (a) $r_P = r_{OP} = \sqrt{(-2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$

$$r_Q = r_{OQ} = \sqrt{(6)^2 + (5)^2 + (4)^2} = \sqrt{77}$$

$$V_{PQ} = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_P} - \frac{1}{r_Q}\right) = \frac{15 \times 10^{-9}}{4\pi \frac{1}{36\pi} \times 10^{-9}}\left(\frac{1}{\sqrt{14}} - \frac{1}{\sqrt{77}}\right) = 20.7 \text{ V}$$

$$V_{PQ} = V_P - V_Q \quad \Rightarrow \quad 20.7 = V_P - 0 \quad \Rightarrow \quad V_P = 20.7\ V$$

(b) $(r_Q = \infty)$; $(1/r_Q = 0)$; and $r_P = r_{OP} = \sqrt{(-2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$

$$V_P = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_P}\right) = \frac{15 \times 10^{-9}}{4\pi \frac{1}{36\pi} \times 10^{-9}}\left(\frac{1}{\sqrt{14}}\right) = 36 \text{ V}$$

(c) $r_P = r_{OP} = \sqrt{(-2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$

$$r_M = r_{OM} = \sqrt{(2)^2 + (4)^2} = \sqrt{20}$$

$$V_{PQ} = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_P} - \frac{1}{r_Q}\right) = \frac{15 \times 10^{-9}}{4\pi \frac{1}{36\pi} \times 10^{-9}}\left(\frac{1}{\sqrt{14}} - \frac{1}{\sqrt{20}}\right) = 5.89\ V$$

$$V_{PQ} = V_P - V_Q \quad \Rightarrow \quad 5.89 = V_P - 5 \quad \Rightarrow \quad V_P = 5.89 + 5 = 10.89 \text{ V}$$

4.5 POTENTIAL FIELD OF A SYSTEM OF CHARGES: (conservative property)

The potential field of a single point charge Q_1 located at $\mathbf{r}_1$, involves the distance $|\mathbf{r}-\mathbf{r}_1|$ from Q_1 to the point at $\mathbf{r}$. The potential for a zero reference at infinity,

$$V(\mathrm{r}) = \frac{Q_1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|}$$

The potential due to two charges Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is a function only of $|\mathbf{r}-\mathbf{r}_1|$ and $|\mathbf{r}-\mathbf{r}_2|$, the distances from Q_1 and Q_2 to the field point, respectively.

$$V(\mathrm{r}) = \frac{Q_1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_2|}$$

The potential due to n point charges is:

$$V(\mathrm{r}) = \frac{Q_1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_2|} + \cdots + \frac{Q_n}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_n|}$$

or

$$V(\mathrm{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_m|}$$

For continuous charge distributions, we replace Q_m in equation above with charge element $\rho_L dL$ or $\rho_S dS$, or $\rho_v dv$ and the summation becomes an integration, so the potential at $\mathbf{r}$ becomes

$$V(\mathrm{r}) = \frac{1}{4\pi\epsilon_o}\int_L \frac{\rho_L(\mathbf{r}')\,dL'}{|\mathbf{r}-\mathbf{r}'|}$$ for line charge density

$$V(\mathrm{r}) = \frac{1}{4\pi\epsilon_o}\int_S \frac{\rho_S(\mathbf{r}')\,dS'}{|\mathbf{r}-\mathbf{r}'|}$$ for surface charge density

$$V(\mathrm{r}) = \frac{1}{4\pi\epsilon_o}\int_{vol} \frac{\rho_v(\mathbf{r}')\,dv'}{|\mathbf{r}-\mathbf{r}'|}$$ for volume charge density

Example 4.9: Find potential V on the z axis for a uniform line charge ρ_L in the form of a ring $\rho = a$ in the $z = 0$ plane.

Solution: as shown in Fig. 4.4. have

$dL' = \rho d\phi = a d\phi'$; $\mathbf{r} = z\mathbf{a}_z$; $\mathbf{r}' = a\mathbf{a}_\rho$; $|\mathbf{r} - \mathbf{r}'| = \sqrt{a^2 + z^2}$; and

$$V = \frac{1}{4\pi\epsilon_o}\int_{line}\frac{\rho_L\, dL'}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_o}\int_0^{2\pi}\frac{\rho_L a d\phi'}{\sqrt{a^2 + z^2}} = \frac{\rho_L a}{2\epsilon_o\sqrt{a^2 + z^2}}$$

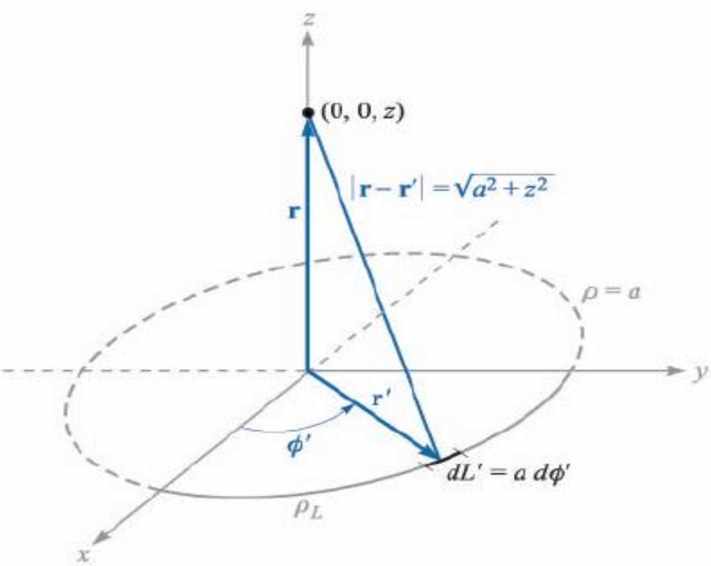

Fig. 4.4.

For a zero reference at infinity, then:

- The potential due to a single point charge is the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, and the work is independent of the path chosen between those two points.
- The potential field in the presence of a number of point charges is the sum of the individual potential fields arising from each charge.
- The potential due to a number of point charges or any continuous charge distribution may therefore be found by carrying a unit charge from infinity to the point in question along any path we choose.

Note: that no work is done in carrying the unit charge around any closed path in static field, or

$$\oint \mathbf{E}\,.\, d\mathbf{L} = 0$$

Example 4.10:

Two point charges $-4\ \mu C$ and $5\ \mu C$ are located at $(2,-1,3)$ and $(0,4,-2)$, respectively. Find the potential at $(1,0,1)$ assuming zero potential at infinity.

Solution:

$$V(\mathrm{r}) = \frac{Q_1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_2|}$$

$$\mathbf{r} = \mathbf{a}_x + \mathbf{a}_z; \qquad \mathbf{r}_1 = 2\mathbf{a}_x - \mathbf{a}_y + 3\mathbf{a}_z; \qquad \mathbf{r}_2 = 4\mathbf{a}_y - 2\mathbf{a}_z$$

$$\mathbf{r}-\mathbf{r}_1 = -\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z; \qquad |\mathbf{r}-\mathbf{r}_1| = \sqrt{6}$$

$$\mathbf{r}-\mathbf{r}_2 = \mathbf{a}_x - 4\mathbf{a}_y + 3\mathbf{a}_z; \qquad |\mathbf{r}-\mathbf{r}_2| = \sqrt{26}$$

$$V(1,0,1) = \frac{1}{4\pi\frac{10^{-9}}{36\pi}}\left[\frac{-4\times10^{-6}}{\sqrt{6}} + \frac{5\times10^{-6}}{\sqrt{26}}\right] = -5.872\ \mathrm{kV}$$

Example 4.11: Consider the force field, $\mathbf{F} = sin\pi\rho\ \mathbf{a}_\phi$. Around a circular path of radius $\rho = \rho_1$, find $\oint \mathbf{F}\,.\,d\mathbf{L}$

Solution: we have $dL = \rho d\phi \mathbf{a}_\phi$, and

$$\oint \mathbf{F}\,.\,d\mathbf{L} = \int_0^{2\pi} sin\pi\rho\ \mathbf{a}_\phi\ .\rho d\phi \mathbf{a}_\phi = 2\pi\rho_1 sin\pi\rho_1$$

The integral is zero if $\rho_1 = 1, 2, 3, \ldots$, etc., but it is not zero for other values of ρ_1, or for most other closed paths, and the given field is not conservative.

Example 4.12:

A total charge of $\frac{40}{3}$ nC is uniformly distributed in the form of a circular disk of radius 2 m. Find the potential due to this charge at a point on the z-axis, 2 m from the disk. Compare this potential with that which results if all of the charge is at the center of the disk.

Solution:

$$R = \sqrt{4+\rho^2}\ \ (\mathrm{m})$$

$$\rho_s = \frac{Q}{A} = \frac{\frac{40}{3}10^{-9}}{4\pi} = \frac{10^{-8}}{3\pi}\ \ \mathrm{C/m^2}$$

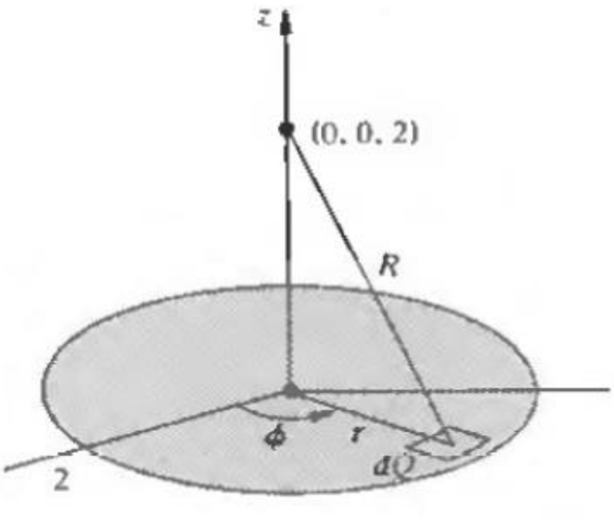

Fig. 4.5

$$V=\frac{1}{4\pi\epsilon_o}\int_S \frac{\rho_S dS}{|\mathbf{r}-\mathbf{r}'|}=\frac{\frac{10^{-8}}{3\pi}}{4\pi\frac{10^{-9}}{36\pi}}\int_0^{2\pi}\int_0^{2}\frac{\rho d\rho d\phi}{\sqrt{4+\rho^2}}$$

$$=\frac{30}{\pi}\int_0^{2\pi}\int_0^{2}\frac{\rho d\rho d\phi}{\sqrt{4+\rho^2}}=49.7\text{ V}$$

If the total charge at the center of the disk, the expression for the potential of a point charge applies:

$$V=\frac{Q}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|}=\frac{Q}{4\pi\epsilon_o z}=\frac{\frac{40}{3}10^{-9}}{4\pi\times(\frac{10^{-9}}{36\pi})\times 2}=60\text{ V}$$

4.6 POTENTIAL GRADIENT:

Fig. 4.6 shows two points M at (x,y,z) and N at $(x+dx,y+dy,z+dz)$.

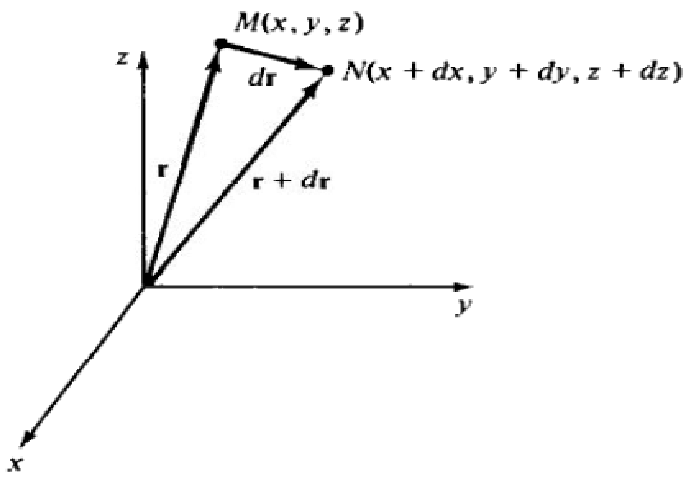

Fig. 4.6

The vector separation of the two points is:

$$d\mathbf{r} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z = d\mathbf{L}$$

The change in V from M to N is given by

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

The Del operator, operating on V gives:

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \text{grad } V$$

It follows that

$$dV = \nabla V . d\mathbf{r}$$

The vector field ∇V (also written grad V) is called the gradient of the scalar function V.

Thus **∇**V lies in the direction of maximum increase of the function V.

If the points M and N to lie on the same equipotential surface, $V(x, y, z) = c_1$, Fig. 4.7. Then $dV = 0$, which implies that ∇V is perpendicular to $d\mathbf{r}$. But $d\mathbf{r}$ is tangent to the equipotential surface; Therefore, ∇V must be along the surface normal at M. Since ∇V in the direction of increasing V, it points from $V(x, y, z) = c_1$, to $V(x, y, z) = c_2$, where $c_2 > c_1$.

The gradient of a potential function is a vector field that is everywhere normal to the equipotential surfaces.

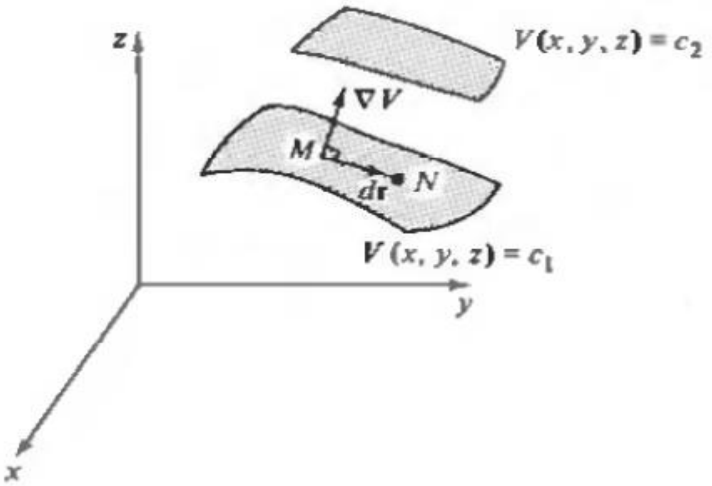

Fig. 4.7.

The gradient V in all coordinate systems:

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{cartesian coordinate}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{cylindrical coordinate}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{spherical coordinate}$$

4.7 RELATIONSHIP BETWEEN E AND V:

From the integral expression for the potential of A with respect to B, the differential of V may be written

$$dV = -\mathbf{E} \,.\, d\mathbf{L}$$

On the other hand,

$$dV = \nabla V . d\mathbf{r}$$

Since $d\mathbf{L} = d\mathbf{r}$ is small displacement, it follows that

$$\mathbf{E} = -\nabla V$$

The electric field intensity **E** may be obtained when the potential function V is known. The gradient was found to be a vector normal to the equipotential surfaces, directed to a positive change in V. With the negative sign here, the **E** field is found to be directed from higher to lower levels of potential V. [3]

Example 4.13:

In spherical coordinates and relative to infinity, the potential in the region $r > 0$ surrounding a point charge Q is $V = \frac{Q}{4\pi\epsilon_o r}$. Find **E**.

Solution:

$$\mathbf{E} = -\nabla V$$

$$-\nabla V = -(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi)$$

$$-\nabla V = -\left(\frac{\partial V}{\partial r}\right)\mathbf{a}_r = -\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_o r}\right)\mathbf{a}_r = \frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r$$

$$\mathbf{E} = -\nabla V = \frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r$$

Example 4.14:

Given the potential $V = \frac{10}{r^2}\sin\theta\cos\phi$,

(a) Find the electric flux density **D** at $(2, \pi/2, 0)$.

(b) Calculate the work done in moving a 10 μC charge from point $A(1, 30°, 120°)$ to $B(4{,}90°, 60°)$.

Solution:

(a)
$$\mathbf{D} = \epsilon_o \mathbf{E}, \quad \text{but}$$

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right)$$

$$\frac{\partial V}{\partial r} = -\frac{20}{r^3}\sin\theta\cos\phi$$

$$\frac{\partial V}{\partial \theta} = \frac{10}{r^2}\cos\theta\cos\phi$$

$$\frac{\partial V}{\partial \phi} = -\frac{10}{r^2}\sin\theta\sin\phi$$

$$\mathbf{E} = -\left[-\frac{20}{r^3}\sin\theta\cos\phi\,\mathbf{a}_r + \frac{1}{r}\frac{10}{r^2}\cos\theta\cos\phi\,\mathbf{a}_\theta - \frac{1}{r\sin\theta}\frac{10}{r^2}\sin\theta\sin\phi\,\mathbf{a}_\phi\right]$$

$$\mathbf{E} = \left[\frac{20}{r^3}\sin\theta\cos\phi\,\mathbf{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\mathbf{a}_\theta + \frac{10}{r^3}\sin\phi\,\mathbf{a}_\phi\right]$$

D at $(2, \pi/2, 0)$

$$\mathbf{D} = \epsilon_o\mathbf{E} = \epsilon_o\left[\frac{20}{8}\mathbf{a}_r - 0\mathbf{a}_\theta + 0\mathbf{a}_\phi\right] = 2.5\epsilon_o\mathbf{a}_r\,\frac{\mathrm{C}}{\mathrm{m}^2} = 22.1\,\mathbf{a}_r\ \ \mathrm{pC/m^2}$$

(b) The work done can be found in two ways, using either **E** or **V**.

Method 1:

$$\mathrm{W} = -\mathrm{Q}\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E}\,.\,d\mathbf{L}$$

Where,
$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta\, d\phi\mathbf{a}_\phi$$

$$A(1, 30^0, 120^0); \xrightarrow[d\mathbf{L}=dr\mathbf{a}_r]{} A'(4, 30^0, 120^0);$$

$$A'(4, 30^0, 120^0); \xrightarrow[d\mathbf{L}=rd\theta\mathbf{a}_\theta]{} B'(4, 90^0, 120^0);$$

$$B'(4, 90^0, 120^0); \xrightarrow[d\mathbf{L}=r\sin\theta\, d\phi\mathbf{a}_\phi]{} B(4, 90^0, 60^0)$$

$$\mathbf{E}\,.\,d\mathbf{L} = \frac{20}{r^3}\sin\theta\cos\phi\,.\,dr - \frac{10}{r^3}\cos\theta\cos\phi\,.\,rd\theta + \frac{10}{r^3}\sin\phi\,.\,d\phi$$

$$W = -Q\left[\int_1^4 \frac{20}{r^3}\sin\theta\cos\phi\,.\,dr\Big|_{\substack{\theta=30\\ \phi=120}} - \int_{30}^{90}\frac{10}{r^2}\cos\theta\cos\phi\,d\theta\Big|_{\substack{r=4\\ \phi=120}} + \int_{120}^{60}\frac{10}{r^3}\sin\phi\,d\phi\Big|_{\substack{r=4\\ \theta=90}}\right]$$

$$W = 28.125\ \mu\text{J}$$

Method 2: Since V is known.

$$\text{W} = -\text{Q}\int_{\text{A}}^{\text{B}} \mathbf{E}\,.\,d\mathbf{L}$$

$$W = Q\,V_{AB} = Q(V_B - V_A)$$

$$W = 10.10^{-6}\left(\frac{10}{4^2}\cos 90^0\cos 60^0 - \frac{10}{1^2}\cos 30^0\cos 120^0\right)$$

$$W = 10.10^{-6}\left(\frac{10}{32} - \frac{(-5)}{2}\right) = 28.125\ \mu\text{J}$$

Example 4.15:

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4,3,6)$, find several numerical values at point P: the potential V, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Solution: The potential at $P(-4,3,6)$, is

$$V_{(P)} = 2(-4)^2 \times 3 - 5 \times 6 = 66\ V$$

we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$\frac{\partial V}{\partial x} = 4xy;\quad \frac{\partial V}{\partial y} = 2x^2;\quad \frac{\partial V}{\partial z} = -5$$

$$\mathbf{E} = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z\ \text{V/m}$$

The value of $\mathbf{E}$ at point P is

$$\boldsymbol{E}_{(P)} = -4(-4)3\mathbf{a}_x - 2(-4)^2\mathbf{a}_y + 5\mathbf{a}_z\ \text{V/m} = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z\ \text{V/m}$$

$$|E_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \text{ V/m}$$

The direction of **E** at P is given by the unit vector

$$\mathbf{a}_{E,P} = \frac{48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z}{57.9} = 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_o \mathbf{E} = -35.4xy\mathbf{a}_x - 17.71x^2\mathbf{a}_y + 44.3\mathbf{a}_z \text{ pC/m}^3$$

Finally, we may use the divergence relationship to find the volume charge density that is the source of the given potential field,

$$\rho_v = \nabla . \mathbf{D} = -35.4y \text{ pC/m}^3$$

At point P, $\rho_v = 106.2 \text{ pC/m}^3$

Example 4.16:

Given that $\mathbf{E} = (3x^2 + y)\mathbf{a}_x + x\mathbf{a}_y$ kV/m, find the work done in moving a $-2\ \mu C$ charge from $(0, 5, 0)$ to $(2, -1, 0)$ by taking the path.

(a) $(0,5,0) \to (2,5,0) \to (2,-1,0)$; (b) $y = 5 - 3x$; (c) $y = 5 - 3x$;

Solution: (a) $(0,5,0) \to (2,5,0) \to (2,-1,0)$

$$A(0,5,0); \xrightarrow[d\mathbf{L}=dx]{} A'(2,5,0)$$

$$A'(2,5,0); \xrightarrow[d\mathbf{L}=dy]{} B(2,-1,0)$$

$$\mathrm{W} = -\mathrm{Q}\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E}\,.\,d\mathbf{L}$$

$$W = -Q\left[\int_0^2 (3x^2 + y)dx\bigg|_{y=5} + \int_5^{-1} xdy\bigg|_{x=2}\right] = -(-2)(18 - 12) = 12\ \mu\text{J}$$

(b) $y = 5 - 3x \quad \Rightarrow \quad dy = -3dx$

$$\mathrm{W} = -\mathrm{Q}\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E}\,.\,d\mathbf{L} = -Q\int_0^2 [(3x^2 + 5 - 3x)dx + x(-3)dx]$$

$$W = -Q\int_0^2 (3x^2 - 6x + 5)dx = -(-2)[8 - 12 + 10] = 12\ \mu\text{J}$$

(c) $$y = 5 - 3x \quad \text{and} \quad x = -\frac{y}{3} + \frac{5}{3}$$

$$W = -Q\int_A^B \mathbf{E}\,.\,d\mathbf{L} = -(-2)\left[\int_0^2 (3x^2 - 3x + 5)dx + \int_5^{-1}\left(-\frac{y}{3} + \frac{5}{3}\right)dy\right] = 12\ \mu\text{J}$$

4.8 AN ELECTRIC DIPOLE AND FLUX LINES:

An electric dipole is formed when two point charges of equal magnitude but opposite sign are separated by a small distance.

Consider the dipole shown in Fig. 4.8. The potential at point $P(r, \theta, \phi)$ is given by:

$$V_{AB} = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = \frac{Q}{4\pi\epsilon_o}\left(\frac{r_B - r_A}{r_A\, r_B}\right)$$

- r_A the distances between P and $+Q$
- r_B the distances between P and $-Q$

If $r >> d$; $r_B - r_A \simeq d\cos\theta$; and $r_A\, r_B = r^2$; then

$$V = \frac{Q}{4\pi\epsilon_o}\left(\frac{d\cos\theta}{r^2}\right)$$

Since, $d\cos\theta = \mathbf{d}.\mathbf{a}_r$, where $\mathbf{d} = d\mathbf{a}_z$, if we define

$$\mathbf{P} = Q\mathbf{d}$$

The dipole moment, may be written as

$$V = \frac{\mathbf{P}.\mathbf{a}_r}{4\pi\epsilon_o r^2}$$

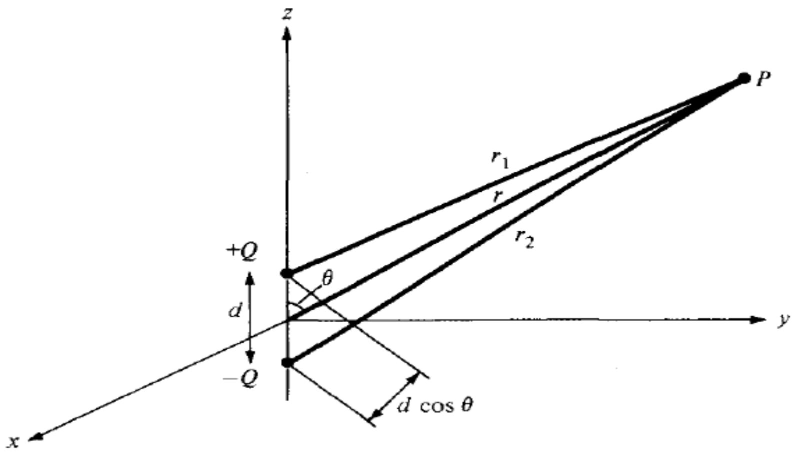

Fig. 4.8

Note that the dipole moment $\mathbf{P}$ is directed from $-Q$ to $+Q$. If the dipole center is not at the origin but at $\mathbf{r}'$, then

$$V(\mathbf{r}) = \frac{\mathbf{P}.(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|^3}$$

The electric field due to the dipole with center at the origin, shown in Fig. 4.8, can be obtained from equations above.[2]

$$\mathbf{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta\right] = \frac{Qd\cos\theta}{2\pi\epsilon_o r^3}\mathbf{a}_r + \frac{Qd\sin\theta}{4\pi\epsilon_o r^3}\mathbf{a}_\theta$$

$$\mathbf{E} = \frac{\mathbf{P}}{4\pi\epsilon_o r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Where, $P = |\mathbf{P}| = Qd$.

Notice that

- A point charge is a monopole and its electric field varies inversely as r^2 while its potential field varies inversely as r.
- The electric field due to a dipole varies inversely as r^3 while its potential varies inversely as r^2.

An electric flux line is an imaginary path or line drawn in such a way that its direction at any point is the direction of the electric field at that point.

Equipotential surface: defined as any surface on which the potential is the same.

Equipotential line: is the intersection of an equipotential surface and a plane results in a path or line.

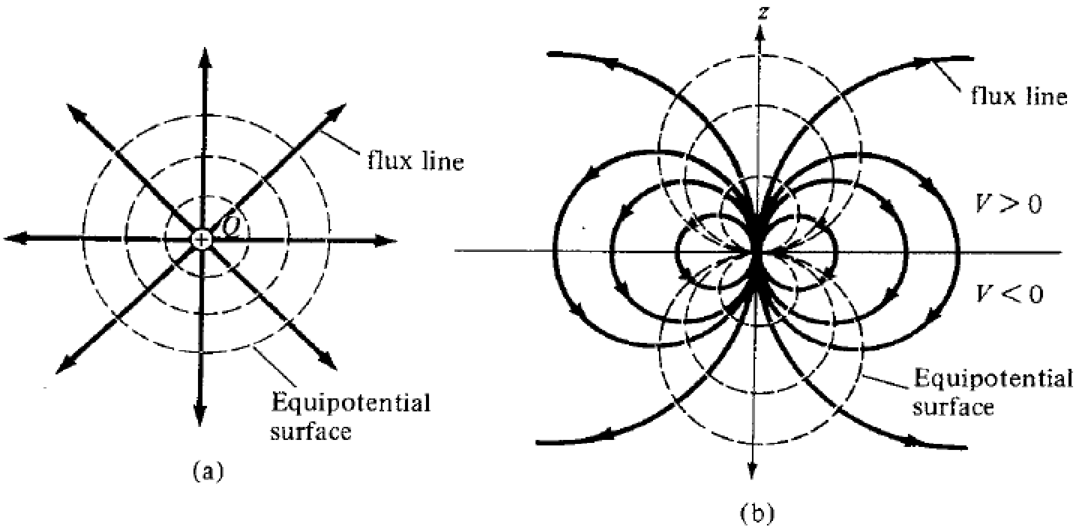

Fig. 4.21 Equipotential surfaces for (a) a point charge and (b) an electric dipole.

Example 4.17:

Two dipoles with dipole moments $-5\mathbf{a}_z$ nC/m and $9\mathbf{a}_z$ nC/m are located at points $(0,0,-2\ )$ and (0,0,3), respectively. Find the potential at the origin.

Solution:

$$V=\sum_{k=1}^{2}\frac{\mathbf{P}_{\boldsymbol{k}}.\mathbf{r}_k}{4\pi\epsilon_o(r_k)^3}=\frac{1}{4\pi\epsilon_o}\left[\frac{\mathbf{P}_{\mathbf{1}}.\mathbf{r}_1}{(r_1)^3}+\frac{\mathbf{P}_{\mathbf{2}}.\mathbf{r}_2}{(r_2)^3}\right]$$

Where,

$\mathbf{P}_{\mathbf{1}}=-5\mathbf{a}_z;$ $\quad\mathbf{r}_1=(0,0,0\ )-(0,0,-2)=2\mathbf{a}_z;$ $\quad r_1=|\mathbf{r}_1|=2$

$\mathbf{P}_{\mathbf{1}}=9\mathbf{a}_z;$ $\quad\mathbf{r}_2=(0,0,0\ )-(0,0,3)=-3\mathbf{a}_z;$ $\quad r_2=|\mathbf{r}_2|=3$

Hence,

$$V=\frac{1}{4\pi\frac{10^{-9}}{36\pi}}\left[\frac{(-5\mathbf{a}_z).(2\mathbf{a}_z)}{(2)^3}+\frac{(9\mathbf{a}_z).(-3\mathbf{a}_z)}{(3)^3}\right]\times 10^{-9}$$

$$V=\frac{9\times 10^{-9}}{10^{-9}}\left[\frac{(-10)}{8}-\frac{27}{27}\right]=-20.25\ V$$

Problems

4.1.[1] Calculate the work done in moving a 4 C charge from $B(1, 0, 0)$ to $A(0,2,0)$ along the path $y = 2 - 2x,\ \ z = 0$ in the field $\mathbf{E}$ if (a) $\mathbf{E} = 5\mathbf{a}_x$ V/m; (b) $\mathbf{E} = 5x\mathbf{a}_x$ V/m; (c) $\mathbf{E} = 5x\mathbf{a}_x + 5y\mathbf{a}_y$ V/m. [Ans: 20 J : 10 J : −30 J]

4.2.[1] Let $\mathbf{E} = y\mathbf{a}_x$ V/m at a certain instant of time, and calculate the work required to move a 3 C charge from $(1, 3, 5)$ to $(2, 0, 3)$ along the straight line segments joining: (a) $(1, 3, 5)$ to $(2, 3, 5)$ to $(2, 0, 5)$ to $(2, 0, 3)$; (b) $(1, 3, 5)$ to $(1, 3, 3)$ to $(1, 0, 3)$ to $(2, 0, 3)$. [Ans: − 9 J : 0]

4.3.[2] An electric dipole of $100\ \mathbf{a}_z$ pC. m is located at the origin. Find V and $\mathbf{E}$ at points (a) (0,0,10) (b) $(1, \pi/3, \pi/2)$

[Ans: (a) 9 mV : 1.8 $\mathbf{a}_r$ mV/m (b) 0.45 V : 0.9 $\mathbf{a}_r$ + 0.7794 $\mathbf{a}_\theta$ V/m]