THEORY OF STRUCTURES

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INFLUENCE LINE of STATICALLY DETERMINATE STRUCTURE

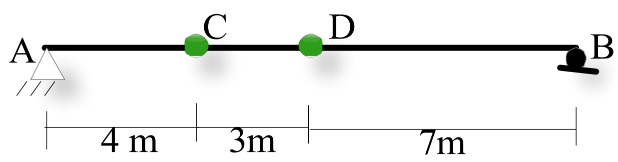
Influence lines have important application for the design of structures that resist large live load.

The (I.L) can be defined as diagram that represents the variation of a certain function such as (REACTION, SHEAR, AND MOMENT) when unit load moves on structure.

AIMS FROM DRAWING I.L

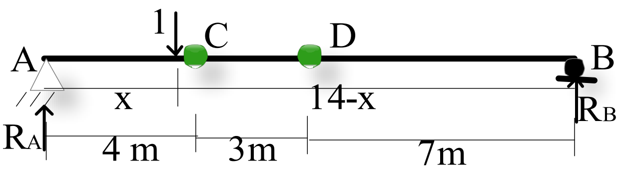
1. **TO PREDICT CRITICAL LOCATION FOR LIVE LOADS OR MOVING LOADS THAT GIVES MAXIMUIM VALUE FOR A CERTAIN FUNCTION**
2. **TO CALCULATE THE MAX VALUE FOR A CERTAIN FUNCTION WHEN THE LIVE LOADS AND MOVING LOADS AT ITS CIRTICAL LOCATIONS.**

Example 1:- draw I.L for reactions at A and B , shear and moment at C and D .



Solution

Put unit load at distance x from point A as shown .



$$\sum M\ B = 0$$

$$R_A = \frac{1*(14-X)}{14} = 1 - \frac{X}{14}$$

$$\sum FY = 0$$

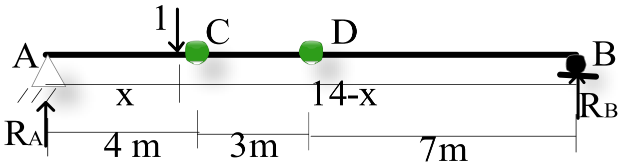
$$R_B = 1 - 1 + \frac{X}{14} = \frac{X}{14}$$

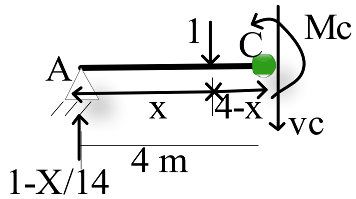
reaction	X=0	X=14
A	1	٠
B	٠	١

To find shear and moment at point c , two cases must be done , first put unit load before point c and the another case by putting unit load after point c as illustrated in figures below

<u>Case 1</u>

$\mathbf{0 \leq x \leq 4}$





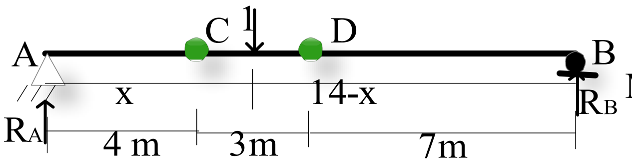
$$\sum fy = 0$$

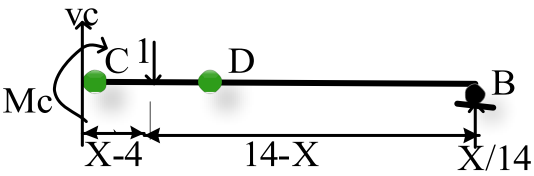
$$V_{C=}1 - \frac{X}{14} - 1 = -\frac{X}{14}$$

$$M_{C=}\left(1 - \frac{X}{14}\right) * 4 - 1(4 - X) = \frac{5X}{7}$$

<u>Case 2</u>

$\mathbf{4 \leq x \leq 14}$





$$\sum fy = 0$$

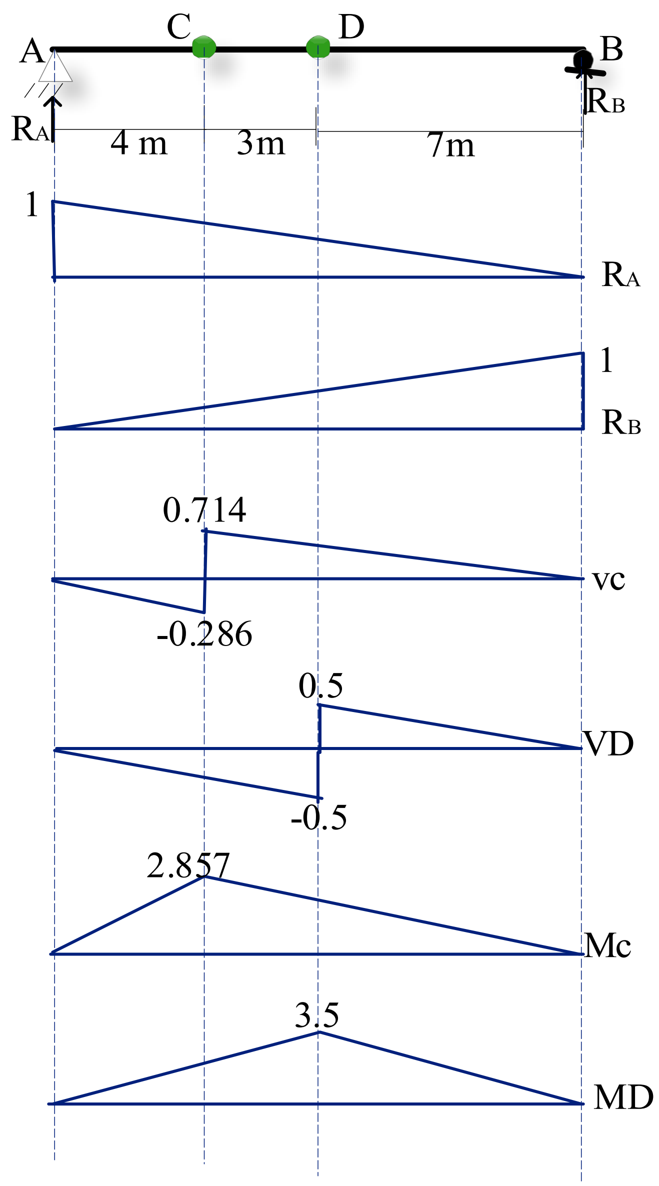
$$V_{C=}1 - \frac{X}{14}$$

$$M_{C=}10 * \frac{X}{14} - 1(X - 4) = -\frac{2X}{7} + 4$$

X	VC	MC
CASE 1		
0	0	0
4	-0.286	2.857
CASE 2		
4	0.714	2.857
14	0	0

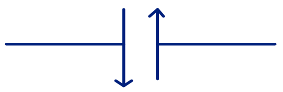
As home work , find shear and moment at point D (do same procedure for point c)

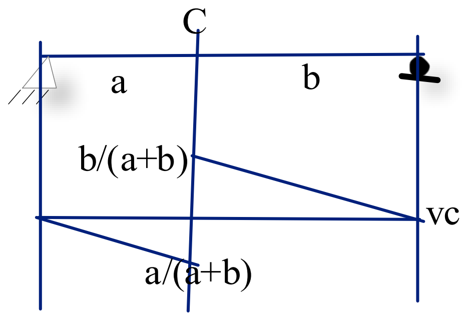
Now , I.L diagram can be constructed depending on equations that obtain previously


C
D
A
B
RB
RA
4 m
3m
7m
1
RA
1
RB
0.714
vc
-0.286
0.5
VD
-0.5
2.857
Mc
3.5
MD

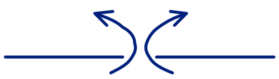
Note :- the previous method that followed to draw influence line is tedious and needs more time and hard work . There is another way to draw influence line for any function easily. This method developed by HEINRICH MULLER in 1886 and known as HEINRICH MULLER principle. This method can be summarized by following steps.

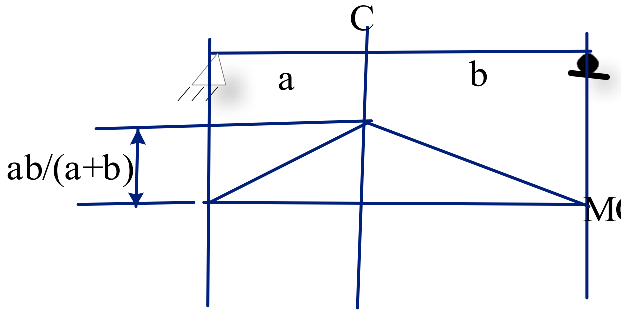
1. **Reaction is considered positive if its direction to upward, to draw i.l for any reaction, move this reaction one unit to upward as shown above for reactions at A and B.**
2. **Positive shear as shown , to draw I.I for shear function , move left piece to downward while right piece to upward . The value of total movement at point must equal 1.0.**





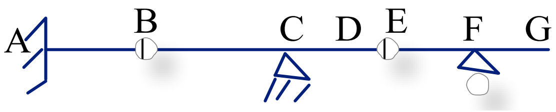
3. **Positive moment causes compression at top (make beam smile) . to draw I.L , rotate left piece counter clockwise while right piece with clockwise (the total value of angle of rotation must be 1.0) . this note is illustrated by following figures.**





Notes about I.L for statically determinate structures

1. **The path of unit load can be divided into pieces. All piece has ends may be (exterior support, free end, interior hinge or cut).**



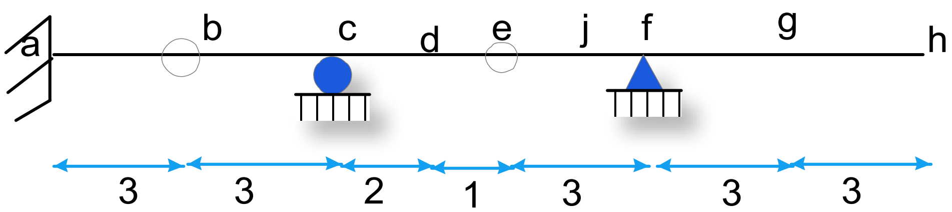
For I.L (R_A) : consider 3 segments (AB , BE, EG)

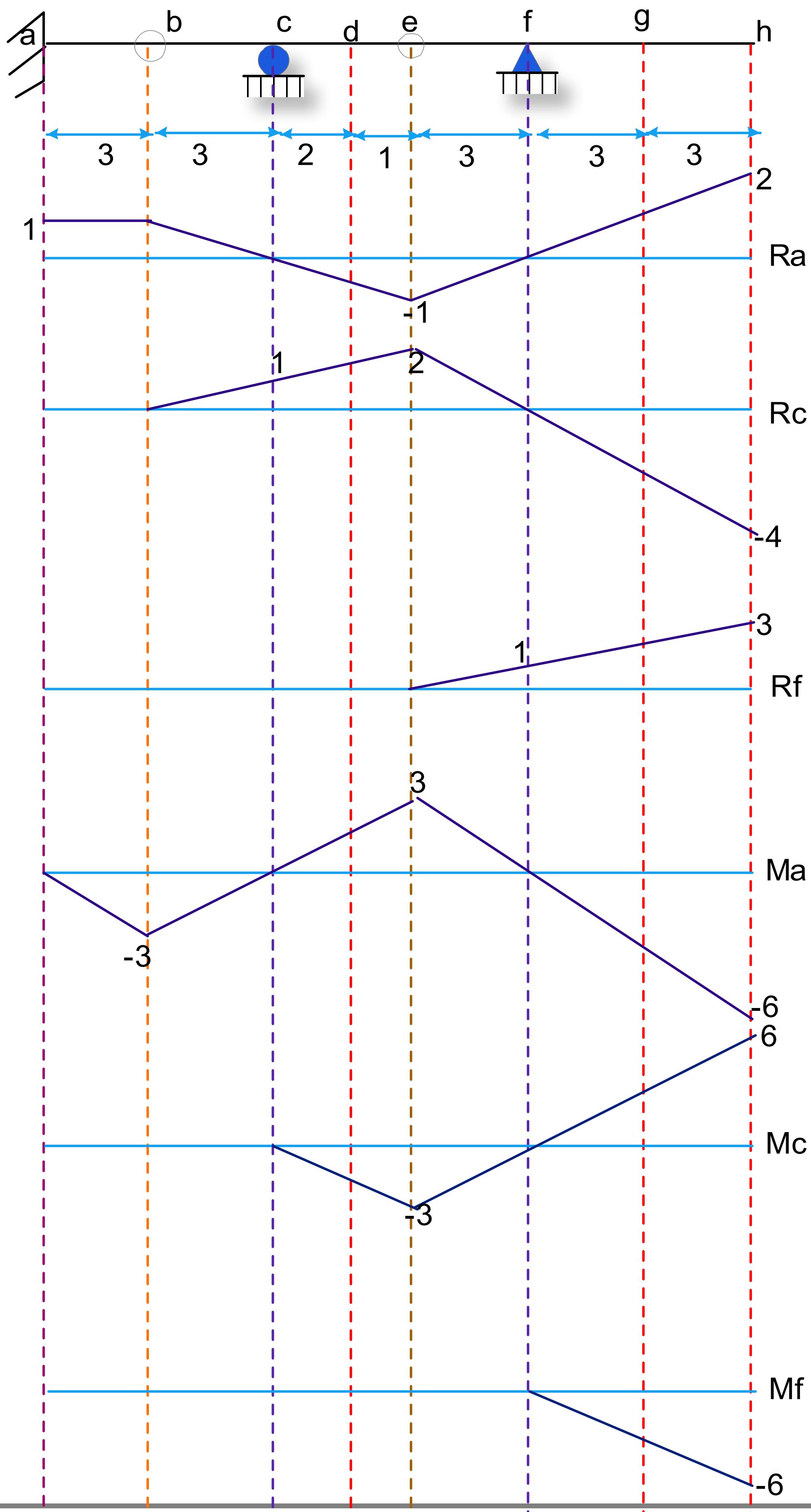
For I.L (V_{CL}) : consider 4 segments (AB ,BC, CE, EG)

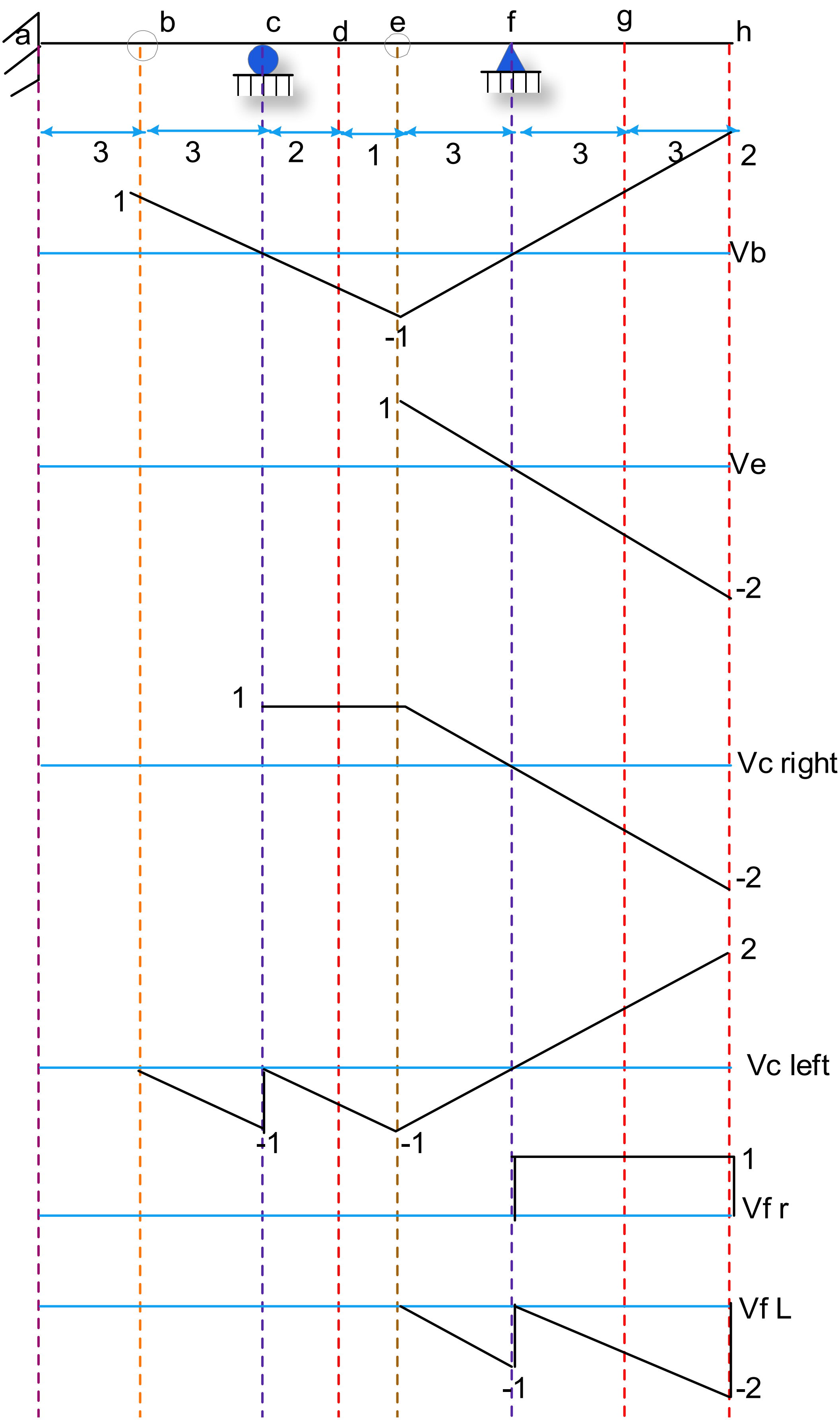
For I.L (M_D) : consider 4 segments (AB , BD, DE,EG)

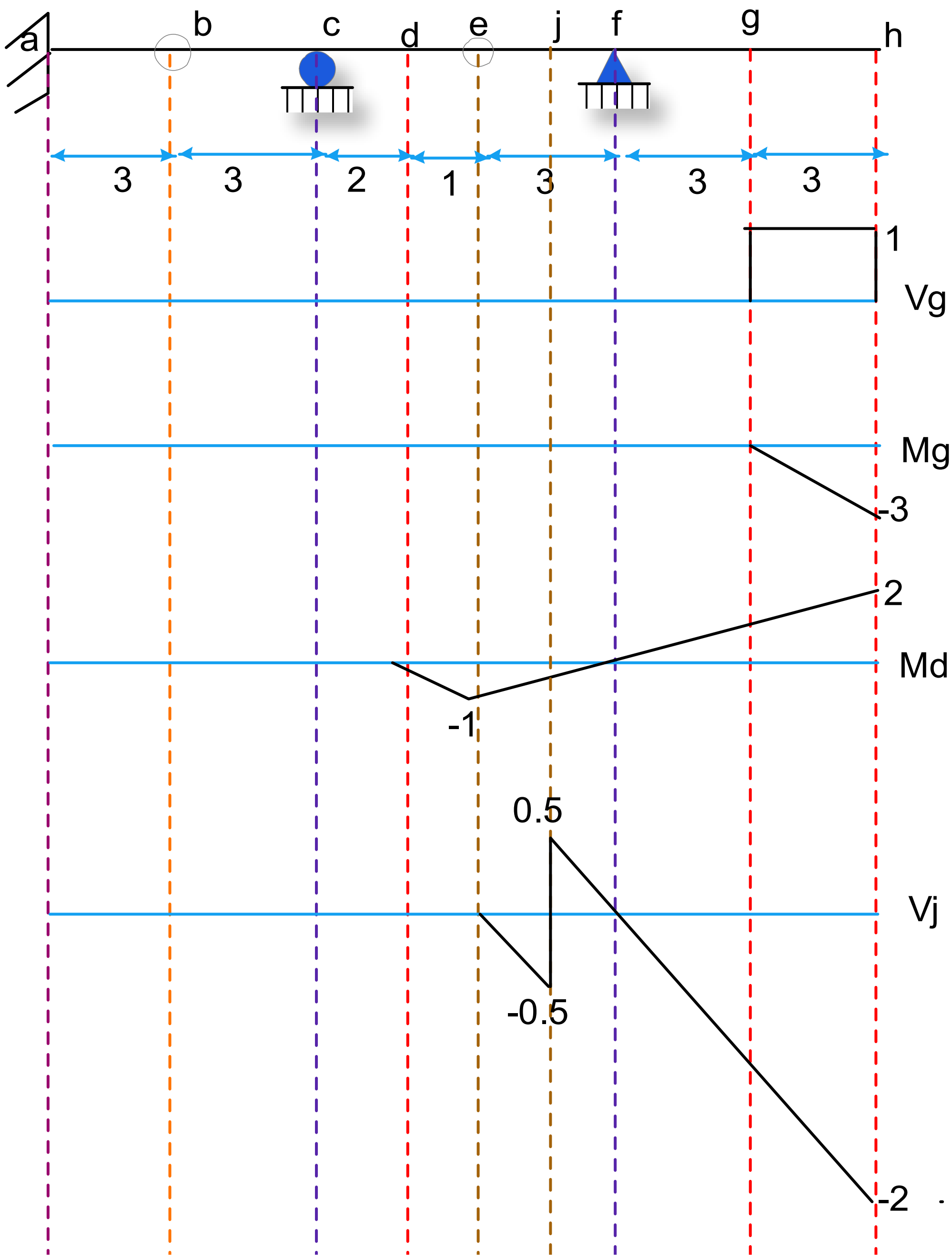
2. **The supports (exterior or interior) are considered fixed point have zero value at all cases except if influence line is required at this point. At this case, the support move upward one unit while others supports stay fix.**
3. **I.L for statically determinate structures is straight lines. Therefore, any piece can draw if the values at its ends are predicated.**
4. **One piece cannot break or bend but it always stay straight.**
5. **Interior hinge has ability to move if the adjacent piece permits that.**
6. **The distance of two points of shear equals to 1.0. If right point fixed at its location , then left point drops one unit . If left point fixed at its location then right point rises one unit.**
7. **Two points of moment stay together at all cases. If one, of them cannot rotate then apply moment to another point and the angle of rotation will be 45.**

Example (2):- draw I.L for reactions at a , c, and f , shear at b, e,c , f , g , and j, and moment at a,c,f , g and d.




a
b
c
d
e
f
g
h
3
3
2
1
3
3
3
1
2
-1
Ra
1
2
-4
Rc
3
1
Rf
3
-3
Ma
-6
6
Mc
-3
Mf
-6


a
b
c
d
e
f
g
h
3
3
2
1
3
3
3
2
1
Vb
-1
1
Ve
-2
1
Vc right
-2
2
Vc left
-1
-1
1
Vf r
Vf L
-1
-2


a
b
c
d
e
j
f
g
h
3
3
2
1
3
3
3
1
Vg
Mg
-3
2
Md
-1
0.5
Vj
-0.5
-2

Maximum Value of Function

Once the I.L for a function has been constructed, it will then possible to position the live load on the beam that will produce the max value of the function. Two types of loadings will now be considered.

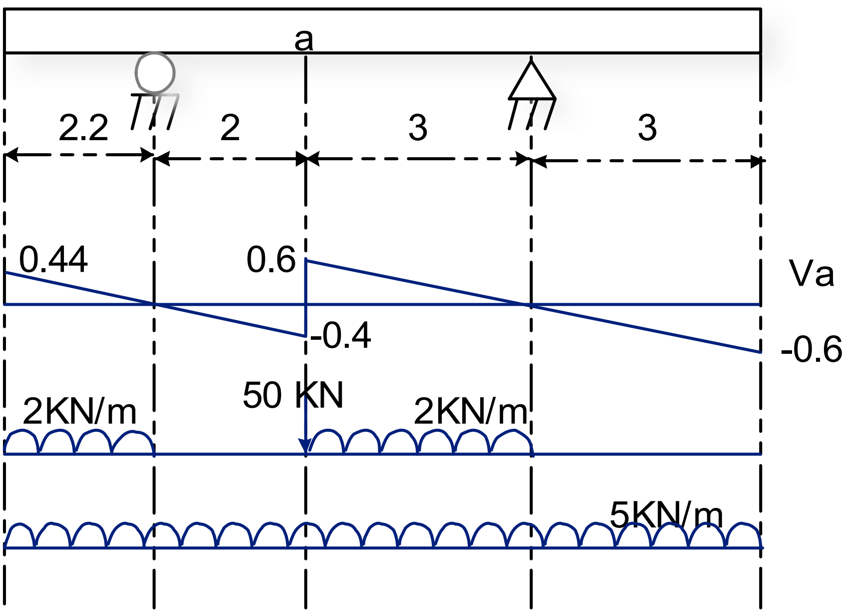
A. Concentrated load P(KN)

F_{max}= P * max ordinate of I.L

B. Uniform load w(KN/m)

F_{max}= w* area of I.L

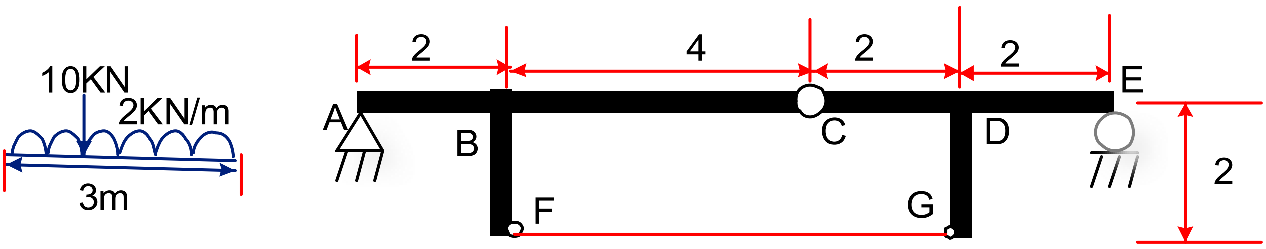
Example(3):- find max positive shear force at point a due to a moving single concentrated load of 50 KN, uniform live load of 2 KN/m and a uniform dead load of 5 KN/m.

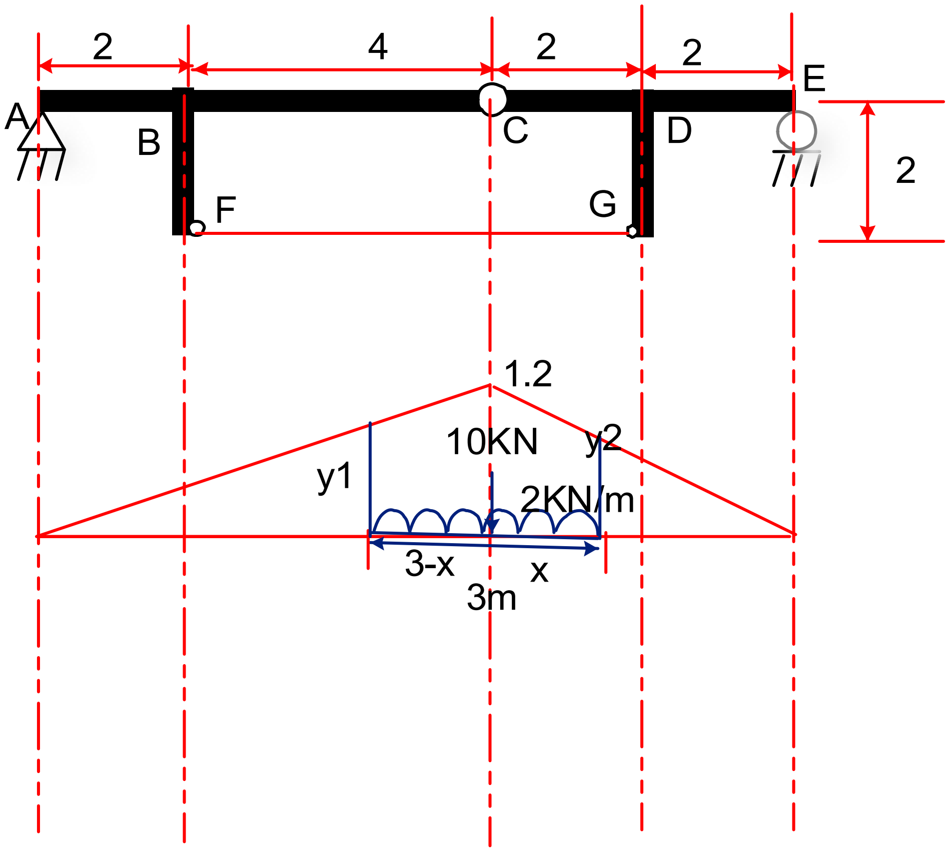


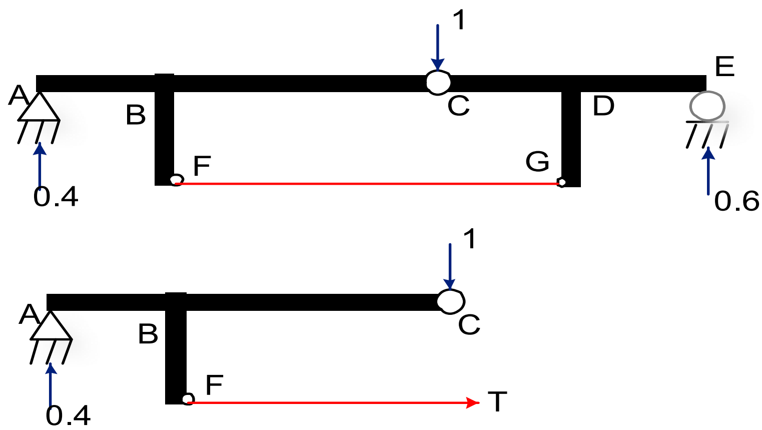
Solution

$$V_{a\,live} = 50(0.6) + 2\left(\frac{0.6*3}{2}\right) + 2\left(\frac{0.44*2.2}{2}\right) + 5\left(\frac{-0.6*3}{2} + \frac{0.6*3}{2} + \frac{-0.4*2}{2} + \frac{0.44*2.2}{2}\right) = 33.2\ KN$$

Example (4):- find max. Tensile force in cable (FG) due to the moving load shown







∑Mc=0

2T-0.4*6=0 T=1.2KN

$$\frac{X}{3-X}=\frac{4}{6}$$

$$X=1.2$$

$$\frac{1.2}{6} = \frac{y1}{4.2}$$

$$y1 = 0.84$$

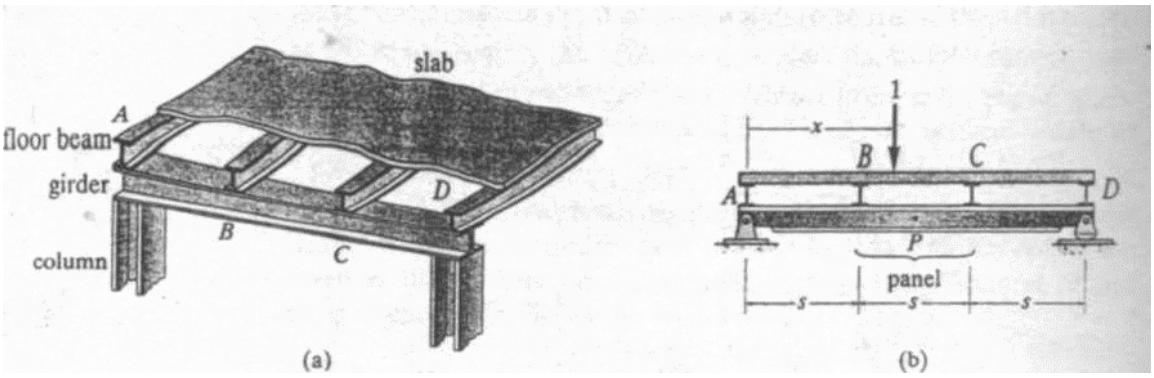
$$\frac{1.2}{4} = \frac{y2}{2.8}$$

$$y2 = 0.84$$

$$FG = 10 * 1.2 + 2\left(0.84 * 3 + \frac{0.36 * 3}{2}\right) = 18.12KN$$

Influence lines for floor girders

Occasionally, floor systems are constructed as shown in figure below. Where it can be seen that floor loads are transmitted from slabs to floor beams, then to side girders, and finally supporting column. An idealized model of this system is shown in plane view shown below. Here the slab is assumed to be a one-way slab and is segmented into simply supported spans resting on the floor beams. Furthermore, the girder is simply supported on the columns.

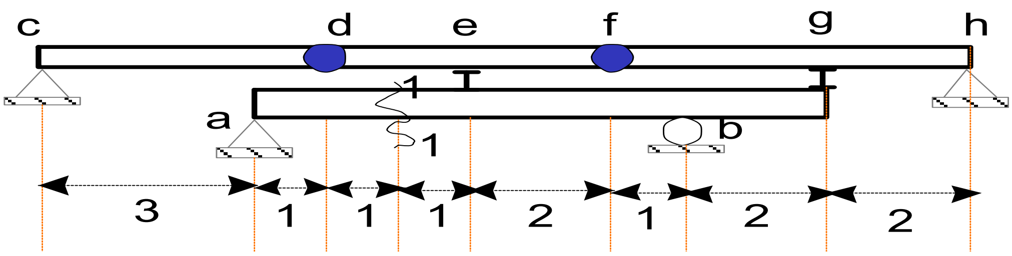


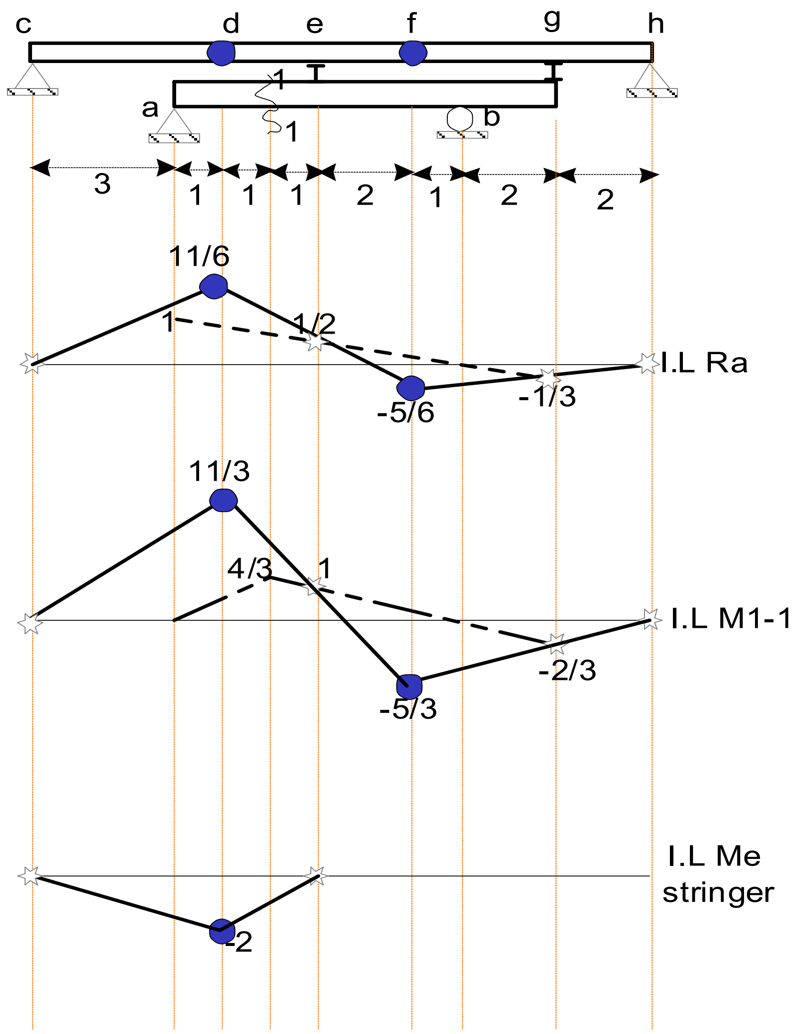
Steps of drawing influence line of girder –floor beam system

1. Only girder is considered existing.
2. Draw I.L for required function on basis that girder is ordinary beam.
3. Drop floor beams on drawn I.L at location of its existing.
4. Points of floor beams are connected according to piece of stringer or slab. Exterior supports of stringers are fix point have zero value at datum.
5. In case of drawing I.L of a certain function for stringer itself , the drawing will perform on basis stringer is ordinary beams having supports represented by floor beams

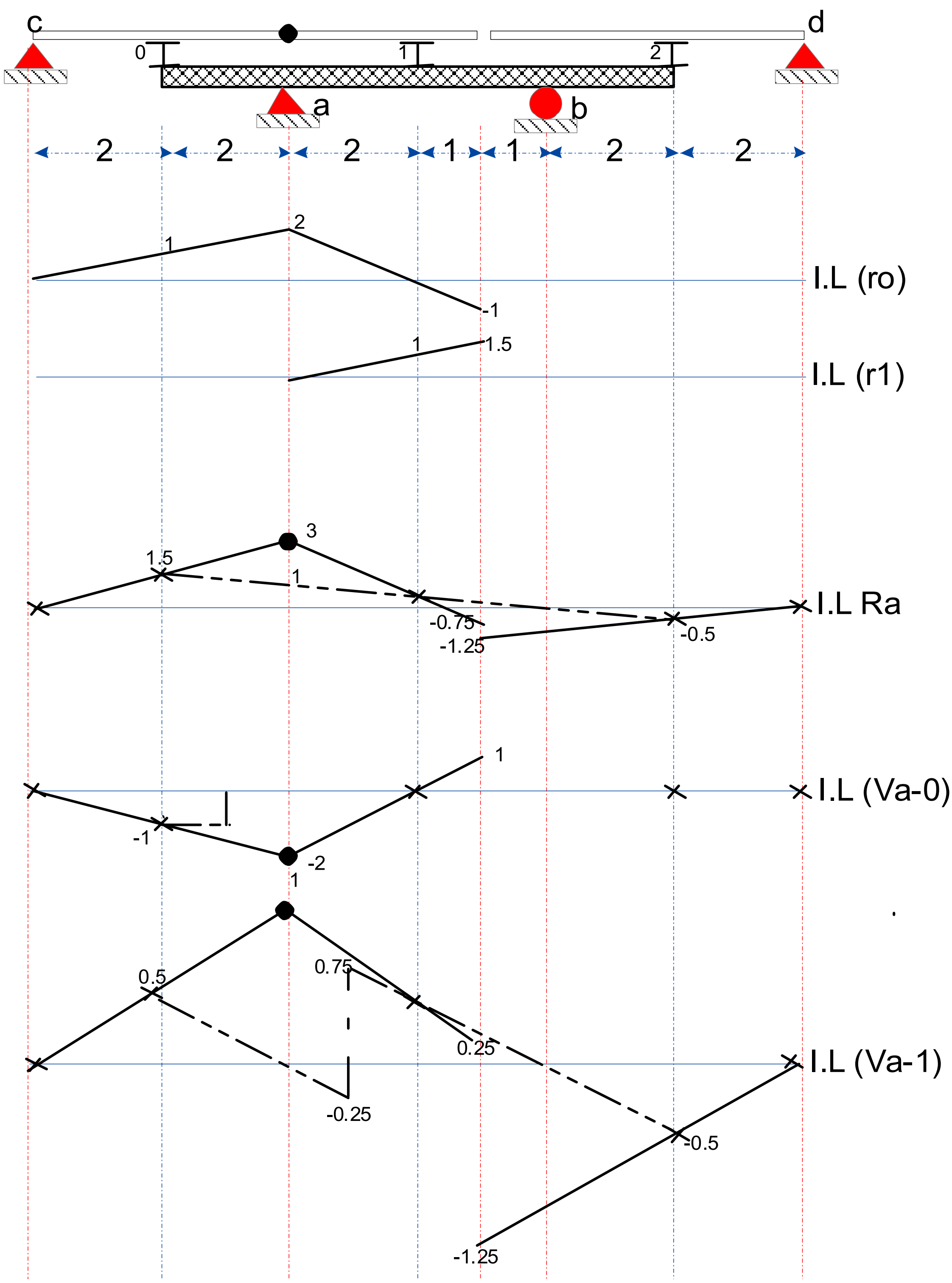
Example :- for the girder- floor- beam – stringer system shown in fig.

1. Draw I.L for Ra & M1-1 in girder
2. Draw I.L for Me in stringer.





Example: - draw I.L for r_0 , r_1 , Ra, $V_{o\text{-}a}$, and $V_{a\text{-}1}$


c
d
0
1
2
a
b
2
2
2
1
1
2
2
I.L (ro)
I.L (r1)
I.L Ra
I.L (Va-0)
I.L (Va-1)
1.5
3
-0.75
-1.25
-0.5
-1
-2
0.5
0.75
0.25
-0.25

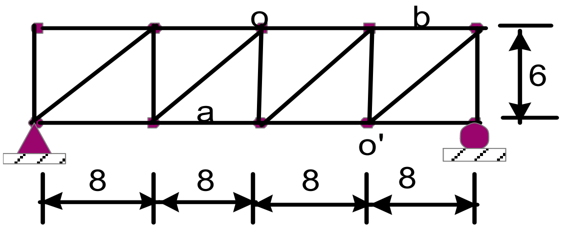
Influence line for trusses

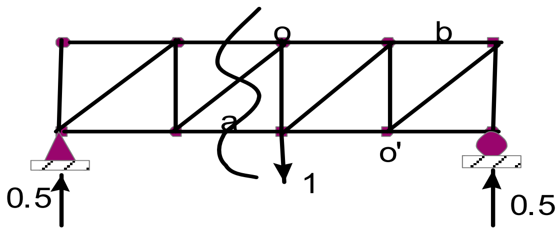
I.L can be constructed for the bar forces in truss members and are important in determining the location of live loads leading to maximum bar forces in truss members as well as for computing the actual values of these maximum bar forces .

Generally, there are three cases in drawing I.L for truss. These can be summarized as

Case 1:- if the force of bar that required to draw I.L to it can be determined from cut and using ∑M=0 about a certain point , then I.L likes I.L of moment about this point with dealing the truss as beam. However, if the location of point is above then the drawing will be same as beam, while if the point is down then the drawing will be inversely to that of beam.

Example: - draw I.L for bar a & b.

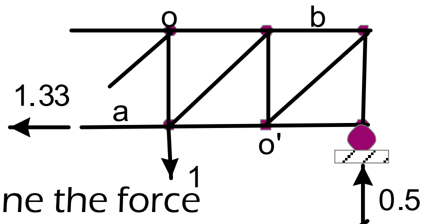




The force at bar a can be determined directly

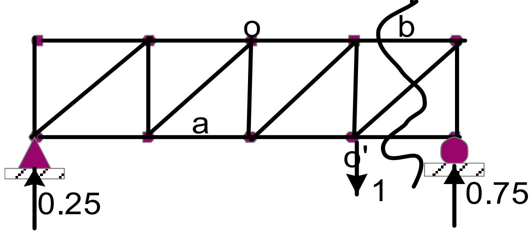
By applying moment equation about point o

Therefore put unit load at point o and determine the force



$$\sum Mo = 0$$

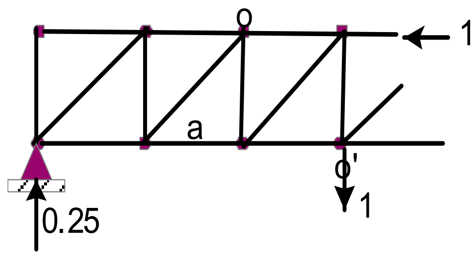
$$F_a = \frac{0.5 \; x \; 16}{6} = 1.33 \; T$$

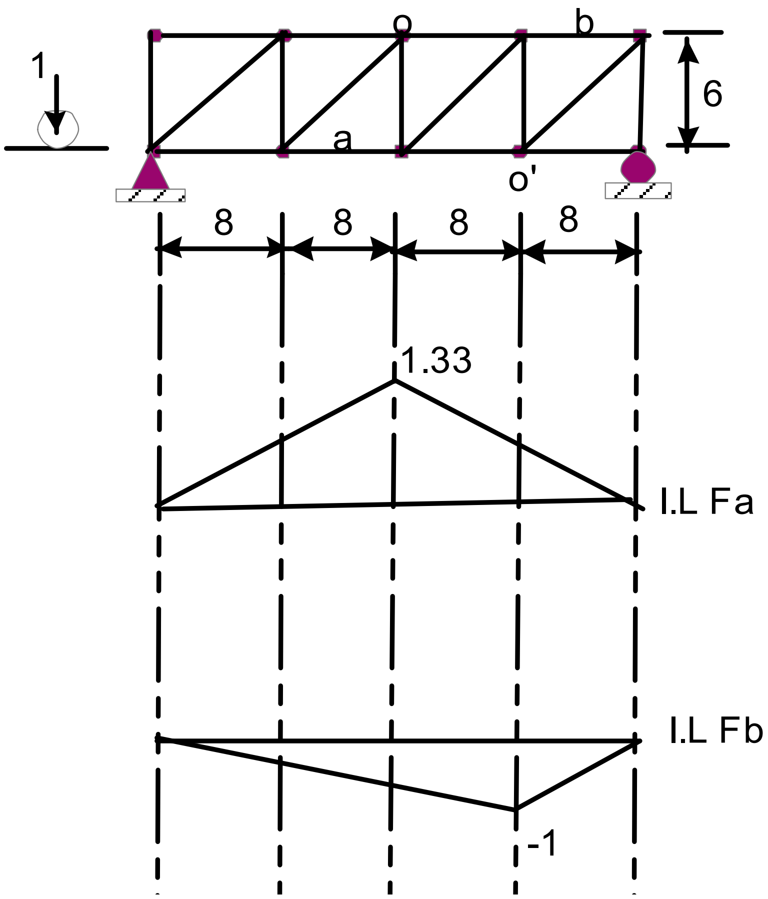


Same procedure is applied to bar b

$$\sum Mo' = 0$$

$$F_b = \frac{0.25 \; x \; 24}{6} = 1.00 \; C$$

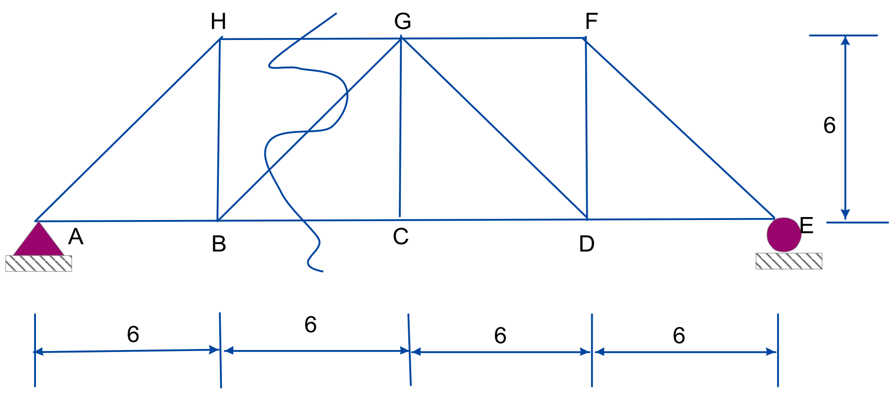


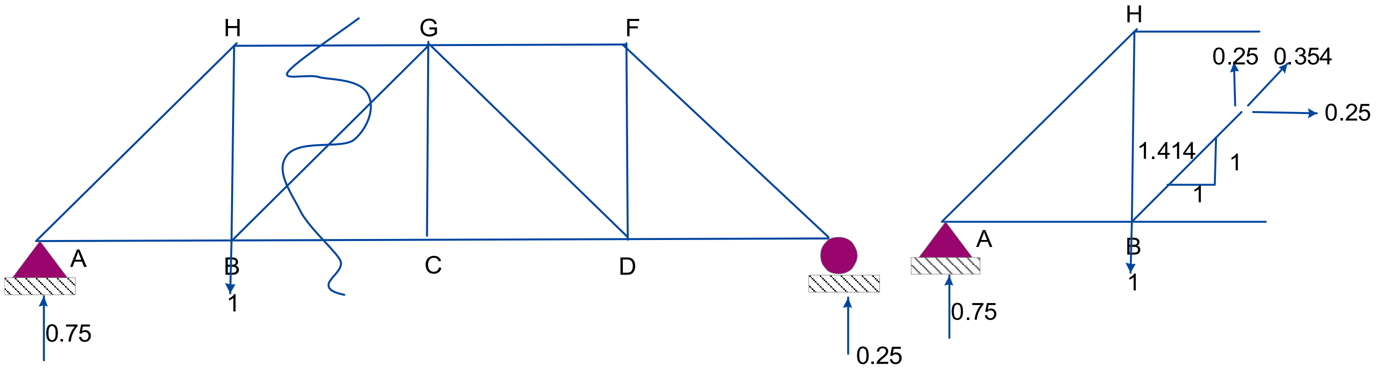


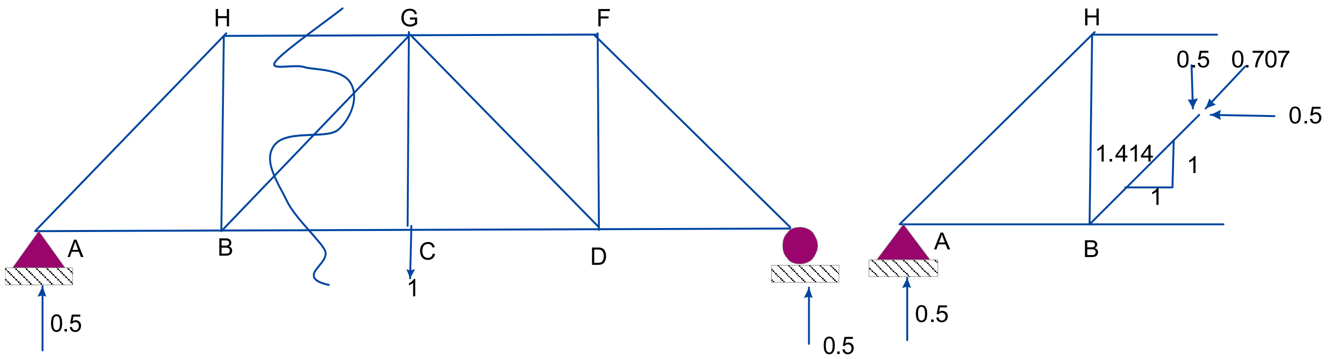
Case 2:- if the force at a certain bar can be determined by cut and using ∑Fy=0 , then I.L can be constructed by following steps

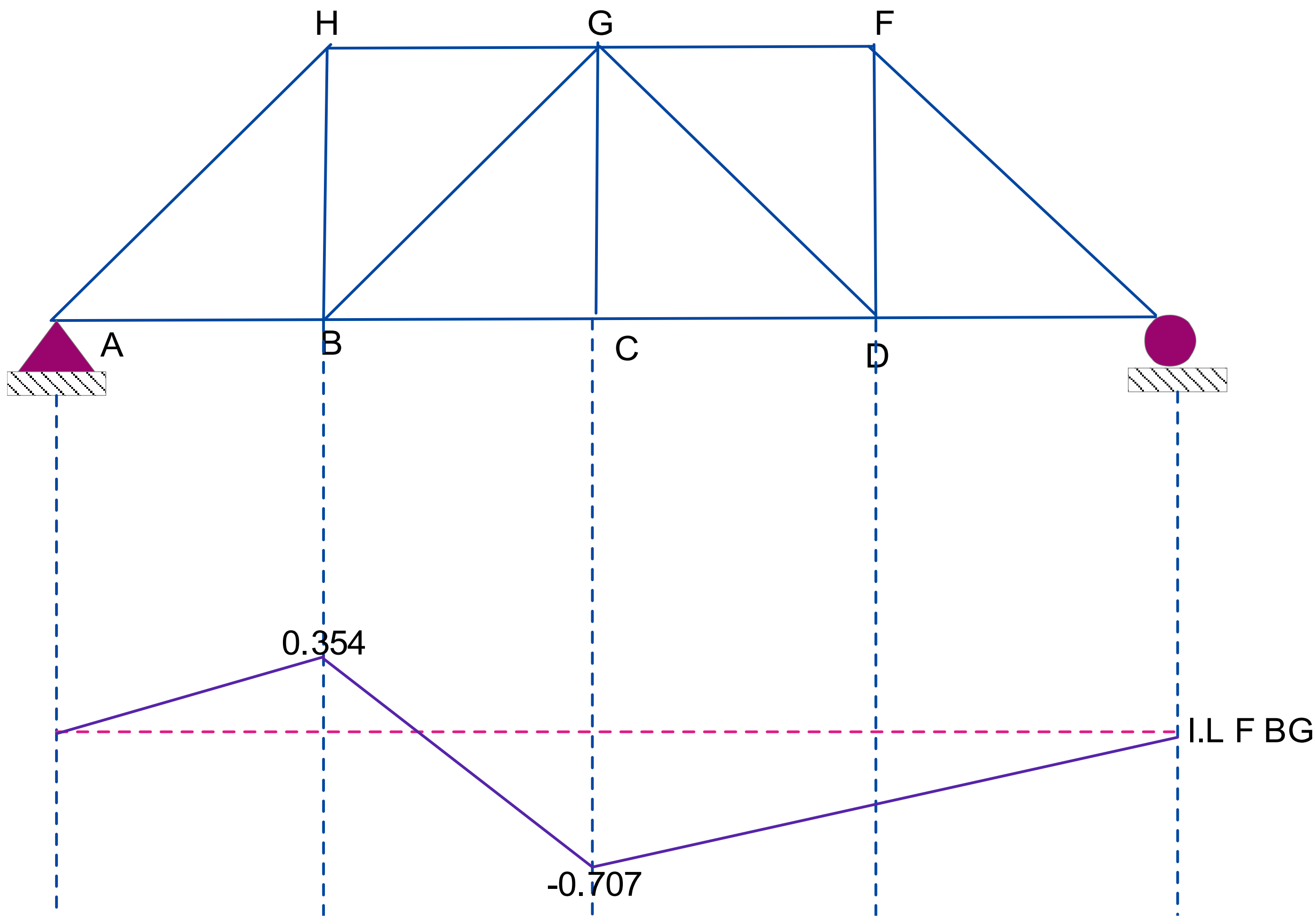
1- Supports are fixed point
2- Analysis is carried out twice. First by putting unit load at first joint on left cut directly. Second by putting load at first joint on right cut directly.
3- Connect two points and points of supports by straight line . the slope of first line equals of slope of third line.

Example:- draw influence line for member BG




H
G
F
A
B
1
C
D
0.75
0.25
H
0.25
0.354
0.25
1.414
1
1
A
B
1
0.75

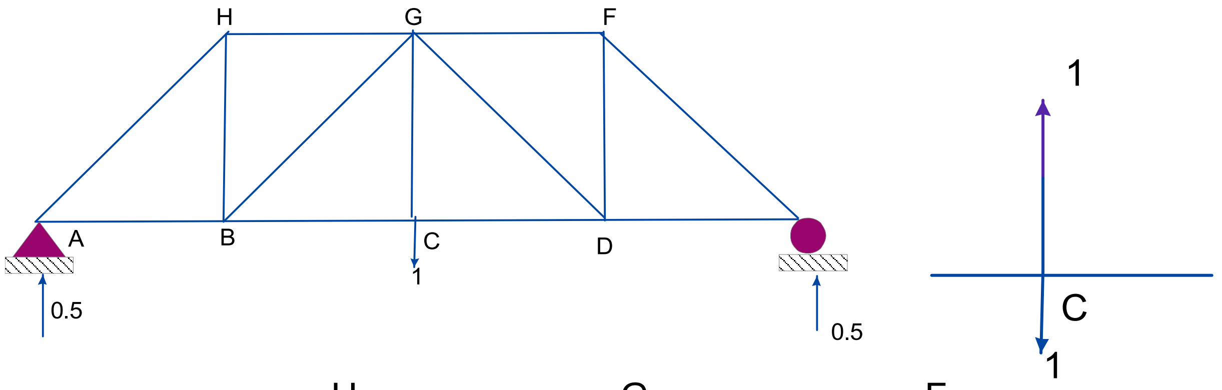

H
G
F
A
B
C
1
D
0.5
0.5
H
0.5
0.707
0.5
1.414
1
1
A
B
0.5

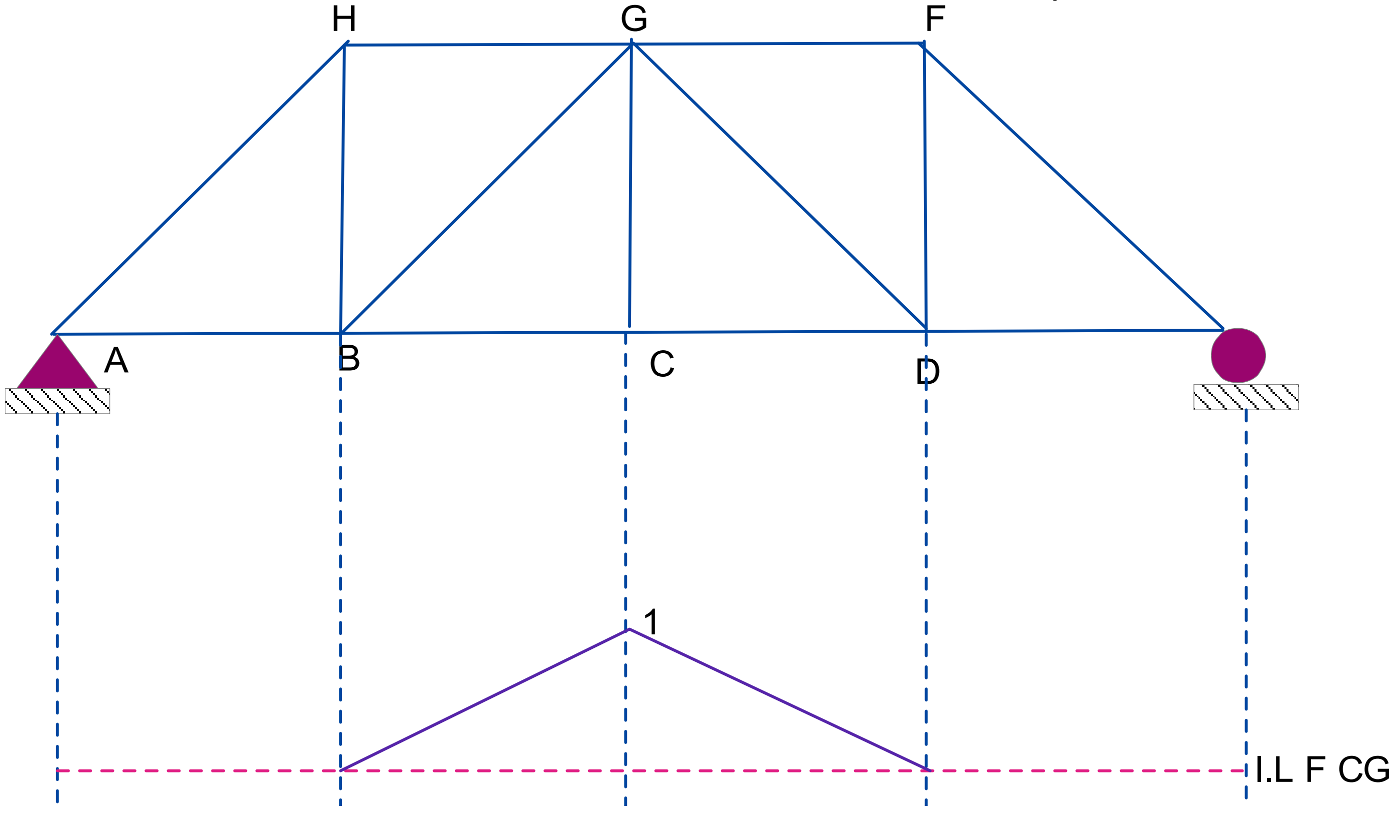

H
G
F
A
B
C
D
0.354
I.L F BG
-0.707

Case 3:- if cases 1 &2 cant applied then I.L can be constructed by putting unit load at each joint on path of motion of moving load.

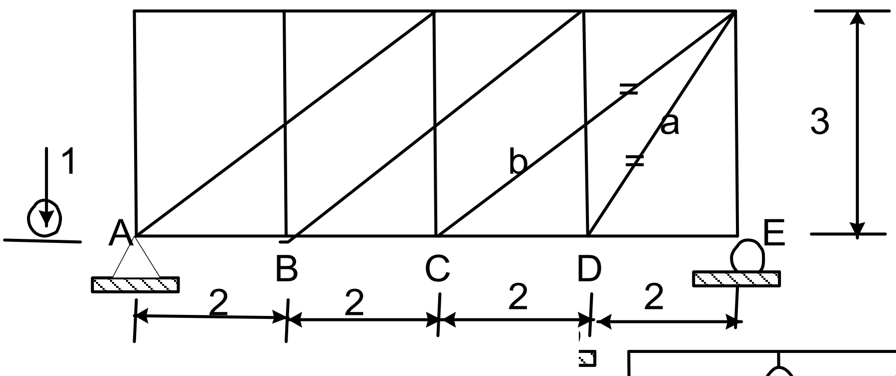
Example: - draw I.L for member CG for above truss

The force at member CG will be zero except when moving load at joint c



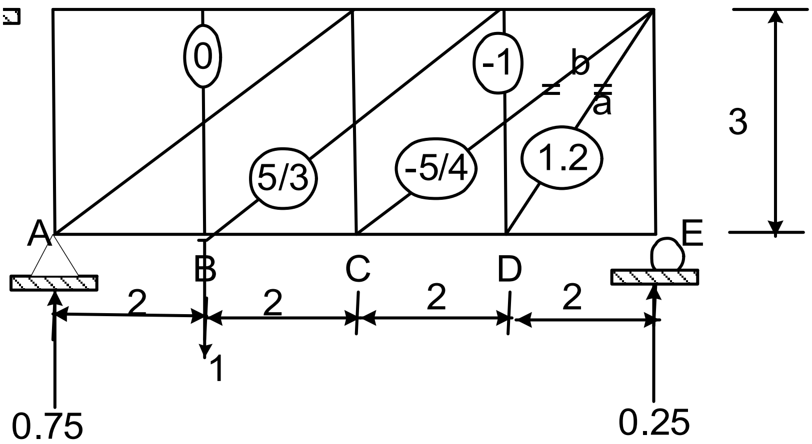


Example:- draw I.L for bar forces a & b



- **1@ A & E**

Fa=Fb=0



- **1@B**

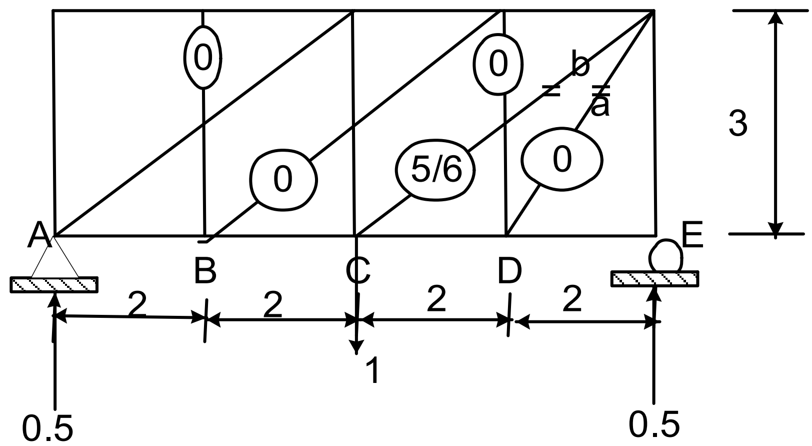
Fa=1.2

Fb=-5/4

- **1@ c**

Fa=0

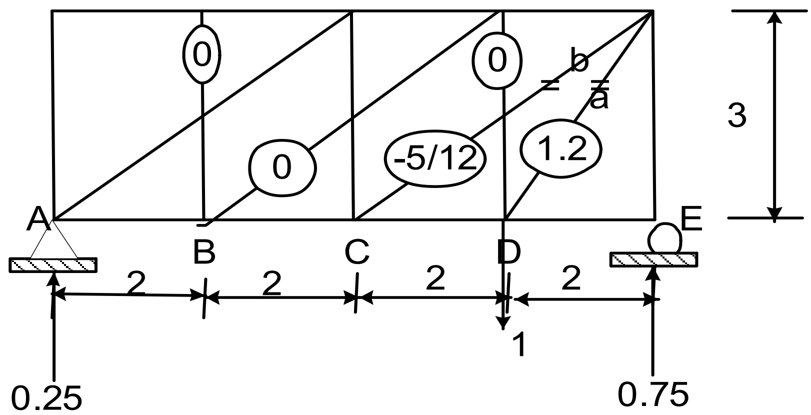
Fb=5/6

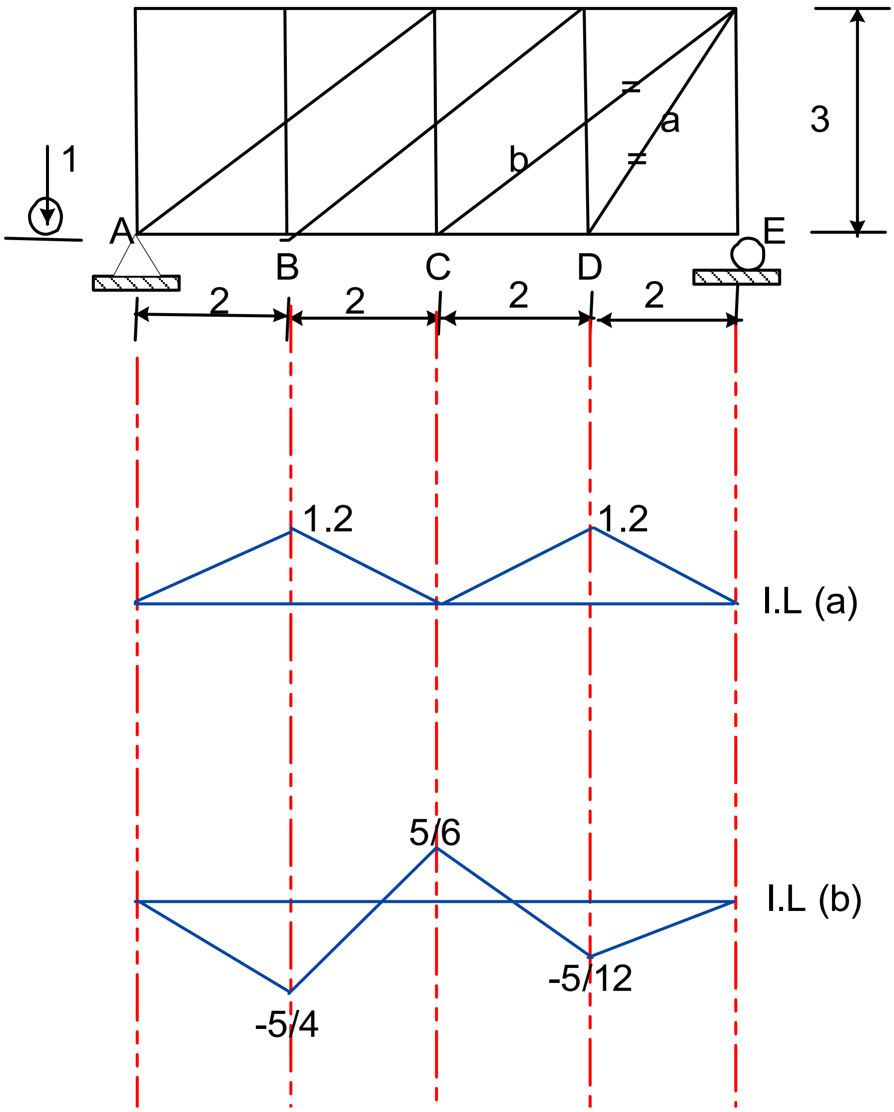


- **1@ D**

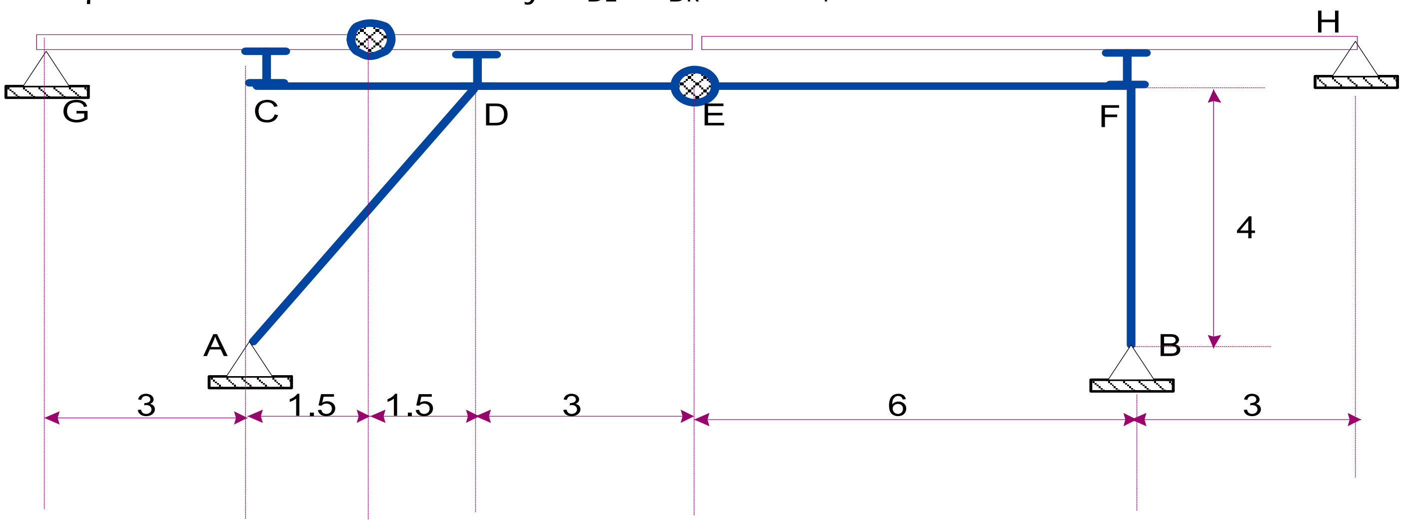
Fa=1.2

Fb=-5/12





Example:- draw I.L for reaction By, V_{DL} , V_{DR}, and M_f



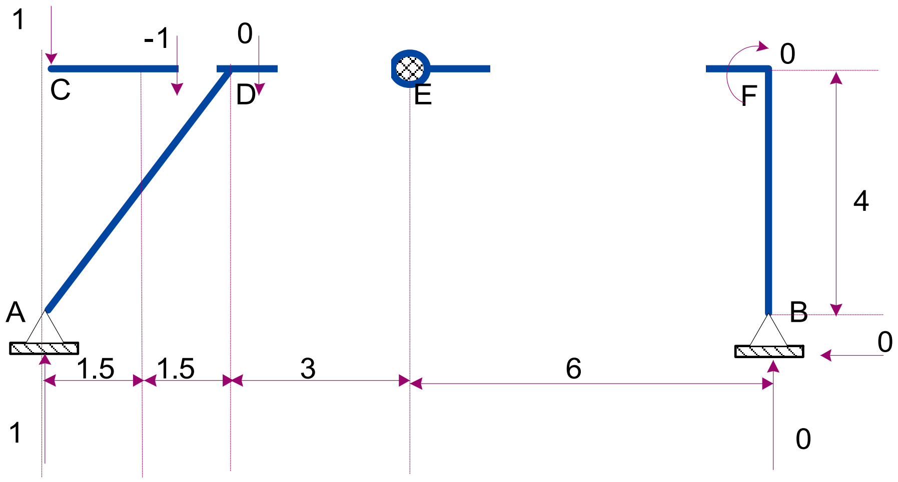
1@ c

$B_y=0$

$V_{DL}=-1$

$V_{DR}=0$

$M_F=0$



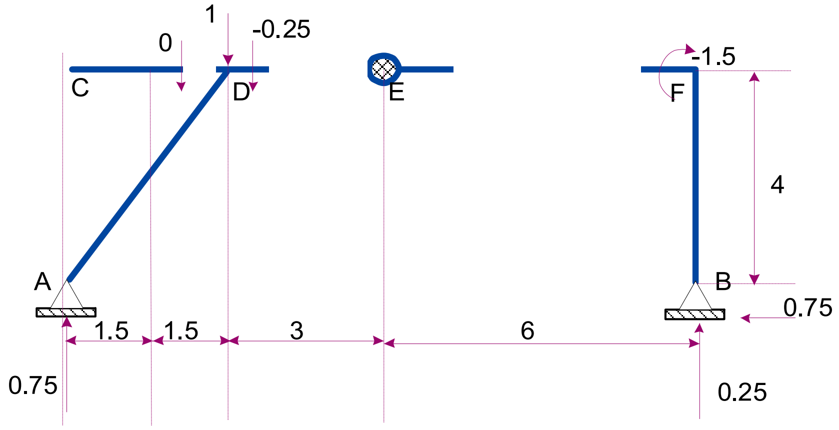
1@D

$B_Y=0.25$

$V_{DL}=0$

$V_{DR}=-0.25$

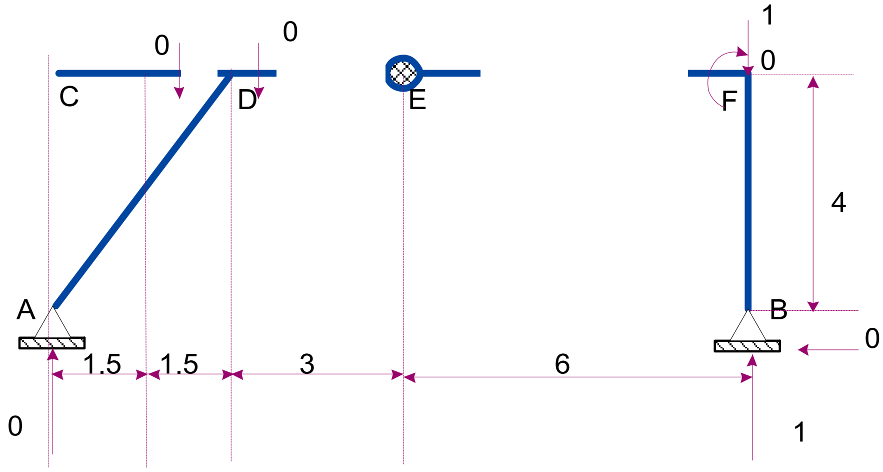
$M_F=-1.5$

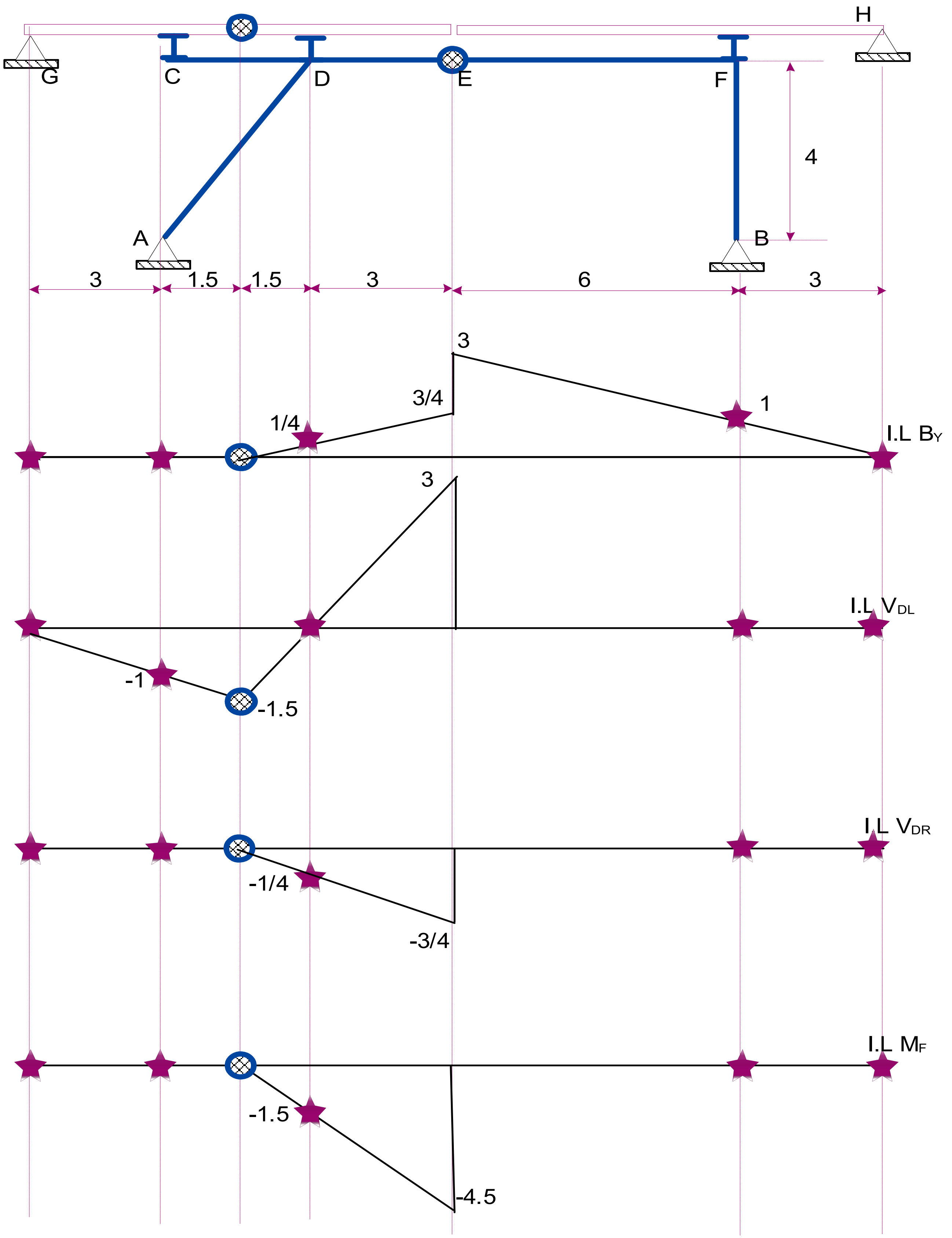


1@F

$B_Y=1$

$V_{DL}=V_{DR}=M_F=0$




H
G
C
D
E
F
4
A
B
3
1.5
1.5
3
6
3
3
3/4
1/4
1
I.L B_Y
3
I.L V_{DL}
-1
-1.5
I.L V_{DR}
-1/4
-3/4
I.L M_F
-1.5
-4.5

Maximum influence at a point due to a series of concentrated loads

Once the influence line of a function has been established for a point in a structure, the maximum effect caused by a live concentrated force is determined by multiplying the peak ordinate of the influence line by the magnitude of the force. In some cases,however, several concentrated forces must be placed on the structure. In order to determine the maximum effect in this case , a trial and error procedure can be used.

Example:- find maximum positive shear at point c due to the moving locomotive left to right

Case 1

$$V_C = 4.5(0.75) + 18 * \frac{0.75}{9} * (9 - 1.5) + 18 * \frac{0.75}{9} * (9 - 3) = 23.63\ KN$$

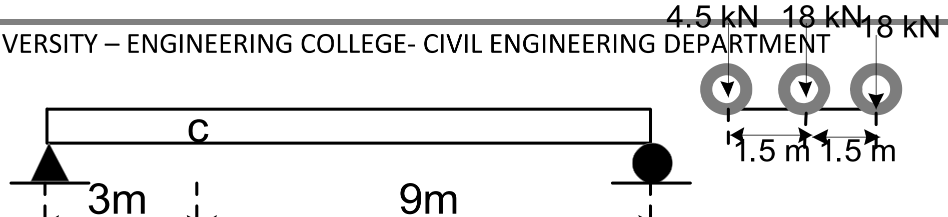
Case 2

$$V_C = 4.5\frac{-0.25}{3}(3 - 1.5) + 18 * 0.75 + 18 * \frac{0.75}{9} * (9 - 1.5) = 24.19\ KN$$

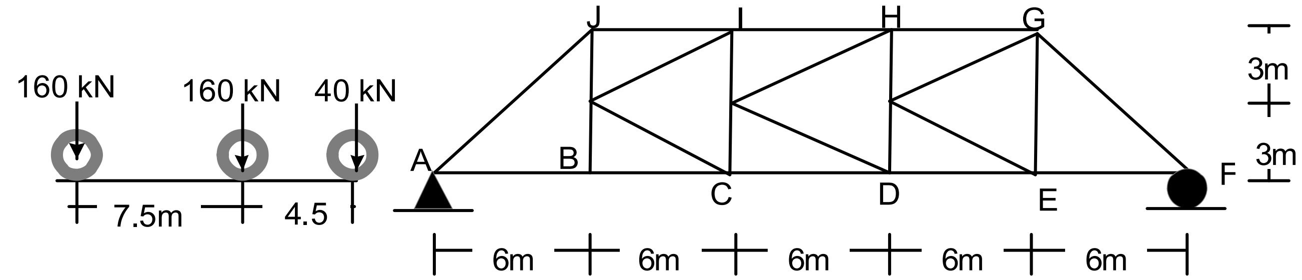
Case 3

$$V_C = 4.5\frac{-0.25}{3}(3 - 3) + 18 * \frac{-0.25}{3}(3 - 1.5) + 18 * 0.75 = 11.25\ KN$$

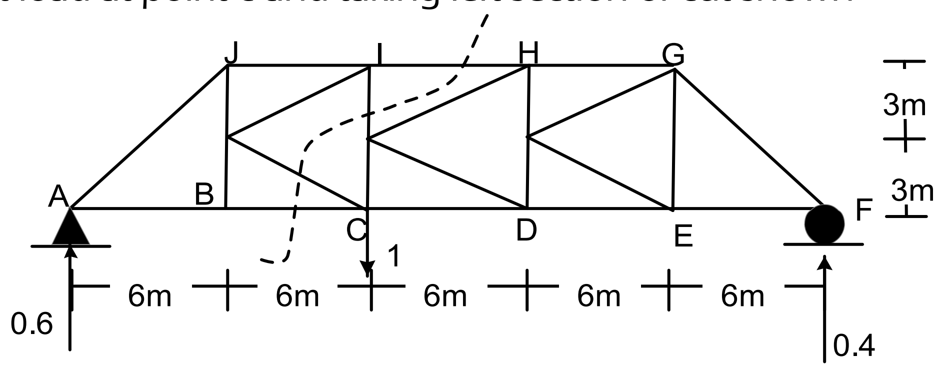
Maximum positive shear at point c equals to 24.19 kN (case 2)



Example:- draw the I.L for the force in member IH of the bridge truss. Compute the maximum live force that can be developed in this member due to a truck having the wheels loads shown. Assume the truck can travel in either direction along the center of the deck, so that half its load is transferred to each of the two side trusses.



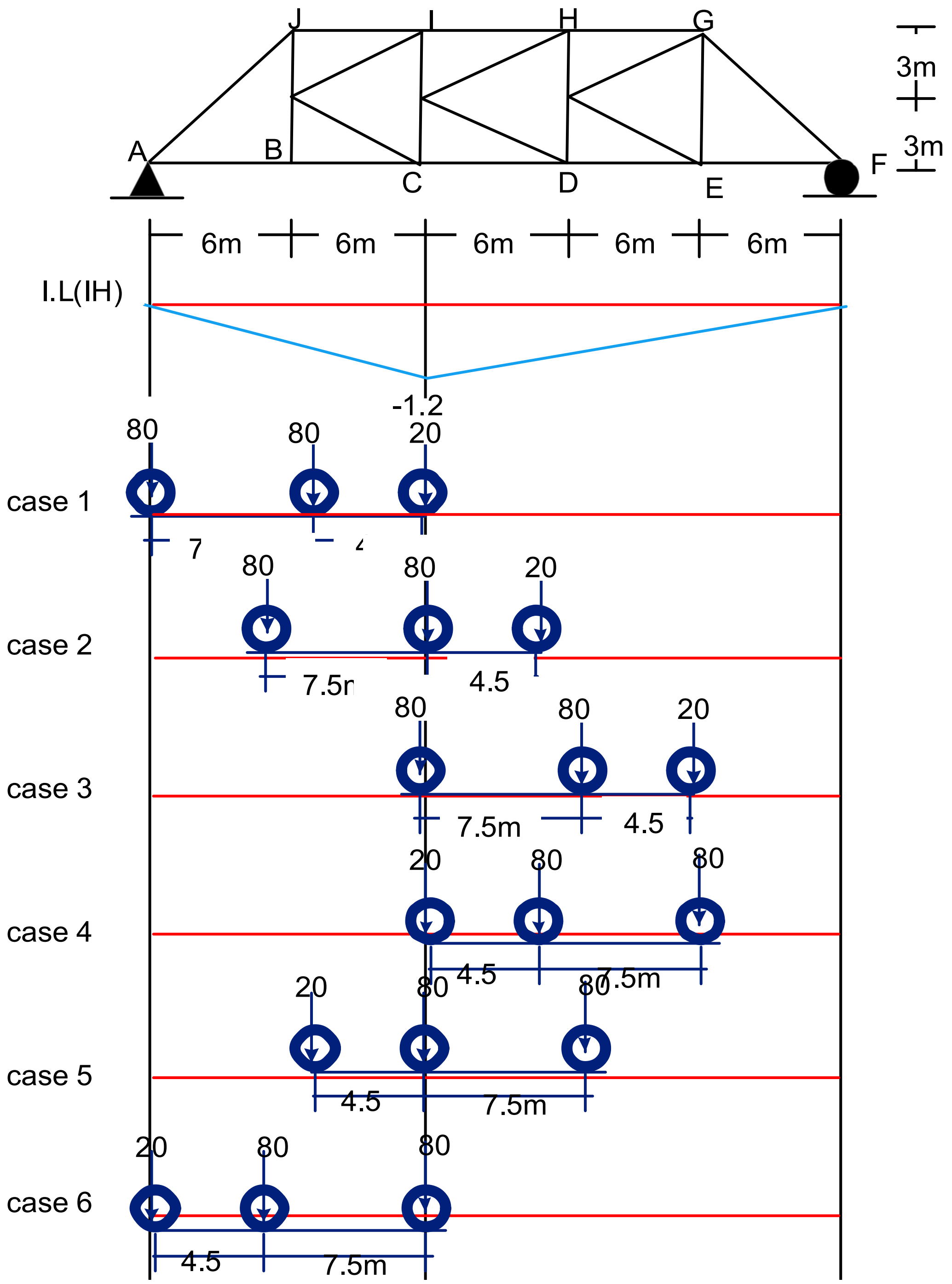
Apply unit load at point c and taking left section of cut shown



$$\sum Mc = o$$

$$F_{IH} = \frac{0.6(12)}{6} = 1.2\ comp.$$

Now, I.L can be drawn and half load must be taken then apply trial and error method


J
I
H
G
3m
3m
A
B
C
D
E
F
6m
6m
6m
6m
6m
I.L(IH)
-1.2
case 1
80
80
20
7
case 2
80
80
20
7.5m
4.5
case 3
80
80
20
7.5m
4.5
case 4
20
80
80
4.5
7.5m
case 5
20
80
80
4.5
7.5m
case 6
20
80
80
4.5
7.5m

Case 1

$$F_{IH} = 20(-1.2) + 80\frac{-1.2}{12}(12 - 4.5) = -84\ kN$$

Case 2

$$F_{IH} = 20\frac{-1.2}{18}(18 - 4.5) + 80(-1.2) + 80\frac{-1.2}{12}(12 - 7.5) = -150\ kN$$

Case 3

$$F_{IH} = 20\frac{-1.2}{18}(18 - 12) + 80\frac{-1.2}{18}(18 - 7.5) + 80(-1.2) = -160\ kN$$

Case 4

$$F_{IH} = 20(-1.2) + 80\frac{-1.2}{18}(18 - 4.5) + 80\frac{-1.2}{18}(18 - 12) = -128\ kN$$

Case 5

$$F_{IH} = 20\frac{-1.2}{12}(12 - 4.5) + 80(-1.2) + 80\frac{-1.2}{18}(18 - 7.5) = -167\ kN$$

Case 6

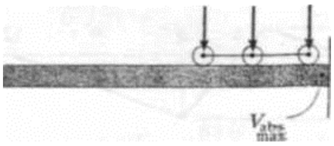
$$F_{IH} = 80(-1.2) + 80\frac{-1.2}{12}(12 - 7.5) = -132\ kN$$

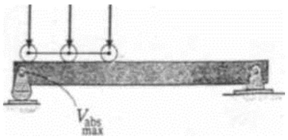
Critical case is case 5

Absolute maximum shear and moment

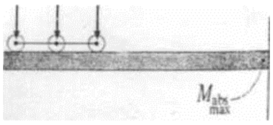
Trial and error method is developed for computing the maximum shear and moment at a specified point in a beam and truss due to a series of concentrated moving loads. A more general problem involves the determination of both the location of the point in the beam and the position of the loading on the beam so that one can obtain the absolute maximum shear and moment caused by the loads. If the beam is cantilevered or simply supported, this problem can be readily solved.

- ☒ Shear: for a cantilevered beam the absolute maximum shear will occur at a point located just next to the fixed support. For simply supported beams the absolute maximum shear will occur just next to one of the supports.





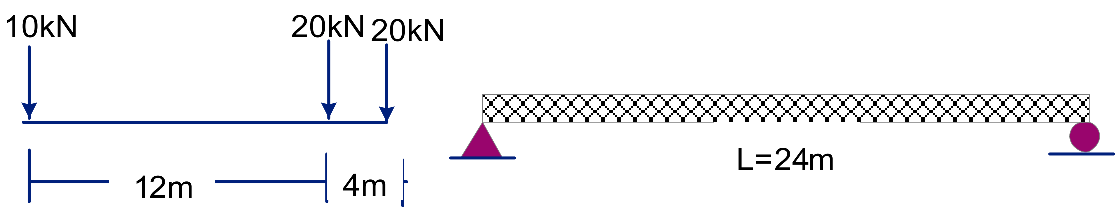
- ☒ Moment. The absolute maximum moment for a *cantilevered beam* occurs at the same point where absolute maximum shear occurs, although in this case the concentrated loads should be positioned at *the far end* of the beam, as shown.



For a simply supported beam the critical position of the loads and the associated absolute maximum moment cannot, in general, be determined by inspection. We can, however, determine the position analytically by following the steps

1. Determine the resultant and its location
2. Put center line of a beam at half distance between the resultant and nearest haviest load.
3. If the nearest load to resultant is light the try two cases
4. If the length of loads is equal or greater than length of beam then remove immoderate load commonly light load .

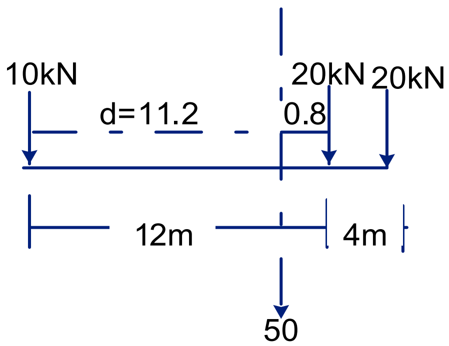
Example : calculate the absolute maximum bending moment in a simply span of 24 m under the action of the shown locomotive .

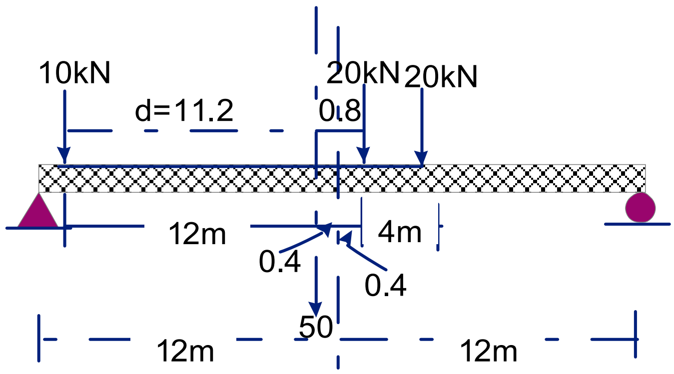


R(resultant)=10+20+20=50 kN

Location of resultant from left point

d=(20*12+20*16)/50=11.2 m



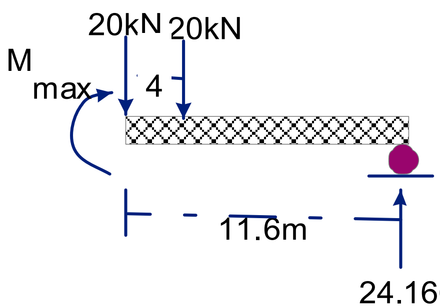


هنا العزم الاكبر سوف يظهر في نقطة تسليط الحمل الاكبر والاقرب الى المحصلة

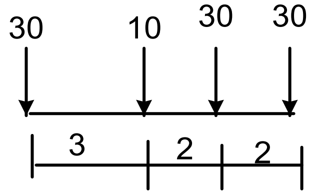
To find right support moment about left support will be taken

R_{right}=50(12-0.4)/24=24.166 kN

M_{max}=24.166(11.6)-20(4)=200.333 kN.m



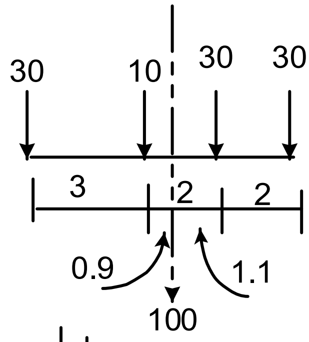
Example : calculate the absolute maximum bending moment in a simply span of 14 m under the action of the shown locomotive



R=30*3+10=100

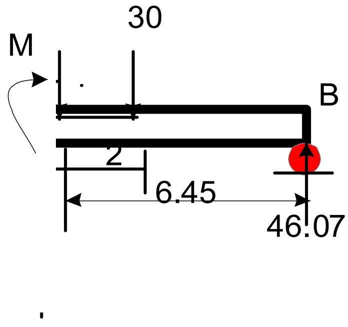
d=(10*3+30*5+30*7)/100=3.9m

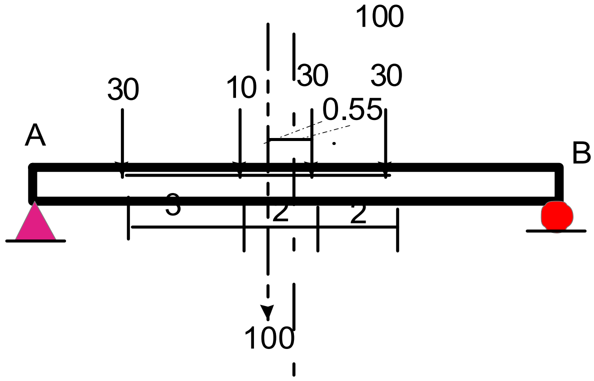
Here the nearest load to resultant is not heaviest, so that two cases must be depended.



Case 1(center line of beam between R & 30kN)

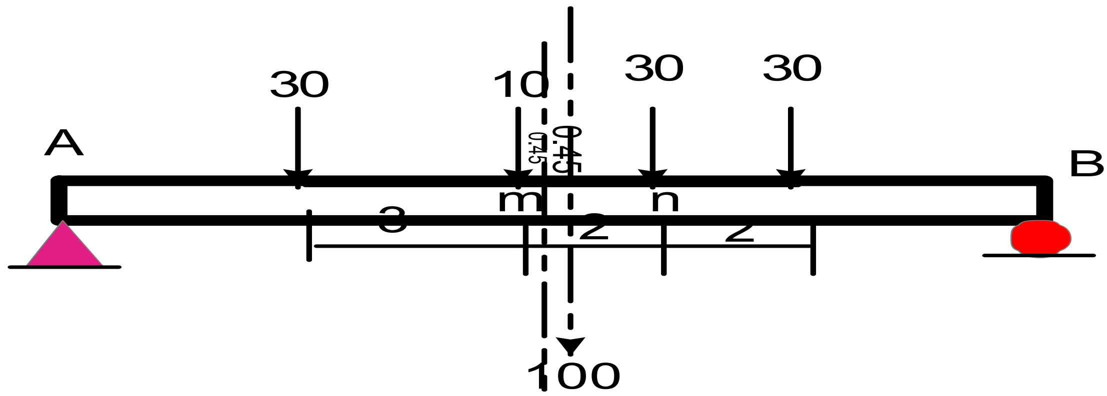
By=100(7-0.55)/14=46.07





M=46.07(6.45)-30(2)=237.15 kN.m

Case 2(center line of beam between R & 10kN)



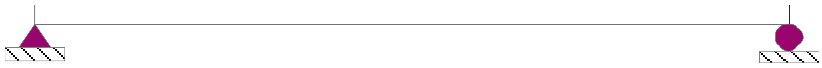
By=100(7+0.45)/14=53.32kN

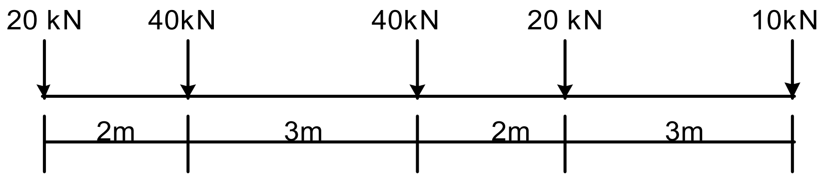
Mn=53.32(7-(2-0.45))-30(2)=230.6 kN.m

Mm=53.32(7+0.45)-30(2)-30(4)=216.41 kN.m

Absolute M=237.15 kN.m(case 1 governs)

Example : calculate the absolute maximum bending moment in a simply span of 10m under the action of the shown locomotive

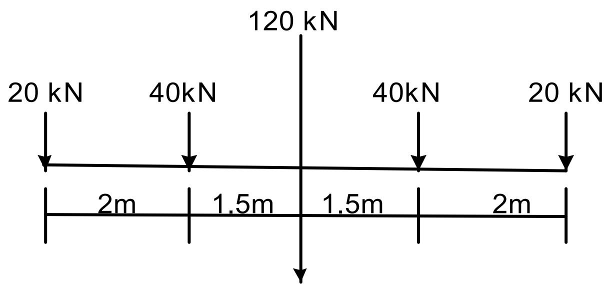


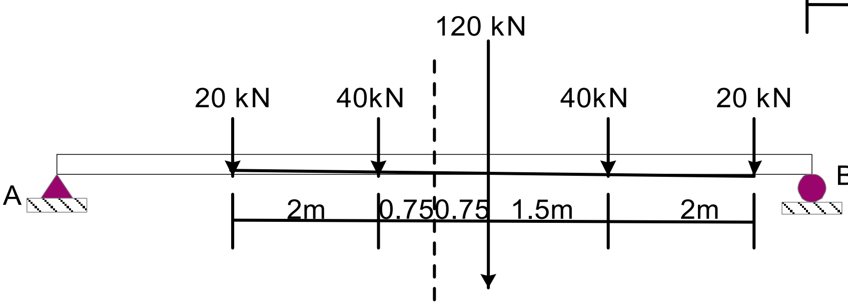


Length of load equals to length of beam therefore it must reduce to become smaller than length of beam. Smallest load of end loads will remove.

R=20*2+40*2=120kN

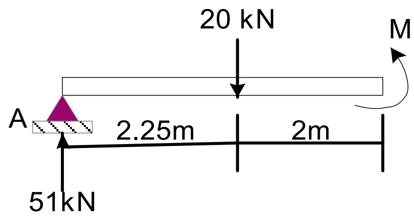
d=(40*2+40*5+20*7)/120=3.5





R_A=120(5-0.75)/10=51kN

M=51(2.25+2)-20(2)=176.75 kN.m



Analysis of statically indeterminate structures

Structure of any type is classified as statically indeterminate when the number of unknown reactions or internal forces exceeds the number of equations available for its analysis.

Advantages of statically indeterminate structures

1- For a given loading the maximum stress and deflection of an indeterminate structure are generally smaller than those of its statically determinate counterpart.
2- Statically indeterminate structures can redistribute its load to its redundant supports in cases where faulty design or overloading occurs.

Although from these advantages of selecting statically indeterminate structures, also; there are some disadvantages such as deformations caused by relative support displacement, or changes in member length caused by temperature or fabrication errors will introduce additional stresses in the structure, which must be considered when designing indeterminate structures.

Methods of analysis

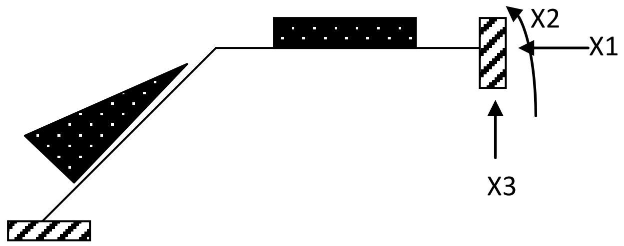
When analyzing any indeterminate structures, it is necessary to satisfy equilibrium, compatibility, and force-displacement requirements for the structure. Equilibrium is satisfied when the reactive forces hold the structure at rest, and compatibility is satisfied when the various segments of the structures fit together without intentional breaks or overlaps. the force-displacements requirements depends upon the way of material responds. Generally, there are two different methods to satisfy these requirements: the force or flexibility method, and the displacement or stiffness method.

In this stage, the following methods will be studied:

1- Consistent method
2- Slope-deflection method
3- Moment distribution method
4- Stiffness matrix method

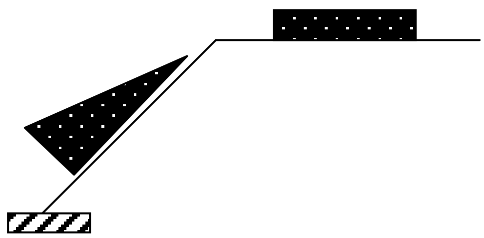
1-CONSISTENT METHOD

1- Determine the degree of indeterminacy of structures.



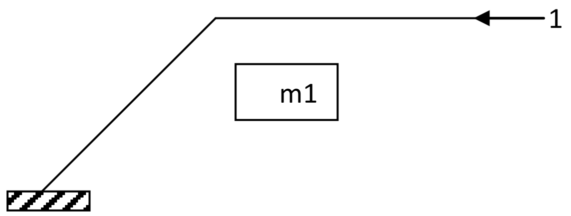
The above structure is statically indeterminate to third degree.

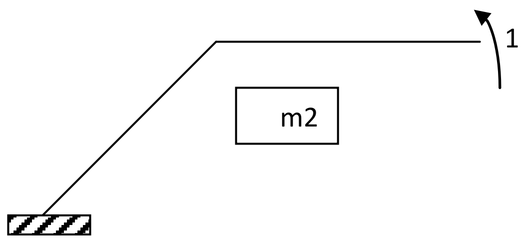
2- Remove redundant to convert structure from statically indeterminate to stable statically determinate structure , this structure is named as primary structure.

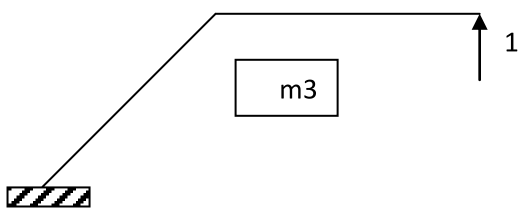


Primary structure

3- Determine internal moment at each part of primary structure (M)
4- Apply unit load in position of removed redundant, then internal moment at each part of structure is calculated as shown in figures.







5- Determine redundant by solving the following simultaneous equations

$\Delta_{10} + s_{11}x_1 + s_{12}x_2 + s_{13}x_3 = o\ or\ settlment\ at\ support\ 1$-----1

$\Delta_{20} + s_{21}x_1 + s_{22}x_2 + s_{23}x_3 = o\ or\ rotational\ at\ support\ 2$---2

$\Delta_{30} + s_{31}x_1 + s_{32}x_2 + s_{33}x_3 = o\ or\ settlment\ at\ support\ 3$---3

Where

$$\Delta_{10} = \int \frac{Mm1\ dx}{EI}$$

$$\Delta_{20} = \int \frac{Mm2\ dx}{EI}$$

$$\Delta_{30} = \int \frac{Mm3\ dx}{EI}$$

$$s_{11} = \int \frac{m1.m1\ dx}{EI}$$

$$s_{12} = \int \frac{m1.m2\ dx}{EI}$$

$$s_{13} = \int \frac{m1.m3\ dx}{EI}$$

$$s_{22} = \int \frac{m2.m2\ dx}{EI}$$

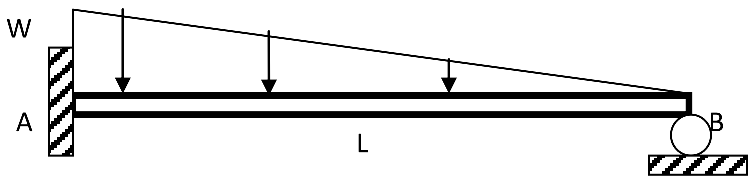
$$s_{33} = \int \frac{m3.m3dx}{EI}$$

S12=S21 , S31=S13, S32=S23

<u>APPLICATIONS OF CONSISTENT METHOD</u>

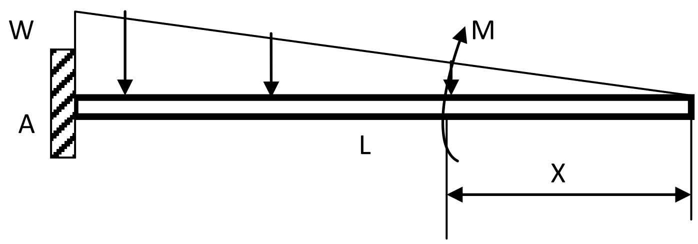
1-BEAMS

EXAMPLE(1):- DETERMINE THE REACTION AT THE SUPPORTS



Solution

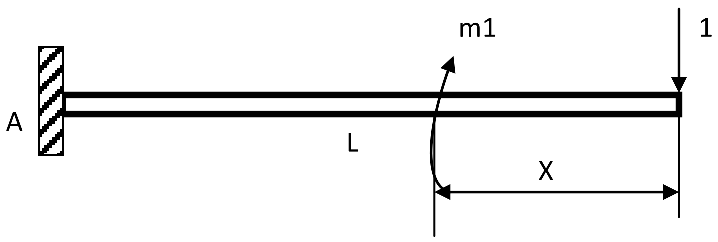
The beam is statically indeterminate to first degree, the roller is considered as redundant and is removed



$$load\ at\ distance\ x\ from\ b = \frac{WX}{L}$$

$$M = -X\frac{WX}{L}.\frac{1}{2}.\frac{X}{3} = -\frac{WX^3}{6L}$$

Apply unit load at position of removed support



$$m1 = -X$$

$$\Delta_{10} + s_{11}x_1 = o$$

$$\Delta_{10} = \int \frac{Mm1\ dx}{EI} = \int_o^l \frac{-wx^3}{6l} * \frac{-xdx}{EI} = \frac{wl^4}{30EI}$$

$$s_{11} = \int \frac{m1.m1\ dx}{EI} = \int_0^l -x\frac{-xdx}{EI} = \frac{l^3}{3EI}$$

$$\frac{wl^4}{30EI} + \frac{l^3}{3EI}x_1 = o$$

$$x_1 = -\frac{wl}{10} = \frac{wl}{10} \uparrow$$

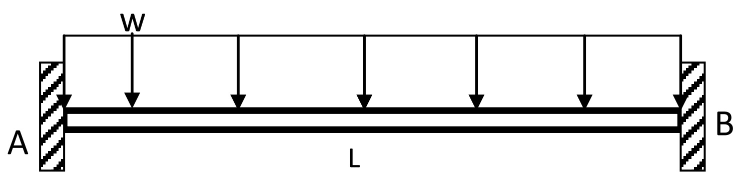
After the reaction at roller is known , remaining reactions can be determined by applying equilibrium equations.

$$R_{yA} = \frac{wl}{2} - \frac{wl}{10} = \frac{2}{5} wl \uparrow$$

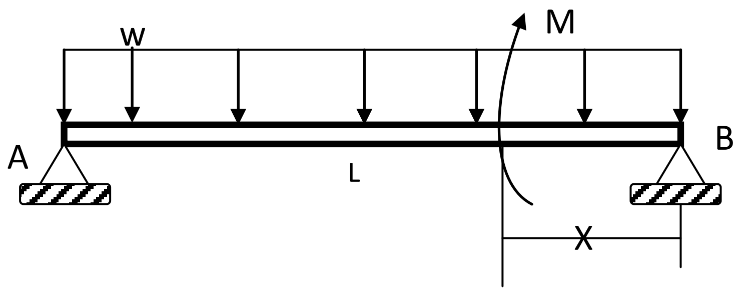
$$M_A = \frac{wl}{10} l - \frac{wl}{2} * \frac{1}{3} = -\frac{wl^2}{15} = \frac{wl^2}{15} \ counterclock\ wise$$

H.W: repeat previous example and choose end moment at A as redundant

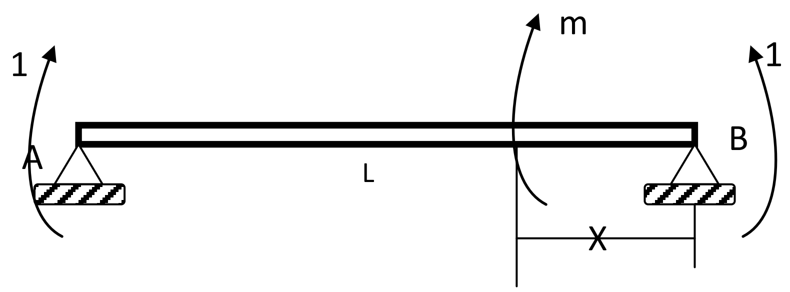
Example (2): analysis the following beam



The beam is statically indeterminate to first degree due to symmetry



$$M = \frac{WL}{2} X - \frac{WX^2}{2} = \frac{W}{2} [LX - X^2]$$



m=1

$$\Delta_{10} = \int \frac{Mm1\,dx}{EI} = \int_0^l \frac{w}{2EI}[lx - x^2]\,dx = \frac{wl^3}{12EI}$$

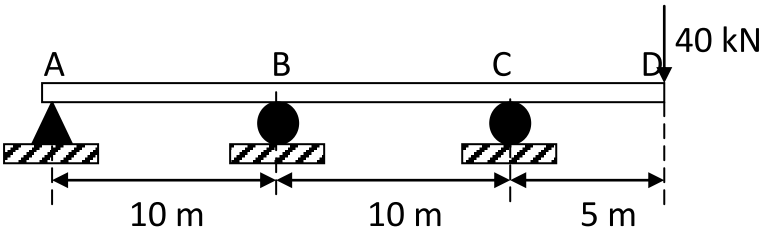
$$s_{11} = \int \frac{m1.m1\,dx}{EI} = \int_0^l \frac{dx}{EI} = \frac{l}{EI}$$

$$\Delta_{10} + s_{11}x_1 = o$$

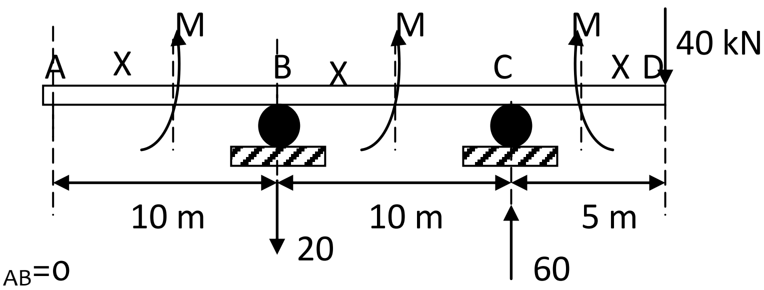
$$\frac{wl^3}{12EI} + \frac{l}{EI}x_1 = o$$

$$x_1 = -\frac{wl^2}{12}$$

Example (3):- determine the internal moments acting in the beam at support B and c . the wall at A moves upward 30mm. take E=200GPa,I=90(10^6)mm^4.



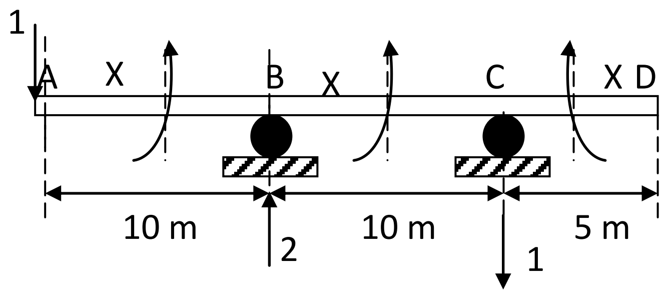
The structure is statically indeterminate to first degree.



M_{AB}=0

M_{BC}=-20X

M_{CD}=-40X



m_{AB}=-X

m_{BC}=-(10+X)+2X=-10+X

m_{CD}=0

$$\Delta_{10}= \int \frac{Mm1\ dx}{EI} = \frac{1}{EI}\left[\int_0^{10} -20x(-10+x)dx\right] = \frac{10000}{3EI} kN^2.m^3$$

$$s_{11} = \int \frac{m1^2\ dx}{EI} = \frac{1}{EI}\left[\int_0^{10} x^2 dx + \int_0^{10} (-10+x)^2 dx\right] = \frac{2000}{3EI} kN^2.m^3$$

$$\Delta_{10} + s_{11}x_1 = settlement$$

$$\frac{10000}{3*200*90} + \frac{2000}{3*200*90}x_1 = -30*10^{-3}$$

$$x_1 = -5.81\ kN$$

R_A=5.81 KN upward

R_B=-20+2*-5.81=-31.62=31.62 downward

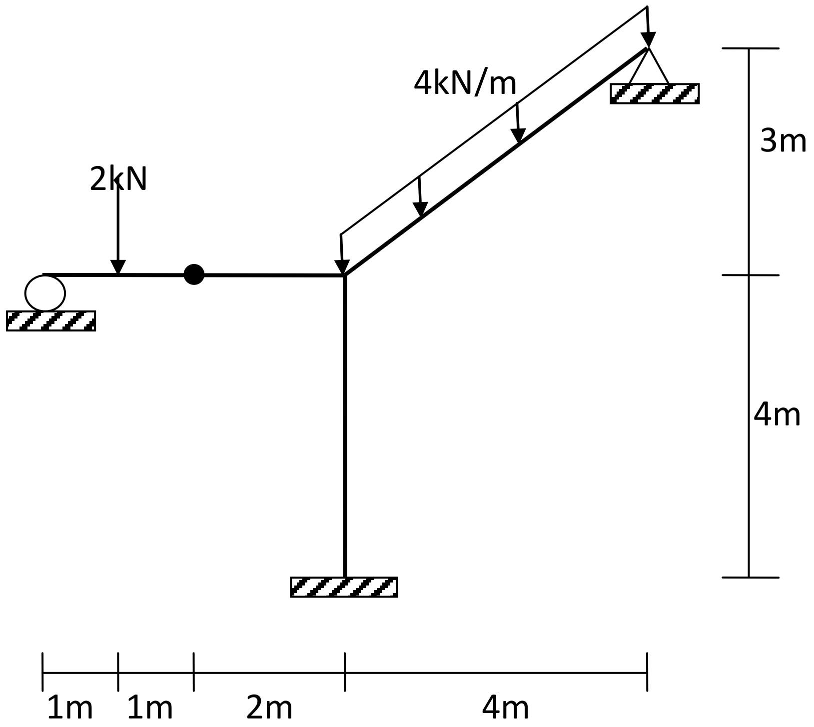
R_C=60-1*5.81=65.81 upward

M_B=5.81*10=58.1 kN.m

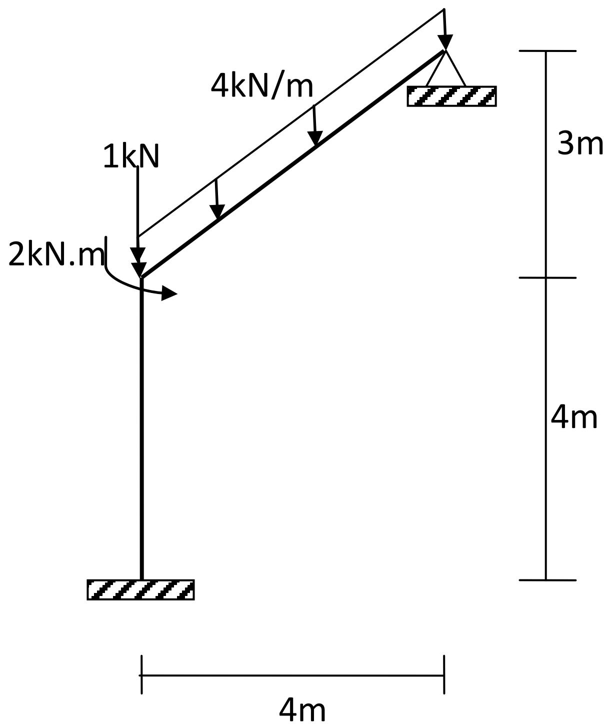
M_C=40*5=200 kN.m

2-FRAME

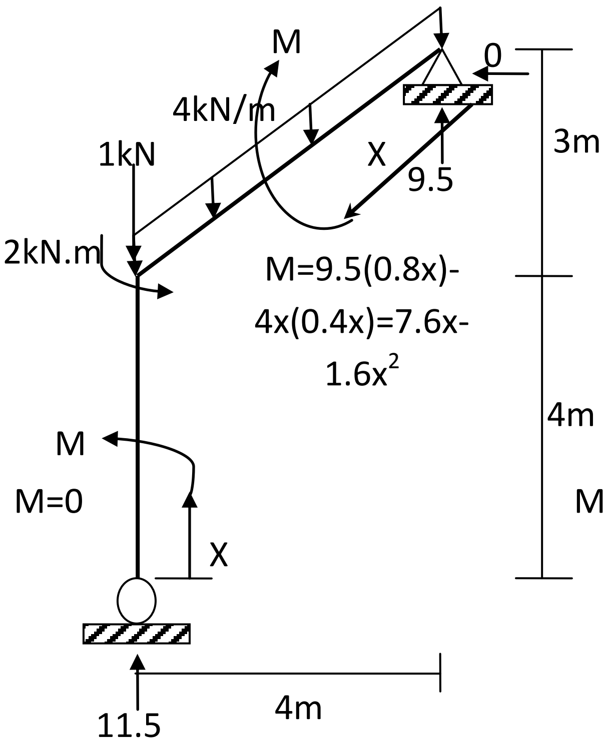
Example(4):- Analysis the frame shown. There is rotational slip of 0.003 rad counter clockwise at support B. take EI=10^4 kN.m^2

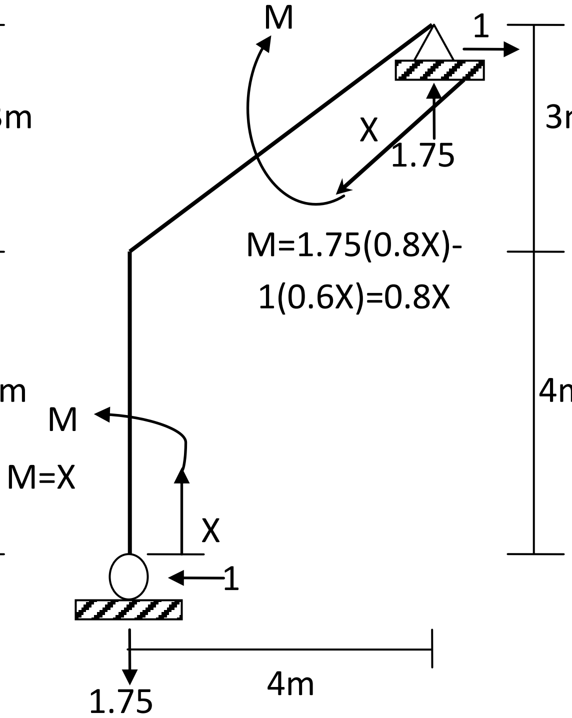


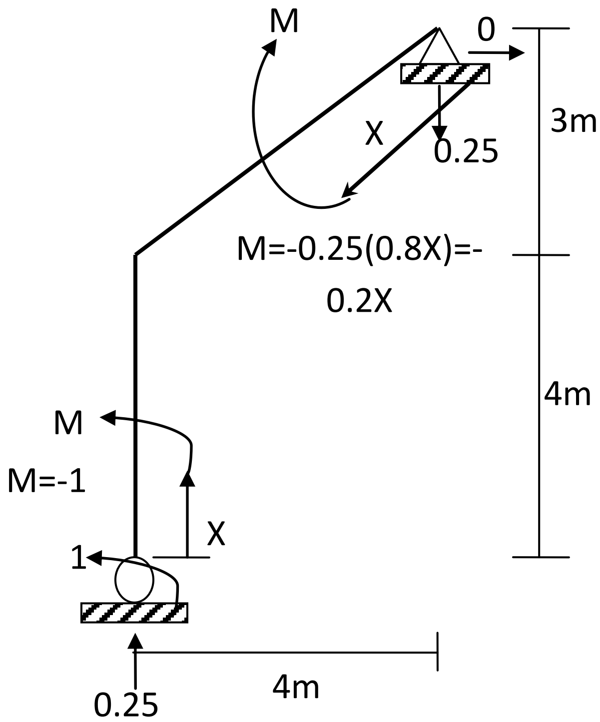
The structure above is statically indeterminate to 2^{nd} degree, there is part of structure is statically determinate



Primary structure







$$\Delta_{10} + s_{11}x_1 + s_{12}x_2 = o$$

$$\Delta_{20} + s_{21}x_1 + s_{22}x_2 = 0.003$$

$$\Delta_{10} = \int \frac{Mm1\,dx}{EI} = \frac{1}{EI}\left[\int_0^5 (7.6X - 1.6X^2)(0.8X)dx\right] = \frac{53.33}{EI}$$

$$\Delta_{20}=\int \frac{Mm1\,dx}{EI}=\frac{1}{EI}\left[\int_0^5 (7.6X-1.6X^2)(-0.2X)dx\right]=\frac{-13.33}{EI}$$

$$s_{11}=\int \frac{m1^2\,dx}{EI}=\frac{1}{EI}\left[\int_0^5 (0.8X)^2dx+\int_0^4 (x)^2dx\right]=\frac{48}{EI}$$

$$s_{12}=\int \frac{m1.m2\,dx}{EI}=\frac{1}{EI}(\int_0^5 (0.8X)(-0.2X)dx+\int_0^4 -Xdx)=\frac{-14.67}{EI}$$

$$s_{22}=\int \frac{m2^2\,dx}{EI}=\frac{1}{EI}\left[\int_0^5 (-0.2X)^2dx+\int_0^4 (-1)^2dx\right]=\frac{5.67}{EI}$$

$$\frac{53.33}{EI}+\frac{48}{EI}X_1-\frac{14.67}{EI}X_2=0$$

$$48X_1-14.67X_2=-53.33-----1$$

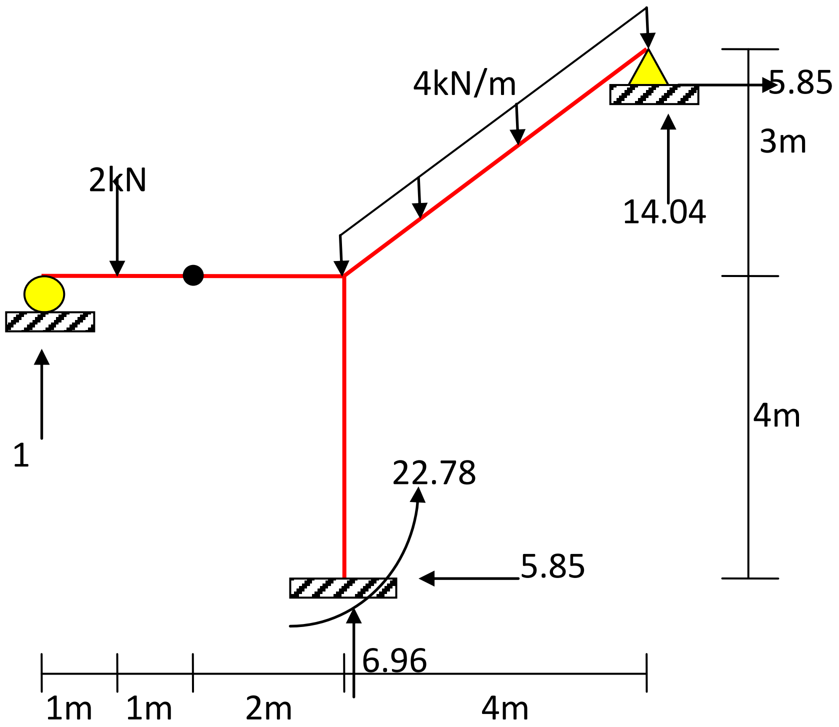
$$\frac{-13.33}{EI}-\frac{14.67}{EI}X_1+\frac{5.67}{EI}X_2=0.003$$

$$-14.67X_1+5.67X_2=43.33-----2$$

Solving Eqs 1&2,gives

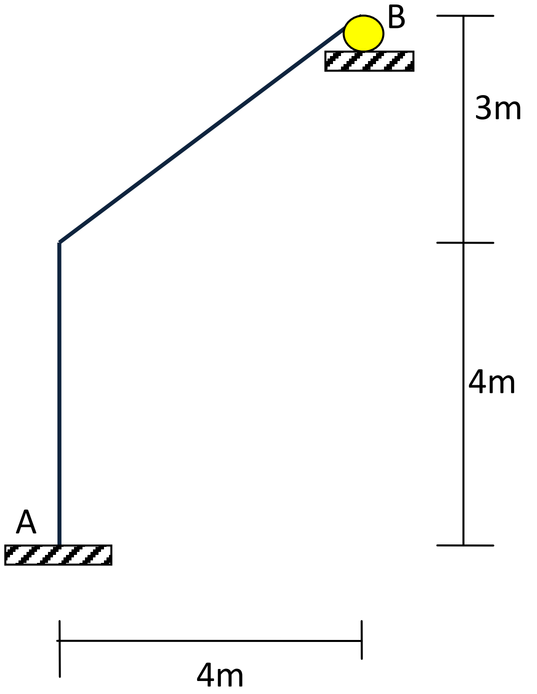
X_1=5.85kN , X_2=22.78 kN.m

Final result

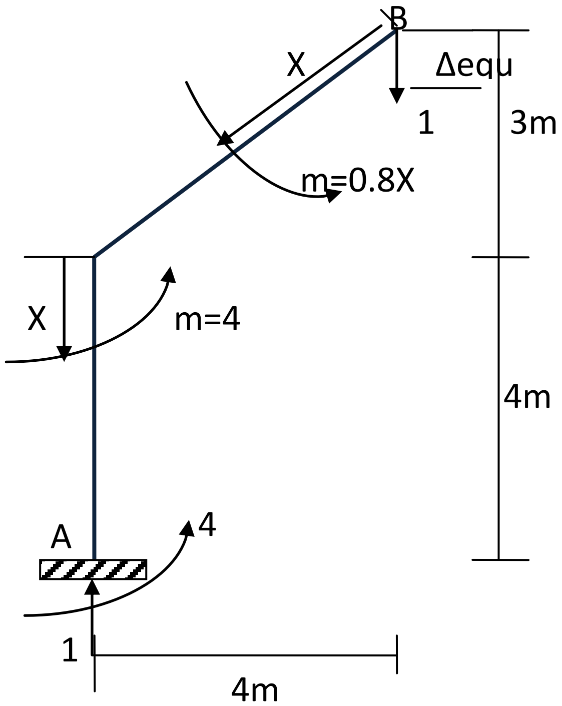


Example(5):- using method of consistent deformation, analysis the frame shown

EI=10^4 kN.m^2, settlement at support A=0.002m ↓and 0.003m← , rotational slip at A=0.003 rad counter clock wise , settlement at B=0.004m ↓



The structure is statically indeterminate to 1^{st} degree, the deformations of support A must be included by using virtual work



$1x\Delta_{eq} = 4x0.003 - 1x0.002 + 0x0.003 = 0.01$

$\Delta_{total} = 0.01 + 0.004 = 0.014m$

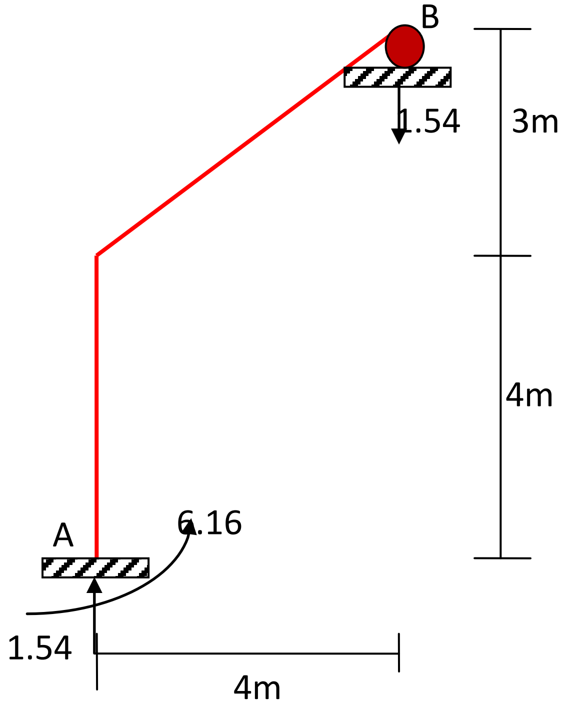
$\Delta_{10} + s_{11}x_1 = 0.014$

$\Delta_{10} = o\ \ no\ load\ applied$

$$s_{11} = \int \frac{m1^2\,dx}{EI} = \frac{1}{EI}\left(\int_0^4 16\,dx + \int_0^5 (0.8x)^2 dx\right) = \frac{90.67}{EI}$$

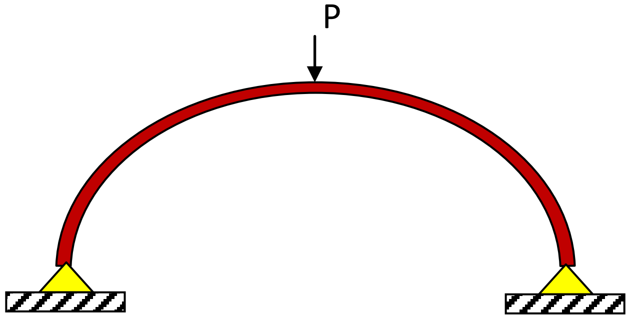
$$\frac{90.67}{EI}x_1 = 0.014$$

$x_1 = 1.54 \downarrow kN$

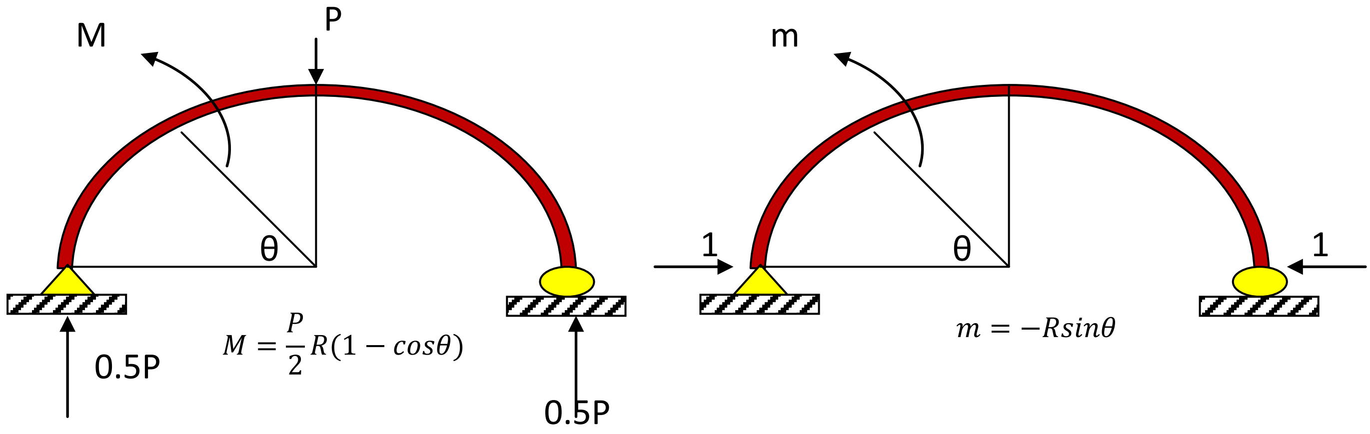


3-Arch

Example(6):- Analysis the semi-circular arch shown in figure by the method of consistent deformation. Radius=R, EI constant



The arch is statically indeterminate to 1st degree



$$\Delta_{10} + s_{11}x_1 = 0$$

$$\Delta_{10} = \int \frac{Mm1\,dx}{EI} = 2\int_0^{0.5\pi} \frac{\frac{P}{2}R(1-cos\theta)(-Rsin\theta)Rd\theta}{EI} = \frac{-PR^3}{EI}[-cos\theta - \frac{sin^2\theta}{2}]_0^{0.5\pi}$$

$$= \frac{-PR^3}{2EI}$$

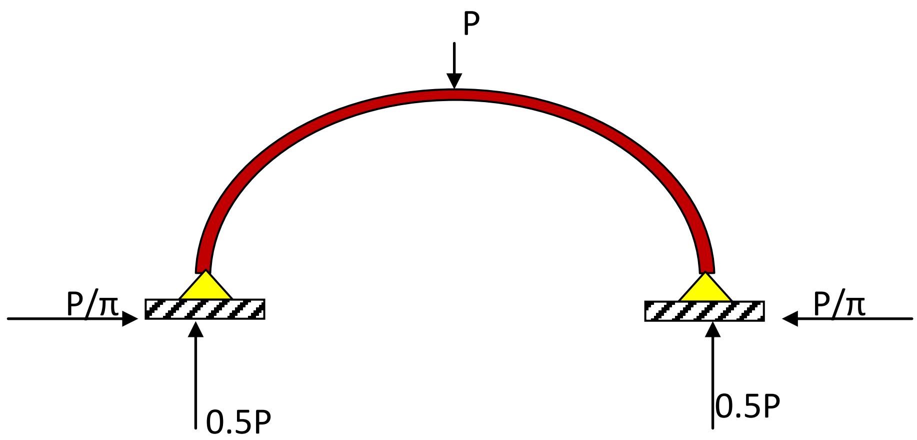
$$s_{11} = \int \frac{m1^2\,dx}{EI}$$

$$= 2\int_0^{0.5\pi} \frac{(-Rsin\theta)^2 Rd\theta}{EI} = \frac{2R^3}{EI}[\int_0^{0.5\pi} \frac{(1-cos2\theta)}{2}d\theta]$$

$$= \frac{2R^3}{EI}[\frac{\theta}{2} - 0.25sin2\theta]^{0.5\pi} = \frac{\pi R^3}{2EI}$$

$$\frac{-PR^3}{2EI} + \frac{\pi R^3}{2EI}x_1 = 0$$

$$x_1 = \frac{P}{\pi}$$



4-Trusses

Same procedure for beam and frame can be followed for truss, the following equations are used in the analysis of trusses. The truss may externally indeterminate or internally indeterminate or both cases

$$\Delta_{10} + s_{11}x_1 + s_{12}x_2 + s_{13}x_3 = o$$

$$\Delta_{20} + s_{21}x_1 + s_{22}x_2 + s_{23}x_3 = o$$

$$\Delta_{30} + s_{31}x_1 + s_{32}x_2 + s_{33}x_3 = o$$

Where

$$\Delta_{10}= \sum \frac{n_1 NL}{EA} + \sum n_1 \propto \Delta TL + \sum n_1 \; error$$

$$\Delta_{20}= \sum \frac{n_2 NL}{EA} + \sum n_2 \propto \Delta TL + \sum n_2 \; error$$

$$\Delta_{30}= \sum \frac{n_3 NL}{EA} + \sum n_3 \propto \Delta TL + \sum n_3 \; error$$

$$s_{11} = \sum \frac{{n_1}^2 L}{EA}$$

$$s_{12} = \sum \frac{n_1 n_2 L}{EA}$$

$$s_{13} = \sum \frac{n_1 n_3 L}{EA}$$

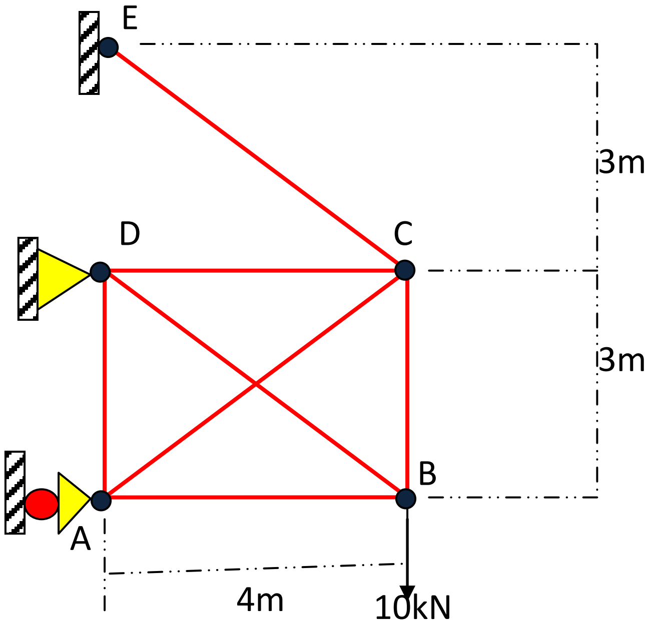
$$s_{22} = \sum \frac{{n_2}^2 L}{EA}$$

$$s_{33} = \sum \frac{{n_3}^2 L}{EA}$$

$$s_{33} = \sum \frac{n_2 n_3 L}{EA}$$

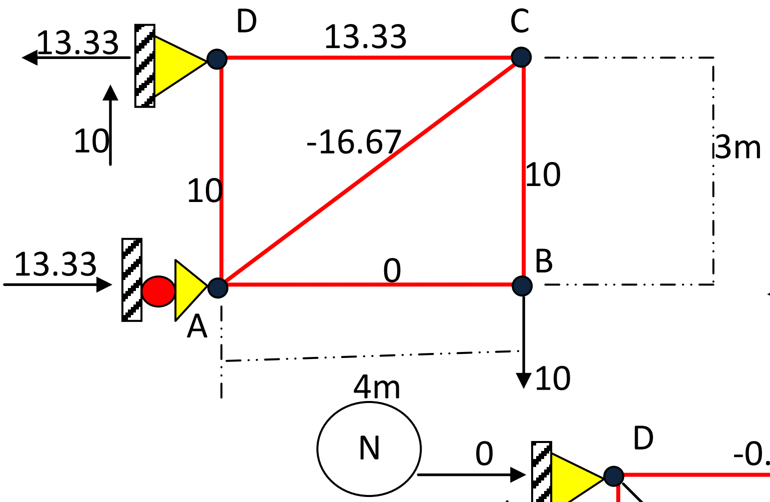
S12=S21 , S31=S13, S32=S23

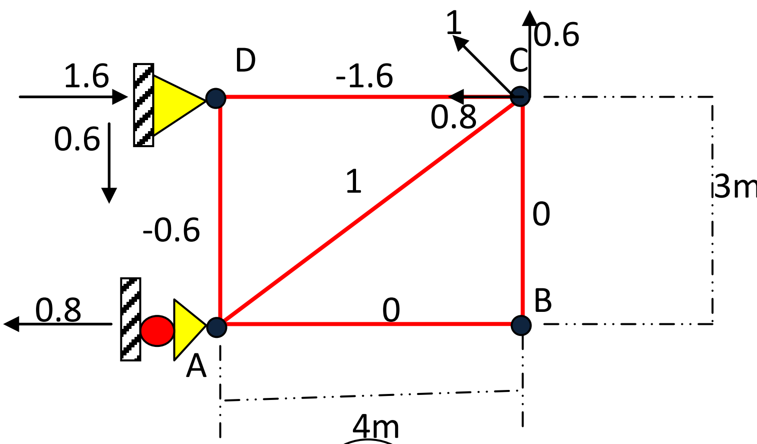
Example(7):- Analysis the truss shown. EA constant

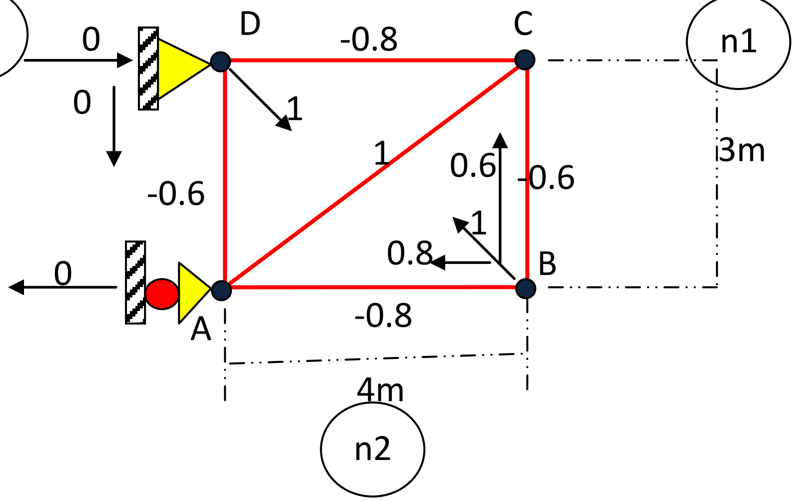


b+r	2j
6+4	2(4)
10	8

The truss is statically indeterminate to 2[nd] degree; it is internally and externally indeterminate







member	L(m)	N	n_1	n_2	$N\, n_1\, L$	$N\, n_2\, L$	$n_1^2\, L$	$n_2^2\, L$	$n_1\, n_2\, L$
DC	4	13.33	-1.6	-0.8	-85.312	-42.66	10.24	2.56	5.12
AB	4	0	0	-0.8	0	0	0	2.56	0
AD	3	10	-0.6	-0.6	-18	-18	1.08	1.08	1.08
BC	3	10	0	-0.6	0	-18	0	1.08	0
AC	5	-16.67	1	1	-83.35	-83.35	5	5	5
EC	5	0	1	0	0	0	5	0	0
DB	5	0	0	1	0	0	0	5	0
$\sum$					-186.66	-162.01	21.32	17.28	11.2

$$\Delta_{10}=\sum\frac{n_1NL}{EA}=\frac{-186.66}{EA}$$

$$\Delta_{20}=\sum\frac{n_2NL}{EA}=\frac{-162.01}{EA}$$

$$s_{11}=\sum\frac{{n_1}^2L}{EA}=\frac{21.32}{EA}$$

$$s_{12}=\sum\frac{n_1n_2L}{EA}=\frac{11.2}{EA}$$

$$s_{22}=\sum\frac{{n_2}^2L}{EA}=\frac{17.28}{EA}$$

$$21.32x_1+11.2x_2=186.66----1$$

$$11.2x_1+17.28x_2=162.01----2$$

$$x_1=5.8\ ,\ \ x_2=5.6$$

Force in each member=$N+n_1X_1+n_2X_2$

member	F
DC	-0.43
AB	-4.48
AD	3.16
BC	6.64
AC	-5.27
EC	5.8
DB	5.6

Example(8):- repeat example 7, if member BC subjected to rise of temperature 40^{o} take $\alpha=12(10^{-6})$ and member DC has error of 1cm too short.

$$\Delta_{10}=\sum\frac{n_1NL}{EA}+\sum n_1\propto\Delta TL+\sum n_1\ error$$

$$\Delta_{10}=\frac{-186.66}{EA}+0x40x3x12E-6-1.6x(-0.01)=\frac{-186.66}{EA}+0.016$$

$$\Delta_{20}=\sum\frac{n_2NL}{EA}+\sum n_2\propto\Delta TL+\sum n_2\ error$$

$$\Delta_{20}=\frac{-162.01}{EA}-0.6x40x3x12E-6-0.8x(-0.01)=\frac{-162.01}{EA}+0.00713$$

Take $EA=10^5$

$$21.32x_1+11.2x_2=186.66-0.016x10E5----1$$

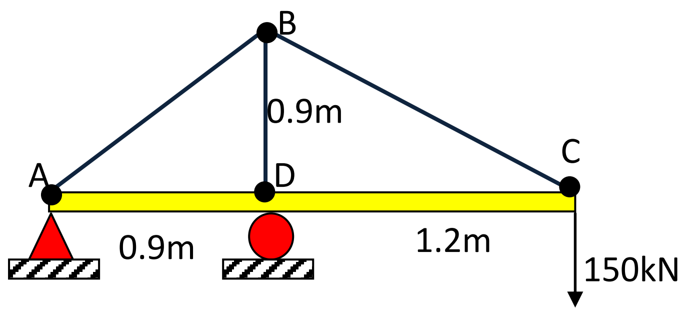
$$11.2x_1+17.28x_2=162.01-0.00713x10E5----2$$

$X_1=-75.11$ $X_2=16.801$

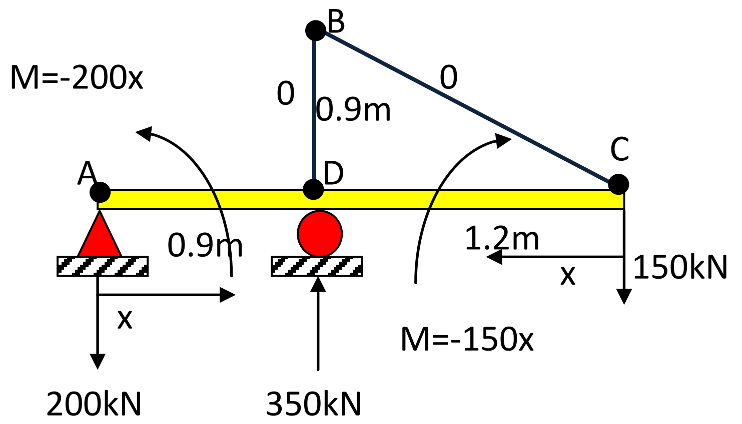
Then find the force in each bar by same procedure of Example 7

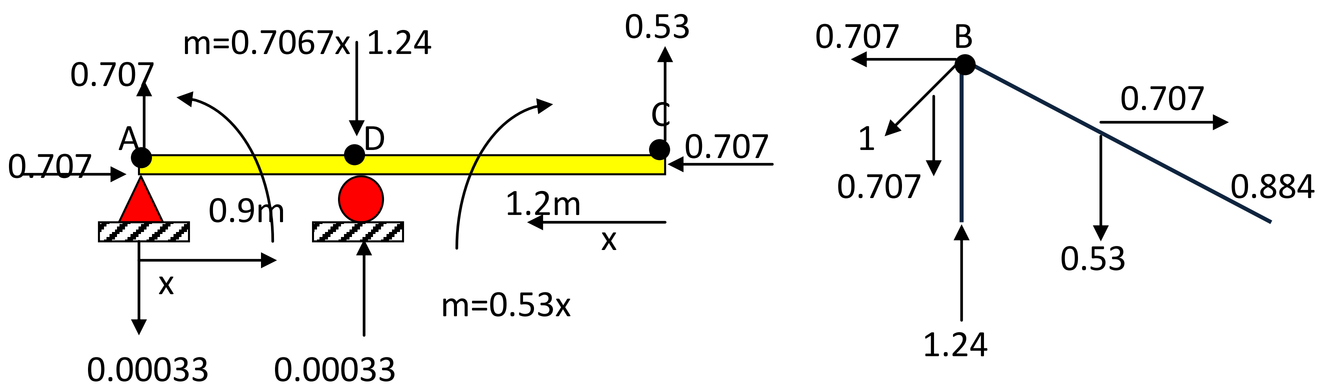
5-COMPOSITE STRUCTURE

Example(8):- Analysis the composite structure shown in figure. The beam has moment of inertia of $240x10^6 mm^4$,the member AB&BC each have a cross-sectional area of 1250 mm^2, and BD has across-sectional area of 2500 mm^2. Take E=200 GPa.



The structure is statically indeterminate to 1^{st} degree





$$\Delta_{10}=\int\frac{Mm1\,dx}{EI}+\sum\frac{n_1NL}{EA}$$
$$=\frac{1}{200x240}[\int_0^{1.2}(-150X)(0.53X)dX+(-200X)(0.7067X)dX]$$
$$=-1.67x10^{-3}$$

$$s_{11}=\int\frac{m1^2\,dx}{EI}$$
$$+\sum\frac{{n_1}^2L}{EA}$$
$$=\frac{1}{200x240}[\int_0^{1.2}(0.53X)^2dX+(0.7067X)^2dX]+\frac{0.884^2x1.5}{200x1250}+\frac{1.24^2x0.9}{200x2500}$$
$$+\frac{1^2x1.27}{200x1250}=1.84x10^{-5}$$

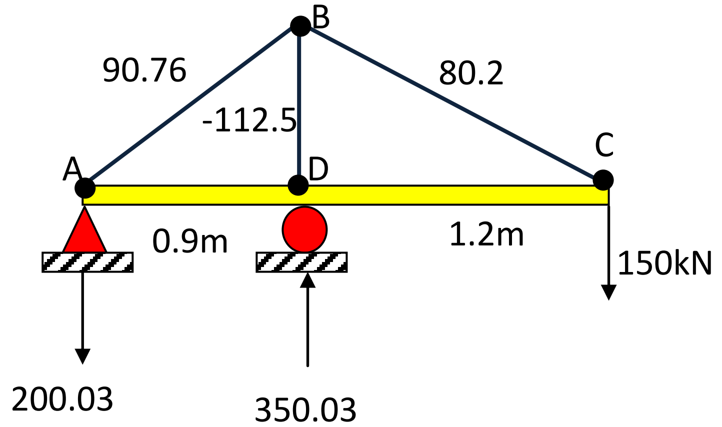
$\Delta_{10}+s_{11}x_1=0$

$x_1=90.76\,T$

F_{AB}=90.76T

F_{BD}=1.24x90.76=112.5 C

F_{BC}=0.884x90.76=80.2 T



The Slope-Deflection Method

Introduction

As pointed out earlier, there are two distinct methods of analysis for statically indeterminate structures depending on how equations of equilibrium, load displacement and compatibility conditions are satisfied: 1) force method of analysis and (2) displacement method of analysis. In the last module, force method of analysis was discussed. In this module, the displacement method of analysis will be discussed. In the force method of analysis, primary unknowns are forces and compatibility of displacements is written in terms of pre-selected redundant reactions and flexibility coefficients using force displacement relations. Solving these equations, the unknown redundant reactions are evaluated. The remaining reactions are obtained from equations of equilibrium.

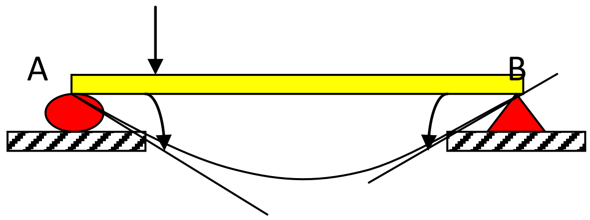
As the name itself suggests, in the displacement method of analysis, the primary unknowns are displacements. Once the structural model is defined for the problem, the unknowns are automatically chosen unlike the force method. Hence this method is more suitable for computer implementation. In the displacement method of analysis, first equilibrium equations are satisfied. The equilibrium of forces is written by expressing the unknown joint displacements in terms of load by using load displacement relations. These equilibrium equations are solved for unknown joint displacements. In the next step, the unknown reactions are computed from compatibility equations using force displacement relations. In displacement method, three methods which are closely related to each other will be discussed.

1. Slope-deflection method
2. Moment distribution
3. Direct stiffness method

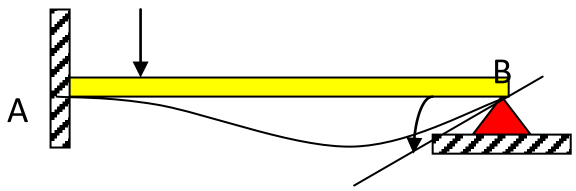
Degrees of Freedom

In the displacement method of analysis, primary unknowns are joint displacements, which are commonly referred to as the degrees of freedom of the structure. It is necessary to consider all the independent degrees of freedom while writing the equilibrium equations. These degrees of freedom are specified at supports, joints and at the free ends

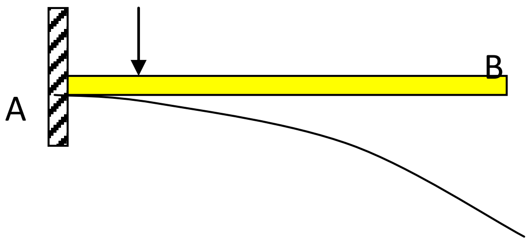
The following examples illustrated who to determine the degree of freedom



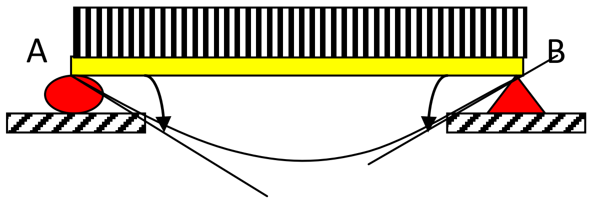
Is kinematically indeterminate to 2nd degree, unknowns are θA,and θB



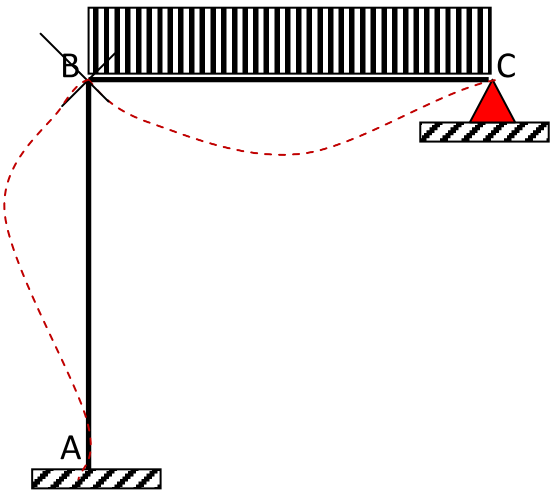
Is kinematically indeterminate to 1st degree, unknowns is θB



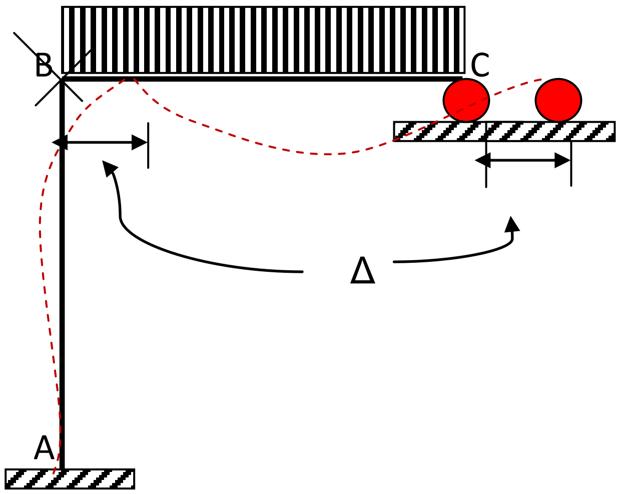
Is kinematically indeterminate to 2nd degree, unknowns are θB, and ΔB



Is kinematically indeterminate to 1st degree, unknowns is θA,because θB=-θA due to symmetry



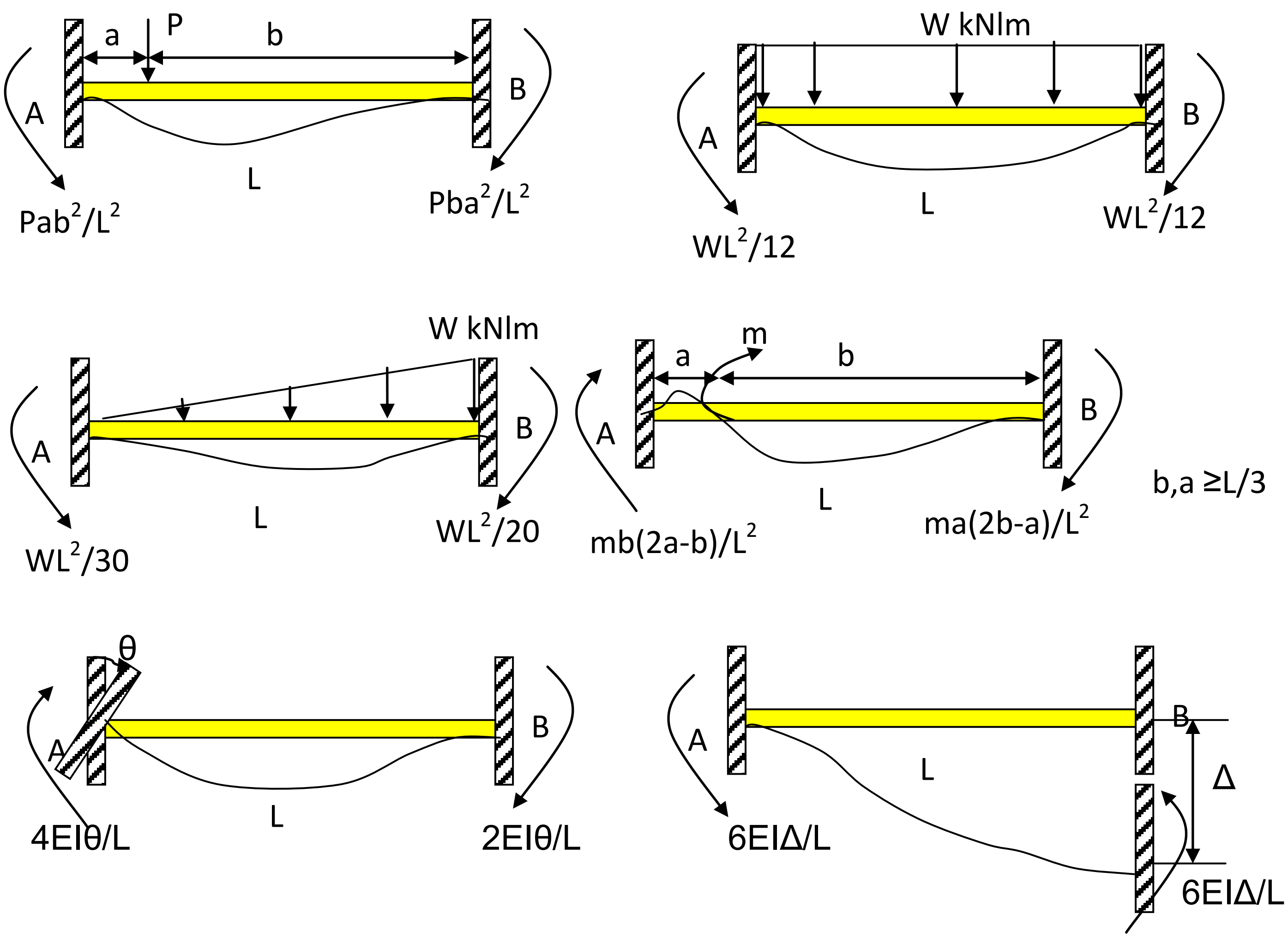
Is kinematically indeterminate to 2nd degree, unknowns are θB,and θC



Is kinematically indeterminate to 3rd degree, unknowns are θB, θC and Δ

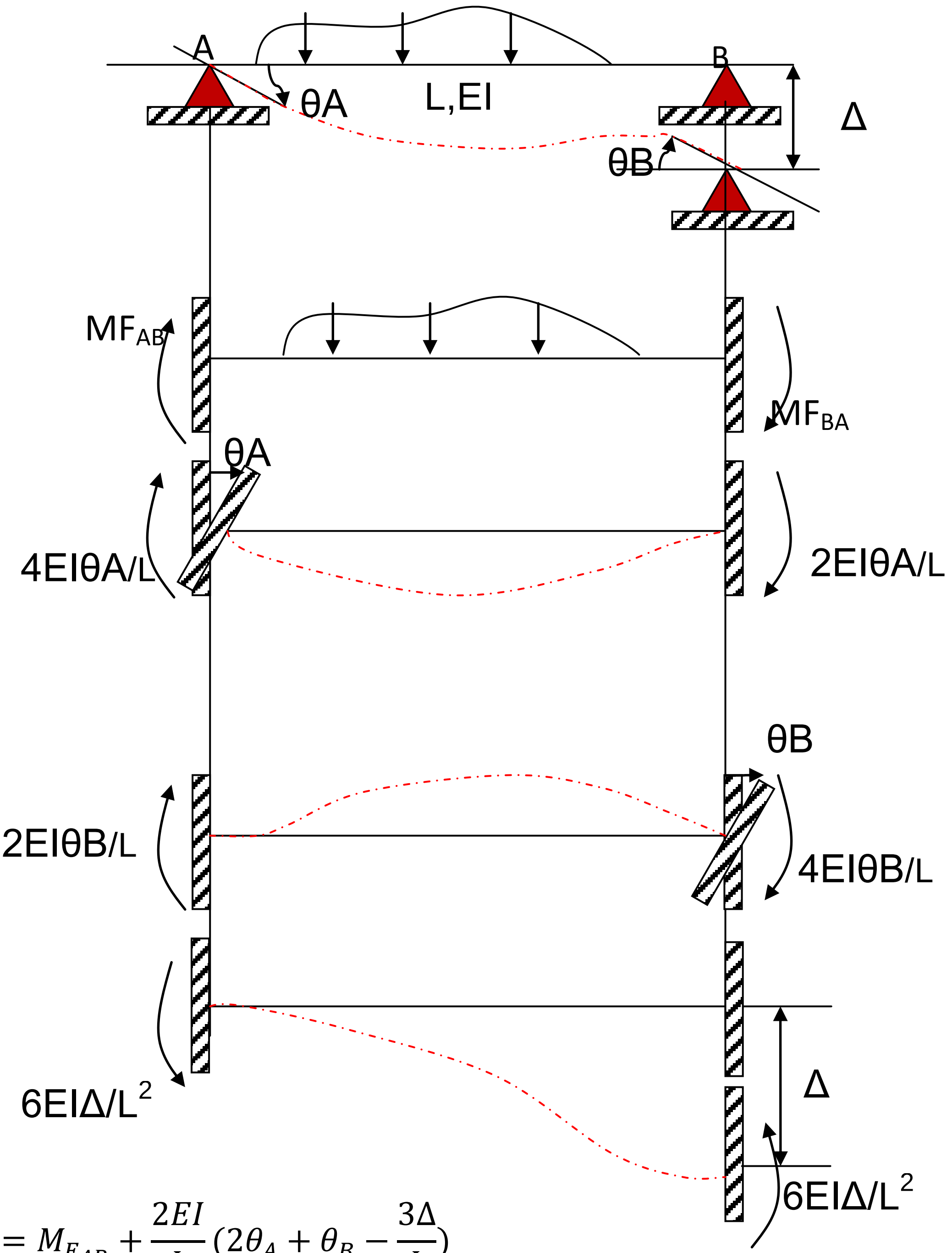
Fixed-end moments

Moment with clockwise will be considered positive moment



SLOPE-DEFLECTION METHOD

This method is applicable to indeterminate beams and frames



$$M_{AB} = M_{F_{AB}} + \frac{2EI}{L}(2\theta_A + \theta_B - \frac{3\Delta}{L})$$

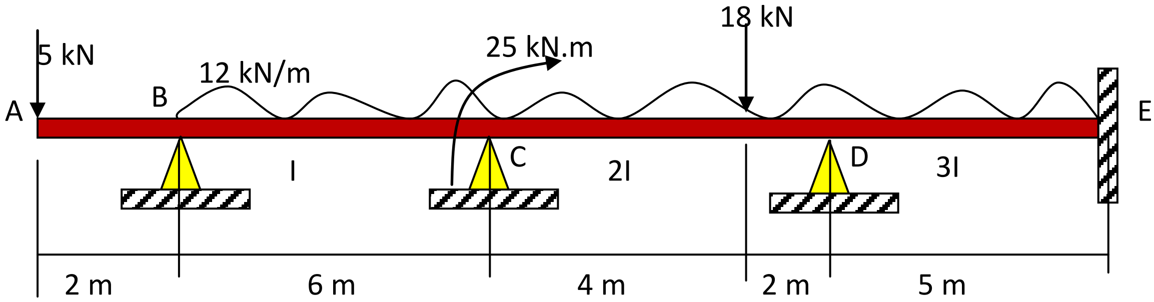
$$M_{BA} = M_{F_{BA}} + \frac{2EI}{L}(2\theta_B + \theta_A - \frac{3\Delta}{L})$$

SIGN CONVENTION

M,θ, Δ/L=+ clockwise

Example(1):- write the F.E.M for the beam shown, due to the applied loading and:

A settlement at support D=18 mm downward , rotational slip at E=0.002 rad C.C.W. use $EI=3(10^4)$ kN.m^2



Solution

1-unknowns θ_B, θ_C and θ_D

2-F.E.M

$$M_{BC} = \frac{-12(6)^2}{12} = -36$$

$$M_{CB} = \frac{12(6)^2}{12} = 36$$

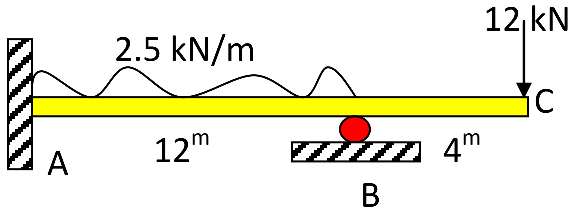
$$M_{CD} = \frac{-12(6)^2}{12} - \frac{18x2^2X4}{6^2} - \frac{6x2x3x10^4x0.018}{6^2} = -224$$

$$M_{DC} = \frac{12(6)^2}{12} + \frac{18x4^2X2}{6^2} - \frac{6x2x3x10^4x0.018}{6^2} = -128$$

$$M_{DE} = \frac{-12(5)^2}{12} - \frac{2x3x3x10^4x0.002}{5} + \frac{6x3x3x10^4x0.018}{5^2} = 291.8$$

$$M_{ED} = \frac{12(5)^2}{12} - \frac{4x3x3x10^4x0.002}{5} + \frac{6x3x3x10^4x0.018}{5^2} = 269.8$$

Example(2):- Analysis the following beam using slope-deflection method. EI constant



Solution

1- Unknown is θ_B
2- F.E.M

$$M_{AB} = \frac{-2.5(12)^2}{12} = -30$$

$$M_{BA} = \frac{2.5(12)^2}{12} = 30$$

3-Slope-deflection equations

$$M_{AB} = -30 + \frac{2EI}{12}\theta_B$$

$$M_{BA} = 30 + \frac{2EI}{12}(2\theta_B) = 30 + \frac{4EI}{12}(\theta_B)$$

3- Joint conditions

Joint B

$M_{BA}+M_{BC}=0$

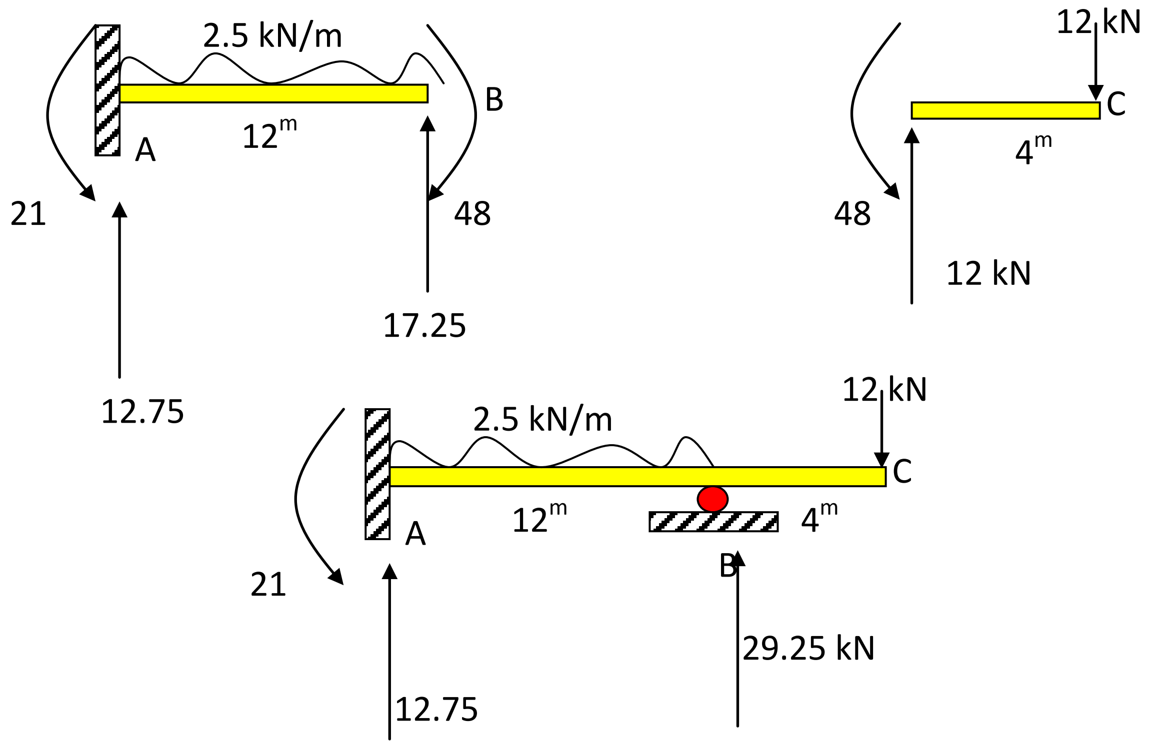
$M_{BC}=-4(12)=-48$(C.C.W)

$$30 + \frac{4EI}{12}(\theta_B) - 48 = 0$$

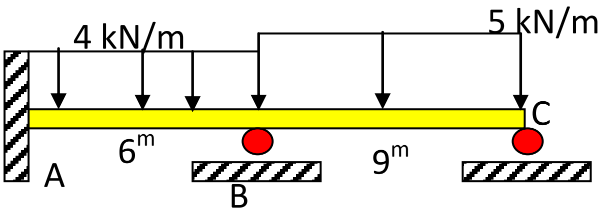
$$EI\theta_B = 54$$

$$M_{BA} = 48$$

$$M_{AB} = -21$$



Example(3):- Analysis the following beam using slope-deflection method. The support at B settles 45mm.E=200GPa and I=3200(10^6) mm^2



Solution

1- Unknown are θ_B, θ_C
2- F.E.M

$$M_{AB} = \frac{-4(6)^2}{12} - \frac{6x200x3200x0.045}{6^2} = -12 - 4800 = -4812$$

$$M_{BA} = 12 - 4800 = -4788$$

$$M_{BC} = \frac{-5(9)^2}{12} + \frac{6x200x3200x0.045}{9^2} = -33.75 + 2133.33 = 2099.58$$

$M_{CB} = 33.75 + 2133.33 = 2167.08$

3-S-D-E

$$M_{AB} = -4812 + \frac{2}{6}EI\theta_B$$

$$M_{BA} = -4788 + \frac{4}{6}EI\theta_B$$

$$M_{BC} = 2099.58 + \frac{4}{9}EI\theta_B + \frac{2}{9}EI\theta_C$$

$$M_{CB} = 2167.08 + \frac{2}{9}EI\theta_B + \frac{4}{9}EI\theta_C$$

4-joint condition

$M_{BA}+M_{BC}=0$

$$\frac{10}{9}EI\theta_B + \frac{2}{9}EI\theta_C = 2688.42 - - - - - - - - - - - - - - - -1$$

$$\frac{2}{9}EI\theta_B + \frac{4}{9}EI\theta_C = -2167.08 - - - - - - - - - - - - - - - -2$$

$EI\theta_B = 3771.96$

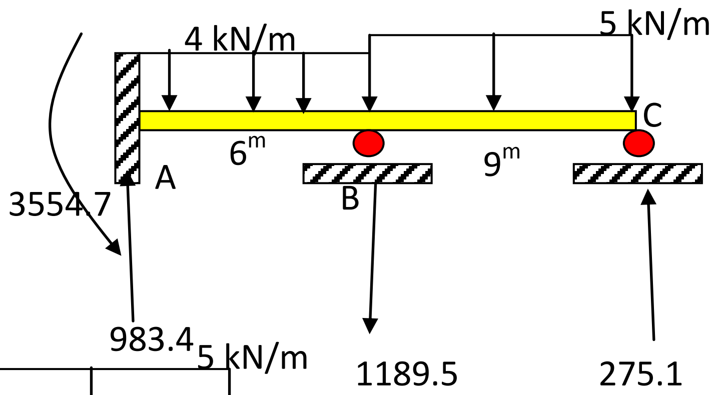
$EI\theta_C = -6761.91$

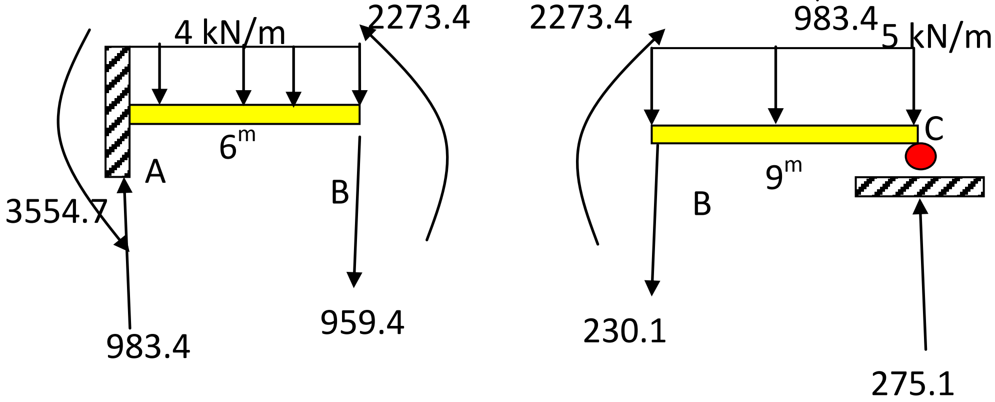
$M_{AB} = -3554.7$

$M_{BA} = -2273.4$

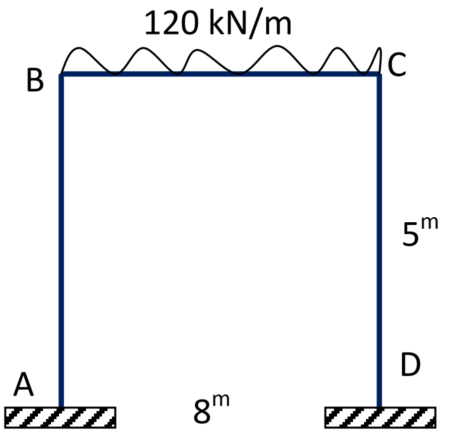
$M_{Bc} = 2273.4$

$M_{CB} = 0$





Example(4):- using slope-deflection method, analysis the frame shown in figure.EI constant



Solution

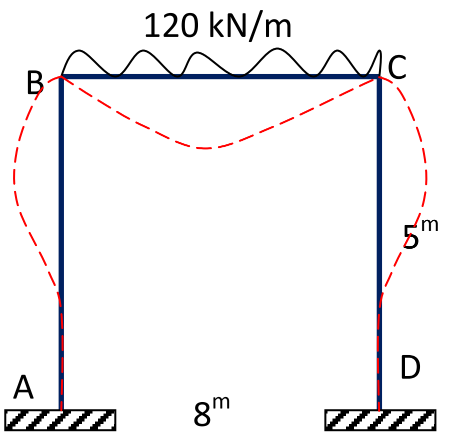
1-unknown, θ

θ_A=- θ_B= θ

2-F.E.M

$$M_{Bc} = \frac{-120(8)^2}{12} = -640$$

3-S.D.E



$$M_{AB} = \frac{2EI}{5}\theta = \frac{2}{5}EI\theta$$

$$M_{BA} = \frac{2EI}{5}(2\theta) = \frac{4}{5}EI\theta$$

$$M_{BC} = -640 + \frac{2EI}{8}(2\theta - \theta) = \frac{1}{4}EI\theta - 640$$

4- joint condition

$$M_{BA} + M_{BC} = 0$$

$$\frac{4}{5}EI\theta + \frac{1}{4}EI\theta - 640 = 0$$

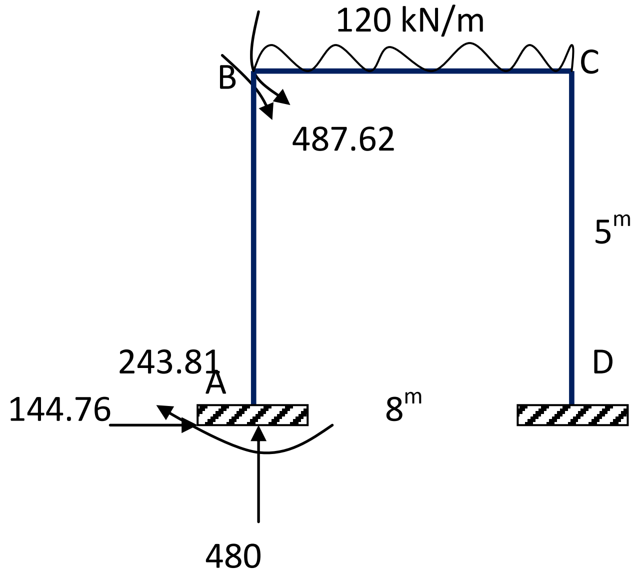
$$EI\theta = 609.52$$

5-final moments

$$M_{BC} = -487.62$$

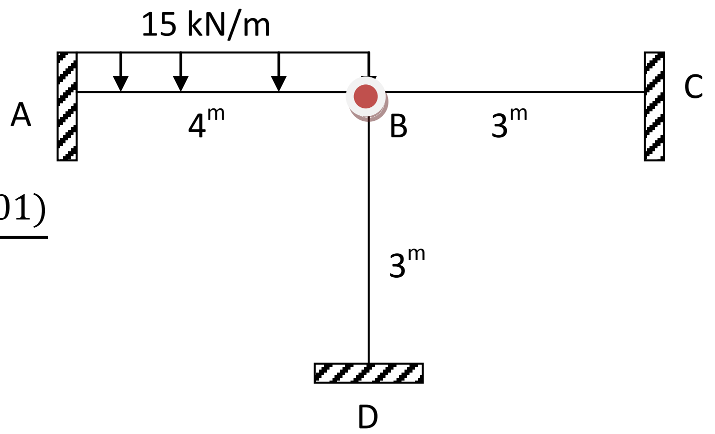
$$M_{BA} = 487.62$$

$$M_{AB} = 243.81$$



Example(5):- Analysis the frame shown using slope-deflection method.E=200GPa, I=800E6 mm^4. The support at D settles 10 mm

solution



1- Unknown θ_{B1}, θ_{B2}, θ_{B3}
2- F.E.M

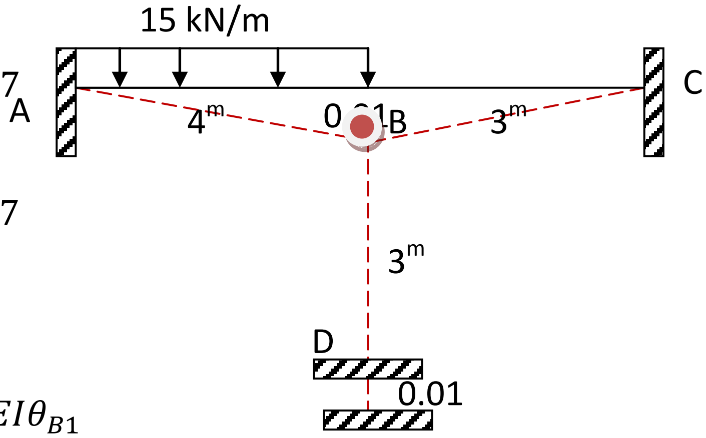
$$M_{AB} = \frac{-15(4)^2}{12} - \frac{6(200x800)(0.01)}{(4)^2}$$

$$M_{AB} = -20 - 600 = -620$$

$$M_{AB} = 20 - 600 = -580$$

$$M_{BC} = -\frac{6(200x800)(-0.01)}{(3)^2} = 1066.7$$

$$M_{CB} = -\frac{6(200x800)(-0.01)}{(3)^2} = 1066.7$$



3-slope deflection equations

$$M_{AB} = -620 + \frac{2EI}{4}\theta_{B1} = -620 + 0.5EI\theta_{B1}$$

$$M_{BA} = -580 + \frac{2EI}{4}2\theta_{B1} = -580 + EI\theta_{B1}$$

$$M_{BC} = 1066.7 + \frac{4}{3}EI\theta_{B3}$$

$$M_{CB} = 1066.7 + \frac{2}{3}EI\theta_{B3}$$

$$M_{BD} = \frac{4}{3}EI\theta_{B2}$$

$$M_{DB} = \frac{2}{3}EI\theta_{B2}$$

4-joint conditions

$$M_{BA} = 0$$

$$-580 + EI\theta_{B1} = 0$$

$EI\theta_{B1} = 580$

$M_{BD} = 0$

$EI\theta_{B2} = 0$

$M_{Bc} = 0$

$1066.7 + \frac{4}{3}EI\theta_{B3} = 0$

$EI\theta_{B3} = -800.025$

5-final moments

$M_{AB} = -330$

$M_{BA} = 0$

$M_{BC} = 0$

$M_{CB} = 533.4$

$M_{BD} = 0$

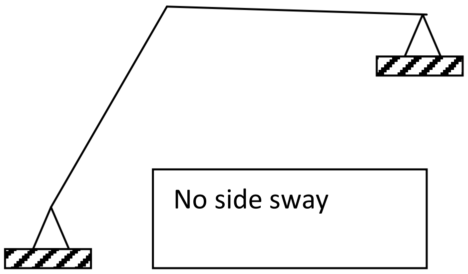
$M_{DB} = 0$

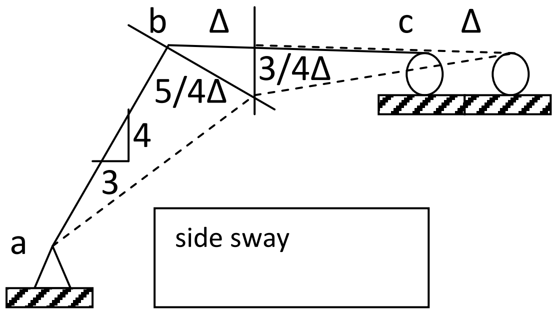
Side sway Frame

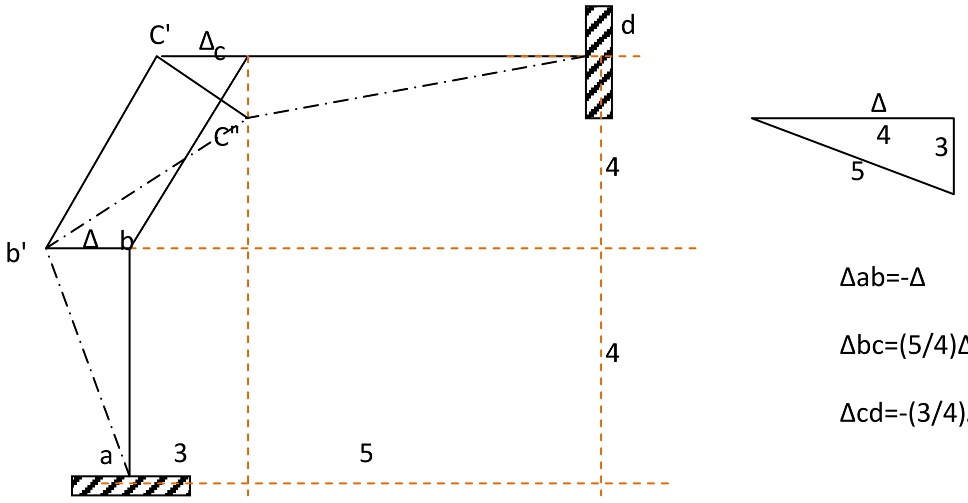
The frame is said to be side sway ,if

1- The frame or loading acting on it is non symmetric
2- There are at least two points in frame can move freely

The following examples to illustrate the side sway topic





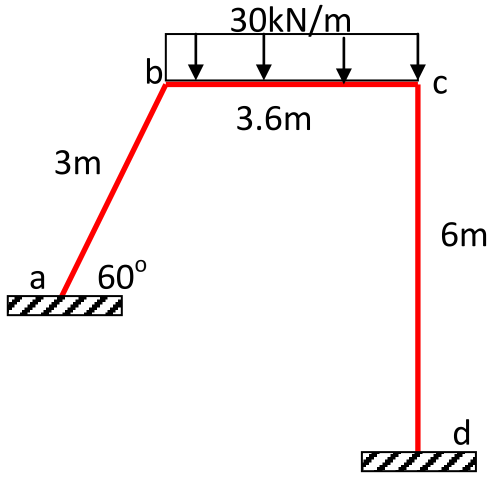


$\Delta ab=-\Delta$

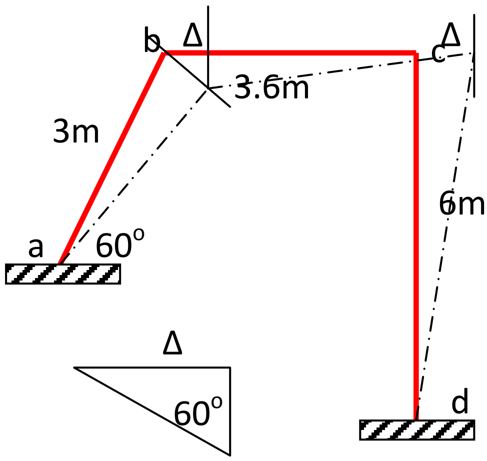
$\Delta bc=(5/4)\Delta$

$\Delta cd=-(3/4)\Delta$

Example:- Analysis the frame shown in figure.EI constant



The frame is sidesway



1- Unknown θ_b, θ_c, Δ
2- F.E.M

$$M_{BC} = -\frac{30(3.6)^2}{12} = -32.4$$

$$M_{CB} = 32.4$$

$$\Delta_{ab} = \frac{\Delta}{sin60} = 1.155\Delta$$

$$\Delta_{bc} = \frac{-\Delta}{tan60} = -0.577\Delta$$

$$\Delta_{cd} = \Delta$$

3-slope deflection equations

$$M_{AB} = \frac{2EI}{3}\left(\theta_b - \frac{3(1.155\Delta)}{3}\right) = 0.667EI\theta_b - 0.77EI\Delta$$

$$M_{BA} = \frac{2EI}{3}\left(2\theta_b - \frac{3(1.155\Delta)}{3}\right) = 1.333EI\theta_b - 0.77EI\Delta$$

$$M_{Bc} = -32.4 + \frac{2EI}{3.6}\left(2\theta_b + \theta_c - \frac{3(-0.577\Delta)}{3.6}\right)$$
$$= -32.4 + 1.111EI\theta_b + 0.556EI\theta_c + 0.267EI\Delta$$

$$M_{Cb} = 32.4 + \frac{2EI}{3.6}\left(\theta_b + 2\theta_c - \frac{3(-0.577\Delta)}{3.6}\right)$$
$$= 32.4 + 0.556EI\theta_b + 1.111\theta_c + 0.267EI\Delta$$

$$M_{Cd} = \frac{2EI}{6}\left(2\theta_c - \frac{3(\Delta)}{6}\right) = 0.667\theta_c - 0.167EI\Delta$$

$$M_{dc} = \frac{2EI}{6}\left(\theta_c - \frac{3(\Delta)}{6}\right) = 0.333\theta_c - 0.167EI\Delta$$

3- joint condition

$$M_{BA} + M_{BC} = 0$$

$1.333EI\theta_b - 0.77EI\Delta + -32.4 + 1.111EI\theta_b + 0.556EI\theta_c + 0.267EI\Delta = 0$

$2.444EI\theta_b - 0.503EI\Delta + 0.556EI\theta_c = 32.4$--------1

$$M_{cb} + M_{cd} = 0$$

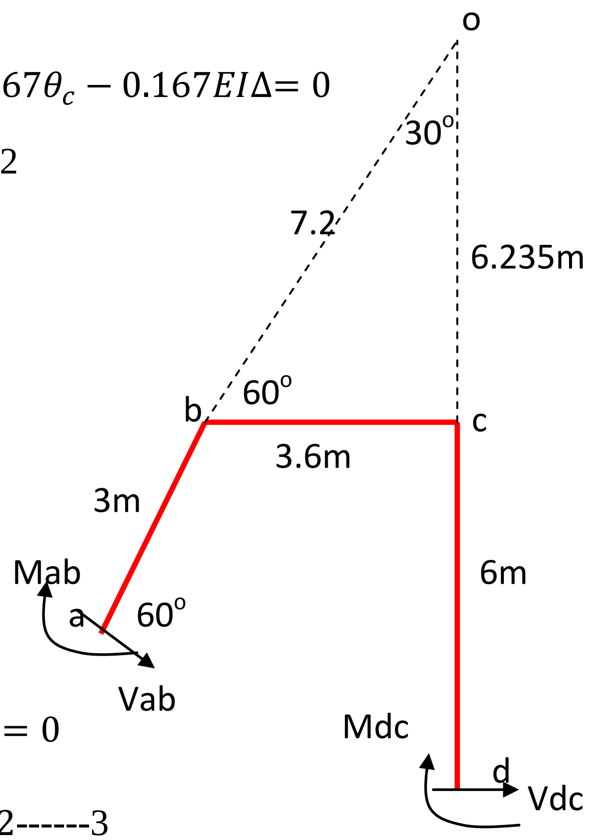
$32.4 + 0.556EI\theta_b + 1.111\theta_c + 0.267EI\Delta + 0.667\theta_c - 0.167EI\Delta = 0$

$0.556EI\theta_b + 1.778\theta_c + 0.1EI\Delta = -32.4$---------2

$$V_{ab} = \frac{M_{ab} + M_{ba}}{3}$$

$$V_{dc} = \frac{M_{dc} + M_{cd}}{6}$$

Moment about o =0



$$M_{ab} + M_{dc} - 10.2V_{ab} - 12.235V_{ab} - \frac{30(3.6)^2}{2} = 0$$

$-1.84EI\theta_b - 0.512EI\theta_c + 1.493EI\Delta = 58.32$------3

Solving equations 1-3,gives

$EI\theta_b = 35.58$

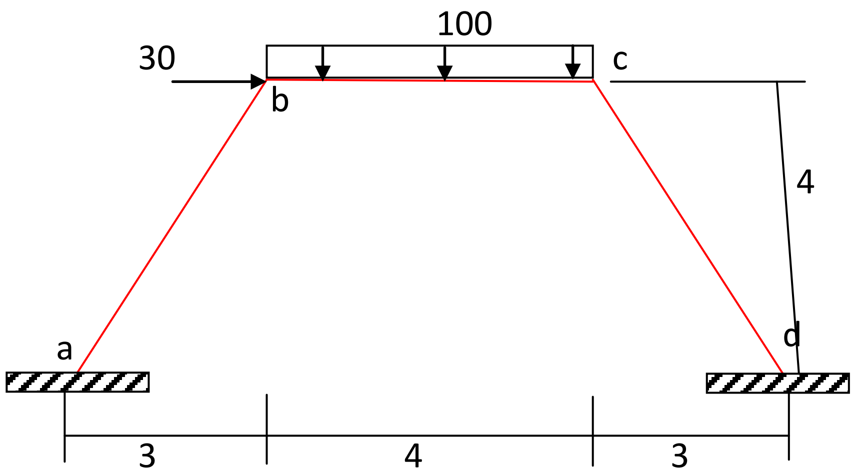
$EI\theta_c = -33.51$

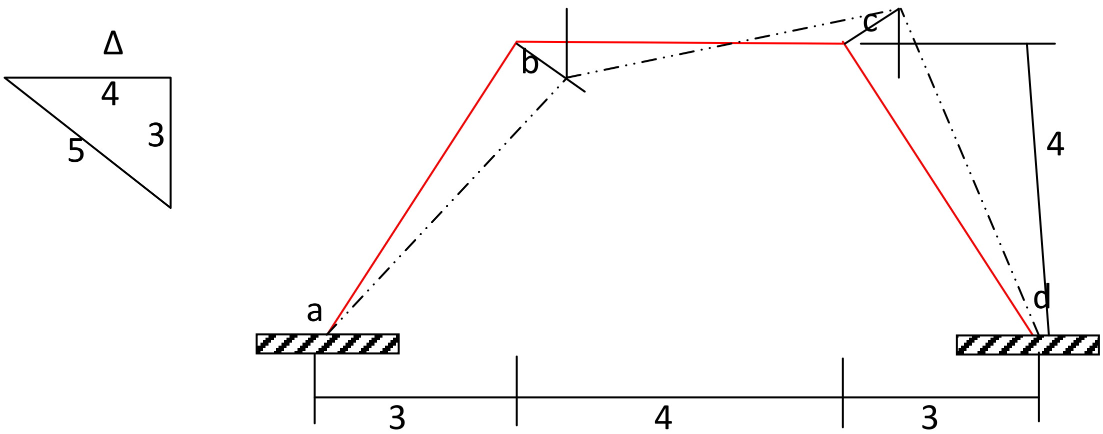
$EI\Delta = 71.42$

$M_{ab} = -31.3$, $M_{ba} = -7.6$, $M_{bc} = 7.6$, $M_{cb} = 34.2$, $M_{cd} = -34.2$, $M_{dc} = -23$

Example:- using slope deflection method, analysis the shown frame

The frame is sidesway





1- Unknown θ_b, θ_c, Δ

2- F.E.M

$$M_{BC} = -\frac{100(4)^2}{12} = -133.33$$

$$M_{CB} = 133.33$$

$$\Delta_{ab} = \frac{5\Delta}{4}$$

$$\Delta_{bc} = \frac{-2 * 3\Delta}{4} = \frac{-6\Delta}{4}$$

$$\Delta_{cd} = \frac{5\Delta}{4}$$

3-slope deflection equations

$$M_{AB} = \frac{2EI}{5}\left(\theta_b - \frac{3(1.25\Delta)}{5}\right) = 0.4EI\theta_b - 0.3EI\Delta$$

$$M_{BA} = \frac{2EI}{5}\left(2\theta_b - \frac{3(1.25\Delta)}{5}\right) = 0.8EI\theta_b - 0.3EI\Delta$$

$$M_{Bc} = -133.33 + \frac{2EI}{4}\left(2\theta_b + \theta_c - \frac{3(-1.5\Delta)}{4}\right)$$
$$= -133.33 + EI\theta_b + 0.5EI\theta_c + 0.563EI\Delta$$

$$M_{Cb} = 133.33 + \frac{2EI}{4}\left(\theta_b + 2\theta_c - \frac{3(-1.5\Delta)}{4}\right)$$
$$= 133.33 + 0.5EI\theta_b + EI\theta_c + 0.563EI\Delta$$

$$M_{Cd} = \frac{2EI}{5}\left(2\theta_c - \frac{3(1.25\Delta)}{6}\right) = 0.8EI\theta_c - 0.3EI\Delta$$

$$M_{dc} = \frac{2EI}{5}\left(\theta_c - \frac{3(1.25\Delta)}{6}\right) = 0.4EI\theta_c - 0.3EI\Delta$$

4- joint condition

$$M_{BA} + M_{BC} = 0$$

$1.8EI\theta_b + 0.263EI\Delta + 0.5EI\theta_c = 133.33$--------1

$M_{cb} + M_{cd} = 0$

$0.5EI\theta_b + 1.8EI\theta_c + 0.263EI\Delta = -133.33$---------2

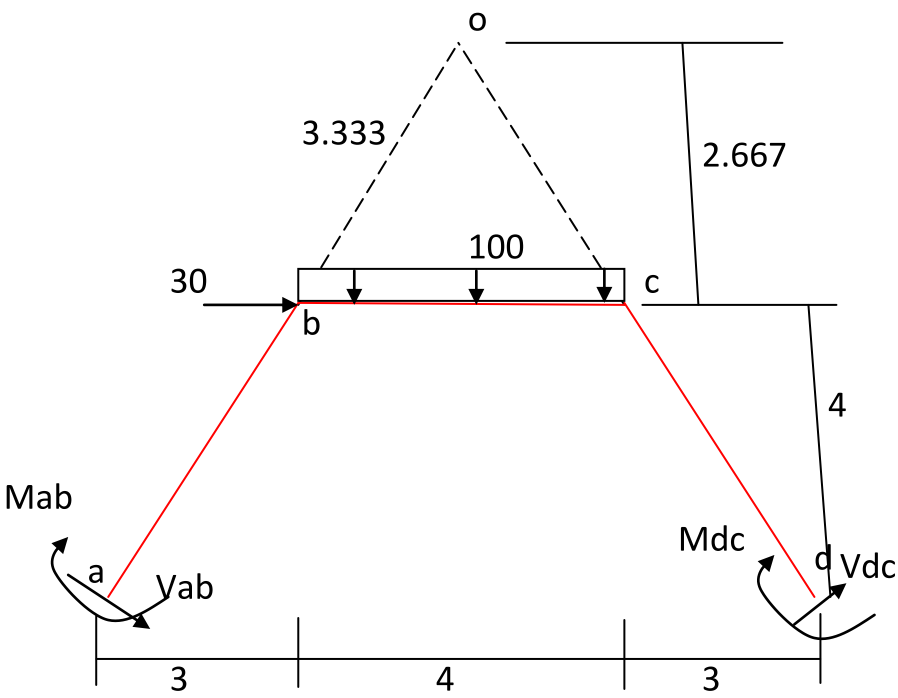
$$V_{ab} = \frac{M_{ab} + M_{ba}}{5}$$

$$V_{dc} = \frac{M_{dc} + M_{cd}}{5}$$

Moment about o =0

$M_{ab} + M_{dc} - 8.333V_{ab} - 8.333V_{dc} - 30(2.667) = 0$

$- - 1.6EI\theta_b - 1.6EI\theta_c + 1.4EI\Delta = 80$------3



Solving equations 1-3,gives

$EI\theta_b = 97.381$

$EI\theta_c = -107.742$

$EI\Delta = 45.302$

$M_{ab} = 25.4$, $M_{ba} = 64.3$, $M_{bc} = -64.3$, $M_{cb} = 99.8$, $M_{cd} = -99.8$, $M_{dc} = -56.7$

MOMENT DISTRIBUTION METHOD

Introduction

In the previous lesson we discussed the slope-deflection method. In slope-deflection analysis, the unknown displacements (rotations and translations) are related to the applied loading on the structure. The slope-deflection method results in a set of simultaneous equations of unknown displacements. The number of simultaneous equations will be equal to the number of unknowns to be evaluated. Thus, one needs to solve these simultaneous equations to obtain displacements and beam end moments. Today, simultaneous equations could be solved very easily using a computer. Before the advent of electronic computing, this really posed a problem as the number of equations in the case of multistory building is quite large. The moment-distribution method proposed by Hardy Cross in 1932, actually solves these equations by the method of successive approximations. In this method, the results may be obtained to any desired degree of accuracy. Until recently, the moment-distribution method was very popular among engineers. It is very simple and is being used even today for preliminary analysis of small structures. It is still being taught in the classroom for the simplicity and physical insight it gives to the analyst even though stiffness method is being used more and more. Had the computers not emerged on the scene, the moment-distribution method could have turned out to be a very popular method.

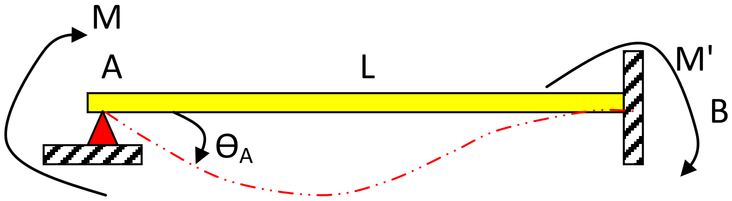
Sign convention

We will establish same sign convention as that established for the slope –deflection equations, clock wise moments that act on members are considered positive , whereas counter clock wise moments are negative.

Member stiffness factor

Can be defined as the amount of moment required to rotate the end of beam 1 rad.

a- Far end is fixed or continuous



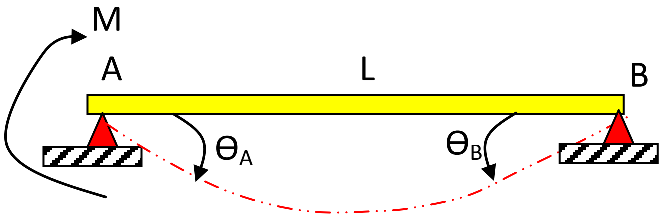
$$M_{AB} = \frac{2EI}{L}(2\theta_A) = \frac{4EI}{L} \quad , \theta_A = 1$$

$$M_{AB} = M = \frac{4EI}{L}$$

$$\therefore K(stiffness\ factor) = \frac{4EI}{L}\ if\ far\ end\ is\ fixed$$

$$M_{BA} = \frac{2EI}{L}(\theta_A) = \frac{2EI}{L} = \frac{1}{2}M \quad , \therefore M' = 0.5\ M$$

b- Far end is pinned



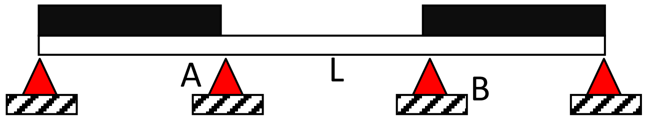
$$M_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A) = 0 \quad , \theta_B = -0.5\theta_A$$

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) = \frac{3EI}{L}\theta_A = \frac{3EI}{L}(because\ \theta_A = 1)$$

$$M_{AB} = M = \frac{3EI}{L}$$

$$\therefore K(stiffness\ factor) = \frac{3EI}{L}\ if\ far\ end\ is\ pinned$$

c-symmetric beam and loading

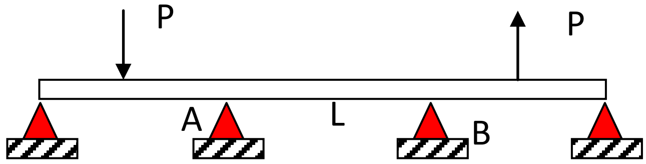


M_{AB}=-M_{BA}

θ_B=- θ_A (due to symmetry)

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) = \frac{2EI}{L}\theta_A \ , \therefore K_{AB} = \frac{2EI}{L}$$

d-Symmetric beam with antisymmetric loading



M_{AB}=M_{BA}

θ_B= θ_A (due to anti symmetry)

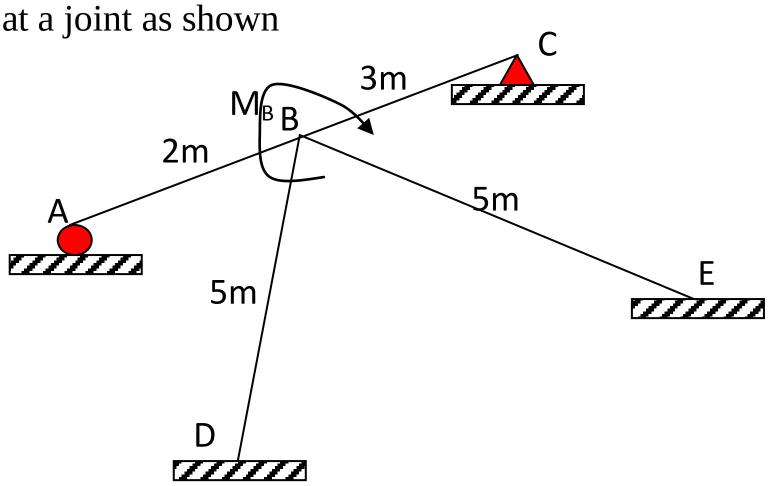
$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) = \frac{6EI}{L}\theta_A \ , \therefore K_{AB} = \frac{6EI}{L}$$

Carry-over factor

The carry-over factor represents the fraction of M that is carried over from the pin to the fixed end. For previous example the carry-over factor is +0.5. the plus sign indicates that both moment acts in the same direction.

Joint stiffness factor

If there are many members are meeting at a joint as shown



To evaluate the total stiffness at joint B(i.e. the amount of moment M_B that required to rotate joint B 1 rad)

From equilibrium at joint B

$M_B = M_{BA} + M_{BD} + M_{BE} + M_{BC}$

As joint B rotates one rad all members that connected at this joint will rotate by same amount , because they are fixed connected to joint B

$$M_{BA} = K_{BA} = \frac{3EI}{L} = \frac{3EI}{2}$$

$$M_{BD} = K_{BD} = \frac{4EI}{L} = \frac{4EI}{5}$$

$$M_{BE} = K_{BE} = \frac{4EI}{L} = \frac{4EI}{5}$$

$$M_{BC} = K_{BC} = \frac{3EI}{L} = \frac{3EI}{3} = EI$$

$M_B = K_B$, $K_B = K_{BA} + K_{BD} + K_{BE} + K_{BC}$

$$K_B = \frac{3EI}{2} + \frac{4EI}{5} + \frac{4EI}{5} + EI = \frac{41EI}{10}$$

Distribution factors

If a moment is applied to fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. That fraction of total resisting moment supplied by the member is called the distribution factor (D.F).to obtain this value, imagine the joint is connected to n members. If an applied moment causes the joint to rotate an amount Ɵ, the each member *i* by this same amount. Return to previous example to evaluate the distribution factor of member AB

$M_{AB} = K_{AB}\, \Theta_B$, $\Theta_B = M_{AB} / K_{AB}$

$M_B = K_B\, \Theta_B$, $\Theta_B = M_B / K_B$

$M_{AB} / K_{AB} = M_B / K_B = M_B / \sum K$

$D.F_{AB} = M_{AB} / M_B = K_{AB} / \sum K$

$$D.F_{BA} = \frac{K_{BA}}{K_B} = \frac{15}{41}$$

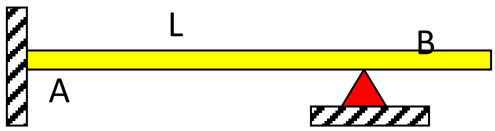
$$D.F_{BD} = \frac{K_{BD}}{K_B} = \frac{8}{41}$$

$$D.F_{BE} = \frac{K_{BE}}{K_B} = \frac{8}{41}$$

$$D.F_{BC} = \frac{K_{BC}}{K_B} = \frac{10}{41}$$

Noting that, $\sum D.F = 1$

Special joints



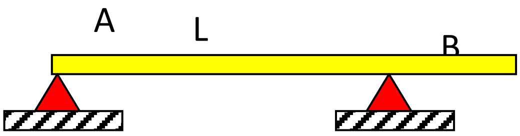
1-If the end is fixed(wall, no rotation allow)

To evaluate the distribution factor at joint A

K_{AB}=3EI/L

$K_{A(wall)}=\infty$

D.F $_{AB}$=(3EI/L)/(3EI/L+∞)=0



2-If the end is pinned(wall, no rotation allow)

To evaluate the distribution factor at joint A

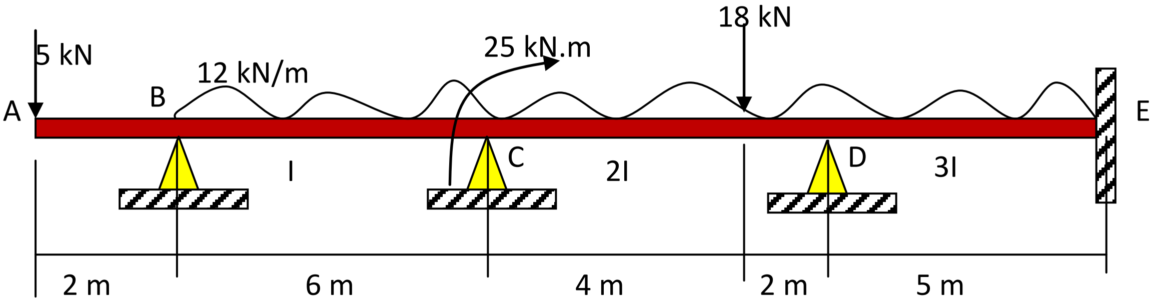
K_{AB}=3EI/L

K_A=0

D.F $_{AB}$=(3EI/L)/(3EI/L+0)=1

Example(1):- using moment distribution method analysis the continuous beam shown , due to the applied loading and:

A settlement at support D=18 mm downward , rotational slip at E=0.002 rad C.C.W. use EI=3(10^4) kN.m^2



Solution

1-F.E.M

$$M_{BC} = \frac{-12(6)^2}{12} = -36$$

$$M_{CB} = \frac{12(6)^2}{12} = 36$$

$$M_{CD} = \frac{-12(6)^2}{12} - \frac{18x2^2X4}{6^2} - \frac{6x2x3x10^4x0.018}{6^2} = -224$$

$$M_{DC} = \frac{12(6)^2}{12} + \frac{18x4^2X2}{6^2} - \frac{6x2x3x10^4x0.018}{6^2} = -128$$

$$M_{DE} = \frac{-12(5)^2}{12} - \frac{2x3x3x10^4x0.002}{5} + \frac{6x3x3x10^4x0.018}{5^2} = 291.8$$

$$M_{ED} = \frac{12(5)^2}{12} - \frac{4x3x3x10^4x0.002}{5} + \frac{6x3x3x10^4x0.018}{5^2} = 269.8$$

2-distribution factor (D.F)

a-for joint c

$$K_{CB} = \frac{3EI}{L} = \frac{3EI}{6} = 0.5EI$$

$$K_{CD} = \frac{4EI}{L} = \frac{4E(2I)}{6} = \frac{4}{3}EI$$

$$K_C = K_{CB} + K_{CD} = \frac{11}{6}EI$$

$$D.F_{CB} = \frac{K_{CB}}{K_C} = 0.273$$

$$D.F_{CD} = \frac{K_{CD}}{K_C} = 0.727$$

b-for joint D

$$K_{DC} = \frac{4EI}{L} = \frac{8EI}{6} = \frac{4}{3}EI$$

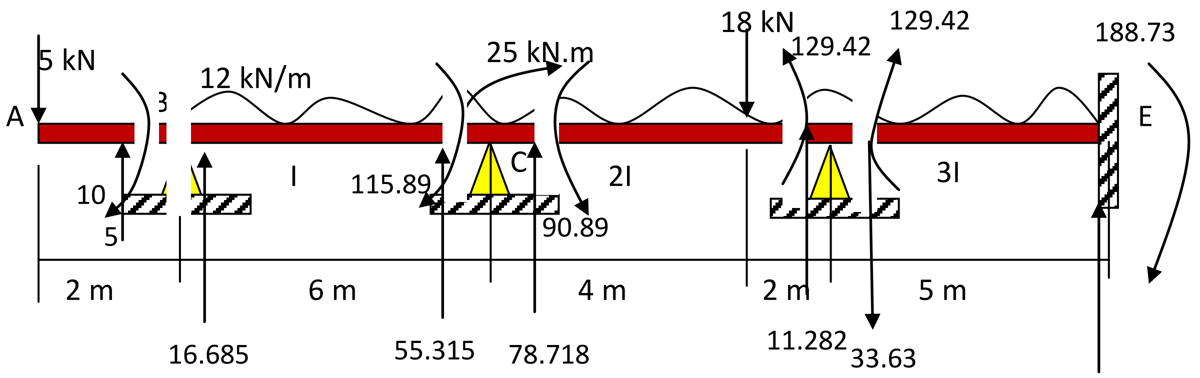
$$K_{DE} = \frac{4EI}{L} = \frac{4E(3I)}{5} = \frac{12}{5}EI$$

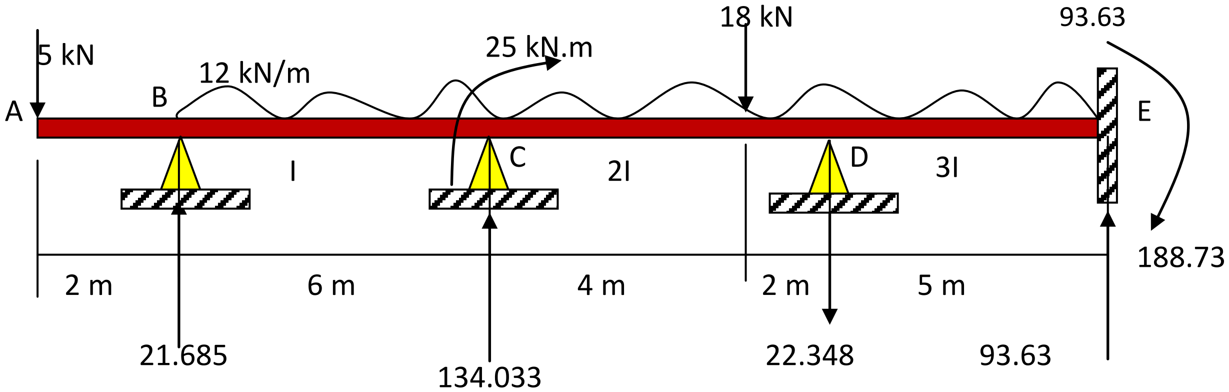
$$K_D = K_{DC} + K_{DE} = \frac{56}{15}EI$$

$$D.F_{DC} = \frac{K_{DC}}{K_D} = 0.357$$

$$D.F_{DE} = \frac{K_{DE}}{K_D} = 0.643$$

joint		B	C		D		E
member		BC	CB	CD	DC	DE	ED
D.F		1	0.273	0.727	0.357	0.643	0
Cycle1	F.E.M	10	-25		-128	291.8	269.8
			36	-224			
		-36					
	Balance moment	26	58.149	154.851	-58.477	-105.323	0
Cycle2	C.O.M		13	-29.239	77.426	0	-52.662
	Balance moment		4.433	11.806	-27.641	-49.785	0
Cycle3	C.O.M		0	-13.821	5.903	0	-24.893
	Balance moment		3.773	10.048	-2.107	-3.796	0
Cycle4	C.O.M		0	-1.054	5.024	0	-1.898
	Balance moment		0.288	0.766	-1.794	-3.23	0
Cycle5	C.O.M		0	-0.897	0.383	0	-1.615
	Balance moment		0.245	0.652	-0.137	-0.246	0
summation		0	115.89	-90.89	-129.42	129.42	188.73





In 2011, Thaar Al-Gasham has developed new procedure in order to reduce calculations necessary for the method of moment distribution. He names this procedure as modified moment distribution. In this method, joints of internal supports are entered in calculations only, joints of external supports either pinned support or fixed support are omitted from calculation, depending upon:

1- External pinned or hinged support have not resisting moment so opposite moment to its fixed end moment is applied to this joint in order to became moment at it equals to zero. Noting, opposite joint to pinned will take 0.5 from applied moment.
2- The moment at external fixed end equals to its fixed end moment plus 0.5 increment of moment at opposite joint.

To illustrate this method numerically, previous example is reanalyzed using this alternative method

Same steps 1&2 for classical moment distribution are considered here

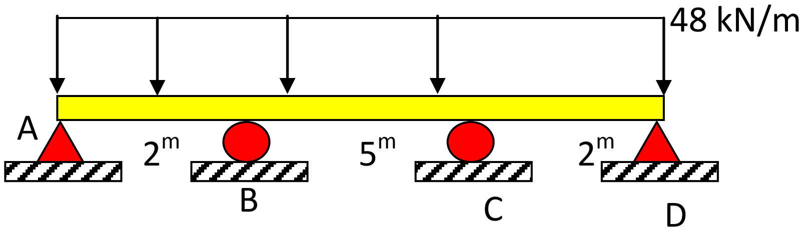
3- Modified F.E.M

$M_{CB} = 36 + 0.5(36 - 10) = 49$

joint		C		D	
member		CB	CD	DC	DE
D.F		0.273	0.727	0.357	0.643
Cycle1	F.E.M	-25		-128	291.8
		49	-224		
	Balance moment	54.6	145.4	-58.477	-105.323
Cycle2	C.O.M	0	-29.289	72.7	0
	Balance moment	7.996	21.293	-25.954	-46.746
Cycle3	C.O.M	0	-12.977	10.647	0
	Balance moment	3.343	9.434	-3.801	-6.846
Cycle4	C.O.M	0	-1.901	4.717	0
	Balance moment	0.519	1.382	-1.684	-3.033
Cycle5	C.O.M	0	-0.842	0.691	0
	Balance moment	0.23	0.612	-0.247	-0.445
summation		115.89	-90.89	-129.41	129.41

$$M_{ED} = 269.8 + 0.5(129.42 - 291.8) = 188.61$$

Example:- using moment distribution, analysis the beam shown. EI is constant



Solution

1-F.E.M

$$M_{AB} = \frac{-48(2)^2}{12} = -16$$

$$M_{BA} = 16$$

$$M_{BC} = \frac{-48(5)^2}{12} = -100$$

2- modification of F.E.M

$$M_{BA} = 16 + 0.5(16) = 24$$

3-D.F

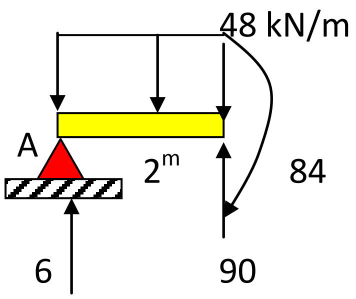
$$K_{BA} = \frac{3EI}{L} = \frac{3}{2}EI$$

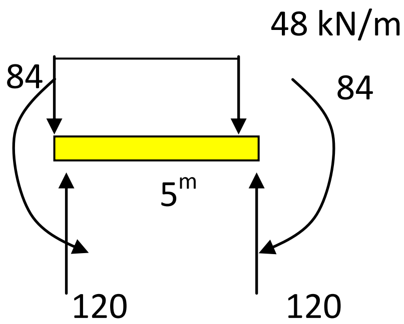
$$K_{BC} = \frac{2EI}{L} = \frac{2}{5}EI$$

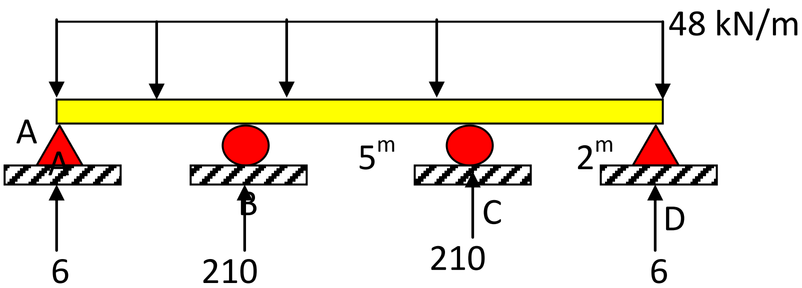
$$D.F_{BA} = \frac{\frac{3}{2}}{\frac{3}{2} + \frac{2}{5}} = 0.789$$

$$D.F_{BC} = \frac{\frac{2}{5}}{\frac{3}{2} + \frac{2}{5}} = 0.211$$

joint		B	
member		BA	BC
D.F		0.789	0.211
Cycle1	F.E.M	24	-100
	Balance moment	59.964	16.036
Cycle2	C.O.M	0	0
	Balance moment	0	0
summation		83.964	-83.964



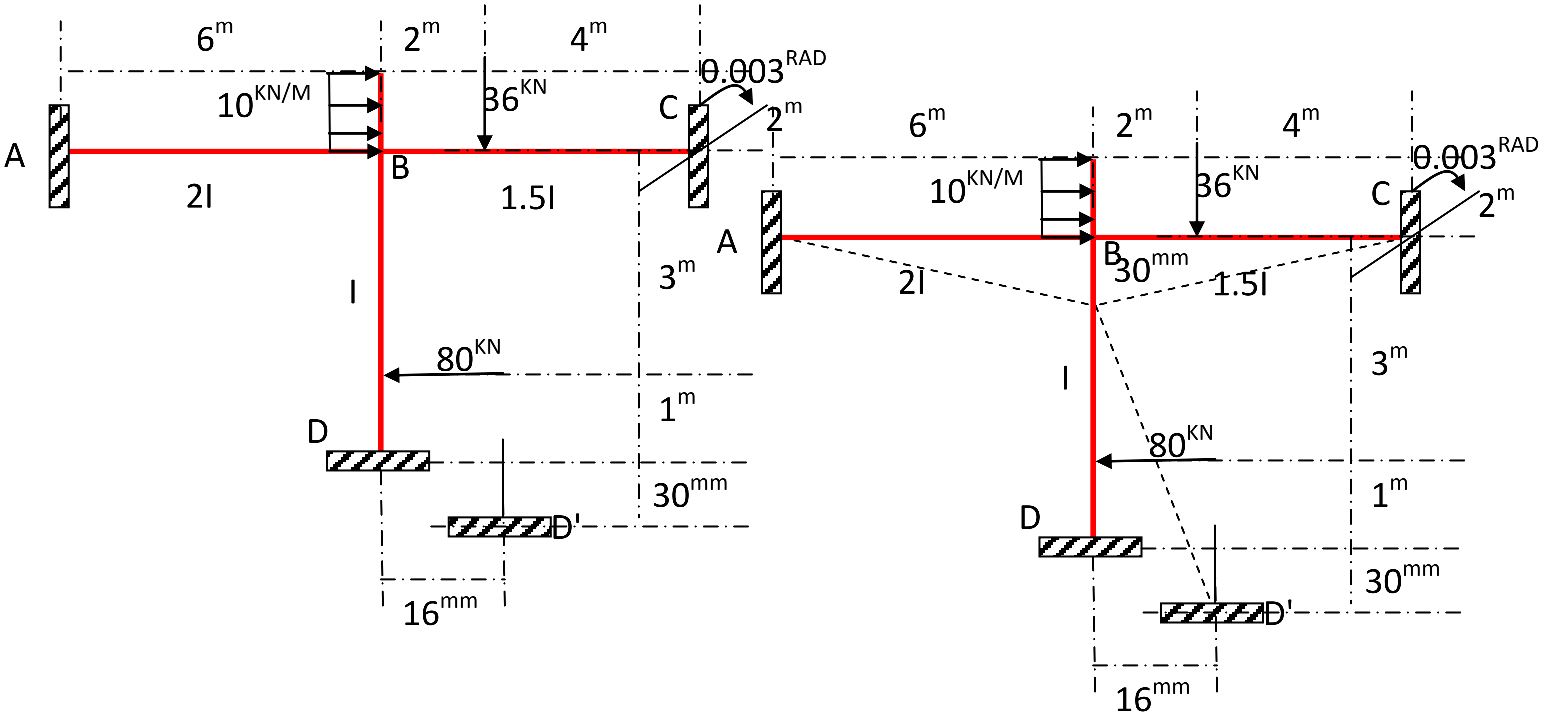




Analysis of Frame

1- Frame without side sway

Example:- using moment distribution method analyze the beam shown, it is subjected to system of forces as well as to rotational slip at C of 0.003 rad C.W, 30mm downward settlement at D and horizontal movement at D of 16 mm toward east. Take EI=10^4 kN.m^2



Solution

1- F.E.M

$$M_{AB} = M_{BA} = \frac{-6(2x10^4)(0.03)}{6^2} = -100$$

$$M_{BC} = \frac{-36x4^2x2}{6^2} + \frac{6(1.5x10^4)(0.03)}{6^2} + \frac{2(1.5x10^4)(0.003)}{6} = -32 + 75 + 15 = 58$$

$$M_{CB} = \frac{36x2^2x4}{6^2} + \frac{6(1.5x10^4)(0.03)}{6^2} + \frac{4(1.5x10^4)(0.003)}{6} = 121$$

$$M_{BD} = \frac{-80x1^2x3}{4^2} + \frac{6(1x10^4)(0.016)}{4^2} = -15 + 60 = 45$$

$$M_{DB} = \frac{80x3^2x1}{4^2} + \frac{6(1x10^4)(0.016)}{4^2} = 105$$

2- D.F

D.F is calculated at B only

$$K_{BA} = \frac{4EI}{L} = \frac{4}{3}EI$$

$$K_{BC} = \frac{4EI}{L} = EI$$

$$K_{BD} = \frac{4EI}{L} = EI$$

$$K_B = \left(\frac{4}{3} + 1 + 1\right)EI = \frac{10}{3}EI$$

$$D.F_{BA} = \frac{\frac{4}{3}}{\frac{10}{3}} = 0.4$$

$$D.F_{BC} = \frac{1}{\frac{10}{3}} = 0.3$$

$$D.F_{BD} = \frac{1}{\frac{10}{3}} = 0.3$$

<table>
<tr><td colspan="2">joint</td><td colspan="3">B</td></tr>
<tr><td colspan="2">member</td><td>BA</td><td>BC</td><td>BD</td></tr>
<tr><td colspan="2">D.F</td><td>0.4</td><td>0.3</td><td>0.3</td></tr>
<tr><td rowspan="3">Cycle1</td><td rowspan="2">F.E.M</td><td colspan="3">-20</td></tr>
<tr><td>-100</td><td>58</td><td>45</td></tr>
<tr><td>Balance moment</td><td>6.8</td><td>5.1</td><td>5.1</td></tr>
<tr><td colspan="2">summation</td><td>-93.2</td><td>63.1</td><td>50.1</td></tr>
</table>

Moment at fixed ends

$$M_{AB} = -100 + \frac{1}{2}\left(-93.2 - (-100)\right) = -96.6$$

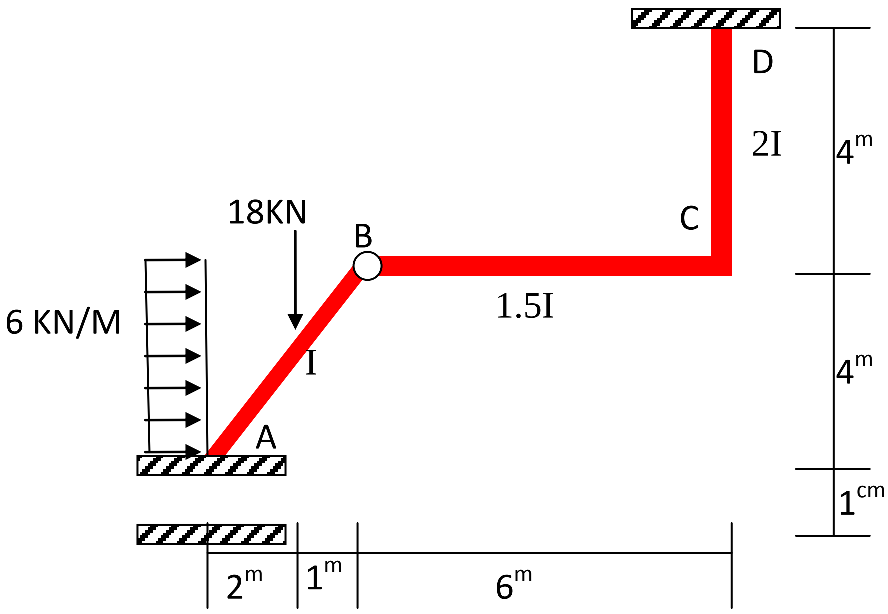
$M_{CB} = 121 + 0.5(63.1 - 58) = 123.55$

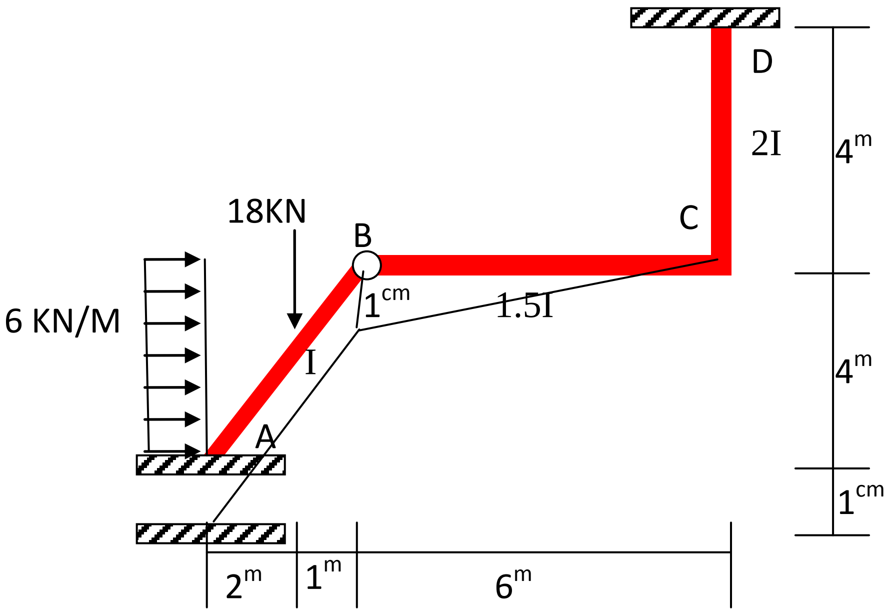
$M_{DB} = 105 + 0.5(50.1 - 45) = 107.55$

Now, all moments are determined. Therefore, all horizontal and vertical reactions can be easily determined.

3- Frame with side sway

Example:- analyze the beam using moment distribution. Take EI=1.5(10^4) kN.m^2





Step 1:- analyze without side sway

1- F.E.M

$$M_{AB} = \frac{-6(4)^2}{12} - \frac{18x2x(1)^2}{(3)^2} = -8 - 4 = -12$$

$$M_{BA} = \frac{6(4)^2}{12} + \frac{18x1x(2)^2}{(3)^2} = 16$$

$$M_{BC} = M_{CB} = \frac{-6(1.5x1.5x10^4)(-0.01)}{6^2} = 37.5$$

2- Modified F.E.M

$$M_{CB} = 37.5 + 0.5(-37.5) = 18.75$$

3- D.F

At joint C

$$K_{CB} = \frac{3EI}{L} = \frac{4.5}{6}EI$$

$$K_{CD} = \frac{4EI}{L} = 2EI$$

$$K_C = K_{CB} + K_{CD} = 2.75EI$$

$$D.F_{CB} = \frac{K_{CB}}{K_C} = 0.273$$

$$D.F_{CD} = \frac{K_{CD}}{K_C} = 0.727$$

joint		C	
member		CB	CD
D.F		0.273	0.727
Cycle1	F.E.M	18.75	0
	Balance moment	-5.119	-13.631
summation		13.631	-13.631

Final moments due to non side sway are

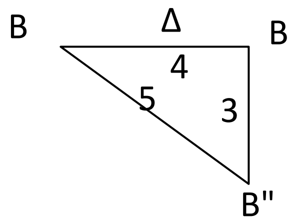
$M_{AB} = -12 + 0.5x - 16 = -20$

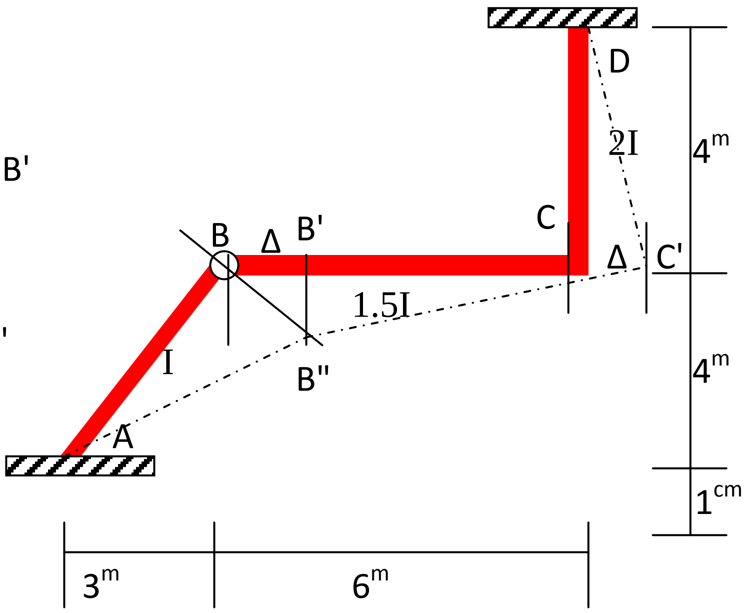
$M_{CB} = 13.631$

$M_{CD} = -13.631$

$M_{DC} = 0 + 0.5(-13.631 - 0) = -6.816$

2- Due to side sway





$\Delta_{AB} = \frac{5}{4}\Delta$, $\Delta_{CB} = -\frac{3}{4}\Delta$, $\Delta_{CD} = -\Delta$

1- F.E.M

Let EIΔ=100f

$$M_{AB} = M_{BA} = \frac{-6EI\Delta}{L^2} = \frac{-6EI(\frac{5}{4}\Delta)}{5^2} = -0.3EI\Delta = -30f$$

$$M_{BC} = M_{CB} = \frac{-6(1.5EI)(-\frac{3}{4}\Delta)}{6^2} = 0.188EI\Delta = 18.8f$$

$$M_{CD} = M_{DC} = \frac{-6(2EI)(-\Delta)}{4^2} = 0.75EI\Delta = 75f$$

2- MODIFIED F.E.M

$M_{CB} = 18.8f - 0.5x18.8f = 9.4f$

joint		C	
member		CB	CD
D.F		0.273	0.727
Cycle1	F.E.M	9.4f	75f
	Balance moment	-23.04f	-61.36f
summation		-13.64f	13.64f

Final moments due to side sway are

$M_{AB} = -30f + 0.5x30f = -15f$

$M_{CB} = -13.64f$

$M_{CD} = 13.64f$

$M_{DC} = 75f + 0.5(13.64f - 75f) = 44.32f$

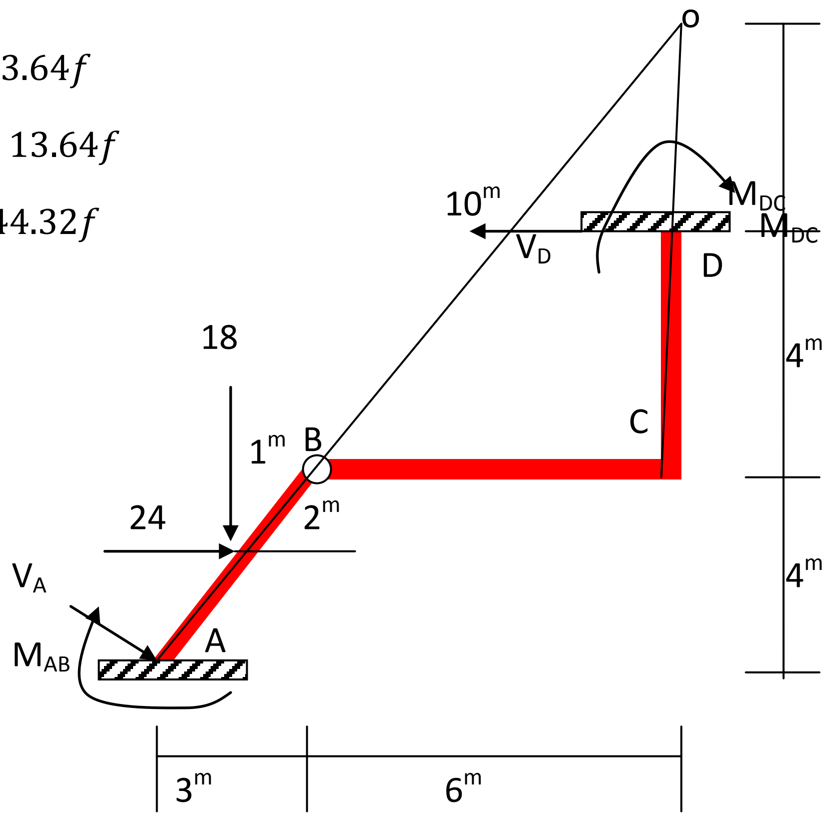
Total moment=M(non side)+ M(side)

$M_{AB} = -20 - 15f$

$M_{CB} = 13.631 - 13.64f$

$M_{CD} = -13.631 + 13.64f$

$M_{DC} = -6.816 + 44.32f$



$$V_A = \frac{M_{AB} - 18(1) - 24(2)}{5} = \frac{M_{AB}}{5} - 13.2 = -17.2 - 3f$$

$$V_D = \frac{M_{DC} + M_{CD}}{4} = -5.112 + 14.49f$$

SUMMATION MOMENT ABOUT O =O

$$M_{AB} + M_{DC} + 4V_D - 15V_A - 24x10 - 18x7 = 0$$

$$-155.264 + 132.28f = 0$$

$$f = 1.174$$

Then actual moments are,

$$M_{AB} = -20 - 15f = -37.61$$

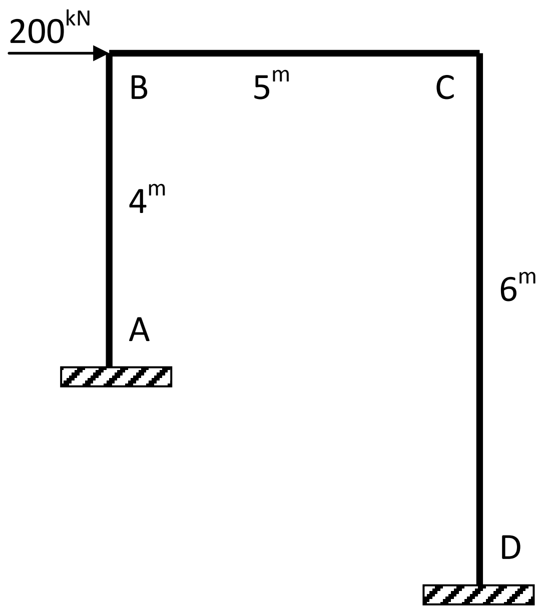
$$M_{CB} = 13.631 - 13.64f = -2.38$$

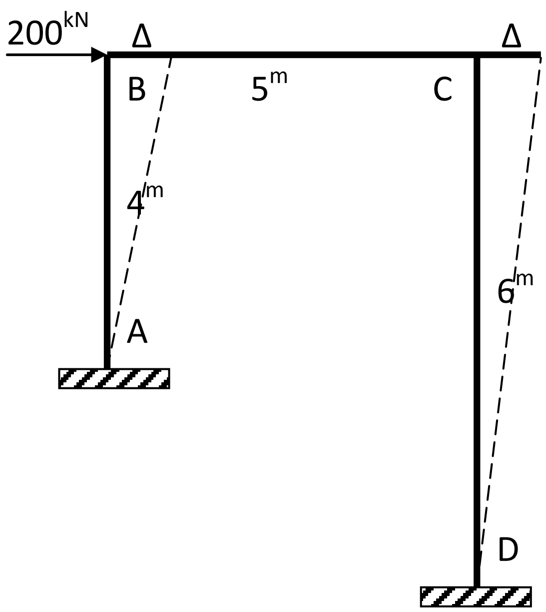
$$M_{CD} = -13.631 + 13.64f = 2.38$$

$$M_{DC} = -6.816 + 44.32f = 45.2$$

Now, all moments are determined. Therefore, all horizontal and vertical reactions can be easily determined.

Example:- determine the moment at each joint of the frame shown. EI is constant





There are no fixed ends moments due to applied loads. Therefore, there is no effect of non-side sway

1-D.F

At joint B

$$K_{BA} = \frac{4EI}{L} = EI$$

$$K_{BC} = \frac{4EI}{L} = \frac{4}{5}EI$$

$$D.F_{BA} = \frac{1}{1+\frac{4}{5}} = 0.556$$

$$D.F_{BA} = \frac{\frac{4}{5}}{1+\frac{4}{5}} = 0.444$$

At joint C

$$K_{CB} = \frac{4EI}{L} = \frac{4}{5}EI$$

$$K_{CD} = \frac{4EI}{L} = \frac{4}{6}EI$$

$$D.F_{CB} = \frac{\frac{4}{5}}{\frac{4}{6}+\frac{4}{5}} = 0.545$$

$$D.F_{CD} = \frac{\frac{4}{6}}{\frac{4}{6}+\frac{4}{5}} = 0.455$$

EFFECT OF SIDE SWAY

Let EIΔ=100f

2-F.E.M

$$M_{AB} = M_{BA} = \frac{-6EI\Delta}{L^2} = \frac{-6EI\Delta}{4^2} = -0.375EI\Delta = -37.5f$$

$$M_{CD} = M_{DC} = \frac{-6EI\Delta}{L^2} = \frac{-6EI\Delta}{6^2} = -0.167EI\Delta = -16.7f$$

joint		B		C	
member		BA	BC	CB	CD
D.F		0.556	0.444	0.545	0.455
Cycle1	F.E.M	-37.5f	0	0	-16.7f
	Balance moment	20.85f	16.65f	9.102f	7.595f
Cycle2	C.O.M	0	4.551f	8.325f	0
	Balance moment	-2.53f	-2.021f	-4.537f	-3.788f
Cycle3	C.O.M	0	-2.269f	-1.011f	0
	Balance moment	1.262f	1.007f	0.551f	0.46f
Cycle4	C.O.M	0	0.276f	0.504f	0
	Balance moment	-0.153f	-0.123f	-0.275f	-0.229f
Cycle5	C.O.M	0	-0.138f	-0.062f	0
	Balance moment	0.077f	0.061f	0.034f	0.028f
summation		-17.994f	17.994f	12.63f	-12.63f

Final moments due to side sway are

$M_{AB} = -37.5f + 0.5(-17.994f + 37.5f) = -27.747f$

$M_{BA} = -17.994f$

$M_{BC} = 17.994f$

$M_{CB} = 12.63f$

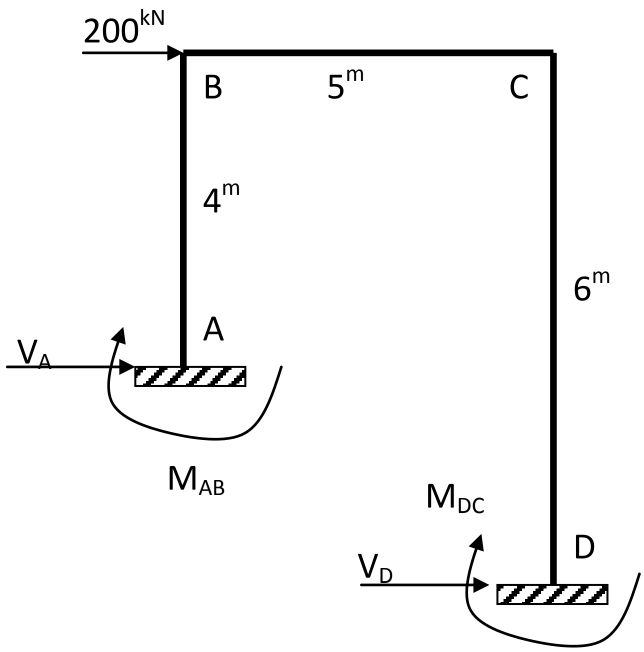
$M_{CD} = -12.63f$

$M_{DC} = -16.7f + 0.5(-12.634f + 16.7f) = -14.667f$

$V_A = \frac{M_{AB}+M_{BA}}{4} = -11.436f$

$$V_D = \frac{M_{DC} + M_{CD}}{6} = -4.55f$$

$$\sum FX = 0$$



$200 + V_A + V_D = 0$

$200 - 11.436f - 4.55f = 0$

$f = 12.511$

$M_{AB} = -27.747f = -347.1$

$M_{BA} = -17.994f = -225.1$

$M_{BC} = 17.994f = 225.1$

$M_{CB} = 12.63f = 158$

$M_{CD} = -12.63f = 158$

$M_{DC} = -14.667f = -183.5$

CONJUGATE-BEAM THEORY

It is one of methods for determining deformations (either displacement, or rotation at a point in a beam). It was developed by Muller-Breslau in 1865, and relies only on the principles of static, and hence its application will be more familiar.

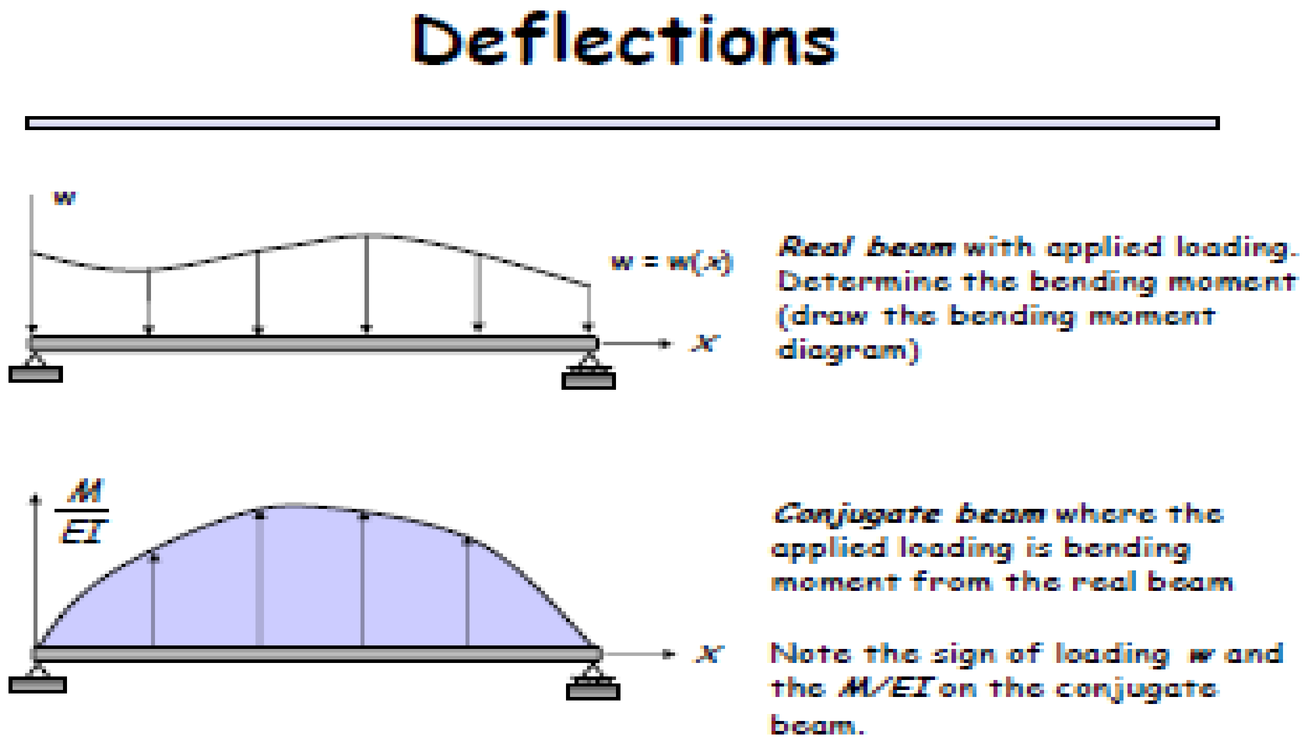
The basis for this method comes from the similarity of the following equations.

$$\frac{dV}{dX} = w \rightarrow \frac{d\theta}{dX} = \frac{M}{EI}$$

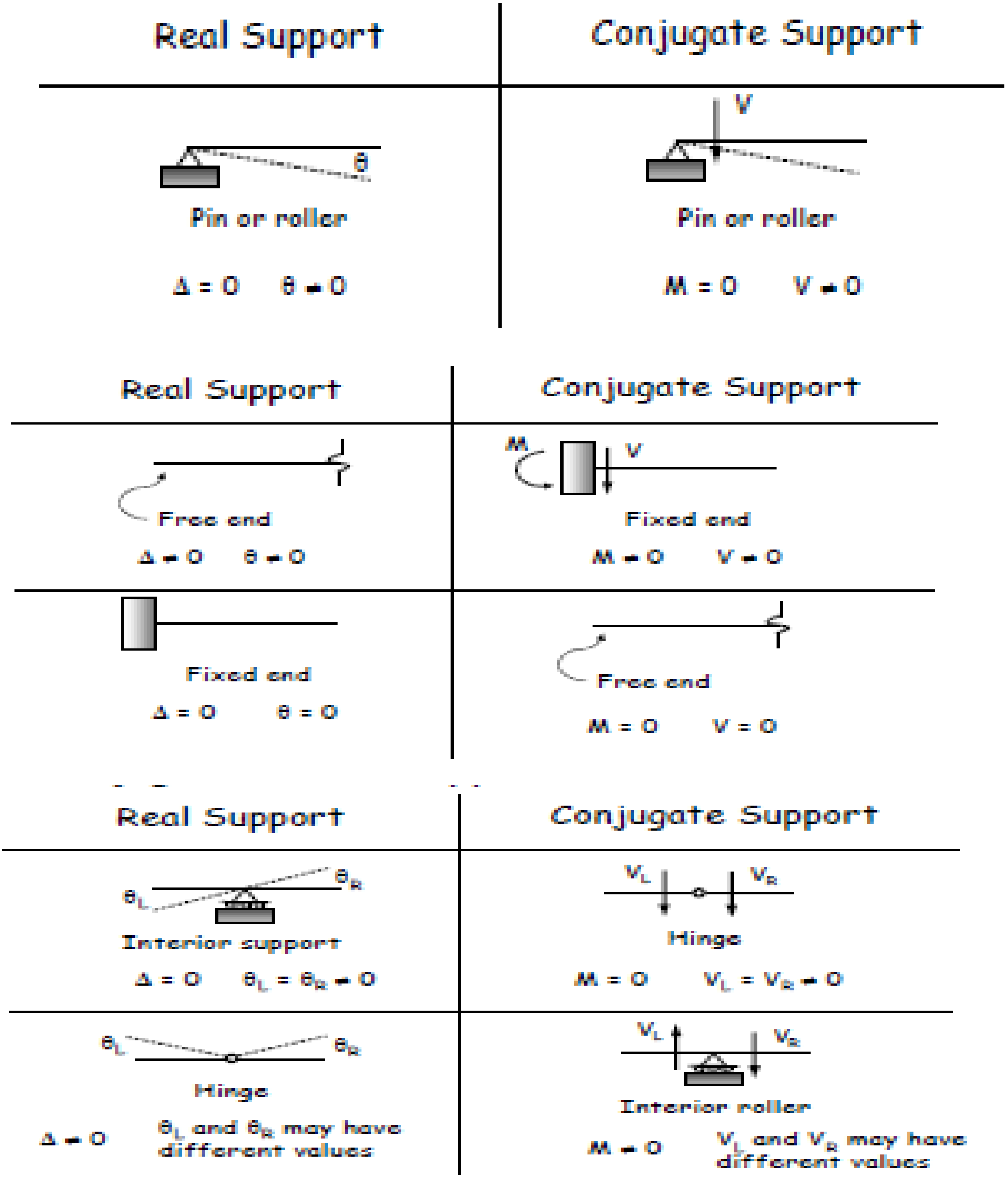
$$\frac{d^2M}{dX^2} = w \rightarrow \frac{d^2y}{dX^2} = \frac{M}{EI}$$

Above equations states that

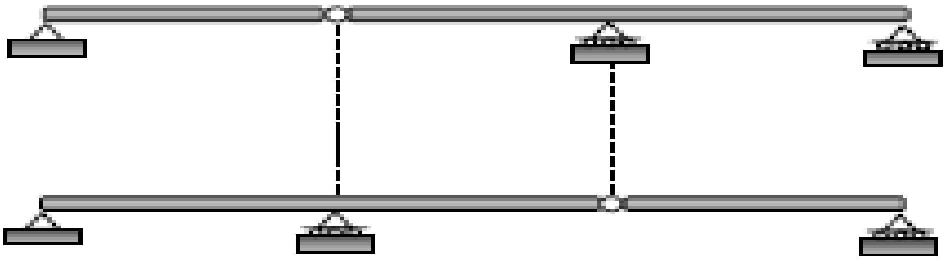
1- The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.
2- The displacement at a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam



Conjugate- Beam supports:- when drawing the conjugate beam it is important that the shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beams at its supports.

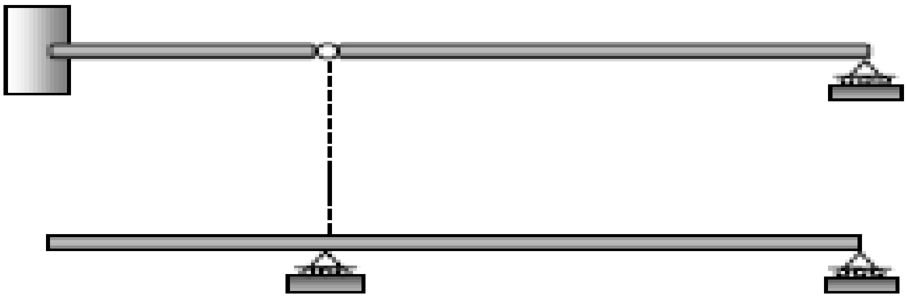


- Draw the conjugate beam, including supports, for the following beams



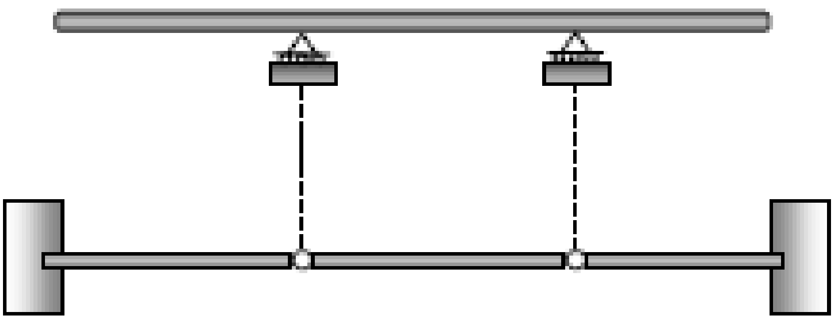
Conjugate beam and supports

- Draw the conjugate beam, including supports, for the following beams

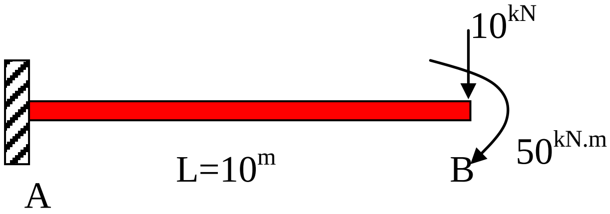


Conjugate beam and supports

- Draw the conjugate beam, including supports, for the following beams



Example:- determine the slope and deflection at point B using conjugate-beam method, EI is constant

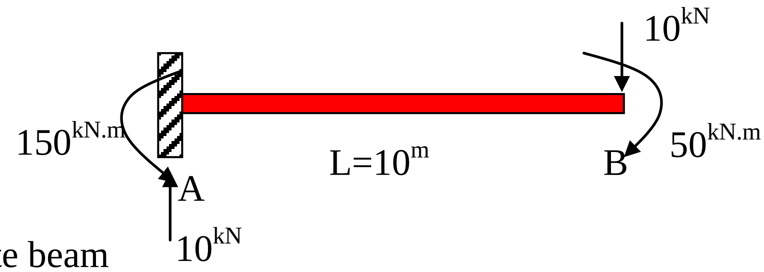


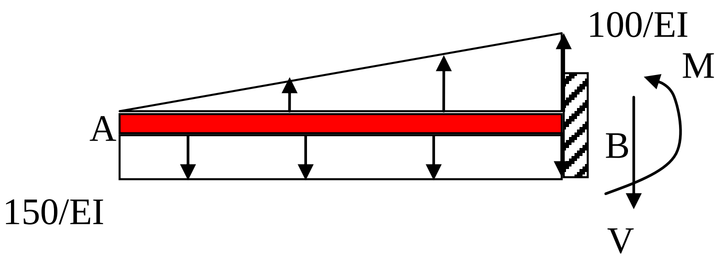
Solution

1- Find the reactions at A

RA=10 ↑ , MA=150 C.C.W

2-draw M/EI for beam,and draw conjugate beam





$$\sum fy = 0$$

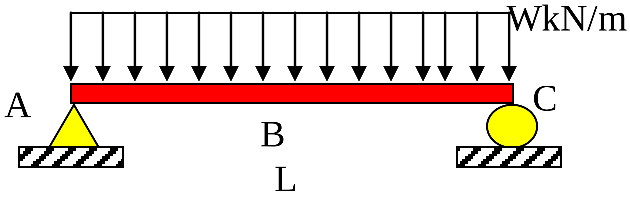
$$V = \theta_B = \frac{100}{EI} x \frac{10}{2} - \frac{150}{EI} x10 = \frac{-1000}{EI}$$

$$\sum M_B = 0$$

$$M = \Delta_B = \frac{100}{EI} x \frac{10}{2} x \frac{10}{3} - \frac{150}{EI} x10x \frac{10}{2} = \frac{-5833.33}{EI}$$

Note that sign negative indicate the slope is measured clockwise and displacement is downward

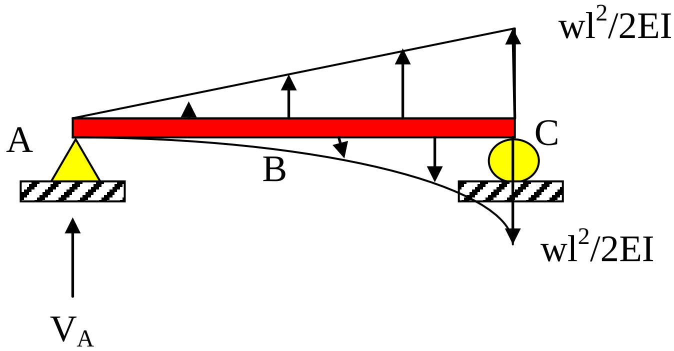
Example:- determine the slope at A and deflection at centre of beam using conjugate-beam method, EI is constant



1- Find the reactions at A,C

RA,RB=wL/2↑

2-draw M/EI for beam,and draw conjugate beam

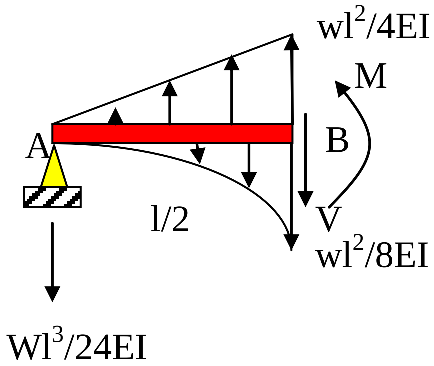


$$\sum M_C = 0$$

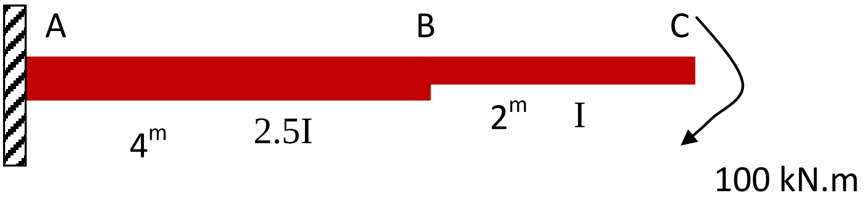
$$V_A = \theta_A = \frac{\frac{-WL^2}{2EI} x \frac{L}{2} x \frac{L}{3} + \frac{WL^2}{2EI} x \frac{L}{3} x \frac{L}{4}}{L} = -\frac{WL^3}{24EI}$$

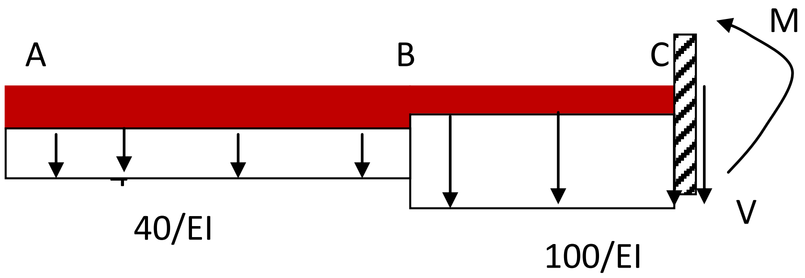
$$\sum M_B = 0$$

$$M = \Delta_B = -\frac{WL^3}{24EI}\left(\frac{L}{2}\right) + \frac{WL^2}{4EI} x \frac{L}{2} x 0.5 x \left(\frac{L}{2} x \frac{1}{3}\right) - \frac{WL^2}{8EI} x \frac{L}{2} x \frac{1}{3} x \left(\frac{L}{2} x \frac{1}{4}\right) = -\frac{5}{384}\frac{WL^4}{EI}$$



Example:- use the conjugate-beam method and determine the deflection and slope of the end C of the cantilever beam. EI=40000kN.m^2





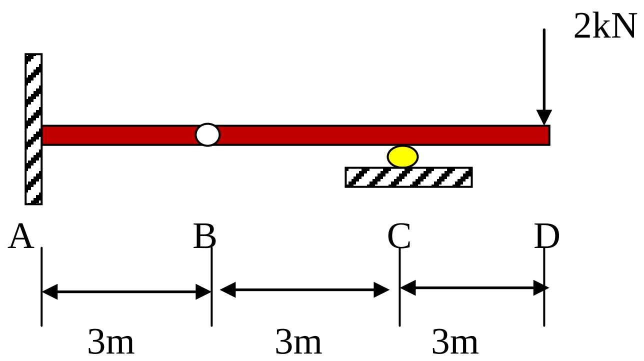
$$\sum M_C = 0$$

$$M = \Delta_C = -\frac{40}{EI}[2+2][4] - \frac{100}{EI}[2][1] = \frac{-840}{EI} = -0.021 = 21mm\ downward$$

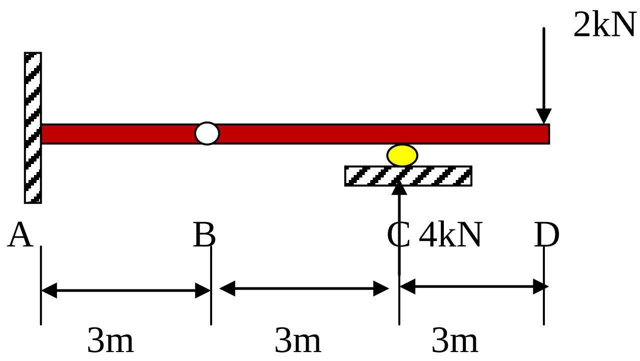
$$\sum fy = 0$$

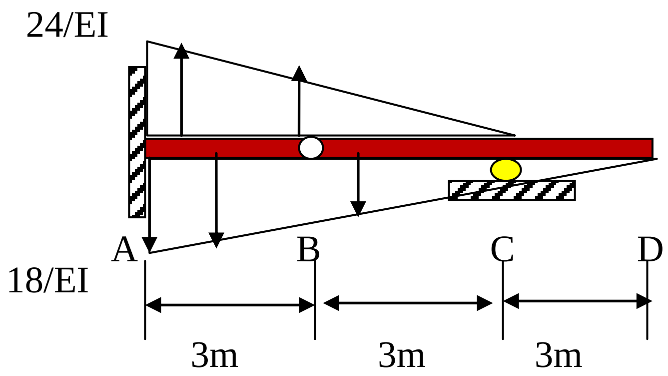
$$V = \theta_C = -\frac{40}{EI}[4] - \frac{100}{EI}[2] = \frac{-360}{EI} = -0.009rad = 0.009\ rad\ C.W$$

Example:- use the conjugate-beam method and determine the deflection at B and slope at C. EI=constant

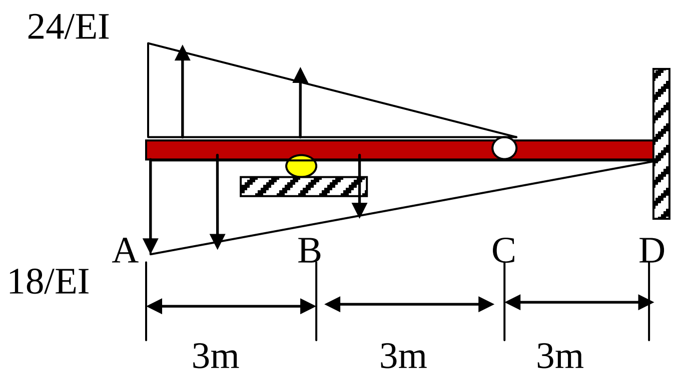


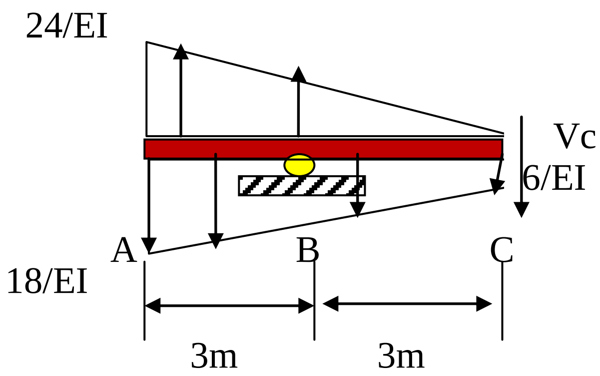
1- Find reaction and draw M/EI diagram





2- Draw conjugate beam

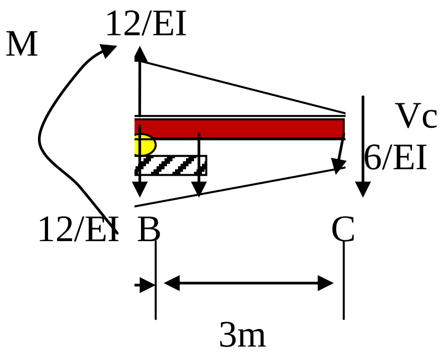




$$\sum M_B = 0$$

$$V_C[3] = -\frac{24}{EI}\left[\frac{6}{2}\right]\left[3-\frac{6}{3}\right] + \frac{12}{EI}\left[\frac{6}{2}\right]\left[3-\frac{6}{2}\right]$$

$$\theta_C = V_C = -\frac{12}{EI}$$



$$\sum M_B = 0$$

$$\Delta_B = \frac{12}{EI}[3] + \frac{12}{EI}\left[\frac{3}{2}\right]\left[\frac{3}{3}\right] - \frac{6}{EI}[3]^2\left[\frac{1}{2}\right] - \frac{6}{EI}\left[\frac{3}{2}\right]\left[\frac{3}{3}\right] = \frac{18}{EI}$$

Deflections using Castigliano's theory

It is method for determining the deflection and slope at a point in structure be is a truss, beam, or frame. It was developed by Italian railroad engineer Alberto Castigliano in 1879; sometime, it is known as least work method.

This method is applied only to structures that have:

1- Constant temperature
2- Unyielding supports
3- Materials are within elastic limits

In this theorem, the displacement or slope at a point in a structure is equal to first partial derivative of elastic energy in the structure with respect to the force or moment acting at this point in the same direction of displacement or slope, respectively.

1- Trusses

$$\Delta = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{EA}$$

Δ=external joint displacement of a truss

P=external force applied to the truss joint in the direction of Δ

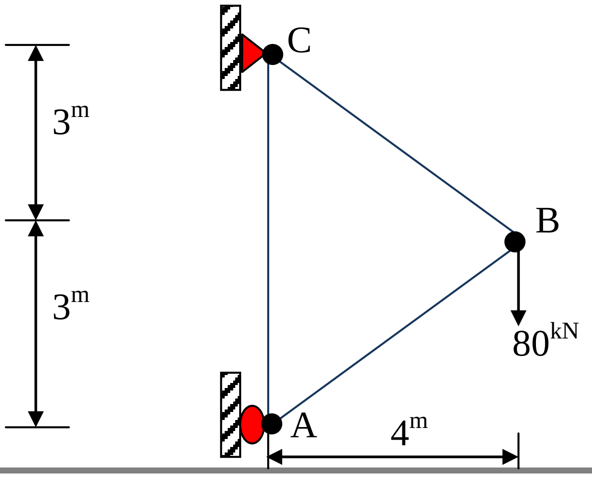
N=internal force in a member caused by both the force P and loads on the truss

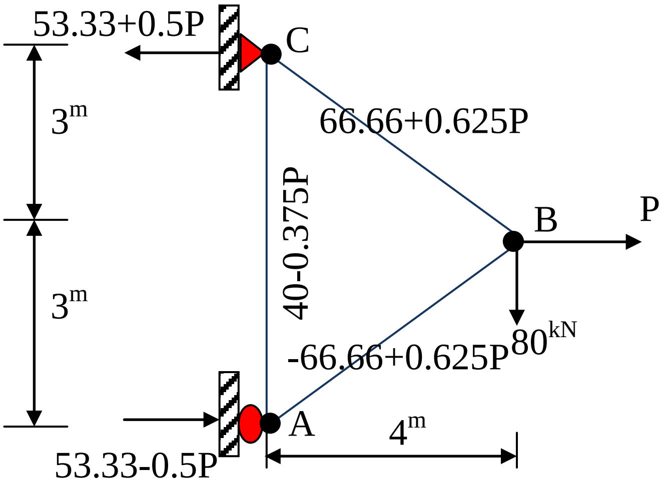
L= length of a member

A=cross-sectional area of a member

E=modulus of elasticity of a member

Example:- using Castigliano theorem, determine the horizontal deflection at joint B of the truss shown in figure. Each member has cross-sectional area of 300 mm^2 and modulus of elasticity of 200GPa.

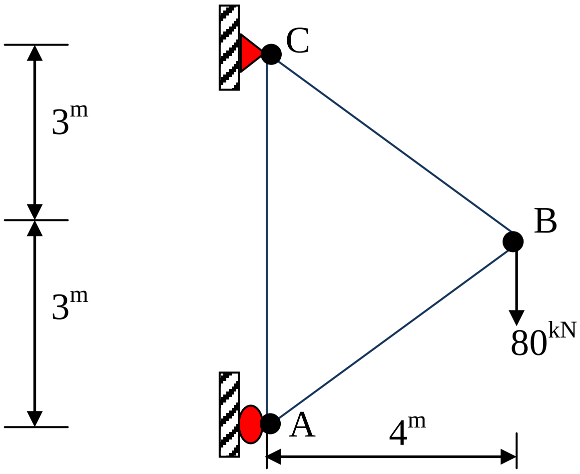


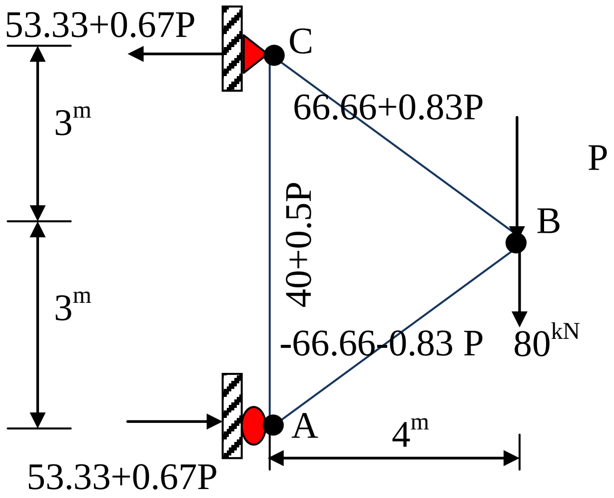


Member	N	δN/ δp	N(P=0)	L	N(δN/ δp)L
AB	-66.66+0.625P	0.625	-66.66	5	-208.31
BC	66.66+0.625P	0.625	66.66	5	208.31
AC	40-0.375P	-0.375	40	6	-90
∑					-90

$$\Delta = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{EA} = \frac{-90}{200(10^6)x300(10^{-6})} = -1.5(10^{-3})\ m = -1.5mm = 1.5mm \leftarrow$$

Example:- using Castigliano theorem, determine the vertical deflection at joint B of the truss shown in figure. Each member has cross-sectional area of 300 mm^2 and modulus of elasticity of 200GPa.





Member	N	δN/ δp	N(P=0)	L	N(δN/ δp)L
AB	-66.66-0.83P	-0.83	-66.66	5	276.64
BC	66.66+0.83P	0.83	66.66	5	276.64
AC	40+0.5P	0.5	40	6	120
∑					673.3

$$\Delta = \sum N(\frac{\partial N}{\partial P})\frac{L}{EA} = \frac{673.3}{200(10^6)x300(10^{-6})} = 0.0112\, m = 11.22 mm \downarrow$$

2- beams and frames

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$$

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

Δ=external displacement of a point caused by the real loads acting on the beam or frame

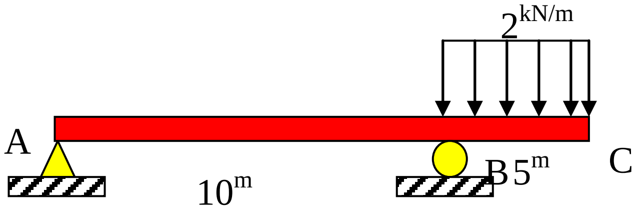
Ө= slope of a point caused by the real loads acting on the beam or frame

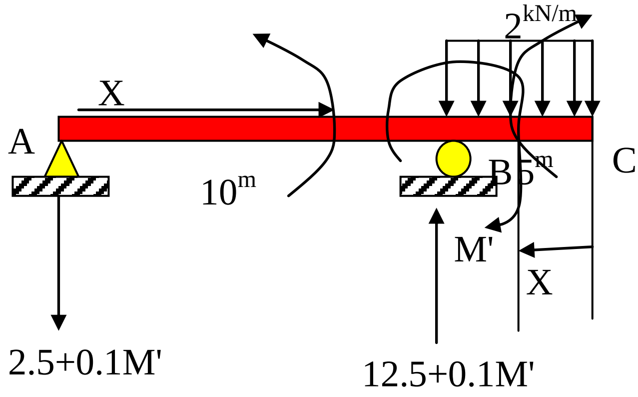
M=internal moment in the beam or frame

P=external loads is applied to the beam or frame at point that required to determine the displacement in the direction of Δ

M'= external moments is applied to the beam or frame at point that required to determine the slope in the direction of ϴ

Example:- use the method of least work and determine the slope at B of the steel beam.E=200GPa and $I=70(10)^6 mm^4$.



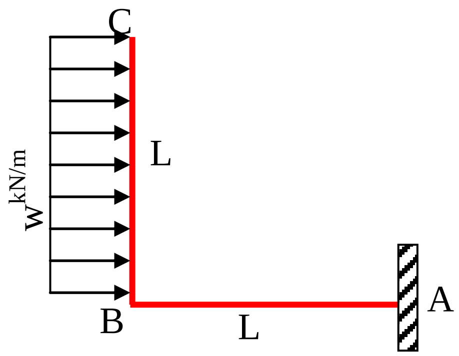


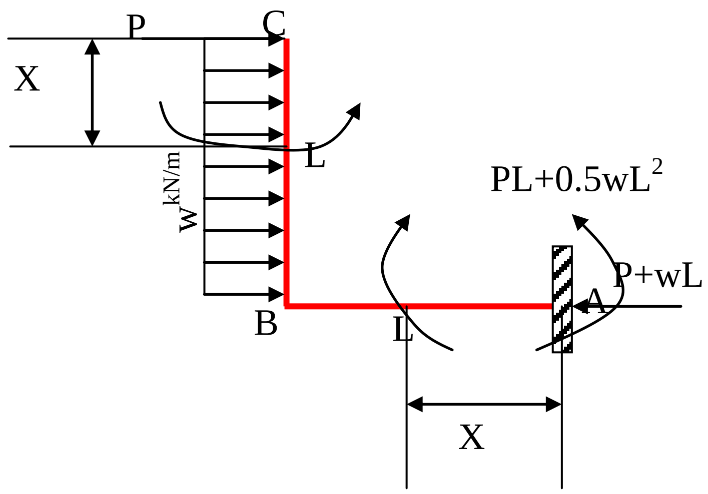
PORTION	ORIGIN	LIMIT	M	dM/dM'	M(M'=0)
AB	A	0-10	(-2.5-0.1M')X	-0.1X	-2.5X
CB	C	0-5	$-X^2$	0	$-X^2$

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI} = \int_0^{10} (-2.5X)\,(-0.1X)\frac{dx}{EI} + \int_0^{10} (-X^2)\,(0)\frac{dx}{EI}$$

$$= \frac{0.25}{200(10)^6 x 70 x 10^6 x 10^{-12}}\left[\frac{X^3}{3}\right]_0^{10} = 0.00595\, rad\ C.W$$

Example;- the L-shaped frame is made from two segments, each length of L and flexural stiffness EI. Determine the horizontal deflection at point C by using castigliano's theorem .





PORTION	ORIGIN	LIMIT	M	dM/dP	M(P=0)
AB	A	0-L	$PL+0.5wL^2$	L	$0.5wL^2$
CB	C	0-L	$PX+0.5wX^2$	X	$0.5wX^2$

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} = \int_0^L (L)\,(0.5wL^2)\frac{dx}{EI} + \int_0^L (X)(0.5wX^2)\frac{dx}{EI} = \frac{wL^3}{2EI}[X]_0^L + \frac{w}{2EI}\left[\frac{X^4}{4}\right]_0^L$$

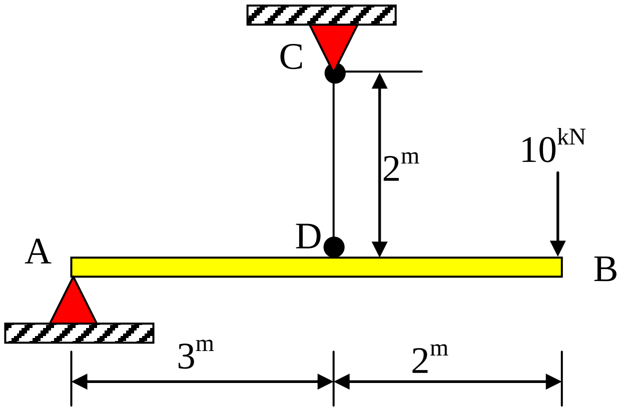
$$= \frac{5wL^4}{8EI} \rightarrow$$

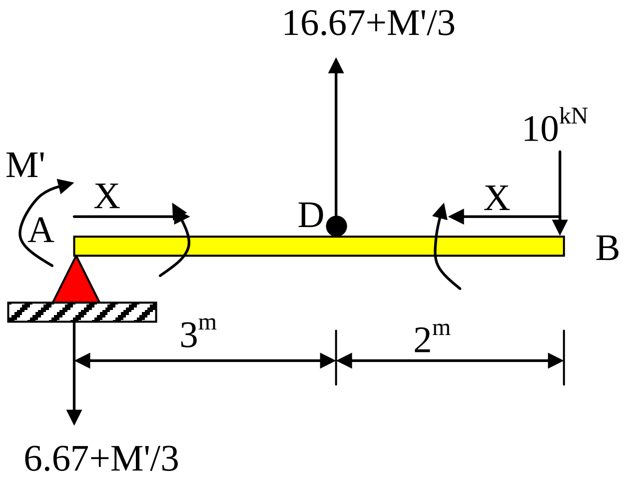
4- Composite structures

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} + \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{EA}$$

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI} + \sum N\left(\frac{\partial N}{\partial M'}\right)\frac{L}{EA}$$

Example:- beam AB has a square cross section of 100mm by 100mm. bar CD has a diameter of 10mm. if both members are made of steel, determine the slope at A due to the loading of 10kN.E=200GPa





$$A = \frac{\pi}{4}10^2 = 78.54mm^2$$

$$I = \frac{100}{12}100^3 = 8.33(10^6)mm^4$$

1-FOR TRUSS MEMBER

Member	N	δN/ δM'	N(M'=0)	L	N(δN/ δ M')L
DC	16.67+M'/3	1/3	16.67	2	11.11

2-for flexural member

PORTION	ORIGIN	LIMIT	M	dM/dM'	M(M'=0)
AD	A	0-3	M'-6.67X-M'/3 X	1-1/3 X	-6.67X
DB	B	0-2	-10X	0	-10X

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI} + \sum N(\frac{\partial N}{\partial M'})\frac{L}{EA} = \int_0^3 \frac{\left(1-\frac{1}{3}X\right)(-6.67X)}{200(8.33)} + \frac{11.11}{200(78.54)}$$
$$= -0.00529RAD = 0.00529\, rad\ C.C.W$$

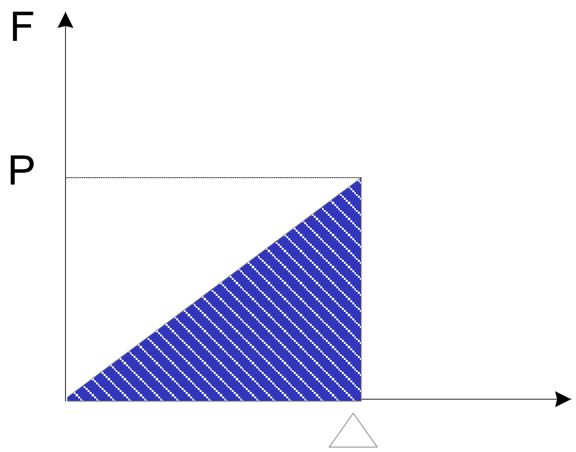
Deflections using energy method

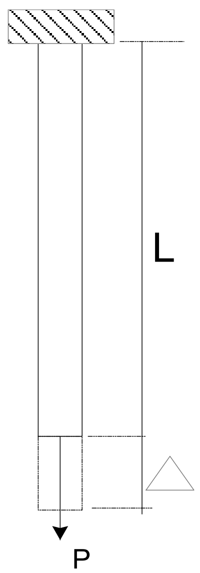
The work done by all external forces acting on a structure, Ue, is transformed to internal work or strain energy,Ui, if the material's elastic limit is not exceeded, the elastic strain energy will return the structure to its undeformed state when the loads are removed.

Ue=Ui

External work: represents the area of the loads(force,moment)-deformation(deflection, rotation) diagrams.

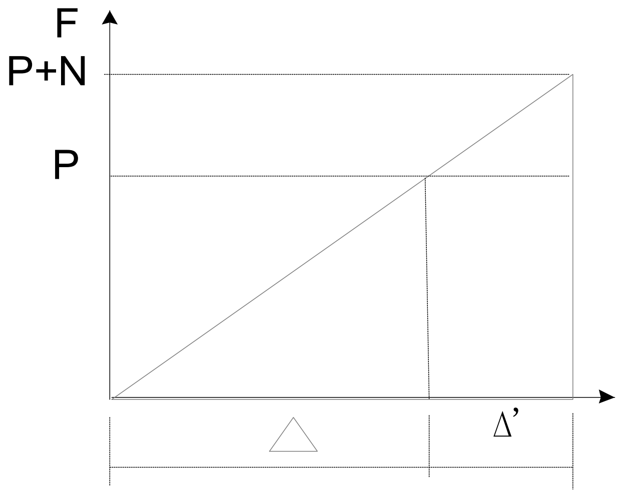
If vertical bar has constant area(A), length(L), and elastic modulus (E) subjected to external force that applied gradually from zero to P, then the diagram of load-Δ can constructed as shown.





$$Ue = \frac{1}{2} P\Delta$$

If the magnitude of applied force is increased by amount N beyond P, then this increment force will cause additional deflection Δ' as shown



The work done by P not N when the bar undergoes the further deflection Δ' is then

Ue=PΔ'

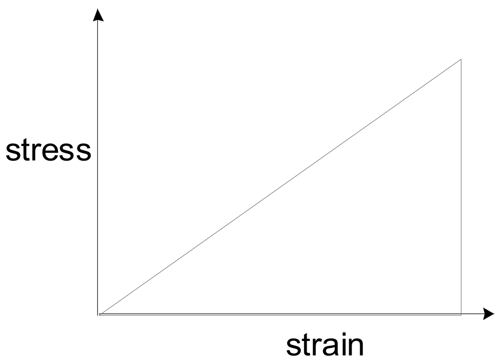
As in the case of force, if the moment is applied gradually to a structure having linear elastic response from zero to M, the external work is then

$$Ue = \frac{1}{2} M\theta$$

However, if the moment is already applied to the structure and other loadings further distort the structure by an amount M rotates θ', and the work is

Ue'=Mθ'

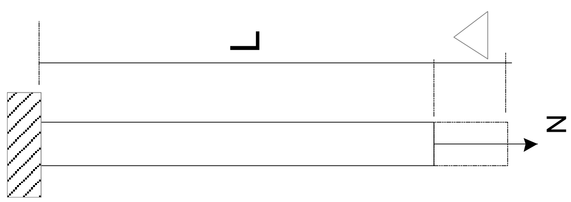
<u>Internal work(strain energy):</u> the area of stress-strain diagram multiplied by volume of structure is called strain energy , if the material within elastic limit , the relation between stress & strain is linear as shown



$$dUi = \frac{1}{2}\sigma\varepsilon * dvol \qquad \therefore Ui = \int_{vol} \frac{\sigma\varepsilon}{2} dvol$$

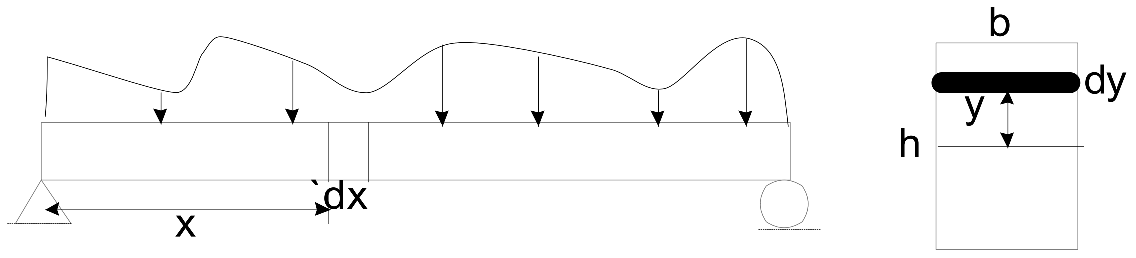
In case of axial member such as (truss, any member

Subjected to axial force only)



$$\sigma = E\varepsilon \ , \quad Ui = \frac{1}{2} E\varepsilon^2 * vol(AL) \ , \varepsilon = \frac{\Delta}{L} \ , \Delta = \frac{NL}{EA} \quad \therefore Ui = \frac{N^2 L}{2EA}$$

In case of flexural member as shown



dvol=b dy dx

$$Ui = \int_0^L \int_{-0.5h}^{0.5h} \frac{\sigma\varepsilon}{2}\, bd_y d_x = \int_0^L \int_{-0.5h}^{0.5h} \frac{\sigma^2}{2E}\, bd_y d_x \quad , \qquad \sigma = \frac{My}{I}$$

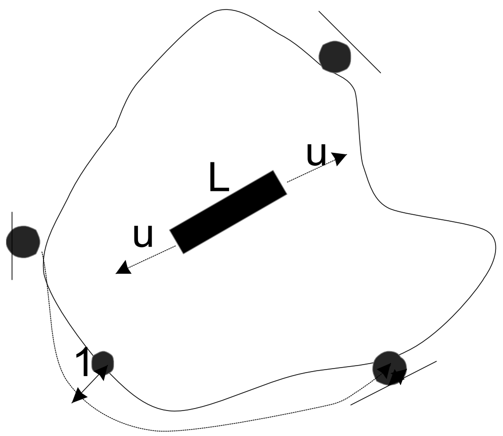
$$I = \int_{-0.5h}^{0.5h} b\, y^2\, dy$$

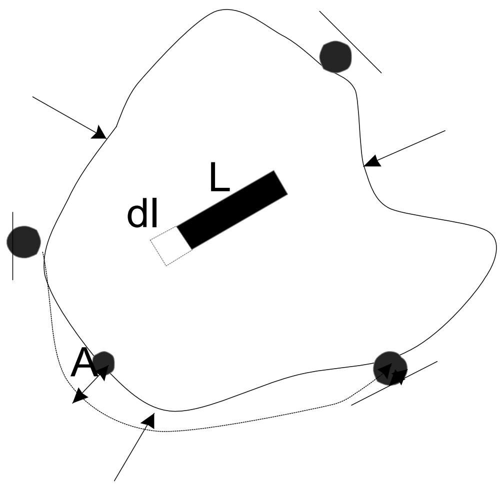
$$Ui = \int_0^L \frac{M^2}{2EI}\, dx$$

Principle of virtual work

The principle of virtual work was developed by john berouli in 1717 and sometime referred to as the unit- load method. It provides a general means of obtaining the displacement and slope at a specific point on a structure, be it a beam , frame or truss.

Assume arbitrary structure subjected to several forces as shown to find displacement at point A , apply virtual force of one unit in direction of displacement as shown





Assume virtual load is applied first then real loads is applied and cause Δ, then external work done by virtual load throughout Δ, is

$U_e = 1 * \Delta$

While internal work is

$U_i = \sum u * dL$

$U_e = U_i$

$1 * \Delta = \sum u * dL$

1=external virtual unit load acting in the direction of Δ.

u=internal virtual load acting on the element in the direction of dL.

Δ=external displacement caused by the real loads.

dL=internal deformation of the element caused by the real loads.

In a similar manner, if the rotational displacement or slope of the tangent at a point on a structure is to be determined , a virtual couple moment M' having a unit magnitude is applied at the point.

$1.\theta = \sum u_\theta .dL$

u_θ=internal virtual load acting on an element in the direction of dL

θ=external rotational displacement or slope in radians caused by the real loads

dL=internal deformation of the element caused by the real loads.

APPLICATION OF UNIT-LOAD METHOD

1. TRUSS

Deflections in trusses are caused by , external loads, temperature ,and fabrication error

$$1.\Delta = \sum \frac{nNL}{EA} + \sum n \propto \Delta TL + \sum n\, \Delta L$$

Where

1=external virtual unit load acting on the truss joint in the state direction of Δ.

n=internal virtual normal force in a truss member caused by the external virtual unit load

N=internal normal force in a truss member caused by the real loads

L=length of member

A=cross-sectional area of a member

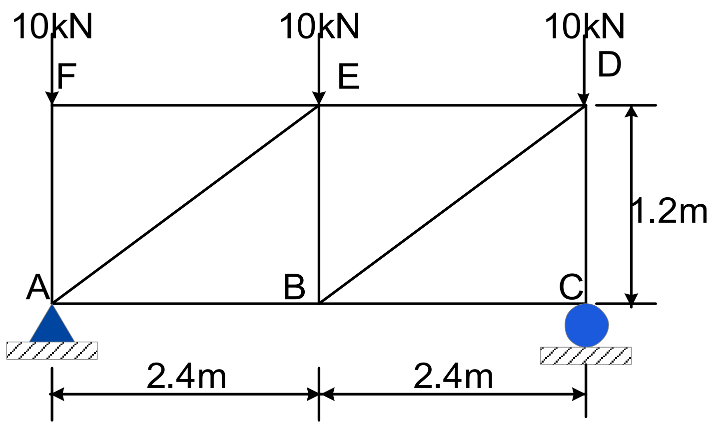
E=modulus of elasticity of a member

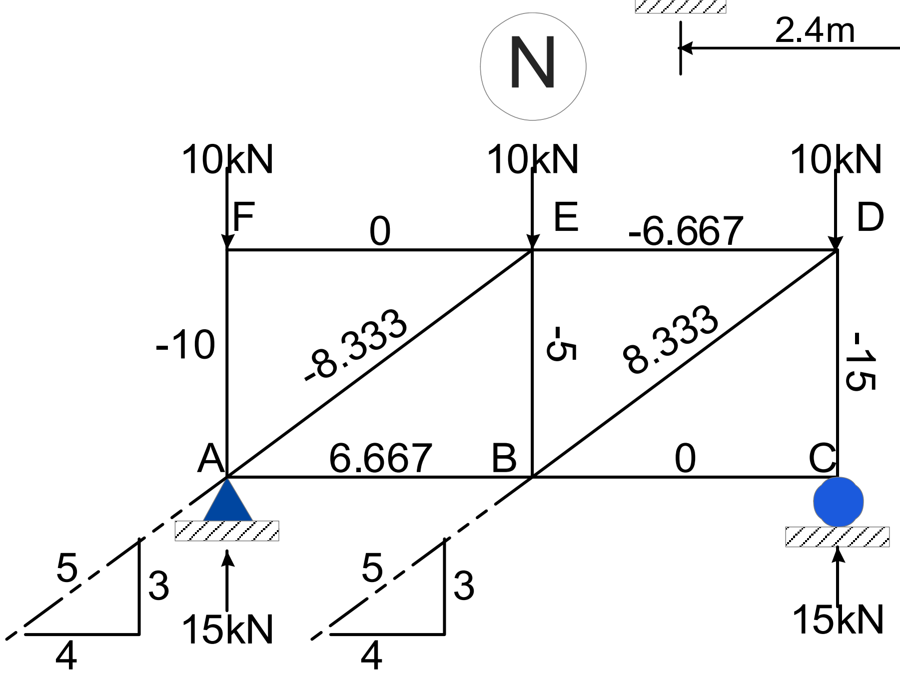
α=coefficient of thermal expansion of member

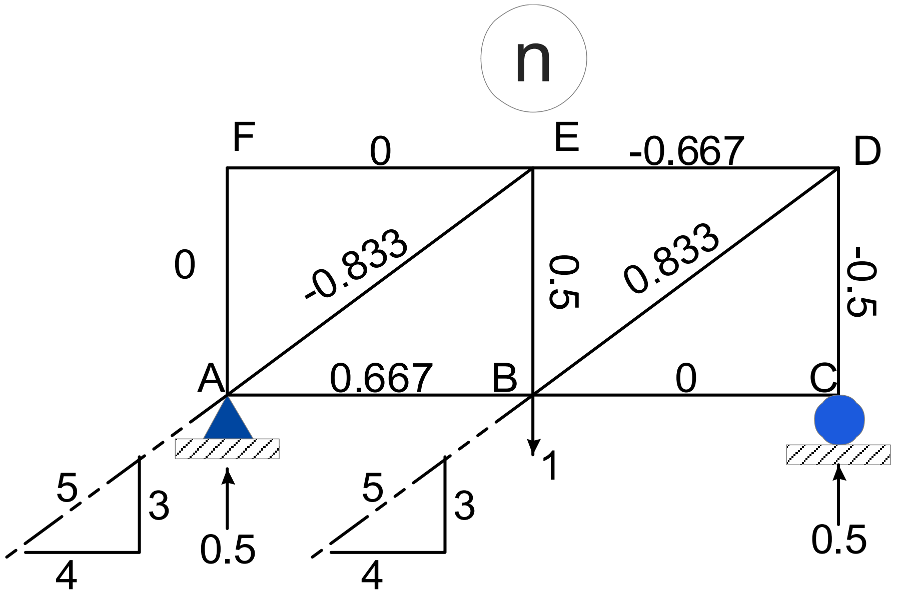
ΔT=change in temperature of member (T_2-T_1)

ΔL=difference in length of the member from its intended size as caused by a fabrication error.(+ if there is increment or - if there is decrement)

Example:- determine the vertical displacement of joint B using unit-load method . for each steel member A=900 mm^2. E=200 Gpa.





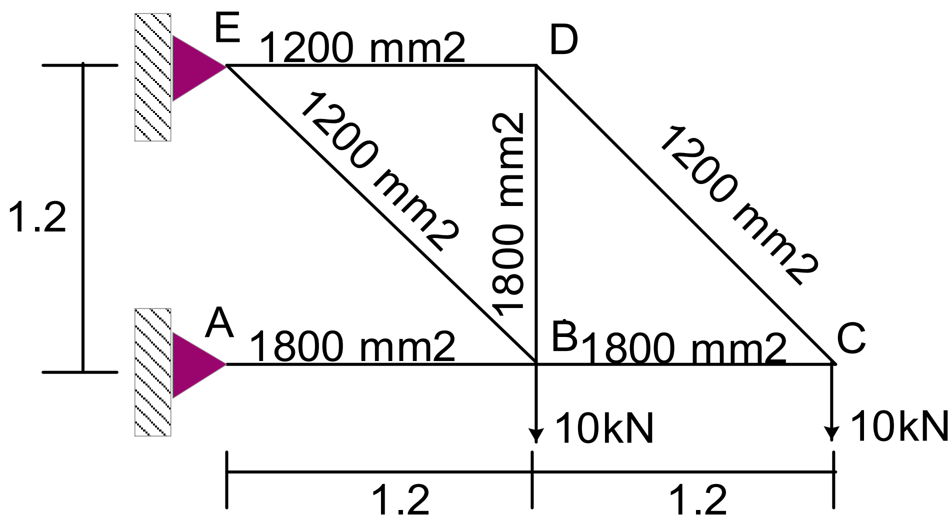


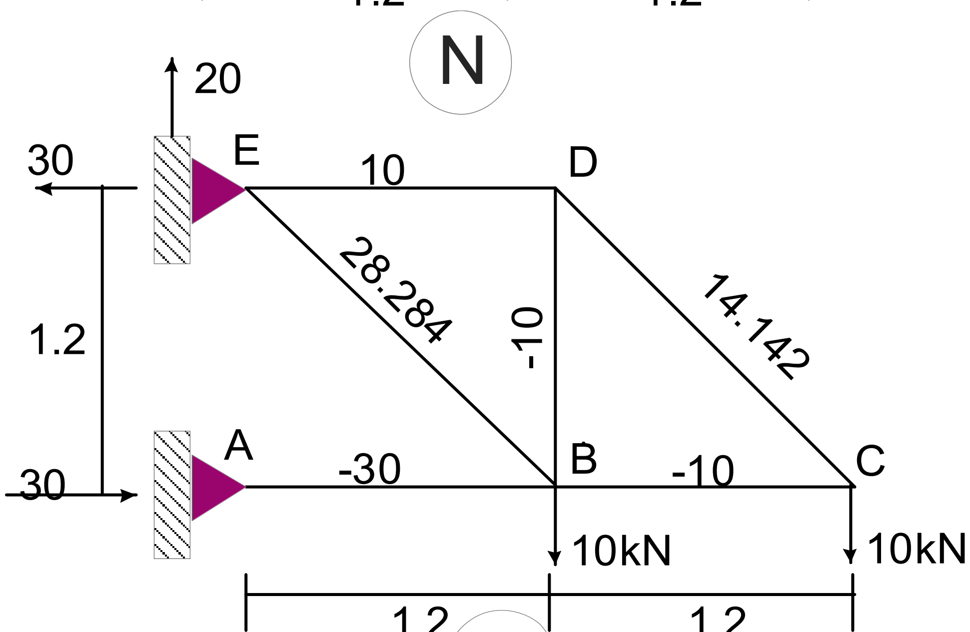
member	L	N	n	nNL
AB	2.4	6.667	0.667	10.673
BC	2.4	0	0	0
FE	2.4	0	0	0
ED	2.4	-6.667	-0.667	10.673
AF	1.8	-10	0	0
BE	1.8	-5	0.5	-4.5
CD	1.8	-15	-0.5	13.5
AE	3	-8.333	-0.833	20.824
BD	3	8.333	0.833	20.824
		$\sum$		71.994

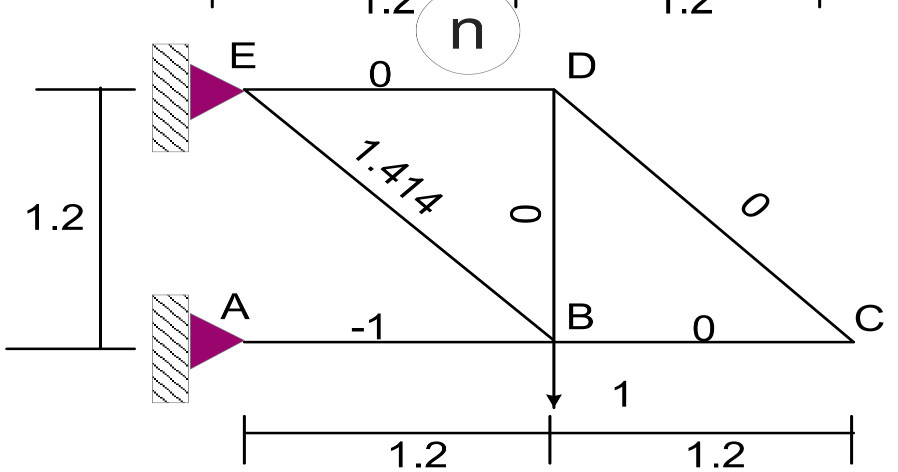
$$\Delta = \sum \frac{nNL}{EA}$$

$$\Delta = \frac{71.994\ x\ 10^3}{900x200x10^3} = 0.0004\ m = 0.4 \downarrow\ mm$$

Example:- by using unit-load method to determine the vertical deflection at point B due to applied loads, increase temperature of 110°c at members AB& BC, and fabrication error at member EB of 19mm too long. Take E=200GPa, and $\alpha=1.8*10^{-6}/^{\circ}c$







Member	L(M)	N(kN)	n	αΔT	ERROR	Amm^2	NnL/A $kN.m/mm^2$	n αΔTL (m)	N ΔL (mm)
AB	1.2	-30	-1	$1.98X10^{-6}$	0	1800	0.02	-0.000238	0
BC	1.2	-10	0	$1.98X10^{-6}$	0	1800	0	0	0
ED	1.2	10	0	0	0	1200	0	0	0
BD	1.2	-10	0	0	0	1800	0	0	0
EB	2.828	28.284	1.414	0	19mm	1200	0.0566	0	26.87
DC	2.828	14.142	0	0	0	1200	0	0	0
$\sum$							0.0766	-0.000238	26.87

$$1.\ \Delta=\sum\frac{nNL}{EA}+\sum n\propto\Delta TL+\sum n\ \Delta L=\frac{0.0766x10^6}{200x10^3}-0.000238x\,10^3+26.87$$
$$=27.015\downarrow\ mm$$

2. .BEAM

The method of unit-load can also be applied to deflection problems involving beams, to determine deflection at any point within beam; one unit is applied at this point. In addition, the tangent (rotation) at any point can be determined by applying unit-moment at this point.

The following laws are applied to determine deflection and rotation, respectively.

$$\Delta=\int_0^L\frac{mM}{EI}dx$$

$$\theta=\int_0^L\frac{m_\theta M}{EI}dx$$

Where,

m= internal virtual moment in the beam or frame, expressed as function of x and caused by the external virtual unit load.

m_θ= internal virtual moment in the beam or frame, expressed as function of x and caused by the external virtual unit load.

Δ,θ=external displacement and rotation of the point caused by the real loads acting on the beam or frame.

M=internal moment in the beam or frame , expressed as a function of x and caused by the real loads.

E=modulus of elasticity of the material

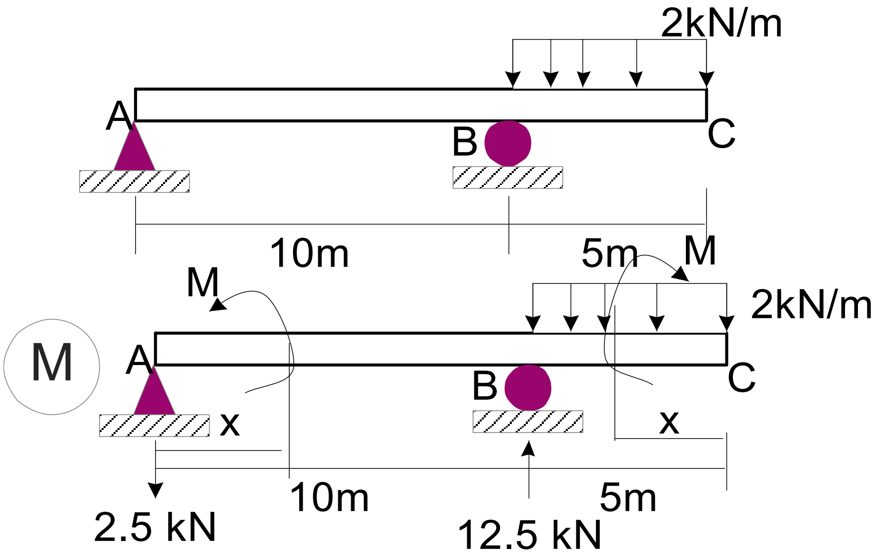
I=moment of inertia of cross-sectional area.

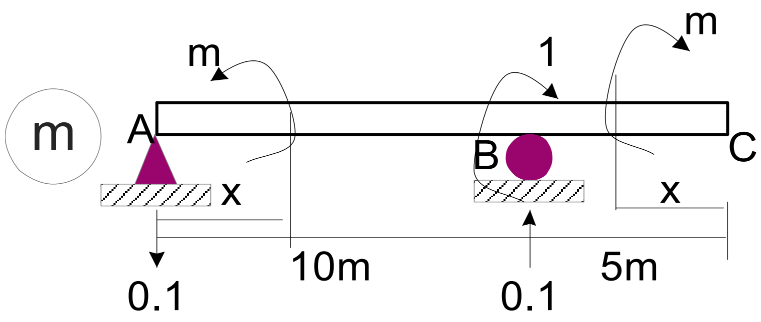
Important note:-

Frame and beam can be divided into pieces. One piece has ends may be,

1- Exterior or interior support
2- Point of applying concentrated load or moment
3- End points of distributed load or moment
4- Point of discontinuity of properties of beam or frame (such as change area, and material)
5- Point of applying virtual unit load or moment
6- Internal hinge

Example: use unit load method; determine the slope at point B of the steel beam. E=200 G Pa , $I=70X10^6 mm^4$



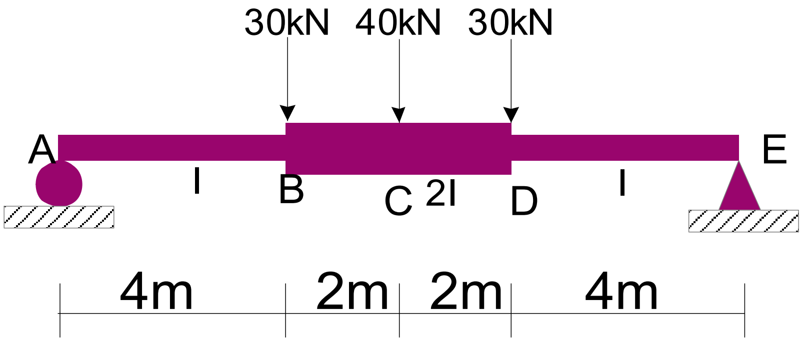


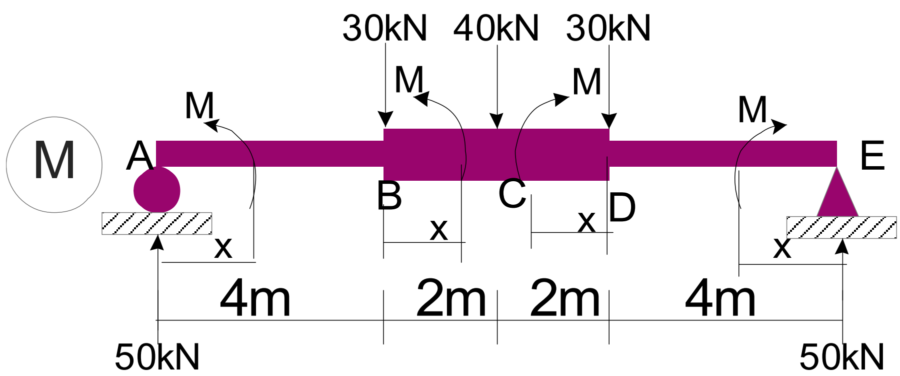
Portion	AB	BC
Origin	A	C
Limits	0---10	0---5
M	-2.5x	$-2X^2/2$
m	-0.1x	0
EI	EI	EI

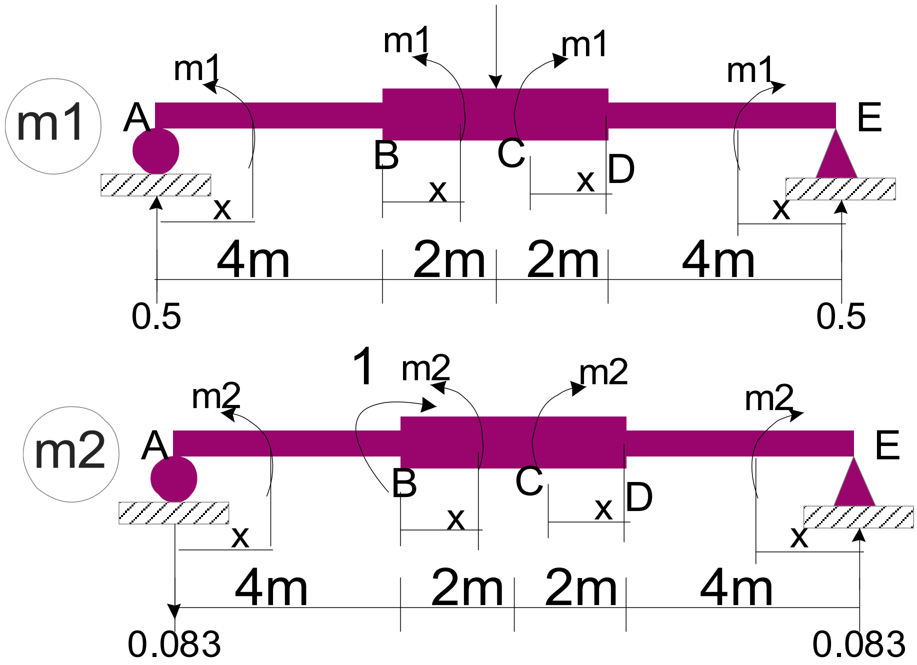
$$1(kN.m).\theta = \int \frac{Mm}{EI} dx = \frac{1}{EI}[\int_0^{10} (-2.5x\,.-0.1x)dx] = \frac{83.333\ kN^2.m^3}{EI}$$

$$\theta = \frac{83.333\ kN\ \ .m^2}{200x10^6(KPa)x70x10^6x10^{-12}m^4} = 0.00595\ radin\ (clockwise)$$

Example: use unit load method; determine the slope at point B ,and deflection at point C of the steel beam. E=200 G Pa ,
$I=270X10^6 mm^4$







Portion	AB	BC	CD	ED
Origin	A	B	D	E
Limits	0-4	0-2	0-2	0-4
M	50x	50(4+x)-30x= 20X+200	50(4+x)-30x= 20X+200	50x
m1	0.5x	0.5(4+x)= 0.5x+2	0.5(4+x)= 0.5x+2	0.5x
m2	-0.083x	-0.083(4+x)+1= 0.917x-0.332	0.083(4+x)= 0.083x+0.332	0.083x
EI	EI	2EI	2EI	EI

$$1kN.\Delta_C = \int \frac{Mm_1}{EI} dx$$

$$= \frac{1}{EI}(\int_0^4 50x * 0.5x\, dx + \frac{1}{2}\int_0^2 (20x + 200)(0.5x + 2)dx$$

$$+ \frac{1}{2}\int_0^2 (20x + 200)(0.5x + 2)dx + \int_0^4 50x * 0.5x\, dx) = \frac{2173.33kN^2.m^3}{EI}$$

$$= \frac{2173.33kN^2.m^3}{200 * 10^6 * 270 * 10^6 * 10^{-12}}$$

$$\Delta_C = 0.04024m = 40.24 \downarrow mm$$

$$1kN.m * \theta_B = \int \frac{Mm_2}{EI} dx$$

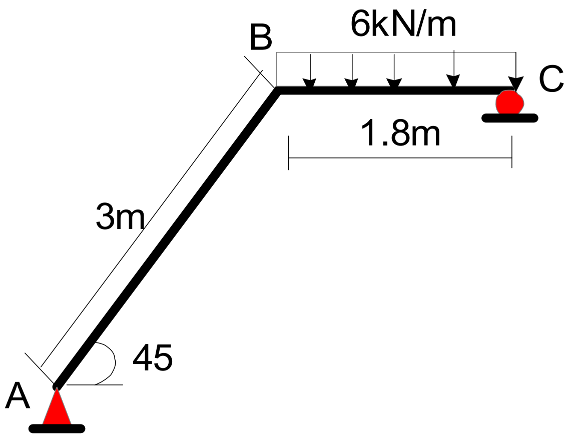
$$= \frac{1}{EI}\left(\int_0^4 50x * -0.083x\, dx + \frac{1}{2}\int_0^2 (20x + 200)(0.917x - 0.332)dx\right.$$

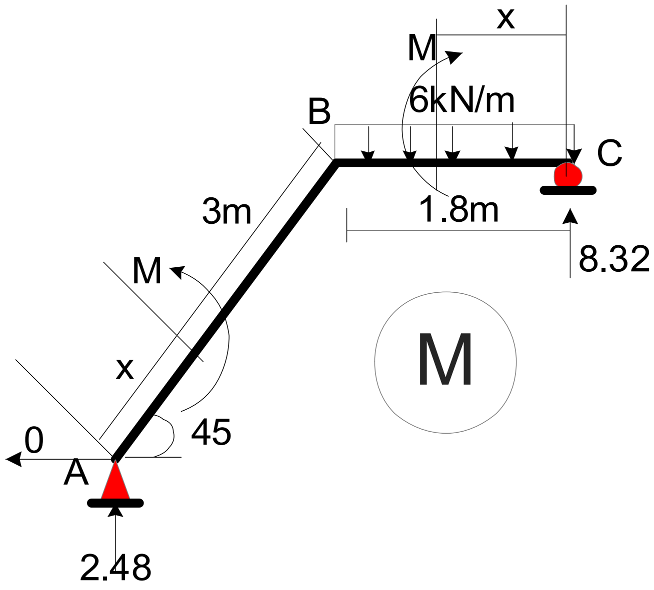
$$+ \frac{1}{2}\int_0^2 (20x + 200)(0.083x + 0.332)dx$$

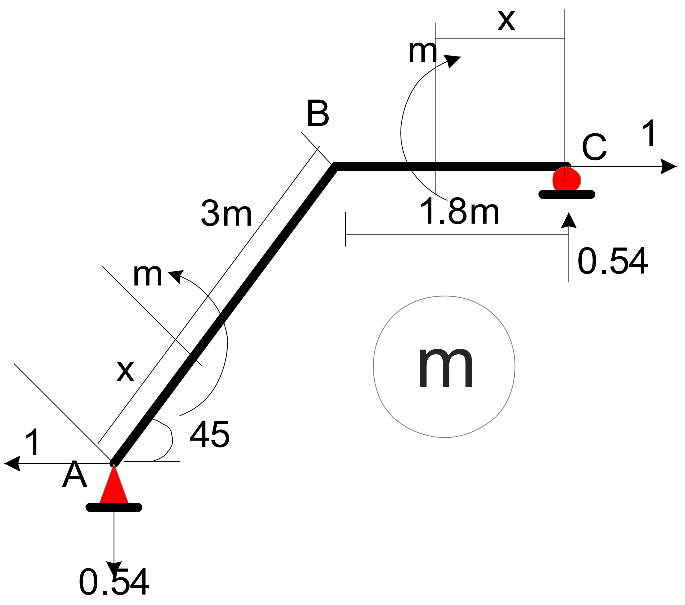
$$\left.+ \int_0^4 50x * 0.083x\, dx\right) = \frac{226kN^2.m^3}{EI} = \frac{226kN^2.m^3}{200 * 10^6 * 270 * 10^6 * 10^{-12}}$$

$$\theta_B = 0.0042\, radin\ (clockwise)$$

Example:- using unit load method determine horizontal deflection at C. EI is constant for all members.







Solution

$$R_c = \frac{6x1.8(\frac{1.8}{2} + 3COS45)}{1.8 + 3COS45} = 8.32 \uparrow$$

$$R_A = 6X1.8 - 8.32 = 2.48 \uparrow$$

REACTION IN CASE m

$$R_c = \frac{1x(3sin45)}{1.8 + 3COS45} = 0.54 \uparrow$$

$$R_A = 0.54 \downarrow$$

Portion	origin	limits	M	m
AB	A	0-3	2.48*xcos45=1.75x	Xsin45-0.54*xcos45=0.33x
CB	C	0-1.8	8.32x-6x^2/2=8.32x-3x^2	0.54x

$$\Delta * 1 = \int_0^3 \frac{1.75x * 0.33x * dx}{EI} + \int_0^{1.8} \frac{(8.32x - 3x^2) * 0.54x * dx}{EI} = \frac{9.68}{EI} \rightarrow$$

$$\Delta = \frac{9.68kN.m^3}{EI} \rightarrow$$

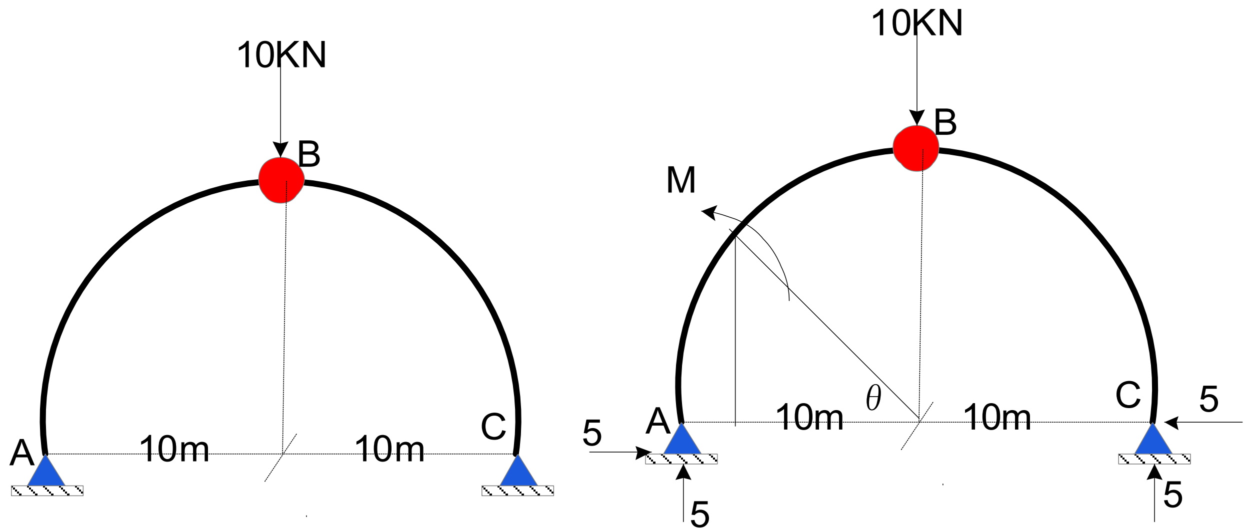
H.w

1. Determine slopes at points A,B,andC
2. If support at c is pin instead of roller and there is internal hinge at point B, determine the slope at point B
3. Determine the vertical displacement at half distance between B&c in both cases.

3-ARCH

Example:- determine the vertical deflection at B, $EI=6*10^4$ kN.m^2

In the arch problem replace dx by Rdθ



M=5(10-10cosθ)-5*10*sinθ=50(1-cosθ-sinθ)

m=M/10

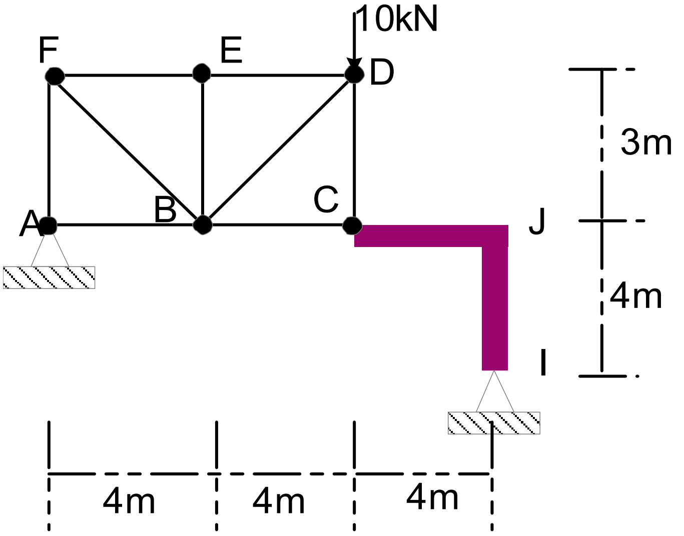
$$\Delta = \int \frac{MmRd\theta}{EI} = 2\int_0^{\frac{\pi}{2}} \frac{2500(1-cos\theta-sin\theta)^2 x10d\theta}{10EI} = \frac{707.96}{6x10^4} = 0.0118m \downarrow$$

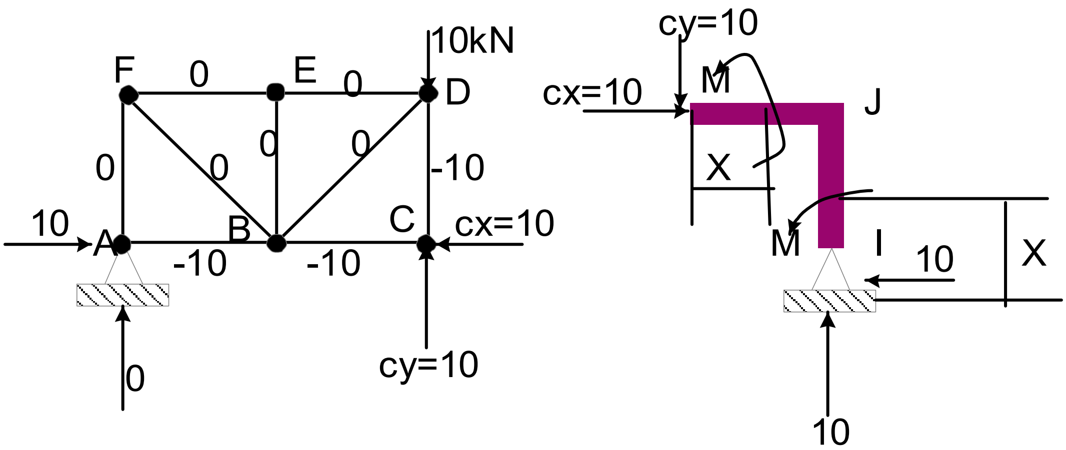
$$= 11.8mm \downarrow$$

4-COMPOSITE STRUCTURE

$$\Delta or\ \theta = \int \frac{Mmdx}{EI} + \sum \frac{nNL}{EA} + \sum n \propto \Delta TL + \sum n\, \Delta L$$

Example:-using unit load method determine vertical deflection at D , EA for axial members is 10^5 kN,and EI for flexural members is 10^4kN.m^2.





From F.B.D of truss

∑MA=0

Cy=10 upward

From F.B.D of frame

∑MI=0

Cx=10 to left

Then all truss member forces can be determined

Here

m=M/10

n=N/10

$$\Delta = \int \frac{Mmdx}{EI} + \sum \frac{nNL}{EA}$$

$$\Delta = \int \frac{M^2dx}{10EI} + \sum \frac{N^2L}{10EA} = \int_0^4 \frac{100x^2dx}{10EI} + \int_0^4 \frac{100x^2dx}{10EI} + \frac{100*3}{10EA} + \frac{100*4}{10EA} + \frac{100*4}{10EA}$$

$$= \frac{426.7}{EI} + \frac{110}{EA} = 0.0054m \downarrow$$