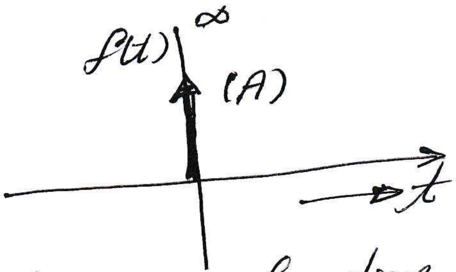
# 4- Time Domain Response :-

## 4-1 Standard Test Signals

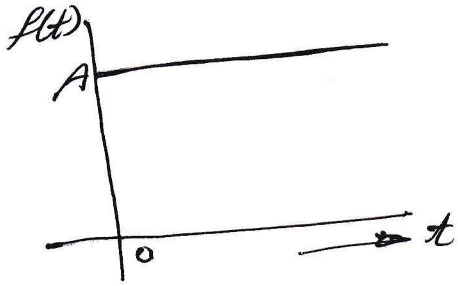
### 4-1-1 Impulse Signal

The impulse function is zero for all $t \neq 0$ and it is infinity at $t=0$. If $A=1$ it is called a unit impulse function.



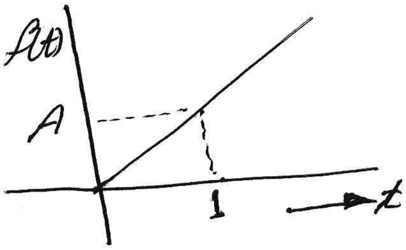

$$f(t) = \begin{cases} A\,\delta(t) & t=0 \\ 0 & t \neq 0 \end{cases}$$

### 4-1-2 Step Signal



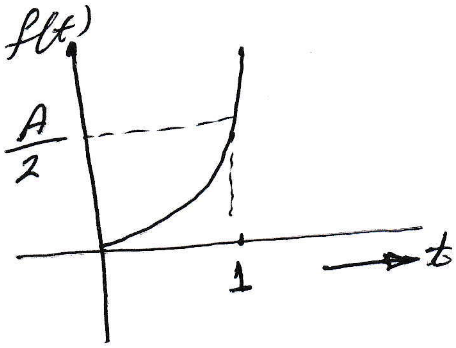

$$f(t) = \begin{cases} A\,u(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

### 4-1-3 Ramp Signal




$$f(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

### 4-1-4 Parabolic Signal




$$f(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

## 4-2 Representation of Systems

The input output description of the system is mathematically represented either as a differential equation or a transfer function.

The differential equation representation is known as a time domain representation and the transfer function is said to be a frequency domain representation

The open loop transfer function of a system is represented in the following two forms.

1 Pole Zero form

$$G(s) = K_1 \frac{(s+Z_1)(s+Z_2)---(s+Z_m)}{(s+P_1)(s+P_2)----(s+P_n)}$$

Zeros occur at $s = -Z_1, -Z_2, ----, -Z_m$

Poles occur at $s = -P_1, -P_2, ----, -P_m$

The poles and zeros may be simple or repeated. Poles and zeros may occur at the origin. In the case where some of the poles occur at the origin, the transfer function my be written as

$$G(s) = \frac{K_1 (s+Z_1)(s+Z_2)--(s+Z_m)}{s^r (s+P_{r+1})(s+P_{r+2})--- (s+P_n)}$$

$\frac{1}{s^r}$ : the poles at the origin.

$\frac{1}{s}$ : represent an integration in the system.

If $r=0$, the system has no pole at the origin and hence is known as a type-0 system.

If $r=1$, there is one pole at the origin and the system is known as a type-1 system.

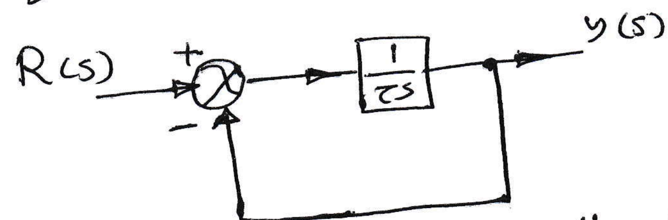
If $r=2$, the system is known as type-2 system.

Thus it is clear that the type of a system is given by number of poles at the origin.

## 4-3 First Order System :-

### 4-3-1 Response to a Unit-Step Input :-

Consider a feedback system with $G(s) = \frac{1}{Ts}$




A first order feedback System.

The closed loop transfer function of the system is given by

$$T(s) = \frac{y(s)}{R(s)} = \frac{\frac{1}{Ts}}{\frac{1}{Ts}+1}$$

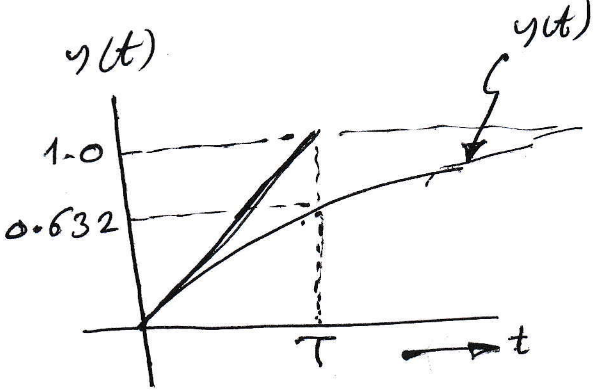
$$\frac{y(s)}{R(s)} = \frac{1}{Ts+1}$$

For a unit step input $R(s) = \frac{1}{S}$ and the output is given by

$$y(s) = \frac{1}{S(\tau s+1)}$$

By taking Laplace transform,

$$y(t) = 1 - e^{-t/\tau}$$




at $t=0 \Rightarrow y(t) = 1-1 = 0$

at $t=\tau \Rightarrow y(t) = 1-e^{-1}$
$= 0.632$

at $t=\infty \Rightarrow y(t) = 1$

which is 63.2 percent of the steady value. This time, $\tau$, is known as the "time constant of the system".

The time constant $\tau$ is indicative of this measure and the speed of response is inversely proportional to the time constant of the system.

Another important characteristic of the system is the error between the desired value and the actual value under steady state condition. this quantity is known as the steady state error of the system and is denoted by "$e_{ss}$".

The error $E(s)$ for a unity feedback system is given by

$$E(s) = R(s) - Y(s)$$
$$= R(s) - \frac{G(s)\,R(s)}{1+G(s)}$$

$$E(s) = \frac{R(s)}{1+G(s)}$$

we have, $G(s) = \frac{1}{Ts}$, $R(s) = \frac{1}{s}$

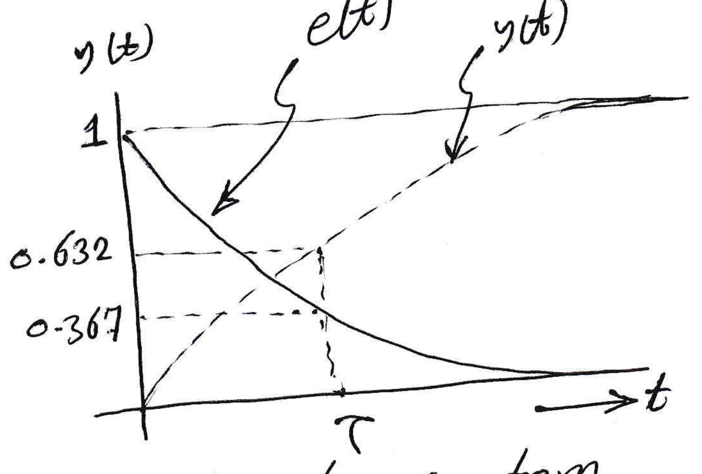
$$E(s) = \frac{\frac{1}{s}}{1+\frac{1}{Ts}} = \frac{T}{Ts+1}$$

$$e(t) = e^{-t/T}$$

at $t=0 \Rightarrow e(t) = 1$

at $t=T \Rightarrow e(t) = e^{-1} = 0.367$

at $t=\infty \Rightarrow e(t) = 0$




Thus the output of the first order system approaches the reference input, which is the desired output, without any error. In other words, we say a first order system tracks the step input without any steadystate error.

## 4-3-2 Response to a Unit Ramp Input or Unit Velocity Input :-

we have the transfer function for the first order is

$$\frac{y(s)}{R(s)} = \frac{1}{Ts+1}$$

for a unit ramp input, for which,

$$R(s) = \frac{1}{s^2},$$

$$y(s) = \frac{1}{s^2(Ts+1)}$$

the time response is obtained by taking inverse laplace transform

$$y(t) = t - T(1-e^{-t/T})$$

$$\frac{dy(t)}{dt^2} = 1 - e^{-t/T}$$

it is ~~seen~~ seen to be identical to equation before, which is the response of the system to a step input, Thus no additional information about the speed of response is obtained by considering a ramp input.

$$\therefore E(s) = \frac{R(s)}{1+G(s)} = \frac{1}{s^2} \cdot \frac{1}{1+\frac{1}{Ts}}$$

$$= \frac{1}{s^2} \cdot \frac{Ts}{Ts+1} \Rightarrow E(s) = \frac{T}{s(Ts+1)}$$

by taking the Laplace transform

$$e(t) = T\left(1-e^{-t/T}\right)$$

$$e_{ss} = \lim_{t\to\infty} e(t) = T$$

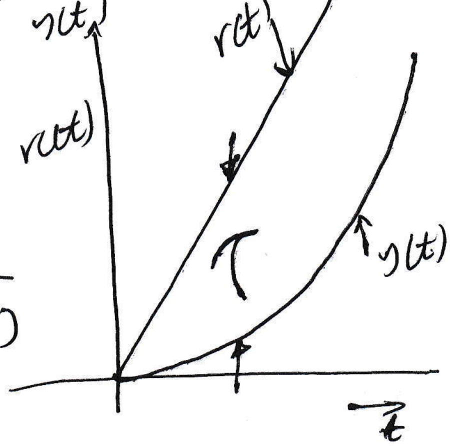
the steady state error is equal to the time
Constant of the system. the first order system, Therefore,
can not track the ramp input without a finite steady
state error. If the time constant is reduce not only the
speed of response increases but also the steady state
error for ramp input decreases.

the steady state value by using the "final value theorem". Thus

$$e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$= \lim_{s\to 0} \cancel{s}\,\frac{Ts}{\cancel{s}(Ts+1)}$$

$$= T$$




Unit ramp response of a first order system.

## 4-3-3 Response to a Unit Parabolic or Acceleration Input.

$$R(s) = \frac{1}{S^3} \quad , \quad Y(s) = \frac{1}{S^3(TS+1)}$$

by taking the inverse Laplace transform, we get

$$y(t) = T^2 - Tt + \frac{t^2}{2} - T^2 e^{-\frac{1}{T}t}$$

Differentiating and get

$$\frac{dy(t)}{dt} = -T + t + Te^{-\frac{t}{T}}$$

$$= t - T(1 - e^{-\frac{1}{T}t})$$

it is seen to be same as equation before in ramp input. Thus subjecting the first order System to a unit parabolic input does not give any additional information regarding transient behaviour of the system.

$$e(t) = r(t) - y(t)$$

$$= \frac{t^2}{2} - T^2 + Tt - \frac{t^2}{2} + T^2 e^{-\frac{1}{T}t}$$

$$e_{ss} = \lim_{t \to 0} e(t) = \infty$$

The steady state error can be easily obtained by using the "final value theorem" as

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{R(s)}{\tau s+1}$$

$$= \lim_{s \to 0} \frac{s}{s^3(\tau s+1)} = \infty$$

- In the first order, the speed of response is inversely proportional to the time constant $\tau$ of the system ~~at~~ step input.

- The ramp and parabolic inputs do not give any additional information regarding the speed of response.

- The steady state error $e_{ss}$.

a) For step input, $e_{ss} = 0$

b) for ramp input or velocity input, the finite error equal to $\tau$., $e_{ss} = \tau$

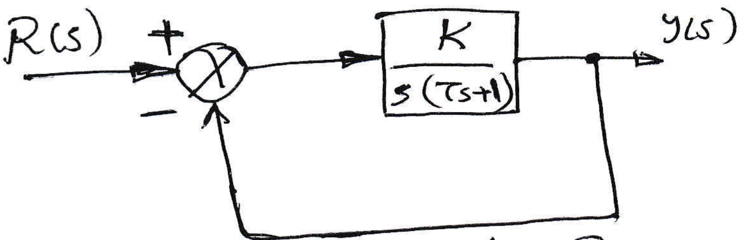
c) for parabolic input or acceleration input,

$e_{ss} = \infty$

## 4-4 Second Order System :-

### 4-4-1 Response to a Unit Step Input :-

Consider a Type 1, second order system as shown in figure below.




second Order System

The closed loop transfer function is given by

$$T(s) = \frac{y(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

The roots of the denominator polynomial in s of T(s) are the poles of the transfer function.

The "Characteristic equation" of the system is,

$$D = Ts^2 + s + K = 0$$

$$T(s) = \frac{K}{T(s^2 + \frac{1}{T}s + \frac{K}{T})} = \frac{K/T}{s^2 + \frac{1}{T}s + \frac{K}{T}}$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where $\omega_n = \sqrt{\frac{K}{T}}$ = natural frequency

$$~~\text{}~~ \quad 2\delta\omega_n = \frac{1}{T} \Rightarrow \delta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{\frac{K}{T}} * T}$$

$$\delta = \frac{1}{2\sqrt{KT}} = \text{damping factor.}$$

The poles of $T(s)$, or, the roots of the characteristic equation,

$$S^2 + 2\delta\omega_n S + \omega_n^2 = 0$$

$$S_{1,2} = \frac{-2\delta\omega_n \pm \sqrt{4\delta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= -\delta\omega_n \pm i\,\omega_n\sqrt{1-\delta^2}$$

$$= -\delta\omega_n \pm i\,\omega_d$$

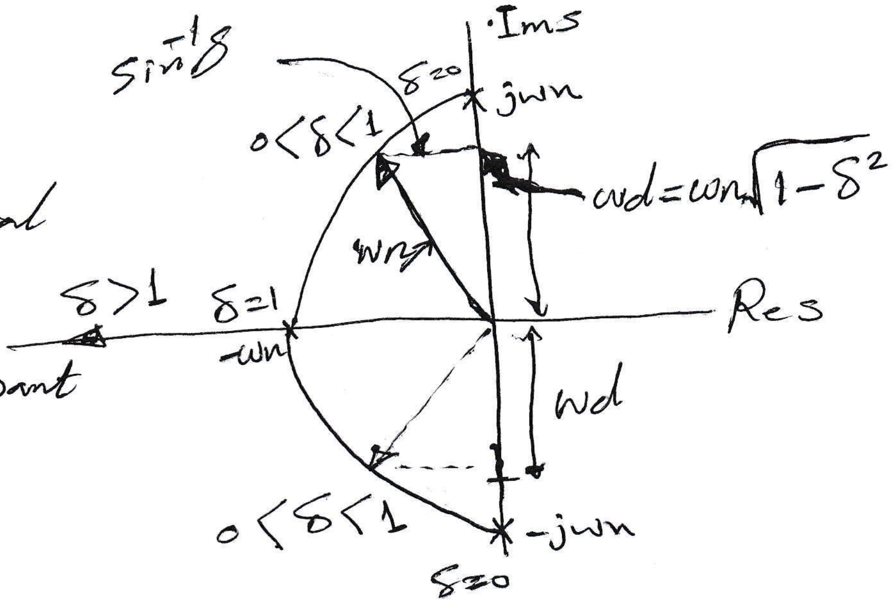
$\therefore$ $\omega_d = \omega_n\sqrt{1-\delta^2}$ «damping natural frequency»

- if $\delta > 1$, the two roots $S_1, S_2$ are real and we have an "over damped system".
- if $\delta = 1$, the system is known as a "critically damped system"
- if $\delta < 1$, is known as the "under damped system".

$S = -\sigma \pm j\omega_d$

$\omega_d$: damped natural frequency

$\sigma$ = negative real part of the pole.




$\sigma = \delta\omega_n$

$\sigma \geq \frac{4}{t_s}$

Locus of the roots of the characteristic equation

- For a unit step input $R(s) = \frac{1}{s}$

$$y(s) = T(s) * R(s) = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2} \cdot \frac{1}{s}$$

By partial fractions, assuming $\delta$ to be less than 1, we have

$$y(s) = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2\delta\omega_n s+\omega_n^2}$$

and solving for $K_1$, $K_2$ and $K_3$

$$y(s) = \frac{1}{s} - \frac{s+2\delta\omega_n}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}$$

$$= \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)} - \frac{\delta\omega_n\sqrt{1-\delta^2}}{\sqrt{1-\delta^2}\,\left((s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)\right)}$$

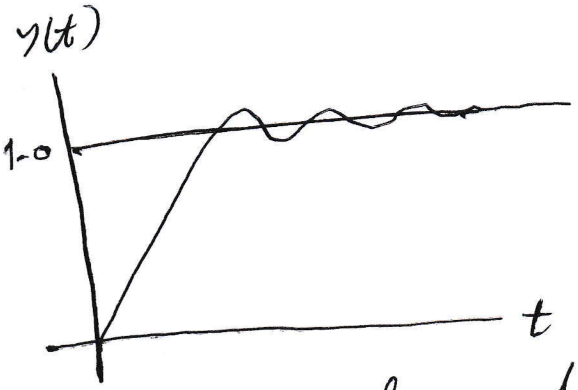
taking inverse Laplace transform and get

$$y(t) = 1 - e^{-\delta \omega_n t}\left[\cos \omega_n \sqrt{1-\delta^2}\, t + \frac{\delta}{\sqrt{1-\delta^2}} \sin \omega_n \sqrt{1-\delta^2}\, t\right]$$

$$y(t) = 1 - \frac{e^{-\delta \omega_n t}}{\sqrt{1-\delta^2}} \sin(\omega_d t + \phi)$$

$$\omega_d = \omega_n \sqrt{1-\delta^2}$$

$$\tan\phi = \frac{\sqrt{1-\delta^2}}{\delta}$$

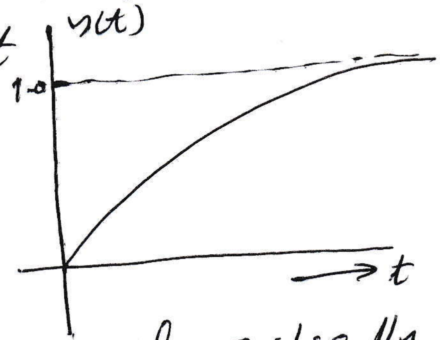



step response of an underdamped second order system.

- If $\delta = 1$, the two roots of the characteristic equation are $S_1 = S_2 = -\omega_n$ and the response is given by

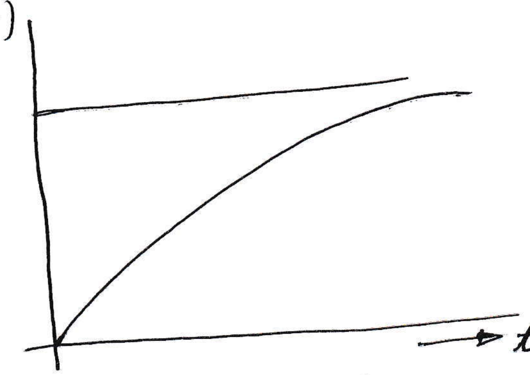
$$y(s) = \frac{\omega_n^2}{(S+\omega_n)^2} \cdot \frac{1}{S}$$

$$y(t) = 1 - e^{-\omega_n t} - t\,\omega_n e^{-\omega_n t}$$




Response of a critically damped second order system

If $\delta > 1$, the roots of the characteristic equation are real and negative, the response approaches unity in an exponential way. The response is known as overdamped response.




Step response of an overdamped second order system

$$y(t) = K_1 e^{-s_1 t} + K_2 e^{-s_2 t}$$

$$s_{1,2} = -\delta w_n \pm w_n \sqrt{\delta^2 - 1}$$

$K_1$ and $K_2$ are constant.

## 4-4-2 Response to a Unit Ramp Input :-

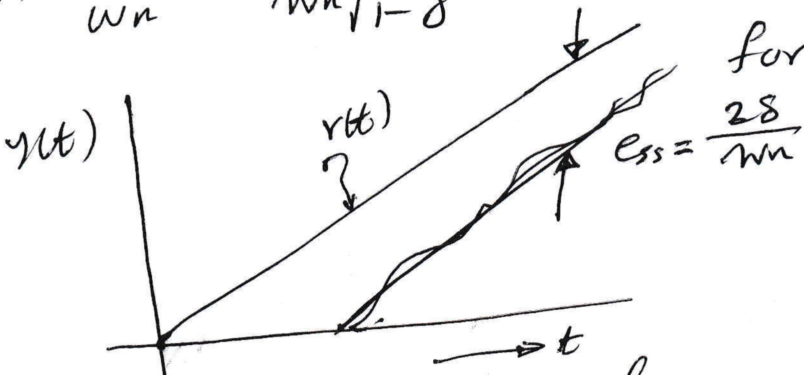
$$R(s) = \frac{1}{s^2}$$

the output $\Rightarrow$ $$y(s) = \frac{w_n^2}{s^2(s^2 + 2\delta w_n s + w_n^2)}$$

By taking the inverse Laplace transform, and get

$$y(t) = t - \frac{2\delta}{w_n} + \frac{e^{-\delta w_n t}}{w_n\sqrt{1-\delta^2}} \sin(w_n\sqrt{1-\delta^2}\, t + \phi)$$

for $\delta < 1$




Unit ramp response of a second order system

$$E(s) = R(s) - Y(s)$$

$$= \frac{1}{s^2} - \frac{\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$= \frac{s^2 + 2\delta\omega_n s + \omega_n^2 - \omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{s^2(s+2\delta\omega_n)}{s^2(s^2+2\delta\omega_n s+\omega_n^2)}$$

$$e_{ss} = \frac{2\delta\omega_n}{\omega_n^2} = \frac{2\delta}{\omega_n}$$

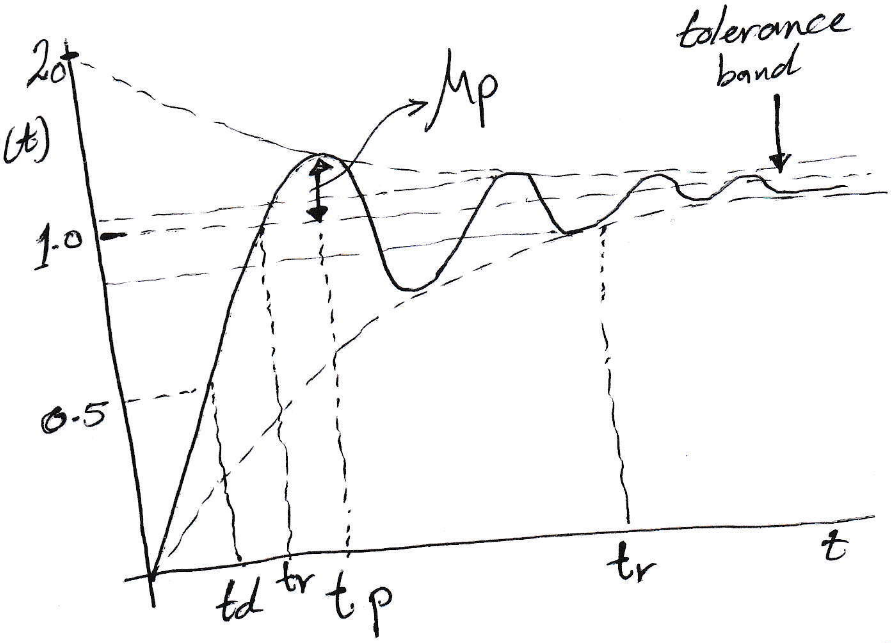
- In the unit parabolic input does not yield any new information about the transient response.

The $e_{ss}$ by the final value theorem is,

$$e_{ss} = \infty$$

## 5- Time Domain Specification of Second order System :-

For an underdamped systems, there are two complex conjugate poles, the two complex conjugate poles nearest to the jω-axis (called dominant poles) are considered and the system is app




Time domain specifications of a second order systems.

The design specifications are :-

1- Delay time $t_d$ :- It is the time required for the response to reach 50% of the steady state value for the first time.

2 - Rise time $t_r$ :- It is the time required for the response to reach 100% of the steady state value for under damped system. Howeve, for over damped systems, it is taken as the time required for the response to rise from 10% to 90% of the steady state value.

$$t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \phi}{\omega_n\sqrt{1-\delta^2}} = \frac{\pi - \tan^{-1}\sqrt{\frac{1-\delta^2}{\delta^2}}}{\omega_n\sqrt{1-\delta^2}}$$

$$t_r \simeq \frac{1.8}{\omega_n}$$

3 - Peak time $t_p$ :- It is the time required for the response to reach the maximum or Peak value of the response.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} = \frac{\pi}{\omega_d}$$

4 - Peak overshoot $M_p$ :- It is defined as the difference between the peak value of the response and the steady state value. It is usually expressed in percent of the steady state value. If the time for the peak is $t_p$, Percent peak overshoot is given by

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100.$$

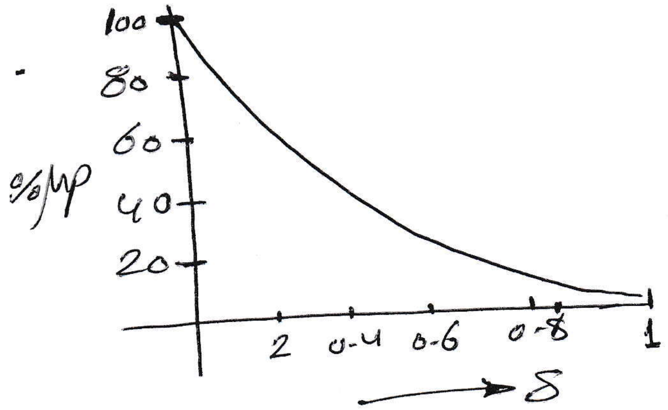
for systems of type 1 and higher, the steady state value $y(\infty)$ is equal to unity, as the same as the input

$$M_p = \frac{e^{-\frac{\delta \omega_n \pi}{\omega_n \sqrt{1-\delta^2}}}}{\sqrt{1-\delta^2}} \sin\phi$$

$$\because \sin\phi = \sqrt{1-\delta^2}$$

$$M_p = 100\, e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} \%$$

the peak overshoot is a function of the damping factor $\delta$ only.

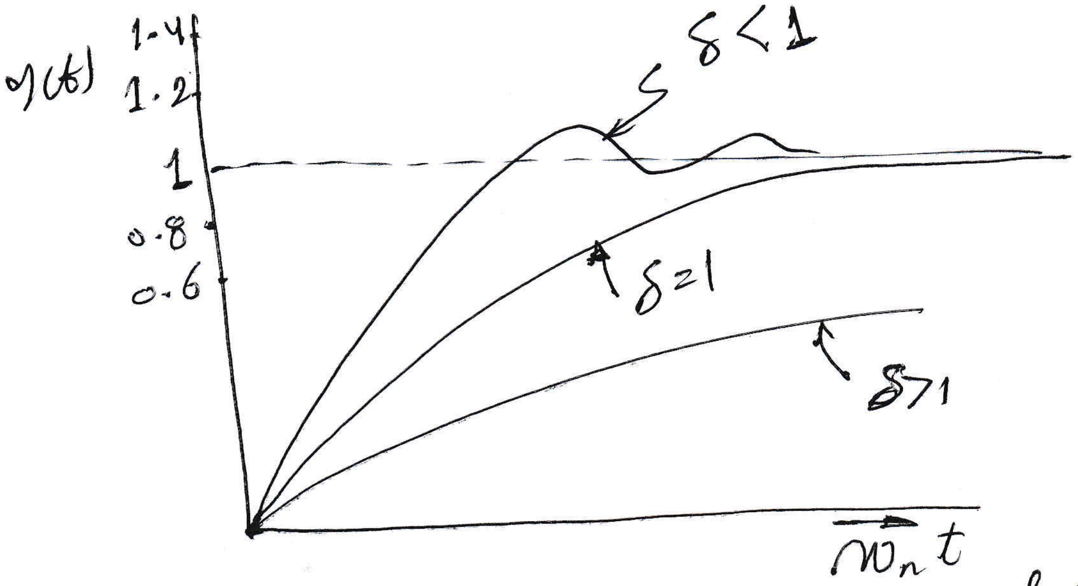



$$\delta = \frac{\ln(M_p)}{\sqrt{\pi^2 + (\ln M_p)^2}}$$

5- Settling time $t_s$ :- It is the time required for the response to reach and remain within a specified tolerance limits (usually $\pm 2\%$ or $\pm 5\%$). around the steady state value.

$$t_s \simeq \frac{4}{\omega_n \delta}$$

$$t_s \simeq \frac{4}{\sigma}$$




plotted for different value of δ.

6- Steady state error $e_{ss}$ :- It is the error between the desired output and the actual output as $t \to \infty$ or under stead state conditions.

$$e_{ss} = \lim_{t \to \infty} [r(t) - y(t)]$$