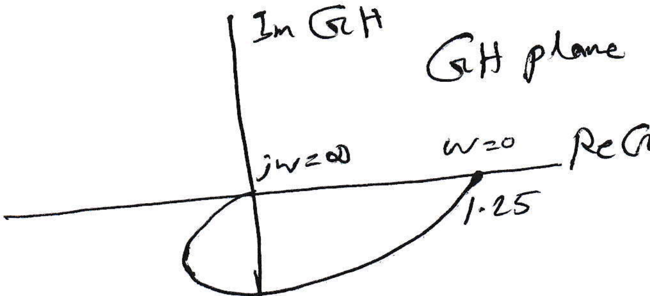


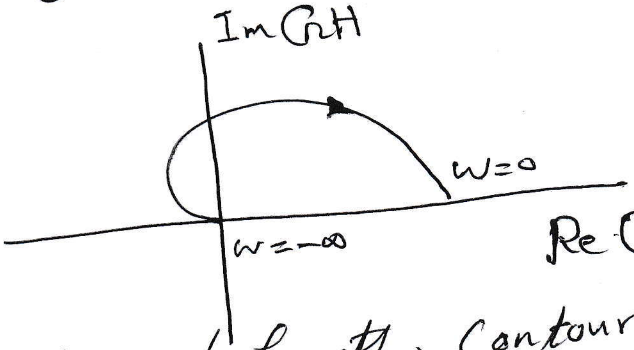
- for the infinite semi circle described by $s = j\infty$ to 0 to $s = -j\infty$ to $\omega = 0$

$$G(s)\,H(s) = \frac{10}{(j\omega+2)(j\omega+4)}$$

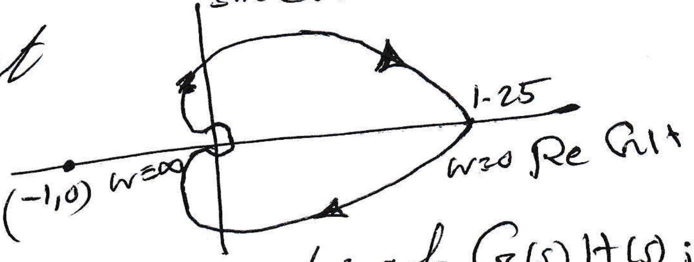
- $s = -j\infty \Rightarrow G(s)H(s) = \dfrac{10}{(-j\infty)(-j\infty)} = 0\angle 180^\circ$

- $s = j0 \Rightarrow G(s)H(s) = \dfrac{1}{(0+2)(0+4)} = 1.25\angle 0$

ω ~~$\to\infty$~~ to 0 $\Rightarrow \theta = +180$ to 0



the Complete Nyquist plot for the Contour Γ_s is



the Nyquist plot does not encircle the point $(-1, j0)$, therefore $N=0$. Since there are no poles of $G(s)\,H(s)$ in the right half of s-plane.

$P = 0$

$N = P - Z$

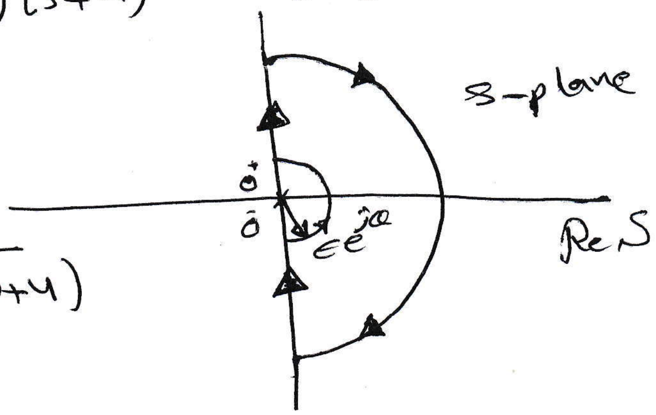
$0 = 0 - Z \Rightarrow Z = 0.$

Thus there are no zeros of $1 + G(s)H(s)$, i.e., poles of the closed loop system in the right half of s-plane.

The system is stable.

EX Determine the stability of the system.

$$G(s)H(s) = \frac{10}{s(s+1)(s+4)}$$



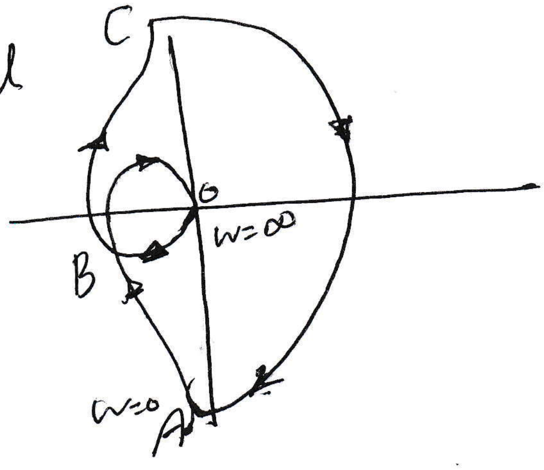
$$G(j\omega)H(j\omega) = \frac{10}{j\omega(j\omega+1)(j\omega+4)}$$

- $s = j0^+$

$$G(s)H(s) = \frac{10}{j0(j0+1)(j0+4)} = \infty \angle -90^\circ$$

- $s = j\infty$

$$G(s)H(s) = \frac{10}{j\infty(j\infty+1)(j\infty+4)} = 0 \angle -270^\circ$$



- For the semicircle $s = \epsilon e^{j\theta}$ around the pole at the origin,

$$G(s)H(s) = \frac{10}{\epsilon e^{j\theta}(\epsilon e^{j\theta}+1)(\epsilon e^{j\theta}+4)}$$

To get the real and imaginary parts.

$$G(j\omega)H(j\omega) = \frac{10(1-j\omega)(4-j\omega)}{j\omega(1+\omega^2)(16+\omega^2)}$$

$$= \frac{10(4-j5\omega-\omega^2)}{j\omega(1+\omega^2)(16+\omega^2)}$$

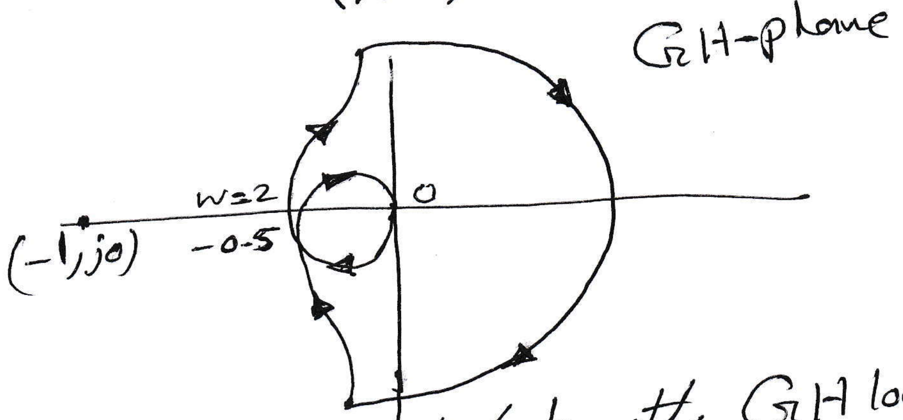
$$= \frac{-50}{(1+\omega^2)(16+\omega^2)} - j\frac{10(4-\omega^2)}{\omega(1+\omega^2)(16+\omega^2)}$$

$$\mathrm{Im}\; G(j\omega)H(j\omega) = 0$$

$$\frac{10(4-\omega^2)}{\omega(1+\omega^2)(16+\omega^2)} = 0$$

$$\omega^2 = 4 \Rightarrow \omega = \pm 2 \text{ rad/sec}$$

$$\omega = 2 \Rightarrow G(j2)H(j2) = \frac{-50}{(1+4)(16+4)} = -0.5$$



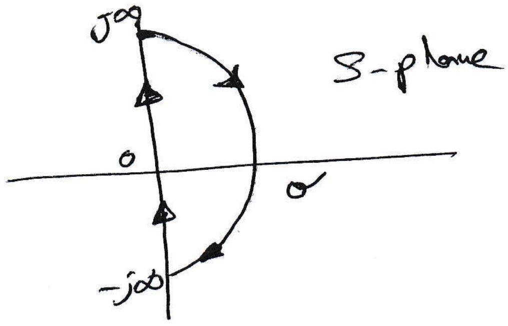
the (-1,j0) point is not encircled by the GH locus.

the system is stable.

$\underline{Ex}$ Determine the stability of

$$G(s)H(s) = \frac{K}{(s+2)(s-1)}$$

the Nyquist path is



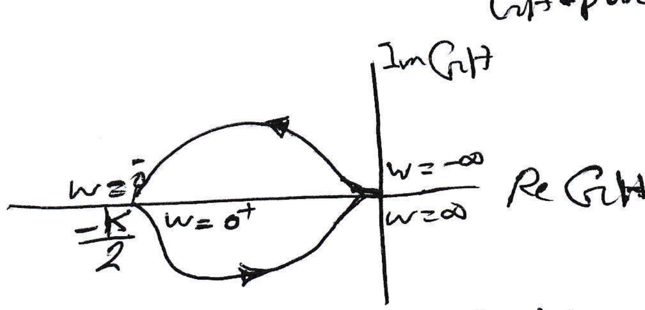
$s = j\omega$

$$G(j\omega)H(j\omega) = \frac{K}{(j\omega+2)(j\omega-1)}$$

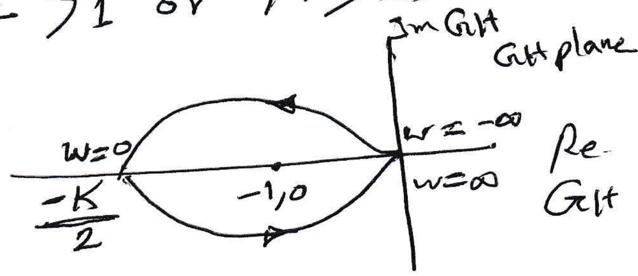
$$- \omega = 0 \Rightarrow G(j0)H(j0) = \frac{K}{(j0+2)(j0-1)} = -\frac{K}{2} \angle 180$$

$$- \omega = \infty \Rightarrow G(j\infty)H(j\infty) = \frac{K}{(j\infty+2)(j\infty-1)} = 0 \angle 180$$

for $s = j\omega$ and $-\infty < \omega < 0$ the GH plot is mirror image of the plot for $0 < \omega < \infty$.



the Nyquist plot encircles the $(-1,0)$ point once in anti clockwise direction, if $\frac{K}{2} > 1$ or $K > 2$,



- for $K > 2$

$N = 1$

there is one pole of $G(s)H(s)$ in the right half of s-plane

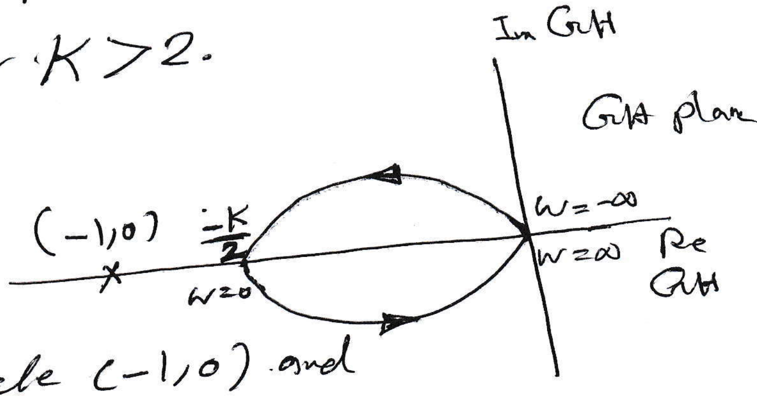
$P = 1$

$N = P - Z$

$1 = 1 - Z$

$Z = 0$

or there are no closed loop poles in the RHS and the system is stable for $K > 2$.



the Nyquist does not encircle $(-1,0)$ and

$N = 0$

$P = 1$

$N = P - Z$

$Z = P = 1$

there is one zero of the characteristic equation, one pole of closed loop system in the RHS and therefore, the closed loop system is unstable for $K < 2$.

Note: Even if the open loop system is unstable the closed loop system may be stable. If some of the open loop poles are in the RHS, for stability the Nyquist plot should encircle the origin, in counter clockwise direction, as many times as there are RHP poles of the open loop system.

Ex: Comment on the stability of the system.

$$G(s)H(s) = \frac{K(s+10)(s+2)}{(s+0.5)(s-2)}$$

$s = j\omega$ and $\omega \to 0$ to ∞

$$G(s)H(s) = \frac{K(j\omega+10)(j\omega+2)}{(j\omega+0.5)(j\omega-2)}$$

$$\omega = 0 \Rightarrow G(s)H(s) = \frac{20K}{-1} = -20K = 20K\angle 180^\circ$$

$$\omega = \infty \Rightarrow G(s)H(s) = \frac{K(j\omega)^2\left(1+\frac{10}{j\omega}\right)\left(1+\frac{2}{j\omega}\right)}{(j\omega)^2\left(1+\frac{0.5}{j\omega}\right)\left(1-\frac{2}{j\omega}\right)}$$

$$= K$$

To find the possible crossing of negative real axis,

$$\operatorname{Im} G(j\omega)H(j\omega) = 0$$

$$\text{Im} \; \frac{K(j\omega+10)(j\omega+2)(-j\omega+0.5)(-j\omega-2)}{(\omega^2+0.25)(\omega^2+4)} = 0$$

$$\text{Im}\,(-\omega^2+20+12j\omega)(-\omega^2-1+1.5j\omega) = 0$$

$$-1.5\,\omega^2+30-12-12\,\omega^2 = 0$$

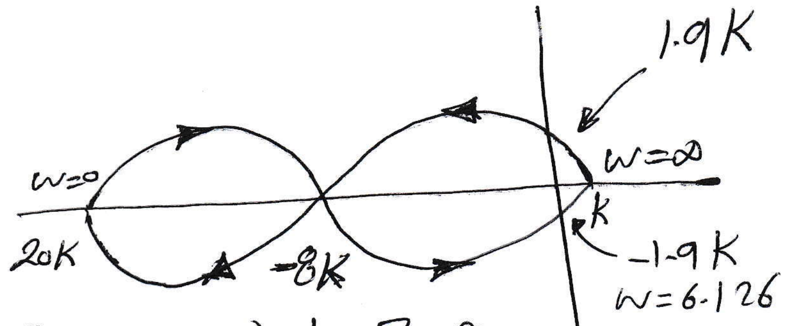
$$\omega^2 = \frac{18}{13.5} = \frac{4}{3}$$

$\omega = 1.1547$ rad/sec.

$$\text{Re}[G(j\omega)H(j\omega)]_{\omega=1.157} = K\,\frac{-(20-\omega^2)(1+\omega^2)-18\omega^2}{(\omega^2+0.25)(\omega^2+4)}\Bigg|_{\omega=1.157}$$

$$= -8K.$$

the Nyquist plot crosses the negative real axis at $-8K$ for $\omega = 1.1547$ rad/sec.



By Real part of $G(j\omega)H(j\omega)$ to zero

$$\text{Re}\,(-\omega^2+20+12j\omega)(-\omega^2-1+1.5j\omega) = 0$$

$$\omega^4+\omega^2-20\omega^2-20-30-18\omega^2 = 0$$

$$\omega^4-37\omega^2-50 = 0$$

$\omega = 6.126$ rad/sec $\Rightarrow \text{Im}[G(j\omega)H(j\omega)] = -1.9K$

It is clear that if $8K > 1$ or $K > 0.125$, $(-1, j0)$ point is encircled once in anticlockwise direction and,

$N = 1$

$P = 1$

$N = P - Z$

$Z = 0$

the system is stable for $K > 0.125$

If $K < 0.125$, the $(-1, j0)$ point is encircled once in the clockwise direction and hence $N = -1$

Since $P = 1$

$N = P - Z$

$Z = 2.$

there are two closed loop poles in the RHP and the system is unstable.

8-5 Relative Stability

The Nyquist criterion tells us whether the system is stable or not by the location of the critical point with respect to the Nyquist plot.

EX Comment the stability of the system

$$G(s)H(s) = \frac{K}{s(s+1)(s+4)}$$

for (a) $K=30$, (b) $K=20$, (c) $K=15$

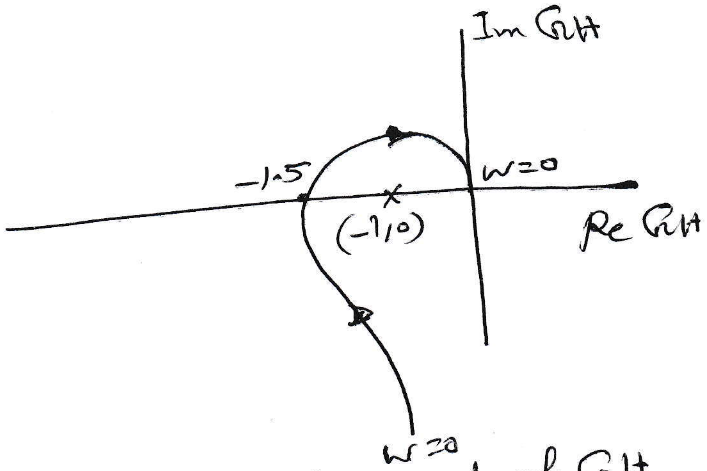
(d) $K=10$.

a) $s = j\omega$

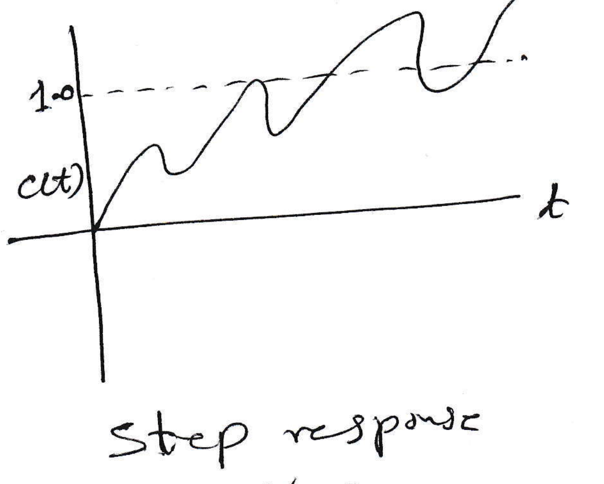
$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+1)(j\omega+4)}$$

$$\omega=0 \Rightarrow |G(j\omega)H(j\omega)| = \infty \angle -90^\circ$$

$$\omega=\infty \Rightarrow |G(j\omega)H(j\omega)| = 0 \angle -270^\circ$$



polar plot of GH for $K=30$



step response for $K=30$

$$G(s)H(s) = \frac{K}{s(s+1)(s+4)}$$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+1)(j\omega+4)}$$

$$= \frac{K}{j\omega(j\omega+1)(j\omega+4)} * \frac{-j\omega(-j\omega+1)(-j\omega+4)}{-j\omega(-j\omega+1)(-j\omega+4)}$$

$$= \frac{-jK\omega(-\omega^2 - j5\omega + 4)}{j\omega(-\omega^2 + j5\omega + 4)\,(-j\omega)\,(-\omega^2 - j5\omega + 4)}$$

$$= \frac{\cancel{jK} + jK\omega^3 - K\omega^2 \times 5 - 4jK\omega}{(-j\omega^3 - 5\omega^2 + 4j\omega)(j\omega^3 - 5\omega^2 - 4j\omega)}$$

$$= \frac{-5K\omega^2 - jK\omega(4 - \omega^2)}{\cancel{j\omega}\,(-5\omega^2 + j(4\omega - \omega^3))(-5\omega^2 - j(4\omega - \omega^3))}$$

$$= \frac{-5K\omega^2 - jK\omega(4 - \omega^2)}{(-5\omega^2 + j(4\omega - \omega^3))(-5\omega^2 - j(4\omega - \omega^3))}$$

$$= \frac{-5K\omega^2 - jK\omega(4 - \omega^2)}{25\omega^4 + (4\omega - \omega^3)^2}$$

$$G(j\omega) = \frac{-5K\omega - jK(4-\omega^2)}{25\omega^3 + \omega(4-\omega^2)^2}$$

Real axis crossing: found by setting the imaginary part of $G(j\omega)$ as zero,

$$Im\,(jK(4-\omega^2)) = 0$$

$$\omega^2 = 4 \Rightarrow \omega = 2$$

$$\therefore Real[G(j\omega)] = \frac{-5K * 2}{25 * 8}$$

$$= \frac{-K}{20}$$

$$\therefore \text{Real part} = \left\{-\frac{K}{20}, j0\right\}$$

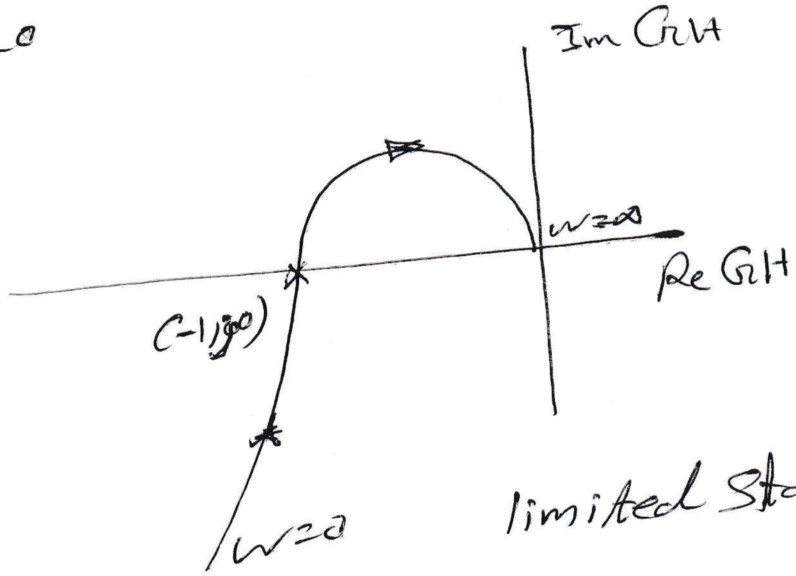
a) $K = 30 \Rightarrow$ Real part $= (-1.5, j0)$

b) $K = 20 \Rightarrow$ Real part $= (-1, j0)$

c) $K = 15 \Rightarrow$ Real part $= (-0.75, j0)$

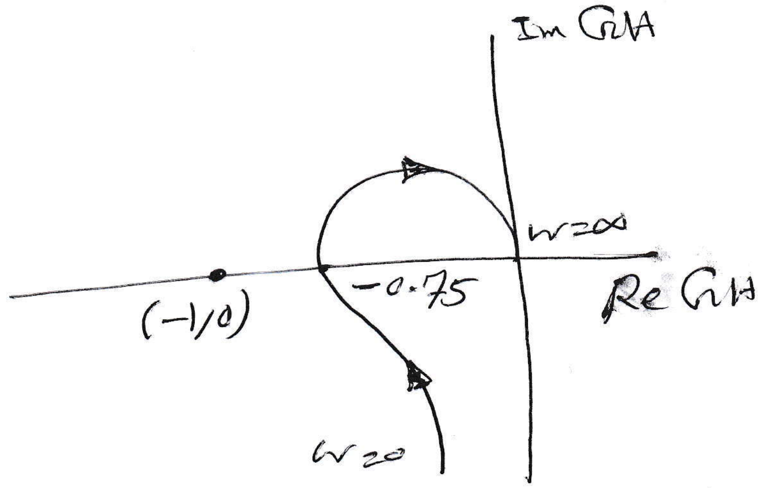
d) $K = 10 \Rightarrow$ Real part $= (-0.5, j0)$

b) K = 20



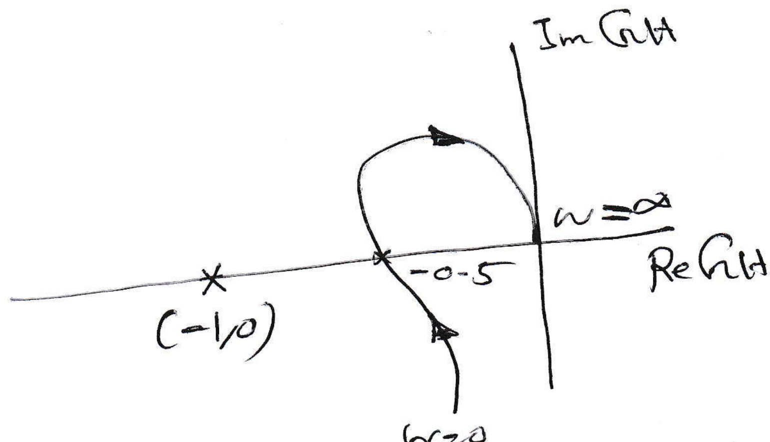
limited stable

c) K = 15



the system is stable

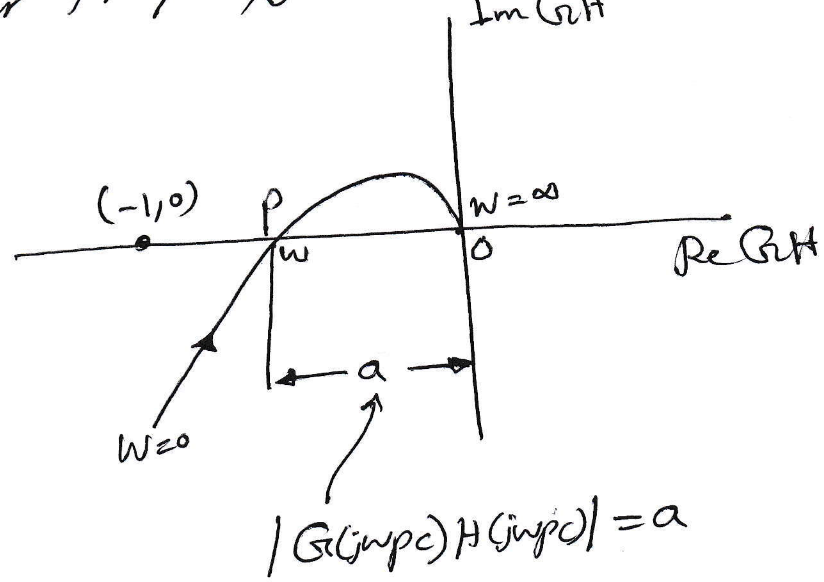
d) K = 10



the system is very stable.

Measures of Relative Stability : Gain Margin

Consider the polar plot shown in figure below - the point where the $G(s)H(s)$ plot cross the negative real axis is known as "the phase cross over frequency".



At $w = w_{cp}$, the $G(s)H(s)$ plot crosses the negative real axis, i.e, the phase angle changes from $-180°$ to $+180°$. This frequency is therefore called as phase crossover frequency w_{pc}. The magnitude of $G(s)H(s)$ at $s = j\omega_{pc}$ is given by OP and is equal to $|G(j\omega_{pc})H(j\omega_{pc})|$. The gain margin is defined as

$$\text{Gain Margin (GM)} = \frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|} = \frac{1}{a}$$

$$\text{Gain margin in db} = 20\log_{10}\frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}$$

$$= 20\log_{10}\frac{1}{a}$$

If $a > 1$, the GM is negative, the polar plot encloses the critical point and the system is unstable. The system is more and more stable if the phase cross over point P is nearer and nearer to the origin. The gain margin is positive for stable systems.

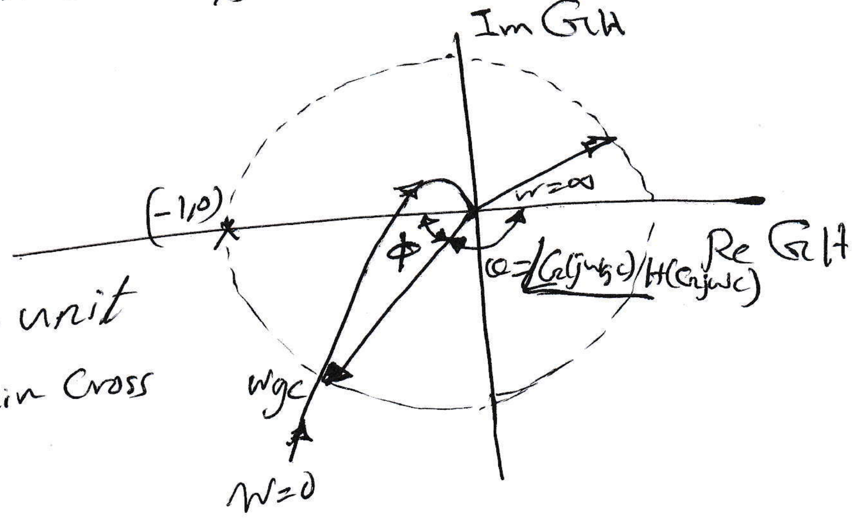
The magnitude of $G(j\omega)H(j\omega)$ at any frequency ω, in particular at $\omega = \omega_{pc}$, will increase as the loop gain K is increased. thus the phase cross over point P moves nearer to the critical point, as the gain K is increased. For a particular value of K the value $|G(j\omega_{pc})H(j\omega_{pc})|$ becomes unity and the system will be on the ~~verge~~ of instability.

"the gain margin can also be defined as the amount of increase in the gain that can be premitted before the system becomes unstable".

phase margin

Draw a circle with origin as centre and unit radius

the frequency at which the polar plot crosses the unit circle, is known as "Gain cross over frequency, ω_{gc}".

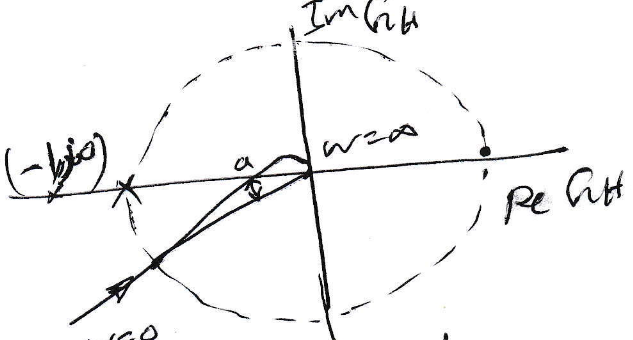


- For frequency greater than w_{gc}, the magnitude of ~~G(s)~~ $G(s)H(s)$ becomes less than unity. If additional phase lag of $\phi = 180 - \angle G(jw_{gc})H(jw_{gc}) = 180 - \theta$ is added without changing the magnitude at this frequency, the polar plot of the system will cross $(-1, j0)$ point. If the polar plot is rotated by an angle equal to ϕ in the clockwise direction, the system becomes unstable. Therefore, the phase margin can be defined as follows.

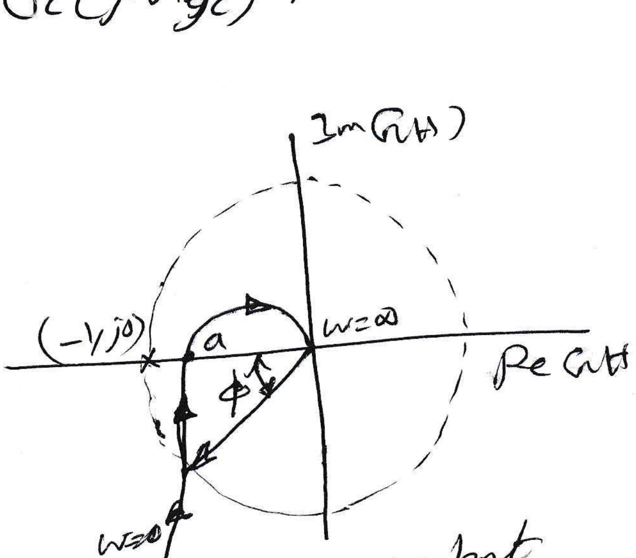
"phase margin is the amount of phase lag that can be introduced into the system at the gain cross over frequency to bring the system to the verge of instability".

$$PM\ \phi = \angle G(jw_{gc})H(jw_{gc}) + 180$$

where the angle of $G(jw_{gc})H(jw_{gc})$ is measured negatively.



good gain margin but low phase margin



good phase margin but poor gain margin