Fourier Series

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The periodic function is one that repeats every T seconds. In other words, a periodic function $f(t)$ satisfies

$$f(t) = f(t + nT) \quad \text{--------} \ (1)$$

Where n is an integer and T is the period of the function. According to the "Fourier theorem" any particular periodic function of frequency "w_o" can be expressed as an infinite sum of Sine or Cosine functions, that are integral multiple of "w_o".

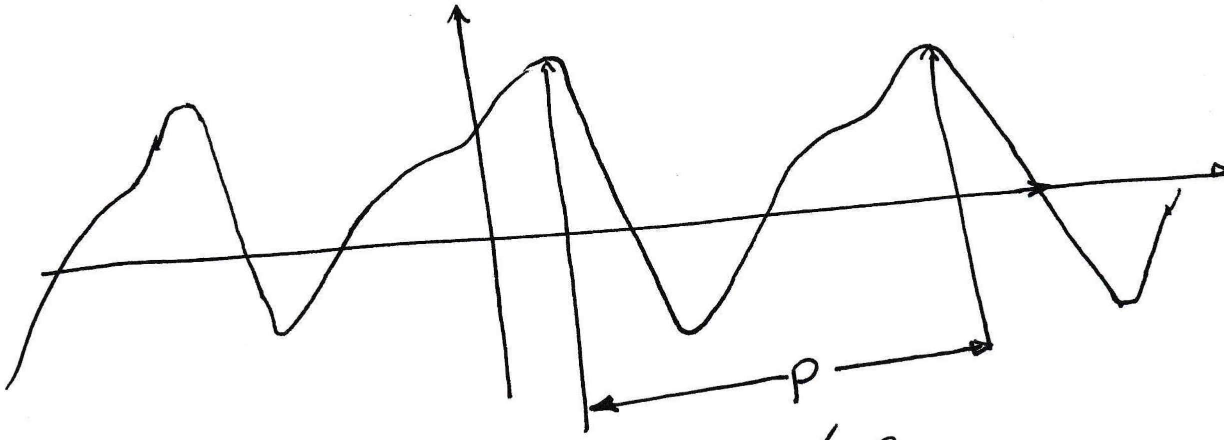
Thus, $f(t)$ can be expressed as:

$$f(t) = a_0 + a_1 \cos w_o t + b_1 \sin w_o t + a_2 \cos 2w_o t + b_2 \sin 2w_o t + a_3 \cos 3w_o t + b_3 \sin 3w_o t + \cdots \quad (2)$$

or

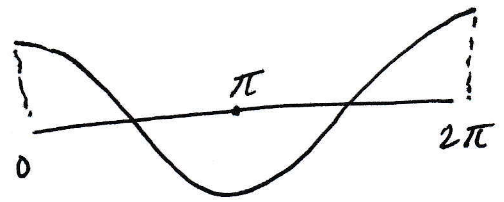
$$f(t) = \underset{dc}{a_0} + \sum_{n=1}^{\infty} (a_n \cos nw_o t + b_n \sin nw_o t) \quad (3)$$

Where $w_o = 2\pi f \Rightarrow w_o = 2\pi / T$ is called "fundamental frequency" in radian persecond.

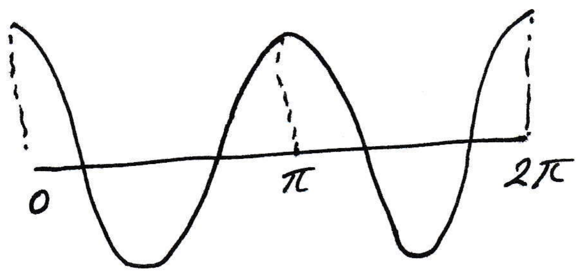


periodic function of period p

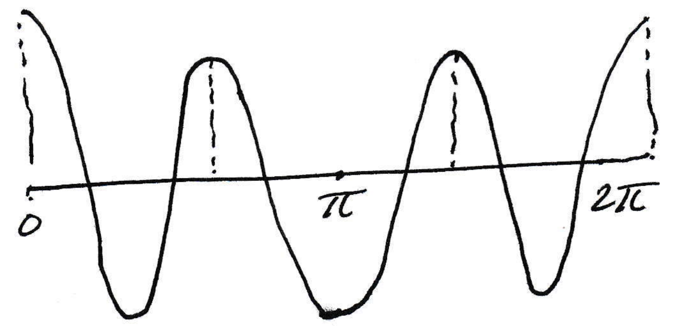
Cos x, $x = 2\pi$



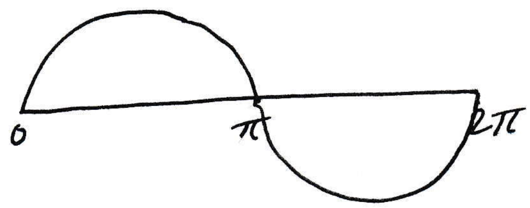
Cos $2x$



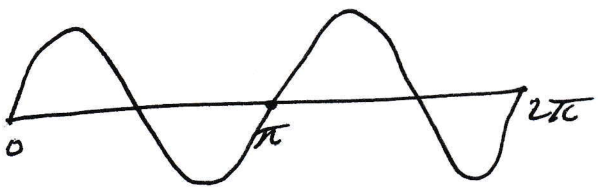
Cos $3x$



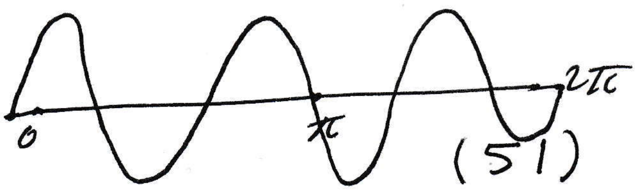
Sin x



Sin $2x$



Sin $3x$



The sinusoid $\sin n\omega_0 t$ or $\cos n\omega_0 t$ is called the "nth" harmonic of $f(t)$" it is an odd harmonic if n is odd and an even harmonic if n is even. Equation (3) is called "trigonometric Fourier Series" of $f(t)$.

The constant "a_n" and "b_n" are the "Fourier Coefficients" the coefficient a_0 is the dc component or the average value of $f(t)$. the coefficients a_n & b_n (for $n \neq 0$) are the amplitude of sinusoids in the "ac component", thus "Fourier Series of aperiodic function $f(t)$ is representation that resolved $f(t)$ into a d.c component & an ac component compurising an infinite series of harmonic sinusoids".

We begin by finding a_0, integral equation (3) of both sides over one period and obtain.

$$\int_0^T f(t)\,dt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right] dt$$

$$= \int_0^T a_0\,dt + \sum_{n=1}^{\infty} \left[\int_0^T a_n \cos n\omega_0 t + \int_0^T b_n \sin n\omega_0 t\right] dt \quad \text{------} (4)$$

$$\int_0^T f(t)\,dt = \int_0^T a_0\,dt = a_0 T$$

$$\therefore \boxed{a_0 = \frac{1}{T_0}\int_0^T f(t)\,dt} \quad \text{------} (5)$$

a_0 is the average value of $f(t)$.

To evaluate a_n, we multiply both sides of Eq (3) by $\cos n\omega_0 t$ integral over one period.

$$\int_0^T f(t)\cos m\omega_0 t\,dt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right] * \cos m\omega_0 t\,dt$$

$$= \int_0^T a_0 \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t \cos m\omega_0 t\,dt + \int_0^T b_n \sin n\omega_0 t \cos m\omega_0 t\,dt\right] dt \quad \text{.....(6)}$$

$$\int_0^T f(t)\cos m\omega_0 t\,dt = a_n \cdot \frac{T}{2} \quad \text{for } m=n$$

$$\therefore \boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt} \quad \text{------(7)}$$

In similar way for b_n and obtain

$$\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt} \quad \text{------(8)}$$

these ~~are~~ call "Fourier Coefficients" of $f(x)$, given by the "Euler formulas".

Note:

Function	Value
$\cos 2n\pi$	1
$\sin 2n\pi$	0
$\cos n\pi$	$(-1)^n$
$\sin n\pi$	0
$\cos n\frac{\pi}{2}$	$\begin{cases} (-1)^{n/2}, & n: \text{even} \\ 0, & n: \text{odd} \end{cases}$
$\sin n\frac{\pi}{2}$	$\begin{cases} (-1)(n-1)/2, & n: \text{odd} \\ 0, & n: \text{even} \end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases} (-1)^{n/2}, & n: \text{even} \\ j(-1)^{(n-1)/2}, & n: \text{odd} \end{cases}$

$$\int \cos at\, dt = \frac{1}{a} \sin at$$

$$\int \sin at\, dt = -\frac{1}{a} \cos at$$

$$\int t \cos at\, dt = \frac{1}{a^2} \cos at + \frac{1}{a} t \sin at$$

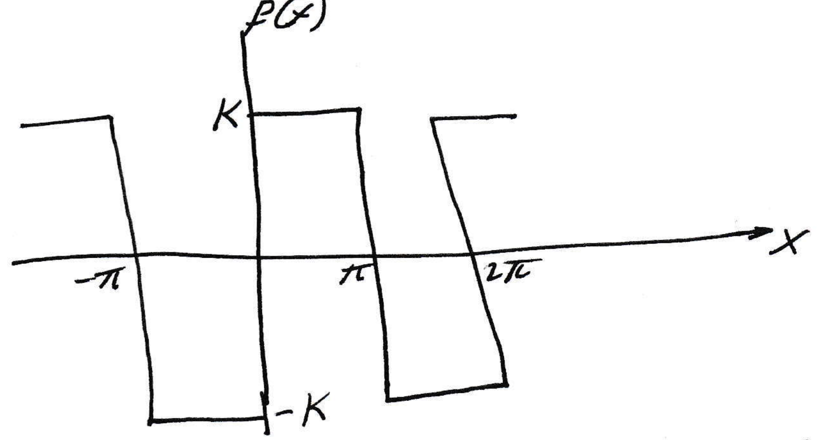
$$\int t \sin at\, dt = \frac{1}{a^2} \sin at - \frac{1}{a} t \cos at.$$

* $\sin at \cos bt = \frac{1}{2}[\sin(a-b)t + \sin(a+b)t]$

* $\sin at \sin bt = \frac{1}{2}[\cos(a-b)t - \cos(a+b)t]$

* $\cos at \cos bt = \frac{1}{2}[\cos(a-b)t + \cos(a+b)t]$.

EX Find the Fourier coefficients of the function $f(x)$ in figure below.



$$f(x) = \begin{cases} -K & \text{if } -\pi < x < 0 \\ K & \text{if } 0 < x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -K\,dt + \int_0^{\pi} K\,dt\right]$$

$$a_0 = \frac{1}{2\pi}\left[\left[-Kt\right]_{-\pi}^{0} + \left[Kt\right]_0^{\pi}\right]$$

$$a_0 = \frac{1}{2\pi}\left[-K[0+\pi] + K[\pi]\right]$$

$$a_0 = \frac{1}{2\pi}\{-K\pi + K\pi\} \Rightarrow a_0 = 0$$

$$a_n = \frac{2}{T}\int_0^T f(t)\,\text{Cos}\, n w_0 t\, dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\cos nx\,dx + \int_0^{\pi} K\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\frac{\sin nx}{n}\Big|_{-\pi}^{0} + K\frac{\sin nx}{n}\Big|_0^{\pi}\right] = 0$$

because $\sin nx = 0$ at $-\pi$, 0 and π for all $n = 1, 2, \ldots$

we see that all these Cosine Coefficients are Zero.

$$n=1 \Rightarrow \frac{1}{\pi}\left[\frac{-K}{1}\{\sin 0 + \sin\pi\} + \frac{K}{1}\{\sin\pi - \sin 0\}\right]$$

$$= 0 + 0 = 0$$

$$n=2 \Rightarrow \frac{1}{\pi}\left[\frac{-K}{2}\{\sin 2*0 + \sin 2*\pi\} + \frac{K}{2}\{\sin 2\pi - \sin 2*0\}\right]$$

$$= 0 + 0 = 0$$

$$n = 3 \Rightarrow 0$$

$$n = 4 \quad = 0$$

that means the Fourier Series has no Cosine terms, it is a "Fourier Sine Series" with Coefficients

$b_1, b_2, \ldots\ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\sin nx\,dx + \int_0^{\pi} K\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[+K\frac{\cos nx}{n}\Big|_{-\pi}^{0} - K\frac{\cos nx}{n}\Big|_0^{\pi}\right]$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{K}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2K}{n\pi}(1 - \cos n\pi)$$

We have $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}, \text{ and thus } 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

$$b_1 = \frac{4K}{\pi},\ b_2 = 0,\ b_3 = \frac{4K}{3\pi},\ b_4 = 0,\ b_5 = \frac{4K}{5\pi}, \ldots$$

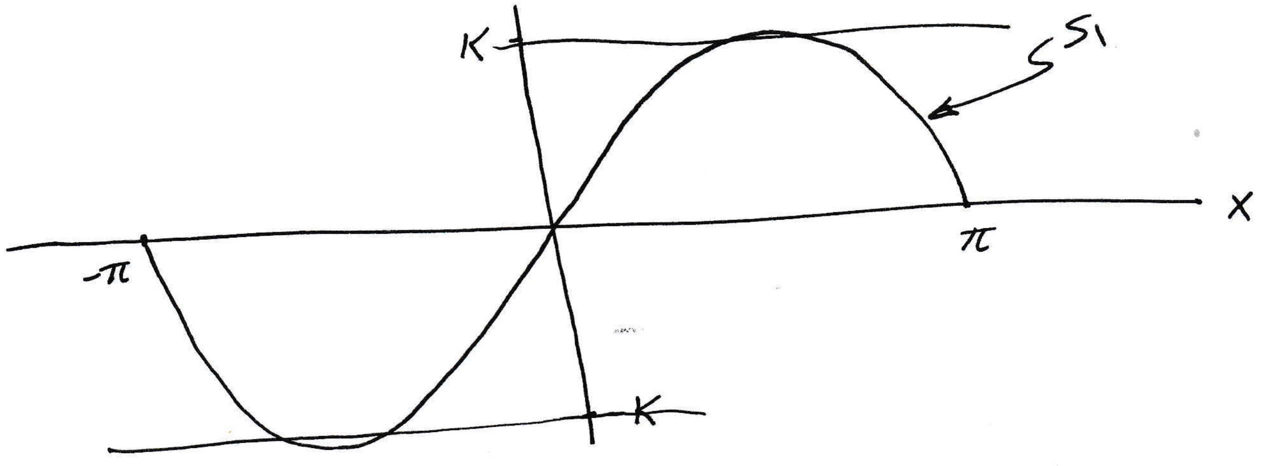
$\therefore$ the Fourier Series of $f(x)$ is

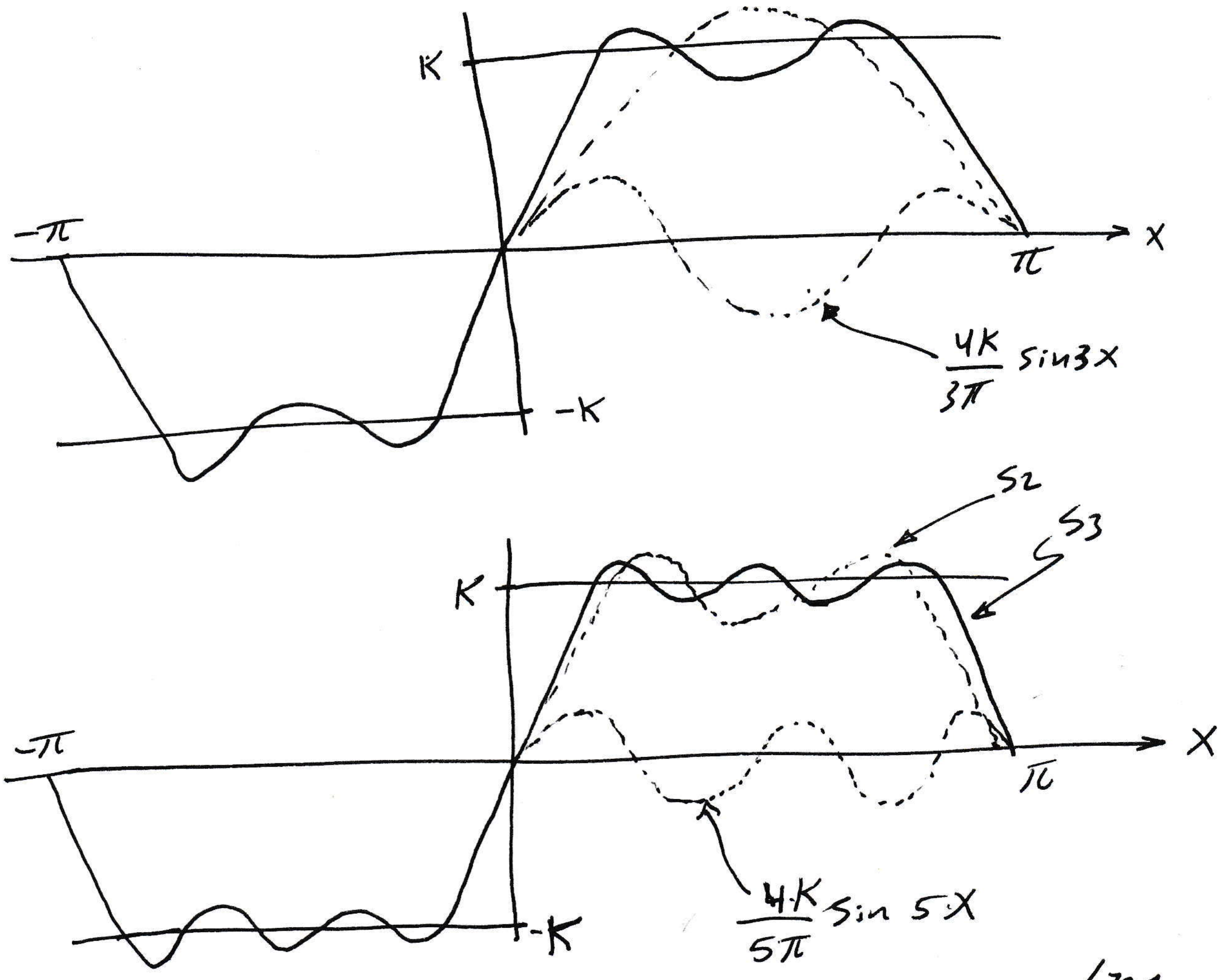
$$\frac{4K}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

the partial Sum are

$$S_1 = \frac{4K}{\pi}\sin x,\quad S_2 = \frac{4K}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$$

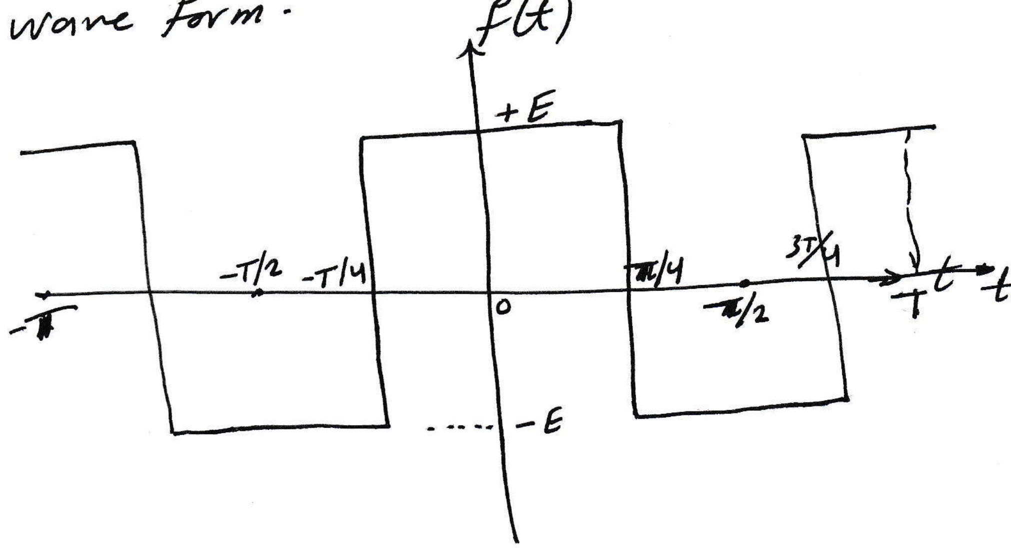
$$S_3 = \frac{4K}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right), \text{etc.}$$





First three partial Sum of the Corresponding Fourier Series.

EX Find the coefficients of Fourier Series for the following voltage wave form.



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_0 = \frac{1}{T}\left[\int_0^{T/4} E\,dt + \int_{T/4}^{3T/4} -E\,dt + \int_{3T/4}^{T} E\,dt\right.$$

$$= \frac{1}{T}\left[Et\Big|_0^{T/4} - Et\Big|_{T/4}^{3T/4} + Et\Big|_{3T/4}^{T}\right]$$

$$= \frac{1}{T}\left[E(T/4) - E(3T/4 - T/4) + E(T - \tfrac{3T}{4})\right]$$

$$= \frac{1}{T}\left[\frac{T}{4}E - \frac{3T}{4}E + \frac{T}{4}E + ET - \frac{3T}{4}E\right]$$

$$= \frac{1}{T}\left[E\left(\frac{T}{4} - \frac{3T}{4} + \frac{T}{4} + T - \frac{3T}{4}\right)\right]$$

$$= \frac{1}{T}\left[E\left(\frac{2T}{4} + T - \frac{6T}{4}\right)\right.$$

$$= \frac{1}{T}\left[E\left(\frac{6T}{4} - \frac{6T}{4}\right)\right. \Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$a_n = \frac{2}{T}\int_{-T/4}^{3T/4} f(t)\cos n\omega_0 t\,dt$$

$$a_n = \frac{2}{T}\left(\int_{-T/4}^{T/4} E\cos n\omega_0 t\,dt + \int_{T/4}^{3T/4} -E\cos n\omega_0 t\,dt\right]$$

$$a_n = \frac{2E}{T}\left[\int_{-T/4}^{T/4}\cos n\omega_0 t\,dt - \int_{T/4}^{3T/4}\cos n\omega_0 t\,dt\right]$$

$$a_n = \frac{2E}{T}\left[\frac{1}{n\omega_0}\left[\sin n\omega_0 t\right]_{-T/4}^{T/4} - \frac{1}{n\omega_0}\left[\sin n\omega_0 t\right]_{T/4}^{3T/4}\right]$$

$$a_n = \frac{2E}{n\omega_0 T}\left\{\left[\frac{\sin \omega_0 nT}{4} - \frac{\sin n\omega_0 T}{4}\right] - \left[\frac{\sin n\omega_0 T}{4} - \frac{\sin n\omega_0 T}{4}\right]\right\}$$

$$\because \sin(-x) = -\sin x, \quad \omega_0 T = 2\pi$$

$$\therefore a_n = \frac{2E}{2n\pi}\left\{\sin\left(\frac{2n\pi}{4}\right) + \sin\left(\frac{2n\pi}{4}\right) - \sin\left(\frac{6n\pi}{4}\right) + \frac{\sin 2n\pi}{4}\right\}$$

$$\therefore a_n = \frac{E}{n\pi}\left\{3\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3}{2}n\pi\right)\right\} \Rightarrow \boxed{a_n = 0 \text{ for all } n}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\cdot\sin n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/4}^{3T/4} f(t)\cdot\sin n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\left[\int_{-T/4}^{T/4} E\sin n\omega_0 t\,dt + \int_{T/4}^{3T/4} -E\sin n\omega_0 t\,dt\right]$$

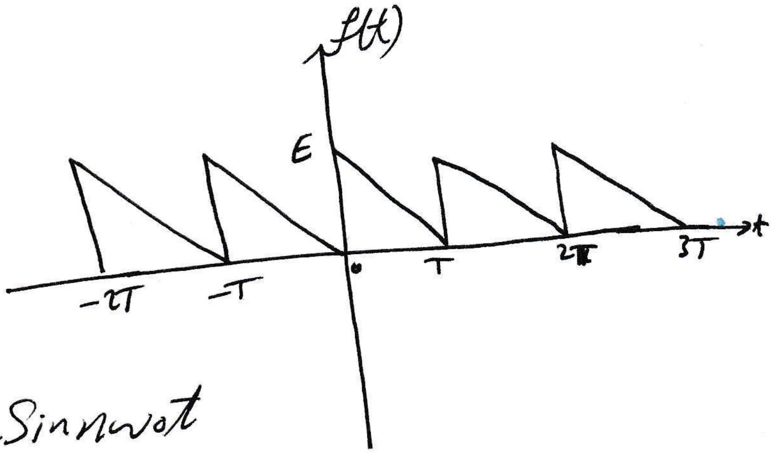
$$b_n = \frac{2E}{T}\left[\int_{-T/4}^{T/4} \sin n w_0 t \, dt - \int_{T/4}^{3T/4} \sin n w_0 t \, dt\right]$$

$$b_n = \frac{2E}{T}\left\{\frac{1}{n w_0}\left[-\cos n w_0 t\right]_{-T/4}^{T/4} - \frac{1}{n w_0}\left[-\cos n w_0 t\right]_{T/4}^{3T/4}\right\}$$

$$b_n = \frac{2E}{n w_0 T}\left\{-\cos\left(\frac{n w_0 T}{4}\right) + \cos\left(\frac{n w_0 T}{4}\right) + \cos\left(\frac{3 n w_0 T}{4}\right) - \cos\left(\frac{n w_0 T}{4}\right)\right\}$$

$$\therefore f(t) = \frac{4E}{\pi}\cos w_0 t - \frac{4E}{3\pi}\cos 3 w_0 t + \frac{4E}{5\pi}\cos 5 w_0 t - \frac{4E}{7\pi}\cos 7 w_0 t + \frac{4E}{9\pi}\cos 9 w_0 t + \cdots$$

EX Find the coefficients of Fourier Series of the Signal shown below.



$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos n w_0 t + B_n \sin n w_0 t$$

$y = mx + b$ m: slope

$f(t) = -\frac{E}{T} t + b$ x: t axis

y = f(t) axis

Sub (0, E) in equation

$E = -\frac{E}{T} * 0 + b \Rightarrow b = E$

or Sub (T, 0) in equation

$0 = -\frac{E}{T} * T + b \Rightarrow b = E$

$$\therefore \boxed{f(t) = -\frac{E}{T} t + E}$$

$$f(t) = -\frac{E}{T} t + E$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt$$

$$a_0 = \frac{1}{T}\int_0^T \left(-\frac{E}{T} t + E\right) dt$$

$$a_0 = \frac{1}{T}\left[\int_0^T -\frac{E}{T} t\, dt + \int_0^T E\, dt\right]$$

$$= \frac{1}{T}\left(\left[-\frac{E}{T}\frac{t^2}{2}\right]_0^T + Et\Big|_0^T\right)$$

$$\therefore \; a_0 = \frac{1}{T}\left(\frac{-\epsilon T + 2\epsilon T}{2}\right) \Rightarrow \boxed{a_0 = \frac{\epsilon}{2}}$$

$$a_n = \frac{2}{T}\int_0^T f(t) \cos n w_0 t \, dt$$

$$= \frac{2}{T}\int_0^T \left(\frac{-\epsilon}{T} t + \epsilon\right) \cos n w_0 t \, dt$$

$$a_n = \frac{2}{T}\left\{\frac{-\epsilon}{T}\int_0^T t \cos n w_0 t \, dt + \int_0^T \epsilon \cos n w_0 t \, dt\right\}$$

$$= \frac{-2\epsilon}{T^2}\int_0^T t \cos n w_0 t \, dt + \frac{2\epsilon}{T}\int_0^T \cos n w_0 t \, dt$$

let $u = t \Rightarrow du = dt$, $dv = \int \cos w_0 t \Rightarrow v = \frac{\sin n w_0 t}{n w_0}$

$$= \frac{-2\epsilon}{T^2}\left[t \cdot \frac{\sin n w_0 t}{n w_0}\right]_0^T - \int_0^T \frac{\sin n w_0 t}{n w_0} dt$$

$$a_n = \frac{-2\epsilon}{T^2}\left\{\left(\frac{t \cdot \sin w_0 n T}{n w_0}\right|_0^T + \frac{\cos n w_0 T}{(n w_0)^2}\Big|_0^T\right.$$

$$= \frac{-2\epsilon}{T^2}\left\{\left(\frac{T \cdot \sin w_0 n T}{n w_0} - \cancel{\frac{0 \times \sin w_0 n \times 0}{n w_0}}\right) - \frac{\{\cos n w_0 T - \cos n w_0\}}{(n w_0)^2}\right.$$

$$+ \frac{2\epsilon}{T n w_0} \sin n w_0 t \Big|_0^T$$

$= \cancel{\frac{-2\epsilon}{T^2}}$ $\quad \sin 2n\pi = 0$ and $\cos 2n\pi = 1$

$$= \frac{-2\epsilon}{T^2}\left[\frac{T \sin w_0 n T}{n w_0} - \frac{\cos n w_0 t}{(n w_0)^2} + \frac{\cos n w_0}{(n w_0)^2}\right]$$

$$- \frac{2\epsilon}{T n w_0}\left[\sin n w_0 T - \sin n w_0 \times (0)\right]$$

$$= \frac{-2E}{T^2}\left[\frac{T \cdot \sin n w_0 n T}{n w_0} - \frac{\cos n w_0 t}{(n w_0)^2} + \frac{1}{(n w_0)^2}\right] - \frac{2E}{T n w_0}\sin n w_0 T$$

$$= \frac{-2E}{T^2}\left[\frac{0}{n w_0} - \frac{1}{(n w_0)^2} + \frac{1}{(n w_0)^2}\right] - \frac{2E}{T n w_0} * 0$$

$$= \frac{-2E}{T^2}\left[\frac{1}{(n w_0)^2} - \frac{1}{(n w_0)^2}\right]$$

$$\boxed{A_n = 0}$$

$$B_n = \frac{2}{T}\int_0^T f(t)\sin w n t \, dt$$

$$= \frac{-2E}{T^2}\int_0^T t \sin n w_0 t \, dt + \frac{2E}{T}\int_0 \sin n w_0 t \, dt$$

let $u = t \Rightarrow du = dt$, $dv = \int \sin n w_0 t$

$$\Rightarrow v = \frac{1}{n w_0} * -\cos n w_0 t$$

$$B_n = \frac{-2E}{T^2}\left[\left(-t \cdot \frac{\cos n w_0 t}{n w_0}\right|_0^T + \frac{1}{n w_0}\int_0^T \cos n w_0 t \, dt - \left(\frac{2E}{T} * \frac{\cos n w_0 t}{n w_0}\right|_0^T\right]$$

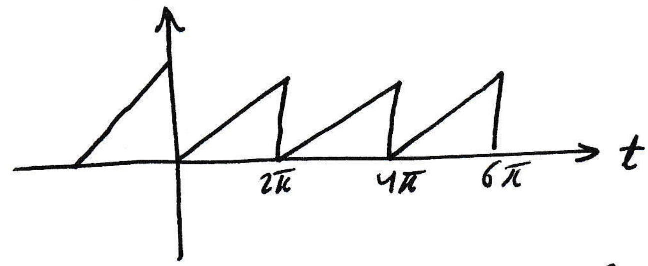
$$= \frac{-2E}{T^2}\left(\left[\frac{-T\cos n w_0 T}{n w_0}\right] + \left[\frac{\sin n w_0 t}{(n w_0)^2}\right|_0^T\right) - \frac{2E}{T}\left(\frac{\cos n w_0 T}{n w_0} - \frac{\cos n w_0}{n w_0} \times 0\right)$$

$$\boxed{B_n = \frac{E}{n\pi}}$$

$$\therefore f(t) = a_0 + A_1 \cos w_0 t + A_2 \cos 2 w_0 t + \cdots + B_1 \sin n w_0 t + B_2 \sin 2 w_0 t \cdots$$

$$= \frac{E}{2} + \frac{E}{\pi}\sin w_0 t + \frac{E}{2\pi}\sin 2 w_0 t + \frac{E}{3\pi}\sin 3 w_0 t + \cdots$$

Ex find the Fourier Series for the function $f(t) = x$ from $x = 0$ to $x = 2\pi$?



$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{T}\int_0^T x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{4\pi^2}{2}\right]$$

$$\therefore \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n w_0 t\,dt \quad , w_0 = 2\pi/T = \frac{2\pi}{2\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

let $u = x \Rightarrow du = 1,\ dv = \int \cos nx\,dx \Rightarrow v = \frac{1}{n} * \sin nx$

$$= \frac{1}{\pi}\left(\left[x \cdot \frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi} \sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[2\pi \cdot \frac{\cancel{\sin 2n\pi}}{2\pi}\right]^{\to 0} + \frac{1}{n^2}\left[\cos nx\right]_0^{2\pi}\right)$$

$$= \frac{1}{\pi n^2}\left[\cos 2n\pi - \cos 0 n\right]$$

$$= \frac{1}{\pi n^2}\left[1 - 1\right] \Rightarrow \boxed{a_n = 0}$$

$b_n = \cancel{\frac{1}{\pi n}[\cos 2n\pi - \cos]}$

$$b_n = \frac{2}{T}\int_0^T f(t)\cdot \sin nt\,dt = \frac{1}{\pi}\int_0^{2\pi} x\cdot \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left(\left(x\cdot \frac{-\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi} \cos nx\,dx\right.$$

$$\therefore b_n = \frac{-2}{n}, \quad b_1 = -2, \quad b_2 = -1, \quad b_3 = -\frac{2}{3}$$

$$\therefore f(t) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots$$

$$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

$$f(t) = \pi - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x + \cdots$$

An alternative form of equation "3" is the amplitude -phase form:-

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n w_0 t + \phi_n)$$

we have $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= a_0 + \sum_{n=1}^{\infty} \underbrace{A_n}_{\text{amplitude}} \cos(n w_0 t + \underbrace{\phi_n}_{\text{phase}})$$

$$= a_0 + \sum_{n=1}^{\infty} \underbrace{A_n}_{\text{amplitude}} \cos\phi_n \cos n w_0 t - A_n \sin\phi \sin n w_0 t$$

Equating the coefficient of the series expression and show that $a_n = A_n\cos\phi_n$, $b_n = -A_n\sin\phi_n$

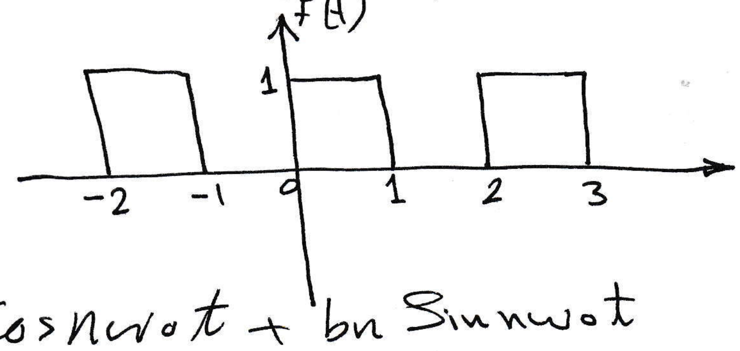
$$\boxed{A_n = \sqrt{a^2 + b^2}} \quad , \quad \boxed{\phi_n = -\tan^{-1}\frac{b_n}{a_n}}$$

It Can be related the terms in Complex form as

$$A_n \angle \phi_n = a_n - j b_n$$

* the plot of the amplitude A_m of the harmonics versus $n\omega_0$ is called the "amplitude Spectrum" of $f(t)$, the plot of the phase ϕ_n versus $n\omega_0$ is the "phase Spectrum" of $f(t)$. Both the amplitude and phase Spectrum form the "frequency spectrum" of $f(t)$.

EX Determine the Fourier Series of the wave form shown in figure below. Obtain the amplitude and the phase spectrum?



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

$T = 2$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$ thus

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 0\,dt\right] \Rightarrow \boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{2}\left[\int_0^1 \cos n\pi t\,dt + \int_1^2 0 * \cos n\pi t\,dt\right.$$

$$= \frac{1}{n\pi}\sin n\pi t\Big|_0^1 = \frac{1}{n\pi}\sin n\pi = 0$$

$$\therefore \boxed{a_n = 0}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

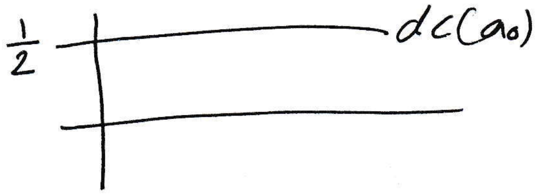
$$= \frac{2}{2}\left[\int_0^1 \sin n\pi t\,dt + \int_1^2 0 * \sin n\pi t\,dt\right.$$

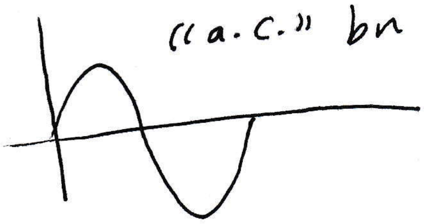
$$= -\frac{1}{n\pi}\cos n\pi t\Big|_0^1 = -\frac{1}{n\pi}\left[\cos n\pi - \cos(0)\right]$$

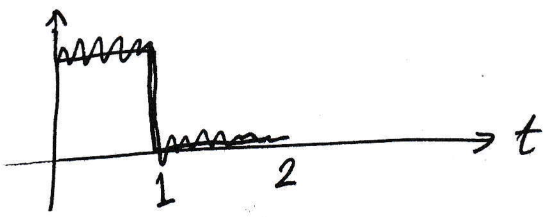
$$= -\frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \frac{2}{n\pi} & n: \text{odd} \\ 0 & n: \text{even} \end{cases}$$

$$\therefore f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t \ldots$$

$$\therefore f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{K=1}^{\infty}\frac{1}{n}\cdot\sin n\pi t, \quad n = 2K-1$$

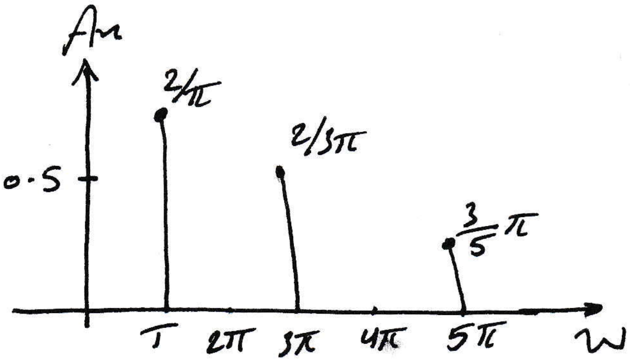


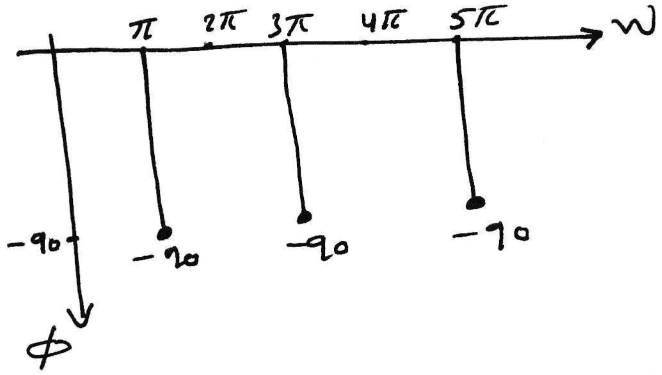




$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \frac{2}{n\pi} \quad \text{if } n\text{: odd}, \quad A_n = 0 \text{ if } n\text{: even}$$

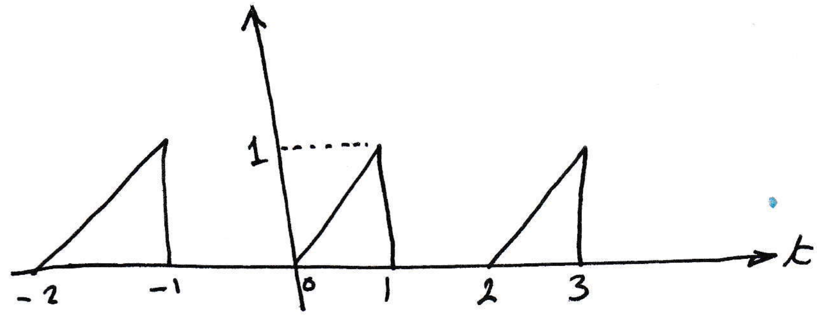
$$\phi_n = -\tan^{-1}\frac{b_n}{a_n} = \begin{cases} -90 & n\text{: odd} \\ 0 & n\text{: even} \end{cases}$$





Ex obtain the Fourier Series for the periodic function in Figure below and the amplitude and phase spectrum.

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$



$T = 2, \; w_0 = 2\pi/T = \pi$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}\left.\frac{t^2}{2}\right|_0^1$$

$$\therefore \boxed{a_0 = \frac{1}{4}}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n w_0 t\,dt = \frac{2}{2}\left[\int_0^1 t\cos n\pi t + \int_1^2 0\cos n\pi t\,dt\right.$$

$$= \frac{1}{n^2\pi^2}\cos n\pi + \frac{1}{n\pi}\sin n\pi t\Big|_0^1 \Rightarrow \boxed{a_n = \frac{(-1)^n - 1}{n^2\pi^2}}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\, \sin n w_o t\, dt = \frac{2}{2}\left[\int_0^1 t \sin n\pi t\, dt + \int_0^1 0 * \sin n\pi t\, dt\right].$$

$$= \frac{1}{n^2\pi^2} \sin n\pi t - \frac{t}{n\pi} \cos n\pi t \Big]_0^1$$

$$\boxed{b_n = \frac{(-1)^{n+1}}{n\pi}}$$

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{[(-1)^n - 1]}{(n\pi)^2} \cos n\pi t + \frac{(-1)^{n+1}}{n\pi} \sin n\pi t\right)$$

To obtain the amplitude and phase spectrum, for even $a_n = 0$, $b_n = -1/n\pi$ so that $A_n \angle \phi = a_n - j b_n = 0 + j\frac{1}{n\pi}$

$$A_n = |b_n| = \frac{1}{n\pi}, \quad n = 2, 4, 6, \cdots$$

$$\phi_n = 90, \quad n = 2, 4, 6, \cdots$$

For odd harmonic $a_n = \frac{-2}{n^2\pi^2}$, $b_n = \frac{1}{n\pi} \Rightarrow A_n \angle \phi = a_n - j b_n$

$$\therefore A_n \angle \phi = \frac{-2}{n^2\pi^2} - j\frac{1}{n\pi}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{\frac{4}{n^4\pi^4} + \frac{1}{n^2\pi^2}} = \frac{1}{n^2\pi^2}\sqrt{4 + n^2\pi^2} \quad n = 1, 3, 5, \cdots$$

$$\phi = 180 + \tan^{-1}\frac{n\pi}{2}, \quad n = 1, 3, 5, \cdots$$