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Complex variables :-

Algebric number for Complex function, we mean that a number of the form:

$$Z = x + iy$$

when x and y are real numbers and i is the so-called "imajinary Unit" $\quad i = \sqrt{-1} \Rightarrow i^2 = -1$

the real number x is called the "real component" or "real part" of Z. the real number y is called the "imajinary Component" or "imajinary part" of Z.

the real part of a Complex number is denoted by $R(z)$, the imajinary part of a Complex number is denoted by $g(z)$:-

$$Z = x + iy \quad , R(z) = x \; , \; g(z) = y$$

$$-z = -x - iy \quad , \text{negative of } Z.$$

If two complex numbers differ only in the sign of their imaginary part either on is said to be the "Conjugate" of the other. the Conjugate of Complex number Z is ~~called~~ usually written $\acute{Z}$ or Z^*

$$Z = x + iy \Rightarrow \acute{Z} = x - iy$$

$$Z = 6 + 3i \Rightarrow \acute{Z} = 6 - 3i$$

$$Z = -3 - 4i \Rightarrow \acute{Z} = -3 + 4i$$

Addition & Subtraction and multiplication of Complex number :-

If $Z_1 = a+ib$; $Z_2 = c+id$, then

$Z_1 + Z_2 = Z_3$, $(a \mp c) + (b \mp d)i$

multiplication :-

$$Z_1 * Z_2 = Z_3 = (a+ib)(c+id)$$
$$= (ac - bd) + (bc + ad)i$$

Division :-

$$\frac{Z_1}{Z_2} = Z_3 = \frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)} * \frac{c-id}{c-id}$$
$$= \frac{(ac+bd) + (bc-ad)i}{c^2+d^2}$$

$$\therefore \frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

Important properties of Conjugate Complex numbers :-

1) $Z\bar{Z} = (X+iy)(X-iy) = X^2+y^2$

2) $Z+\bar{Z} = (X+iy)+(X-iy) = 2X = 2R(Z)$

$$\Rightarrow RZ = \frac{Z_1+Z_2}{2}$$

3) $Z-\bar{Z} = (X+iy)-(X-iy) = 2iy = 2g(Z)$

$$\Rightarrow g(Z) = \frac{Z-\bar{Z}}{2}$$

4) $\overline{Z_1 \pm Z_2} = \bar{Z}_1 \pm \bar{Z}_2$

EX

$Z_1 = 5+2i \quad , \quad Z_2 = 2+3i$

$Z_1+Z_2 = Z_3 = 7+5i$

$\bar{Z}_3 = 7-5i \quad , \bar{Z}_1 = 5-2i \; , \; \bar{Z}_2 = 2-3i$

$\therefore \; \bar{Z}_1+\bar{Z}_2 = 7-5i$

5) $\overline{Z_1 Z_2} = \bar{Z}_1 \times \bar{Z}_2$

$Z_1 = 5+2i \; , \quad Z_2 = 2+3i$

$Z_3 = Z_1 \cdot Z_2 = (5+2i)(2+3i)$

$Z_3 = 4+19i \; , \; \bar{Z}_3 = 4-19i$

$\bar{Z}_1 = 5-2i \; , \; \bar{Z}_2 = 2-3i$

$\bar{Z}_1 \cdot \bar{Z}_2 = (5-2i)(2-3i) = 4-19i$

6) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ at $z_2 \neq 0$

$z_1 = 5+2i$, $z_2 = 2+3i$

$\overline{z_1} = 5-2i$, $\overline{z_2} = 2-3i$

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{5-2i}{2-3i} = \frac{(5-2i)(2+3i)}{(2-3i)(2+3i)} = \frac{16}{13} + i\frac{11}{13}$$

$$\frac{z_1}{z_2} = \frac{5+2i}{2+3i} = \frac{(5+2i)(2-3i)}{(2+3i)(2-3i)} = \frac{16}{13} - \frac{11}{13}i$$

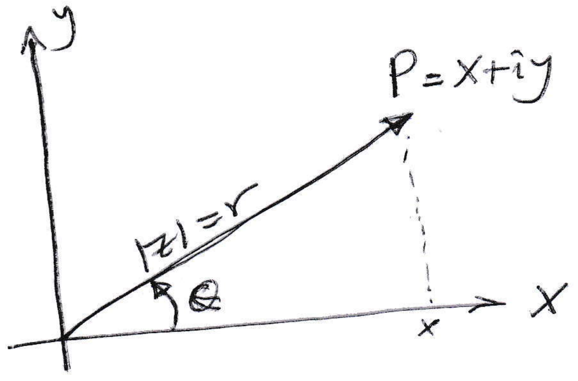
$$\overline{\frac{z_1}{z_2}} = \frac{16}{13} + \frac{11}{13}i$$

The geometric representation of Complex number :-

The vector OP which represents the Complex number $x+iy$, the length "r" equal

$|z| = r = \sqrt{x^2+y^2} \Rightarrow$ absolute value or modulus of Z and its direction angle.

$\arg(z) = \theta = \tan^{-1}\frac{y}{x}$, argument of Z.



from figure we notice

$x = r\cos\theta$ and

$y = r\sin\theta$

we have $Z = x+iy$

$\therefore Z = r\cos\theta + i\, r\sin\theta = r(\cos\theta + i\sin\theta)$

the last term is represnt the polar form of trigonometric form of Complex number.

$= r\angle\phi \leftarrow$ principle value.

<u>EX</u> let $Z = 1+i$ then find modulus & argument of Z.

$|Z| = r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\arg Z = \tan^{-1}\frac{y}{x} \Rightarrow \tan^{-1} 1 = 45^\circ = \frac{\pi}{4}$

<u>EX</u> $Z = 3 + 3\sqrt{3}\, i$, find modulus and argument of Z.

$|Z| = r = \sqrt{9+9*3} = \sqrt{36} = 6$

$\arg Z = \tan^{-1}\frac{3}{3\sqrt{3}} \Rightarrow \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ$

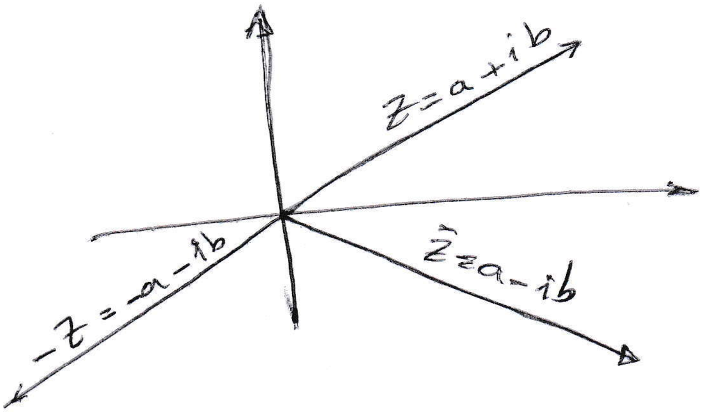
$Z_1 = 6\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

<u>Addition and subtraction of Complex Variables</u>

let $Z = a+ib$

$-Z = -a-ib$

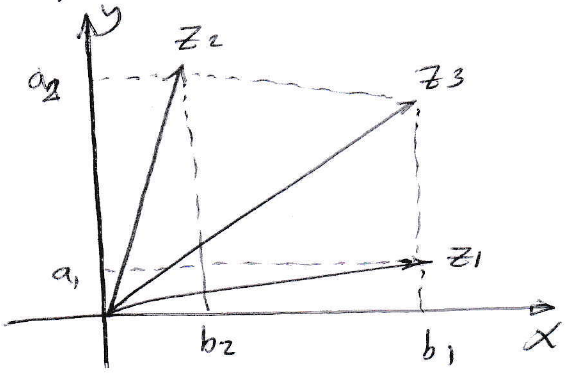
$\bar{Z} = a - ib$



The sum and difference of the complex number:-

$Z_1 = a_1 + ib_1$

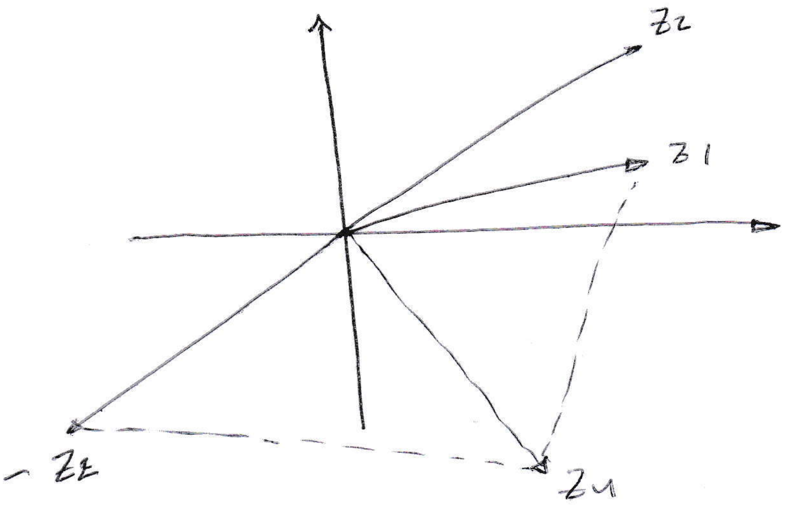
$Z_2 = a_2 + ib_2$



$\therefore Z_3 = Z_1 + Z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$= (a_1 + a_2) + i(b_1 + b_2)$

$Z_4 = Z_1 - Z_2 = (a_1 + ib_1) - (a_2 + ib_2)$



Multiplication of two complex number (polar form):-

$Z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and

$Z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

$Z_1 \cdot Z_2 = r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_1 + \cos\theta_1 \sin\theta_2]$

We have $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$, and

$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1$

$\therefore Z_1 \cdot Z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore |Z_1 Z_2| = |Z_1| \cdot |Z_2|$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$

EX $z_1 = 1+i$, $z_2 = \sqrt{3}+i$, find $z_1 z_2$ in polar form? and z_1/z_2.

$|z_1| = \sqrt{1+1} = \sqrt{2}$, $\arg(z_1) = \theta = \tan^{-1} 1 = 45° = \frac{\pi}{4}$

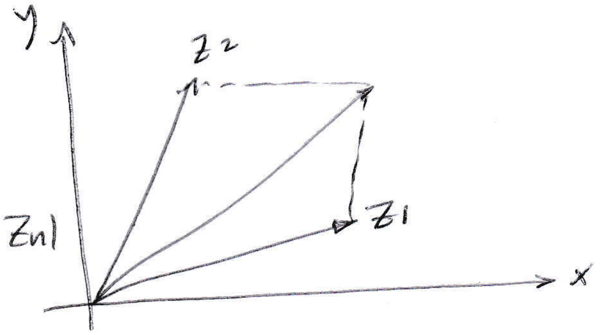
$z_1 = \sqrt{2}\angle 45$ $\therefore r_1 = \sqrt{2}$, $\theta = 45 = \frac{\pi}{4}$

$|z_2| = \sqrt{3+1} = 2$, $\arg(z_2) = \tan^{-1}\frac{1}{\sqrt{3}} = 30° = \frac{\pi}{6}$

$\therefore z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)]$

$z_1 z_2 = 2\sqrt{2}\angle 75° = \frac{5\pi}{12}$

In-equalities :- For any complex numbers, we have the important triangle inequality



$|z_1+z_2| \leq |z_1| + |z_2|$

OR $|z_1+z_2+\cdots| \leq |z_1| + |z_2| + \cdots + |z_n|$

EX Describe the complex number, is equalities or not.

$z_1 = 6+2i$, $z_2 = 2+8i$

$|z_1| = \sqrt{40} = 6.3246$, $|z_2| = \sqrt{68} = 8.2462$

$|z_1| + |z_2| = 14.5708$

$\therefore |z_1+z_2| = \sqrt{(6+2)^2+(2+8)^2} = 12.806$

$|z_1+z_2| \leq |z_1| + |z_2|$

$\therefore$ Inequalities

Integer power of Z, De moiver's Formula. Type 1

$$Z^{\pm n} = r^{\pm n}(\cos n\theta \pm i \sin n\theta) \quad , n = 0, 1, 2, \cdots$$

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$$

EX Calculate $\left(\cos\frac{3\pi}{8} + i \sin\frac{3\pi}{8}\right)^4$

$$\left(\cos\frac{3\pi}{8} + i \sin\frac{3\pi}{8}\right)^4 = \cos\left(4 \cdot \frac{3\pi}{8}\right) + i \sin\left(4 \cdot \frac{3\pi}{8}\right)$$

$$= 0 + i(-1)$$

$$= -i$$

EX Calculate Z^{3} if $Z = \cos\frac{7\pi}{12} + i \sin\frac{7\pi}{12}$

$$= [\cos\left(3 \cdot \frac{7\pi}{12}\right) - i \sin\left(3 \cdot \frac{7\pi}{12}\right)]$$

$$= \cos\frac{7\pi}{4} - i \sin\frac{7\pi}{4} = \cos\left(2\pi - \frac{\pi}{4}\right) - i \sin\left(2\pi - \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

EX Calculat $(\sqrt{3} + i)^{-9}$

$$Z = \sqrt{3} + i \Rightarrow r = \sqrt{3+1} \Rightarrow r = 2$$

$$\cos\theta = \frac{\sqrt{3}}{2}, \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\therefore Z = 2\left[\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right]$$

$$(\sqrt{3}+i)^{-9} = Z^{-9} = 2^{-9}\left[\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right]^{-9}$$

$$= \frac{1}{2^9}\left[\cos\frac{\pi}{6} - i \sin\frac{\pi}{6}\right]^9$$

Ex By Demoiver's Formula, Calculate $(1+i)^{11}$

$Z = 1+i$

$r = \sqrt{x^2+y^2} = \sqrt{1+1} \Rightarrow r = \sqrt{2}$

$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$

$\therefore \theta = \frac{\pi}{4}$

$\therefore Z = r[\cos\theta + i\sin\theta]$

$\therefore Z = \sqrt{2}[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}]$

$$(1+i)^{11} = Z^{11} = (\sqrt{2})^{11}[\cos 11\frac{\pi}{4} + i\sin\frac{11\pi}{4}]$$

$$= 32\sqrt{2}[\cos(2\pi + \frac{3\pi}{4}) + i\sin(2\pi + \frac{3\pi}{4})]$$

$$= 32\sqrt{2}[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}]$$

$$= 32\sqrt{2}[\cos(\pi - \frac{\pi}{4}) + i\sin(\pi - \frac{\pi}{4})]$$

$$= 32\sqrt{2}[-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}]$$

$$= 32\sqrt{2}[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i] = 32(-1+i)$$

$\therefore (1+i)^{11} = 32(-1+i)$

Integer power of Z, Demoiver's Formula. Type 2 :-

if $Z = r[\cos\theta + i\sin\theta)$, then

$$Z^{\frac{1}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}}$$

$$Z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2K\pi}{n}\right) + i\sin\left(\frac{\theta + 2K\pi}{n}\right)\right]$$

$K = 0, 1, 2, \cdots n-1$

EX Find the five Roots of $(\sqrt{3}+i)^2$ by Demoivir's Formula?

$$Z = \sqrt{3} + i \Rightarrow r = \sqrt{3+1} = \sqrt{4} = 2$$

$$\cos\theta = \frac{\sqrt{3}}{2}, \quad \sin\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6}$$

$$\therefore Z = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$Z^2 = 2^2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]^2 =$$

$$= 4\left[\cos 2\cdot\frac{\pi}{6} + i\sin 2\cdot\frac{\pi}{6}\right]$$

$$Z^2 = 4\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$$(\sqrt{3}+i)^{\frac{2}{5}} = Z^{\frac{2}{5}} = (Z^2)^{\frac{1}{5}} = \left(4\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]\right)$$

$$= \sqrt[5]{4}\left[\cos\frac{\frac{\pi}{3} + 2K\pi}{5} + i\sin\frac{\frac{\pi}{3} + 2K\pi}{5}\right]$$

$K = 0, 1, 2, 3, 4$

at $K=0 \Rightarrow Z_1 = \sqrt[5]{4}\left[\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right]$

at $K=1 \Rightarrow Z_2 = \sqrt[5]{4}\left[\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right]$

at $K=2 \Rightarrow Z_3 = \sqrt[5]{4}\left[\cos\frac{13\pi}{15} + i\sin\frac{13\pi}{15}\right]$

at $K=3 \Rightarrow Z_4 = \sqrt[5]{4}\left[\cos\frac{19\pi}{15} + i\sin\frac{19\pi}{15}\right]$

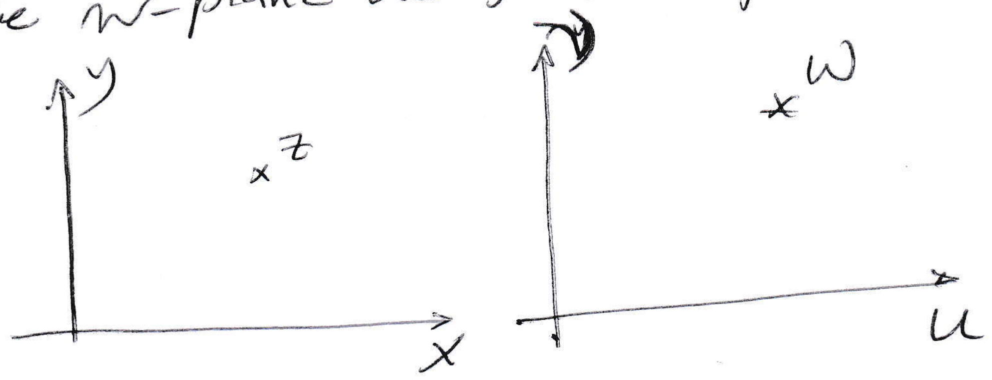
at $K=4 \Rightarrow Z_5 = \sqrt[5]{4}\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]$

Analytic function :-

$w = f(z)$

$w = f(z) = u(x,y) + i\,v(x,y)$

Each point inthe w-plane assigns to a point in the Z-plane.



EX

$w = f(z) = z^2 + 3z$ at $z = 1 + 3i$

$w = f(z) = (x+iy)^2 + 3(x+iy)$

$= x^2 + 2xy\,i - y^2 + 3x + 3iy$

$u(x,y) = Re\,f(z) = x^2 - y^2 + 3x$

$v(x,y) = Im\,f(z) = 2xy + 3y$

$\therefore \quad u(x,y) = u(1,3) = 1-9+3 = -5$

$i\,v(x,y) = v(1,3) = 15$

$\therefore \quad f(z) = f(1-3i) = -5 + i\,15$

EX let $w = f(z) = 2iz + 6\bar{z}$. Find u and v and the value of f at $z = \frac{1}{2} + 4i$.

$f(z) = 2i(x+iy) + 6(x-iy)$ gives $u(x,y) = 6x - 2y$

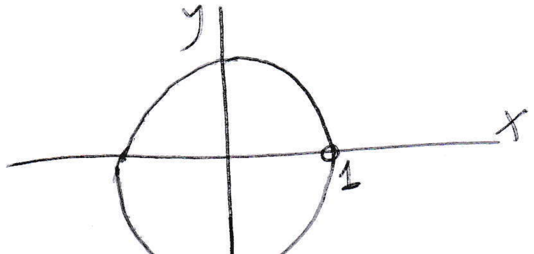
and $v(x,y) = 2x - 6y$.

$f(\frac{1}{2}+4i) = 2i(\frac{1}{2}+4i) + 6(\frac{1}{2}-4i)$

$= i - 8 + 3 - 24i = -5 - 23i$

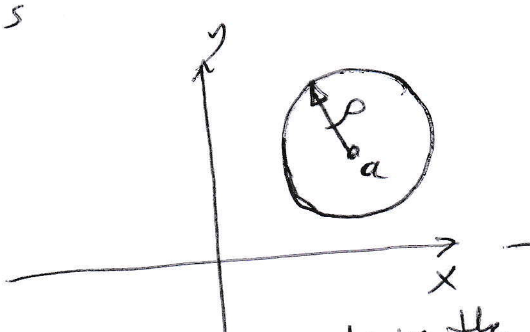
Circle and Disks :-

The unit circle $|z| = 1$ as shown in fig (1), in figure(2) shows a general circle of radius "ρ" and center "a", the equation is



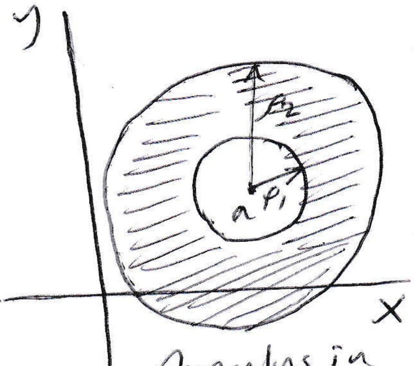
Unit circle

fig(1)



Circle in the Complex plane

fig (2)



Annulus in the Complex plane

fig(3)

$$|z - a| = \rho$$

- An open circular disk $|z-a|<\rho$ is called a "neighborhood" of a or "open circular disk" as figure (331).

- figure(332) shows an open "annulus" (circular ring)

$$\rho_1 < |z-a| < \rho_2 .$$

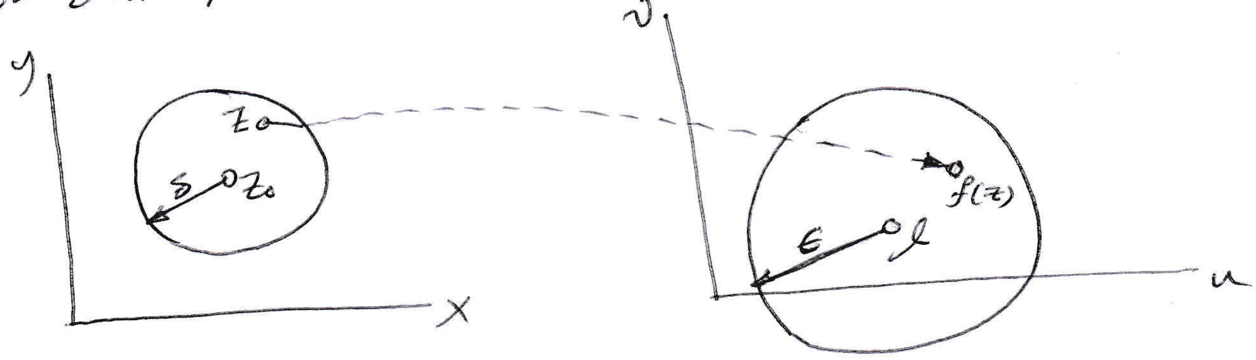
- A "closed circular disk" by $|z-a| \leq \rho$.

Limits of functions :-

A function $f(z)$ is said to have the limit l as z approaches a point z_0, written

$$\lim_{z \to z_0} f(z) = l,$$

if "f" is defined in a neighborhood of z_0 (except perhaps at z_0 itself) and if the values of f are "close" to l for all z "close to z_0", if for every positive real "ϵ" we can find a positive real "δ" such that for all $z \neq z_0$ in the disk $|z-z_0|<\delta$ as in figure below.



- A function $f(z)$ is said to be "continuous" at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

EX if $f(z) = \frac{iz}{2}$, find if it has limit at $z_0 \to 1$

$$f(z_0) = \lim_{z \to 1} f(z) = \frac{i}{2}$$

$\therefore |f(z) - f(z_0)| < \epsilon$, for $|z - z_0| < \delta$

$$\left|\frac{iz}{2} - \frac{i}{2}\right| = \left|\frac{i(z-1)}{2}\right| = \left|\frac{(z-1)}{2}\right| = \epsilon$$

for $z \to z_0$, i.e., $z \to 1$, $|z-1| < \delta$

$$|f(z) - f(z_0)| = \left|\frac{z-1}{2}\right| < \delta$$

$\therefore \epsilon = \frac{\delta}{2}$ - very small

$\therefore$ no limit or final limit.

Derivative

the derivative of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

$$\boxed{\frac{df(z_0)}{dz} = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}}$$

Provided this limit exists. Then f is said to be differentiable at z_0. If we write $\Delta z = z - z_0$, we have $z = z_0 + \Delta z$ and can write the equation above as

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

EX find $f'(z_0)$ for $f(z) = z^2$ at $z_0 = 1$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^2} + 2z_0 \Delta z + \Delta z^2 - \cancel{z_0^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z_0 + \Delta z \Rightarrow 2z_0 \Rightarrow f'(z)$$

$$= 2 * 1 = 2$$

Cauchy-Riemann Condition :-

If a function $f(z)$ is analytic at z_0, then $\lim f(z) = f(z_0)$ & its derivative exists at z_0.

$$\frac{df}{dz} = f'(z) = \lim_{z \to z_0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)] - [u(x,y) + i v(x,y)]}{\Delta x + i \Delta y}$$

Approaching z_0 along path parallel to x-axis, i.e. $\Delta y = 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{——— (1)}$$

Approaching z_0 along a path parallel to y-axis,

i.e., $\Delta x = 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + \frac{v(x, \Delta y + y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{——— (2)}$$

Since the function is analytic

$$\therefore \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\left.\begin{aligned} \therefore \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}\right\} \text{Cauchy–Riemann condition}$$

<u>EX</u> Solve $f(z) = z^2$ for C-R condition

$f(z) = x^2 - y^2 + i\,2xy$, $u = x^2 - y^2$, $v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

C-R condition are satisfied for all z, ie the fn is analytic every where.

EX Is $f(z) = u(x,y) + i v(x,y) = e^{x}(\cos y + i \sin y)$ analytic?

$$u = e^{x}\cos y \quad , \quad v = e^{x}\sin y$$

$$\frac{\partial u}{\partial x} = e^{x}\cos y \quad , \quad \frac{\partial v}{\partial y} = e^{x}\cos y$$

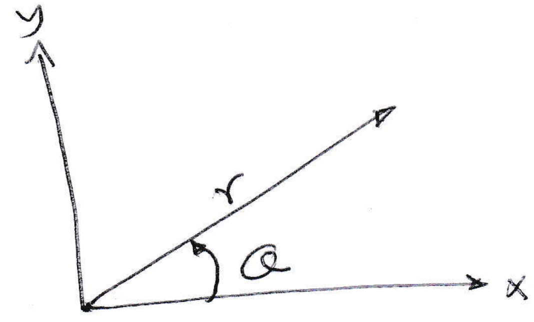
$$\frac{\partial u}{\partial y} = -e^{x}\sin y \quad , \quad \frac{\partial v}{\partial x} = e^{x}\sin y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ we see that the Cauchy-Rieman equations are satisfied that $f(z)$ is analytic for all z,

Polar form :-



$$z = re^{i\theta} = r[\cos\theta + i\sin\theta]$$

$$w = f(z) = u(r,\theta) + i v(r,\theta)$$

$$f'(z) = \lim_{\Delta r \to 0} \frac{u(r+\Delta r,\theta) - u(r,\theta)}{\Delta r\, e^{i\theta}} + \lim_{\Delta r \to 0} i \frac{v(r+\Delta r,\theta) - v(r,\theta)}{\Delta r\, e^{i\theta}}$$

$$= \frac{1}{e^{i\theta}}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) e^{i\theta} \quad \text{------} \quad (1)$$

$$f'(z) = \lim_{\Delta\theta \to 0} \frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{i r \Delta\theta \, e^{i\theta}} + i \lim_{\Delta\theta \to 0} \frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{i r \Delta\theta \, e^{i\theta}}$$

$$= \frac{1}{r}\left(\frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right) e^{-i\theta} \quad \text{—— } (2)$$

from 1 & 2 :-

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\} \text{C-R Conditions in polar form.}$$

Example in (262)

Harmonic function :-

If $f(z) = u + iv$ is analytic fn, then :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—— } (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—— } (2)$$

equations (1) and (2)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{—— } (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \quad \text{—— } (4)$$

from 3 & 4

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- } (5)$$

Similarity from 1 & 2

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \, , \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{--- } (6)$$

Ex

the great impontant of Complex analysis in engineering mathematics results mainly from the fact that both the real part and the imajinary part of an analytic function satisfy Laplace's equation, the most impontant PDE of physics. It occurs in gravitation, electrostatics, fluid flow, heat Conduction and other application

EX for $u = x^2 - y^2$, check the function for harmonic and then find $f(z)$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

i.e $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore u$ is harmonic

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \partial v = \frac{\partial u}{\partial x}\, dy$$

$$\therefore v = \int \frac{\partial u}{\partial x}\, dy$$

$$\therefore v = \int 2x\, dy$$

$$v = 2xy + F(x)$$

$$\frac{\partial v}{\partial x} = 2y + F'(x) = 2y$$

$$\therefore F'(x) = 0, \quad F(x) = C$$

$$\therefore v = 2xy + C$$

$$f(z) = u + iv = x^2 - y^2 + i\,2xy + iC$$

$$= x^2 + 2ixy - y^2 + iC$$

$$\therefore f(z) = z^2 + K$$