Laurent Series :-

Laurent Series generalize Taylor Series. If, in an application, we want to develope a function $f(z)$ in powers of $z-z_0$ when $f(z)$. ~~is~~ is singular at z_0, this called "Laurent Series".

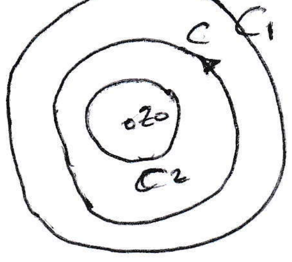
let $f(z)$ be analytic in a domain Containing two Concentric circles C_1 and C_2 with center z_0 and the annulus between them. Then $f(z)$ can be represented by the "Laurent series".

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C (z^*-z_0)^{n-1} f(z^*) dz^*,$$

z^* is denoted ~~by~~ variable of integration



Laurent series

We can write

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\,dz^* \qquad n=0,\pm1,\pm2,\cdots$$

__EX__ Find the Laurent series of $z^{-5}\sin z$ with center 0.

We have $\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$, then

$$z^{-5}\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n-4}}{(2n+1)!}$$

$$=\frac{1}{z^4}-\frac{1}{6z^2}+\frac{1}{20}-\frac{1}{5040}z^2+\cdots\cdots \qquad |z|>0$$

the principal part of the series at 0 is $z^{-4}-\frac{1}{6}z^{-2}$

__EX__ Find the Laurent series of $z^2 e^{\frac{1}{z}}$ with center 0.

We have $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, then

$$z^2 e^{\frac{1}{z}}=\sum_{n=0}^{\infty}\frac{z^{-n+2}}{n!}$$

$$=z^2+\frac{z}{1!}+\frac{1}{2}+\frac{1}{3!\,z}+\cdots$$

EX Develope $\frac{1}{1-z}$ (a) in nonnegative powers of z (b) in negative powers of z.

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{if } |z|<1.$$

(b) $$\frac{1}{1-z} = \frac{-1}{z-1} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} \cdots \qquad |z|>1.$$

EX Find all Laurent Series of $1/(z^3-z^4)$ with center 0.

(a) $$\frac{1}{z^3-z^4} = \frac{1}{z^3(1-z)} = \frac{1}{z^3}\frac{1}{1-z}, \text{ then}$$

$$\frac{1}{z^3}\frac{1}{1-z} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots$$
$$0<|z|<1.$$

(b) $$\frac{1}{z^3-z^4} = \frac{1}{z^3(1-z)} = \frac{-1}{z^3(z-1)} = \frac{-1}{z^3 z(1-z^{-1})}, \text{ then}$$

$$\frac{-1}{z^3 z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} + \cdots$$
$$|z|>1$$

EX Find all Taylor and Laurent series of $f(z) = \frac{-2z+3}{z^2-3z+2}$ with center 0.

By partial fraction we can get

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}.$$

for first term

(a) $$-\frac{1}{z-1} = +\sum_{n=0}^{\infty} z^n \qquad |z|<1$$

(b) $$\frac{-1}{z(1-z^{-1})} = \sum_{n=0}^{\infty} \frac{-1}{z^{n+1}} \qquad |z|>1$$

for second term

(c) $$-\frac{1}{z-2} = \frac{1}{2-z} = \frac{1}{2(1-\frac{1}{2}z)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad |z|<2$$

(d) $$-\frac{1}{z-2} = \frac{1}{2-z} = -\frac{1}{z(1-\frac{2}{z})} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad |z|>2$$

(1) from (a) and (c) for $|z|<1$

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n$$

$$= \frac{3}{2} + \frac{5}{4} z + \frac{9}{8} z^2 + \cdots$$

2 from b and c, for $1 < |z| < 2$,

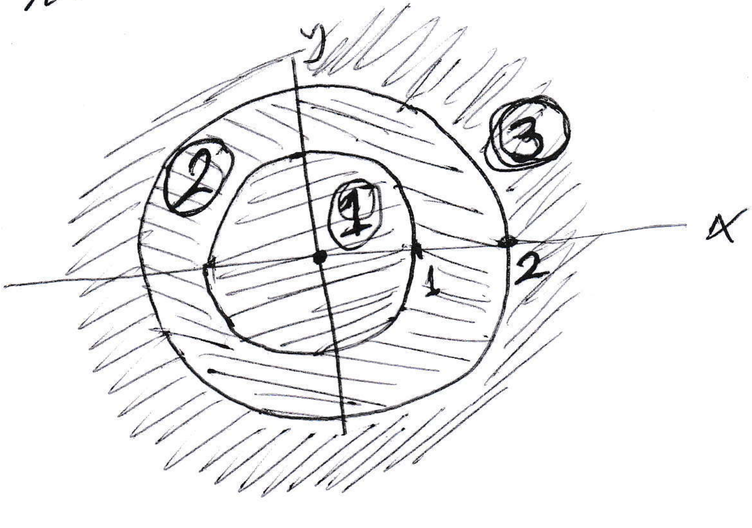
$$f(z) = \sum_{n=0}^{\infty} \frac{-1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

$$= \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots$$

3 from b and d, $|z| > 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{-1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$$= - \sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \cdots$$



Residue theorem :-

We now covered Cauchy method, Taylor series, and Laurent series. Our hardwork will be more in evaluating complex integrals by the residue method.

The purpose of Cauchy's residue integration method is the evaluating of integrals

$$\oint_C f(z)\, dz$$

taken around a simple close path C. The idea is as follows.

If $f(z)$ is analytic every where on C and inside C, such an integral is zero by Cauchy's integral theorem.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

$$\oint_C f(z)\, dz = 2\pi i b_1.$$

The coefficient b_1 is called the "residue" of $f(z)$ at $z = z_0$ and we denoted it by

$$b_1 = \operatorname*{Res}_{z \to z_0} f(z).$$

Ex integrate the function $f(z) = z^{-4}\sin z$ around the unit circle C by means of a Residue.

we have $\sin z = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!}$, then

$$z^{-4}\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-3}}{(2n+1)!}$$

$$= \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{1}{5!} - \frac{z^3}{7!} + \cdots - \cdots$$

which convergence for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of a third order at $z=0$ and the residue $b_1 = -\frac{1}{3!}$

$$\oint_C \frac{\sin z}{z^4}\,dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

Formulas for Residues :-

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can drive formulas for residue once and for all.

ⓐ Simple Poles at z_0.

A first formula for the residue at a simple pole is

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0)\, f(z).$$

A second formula for the residue at a simple pole is

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

EX find the Residue at a simple pole.

$$f(z) = \frac{9z+i}{z^3+z}$$

$z^3+z=0 \Rightarrow z(z^2+1) \Rightarrow z^2+1=(z+i)(z-i)$, then

it has a simple pole at i.

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0) f(z).$$

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \lim_{z\to i} (z-i) \frac{9z+i}{z(z+i)(z-i)} = \left[\frac{9z+i}{z(z+i)}\right]_{z=i}$$

$$= \frac{10i}{-2} = 5i$$

OR $$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

$$p(i) = 9i+i \Rightarrow p(i) = 10i$$

$$q'(z_0) = 3z^2+1 \Rightarrow q'(i) = -3+1 = -2$$

$$\therefore \frac{p(z_0)}{q'(z_0)} = \frac{10i}{-2} = 5i$$

B) Poles of Any Order at z_0.

The residue of $f(z)$ at an mth-order pole at z_0

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}.$$

for a second-order pole $(m=2)$

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \left\{ \left[(z-z_0)^2 f(z) \right]' \right\}.$$

EX. Residue at high order

$f(z) = 50z / (z^3 + 2z^2 - 7z + 4)$ ~~Res~~

$$f(z) = \frac{50\, z}{(z+4)(z-1)^2}$$

it has a pole of second order at $z=1$.

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \left\{ \left[(z-z_0)^2 f(z) \right]' \right\}.$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz} \left[(z-1)^2 f(z) \right]$$

$$= \lim_{z\to 1} \frac{d}{dz} \left(\frac{50z}{z+4} \right) = \frac{200}{5^2} = 8.$$

Several Singularities inside the Contour :-

Residue integration can be extended from the case of a single singularity to the case of several singularities within the contour C. This is the purpose of the residue theorem. the extension is surprisingly simple.

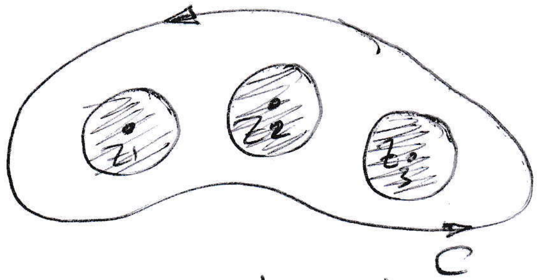
let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_K$ inside C. then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, \cdots, z_K$;

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z).$$

if $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, \cdots, C_K$ and on the entire boundary of D. from Cauchy's integral theorem

$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots \oint_{C_K} f(z)\,dz = 0$$

$$\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z), \quad j=1,\cdots,K,$$



Residue theorem

<u>Ex</u> Evaluate the following integral counterclockwise around any simple closed path.

$$\oint_C \frac{4-3z}{z^2-z}\,dz$$

$$\frac{4-3z}{z(z-1)}\ , \quad z=0\ ,\ z=1$$

the integrand has simple poles at 0 and 1.

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0)\, f(z).$$

$$\operatorname*{Res}_{z=0} \frac{4-3z}{\cancel{z}(z-1)}\cancel{z}\Big|_{z=0} = -4\ , \quad \operatorname*{Res}_{z=1} \frac{4-3z}{z\cancel{(z-1)}}\cancel{(z-1)}\Big|_{z=1} = 1.$$

ⓐ $2\pi i(-4) + 2\pi i(1) = 2\pi i(-4+1) = -6\pi i$ if 0 and 1 inside C

ⓑ $2\pi i(-4) = -8\pi i$ if 0 is inside and 1 outside C

ⓒ $2\pi i(1) = 2\pi i$ if 0 is outside and 1 inside C

ⓓ 0 if 0 and 1 are outside C.

Complex numbers   Homework 1

Q1

if $Z_1 = 8+3i$ and $Z_2 = 9-2i$, then find,

(a) $Z_1 + Z_2$ (b) $Z_1 - Z_2$ (3) $Z_1 Z_2$ (4) $\frac{Z_1}{Z_2}$ (5) $\overline{\frac{Z_1}{Z_2}}$

Q2

let $Z_1 = -2+11i$, $Z_2 = 2-i$, showing the details of your work, find, in the form $x+iy$:

(a) $Z_1 Z_2$ (b) $\overline{Z_1 Z_2}$ (c) $Re(Z_1^2)$, $(Re\, z_1)^2$

d) $Re\left(\frac{1}{Z_2}\right)^2$, $\frac{1}{Re(Z_2^2)}$ (e) $(z_1 - z_2)^2/16$ (f) $\frac{Z_1}{Z_2}$, $\frac{Z_2}{Z_1}$

(9) $Z_1^2 - Z_2^2$ (10) $4(Z_1+Z_2)/(Z_1-Z_2)$.

Q3 sketch the inequality if $Z_1 = 1+i$ and $Z_2 = -2+3i$

Q4 Represent in polar form and graph in Complex plane.

(a) $1+i$ (b) $-4+4i$ (c) $\frac{\sqrt{2}+\frac{i}{3}}{-\sqrt{8}-2\frac{i}{3}}$ (d) $1+\frac{1}{2}\pi i$

Q5 Determin the principle Value of the argument and graph it

(a) $-1+i$ (b) -5, $-5-i$, $-5+i$ (c) $(1+i)^{20}$

Complex numbers   Homework 2

Q1 Find and graph all roots in the complex plane.

ⓐ $\sqrt[3]{1+i}$ ⓑ $\sqrt[3]{216}$ ⓒ $\sqrt[4]{i}$

Q2 Solve and graph the solutions. Show details.

ⓐ $z^2-(6-2i)z+17-6i=0$

ⓑ $z^2+z+1-i=0$

ⓒ $z^2=-12$, ⓓ $z^2-3z+3+i=0$

ⓔ $2z^2-5z+13=0$ ⓕ $z^2+2z+i(2-i)=0$

ⓖ $4z^2+25=0$ ⓗ $z^2-2zi+3=0$

Q3 Find the roots of

ⓐ $-6i$ ⓑ $7+24i$ ⓒ $\dfrac{4}{1-\sqrt{3}i}$

Q4 By using Demoivers formula find the roots of

ⓐ $(1-i)^7$ ⓑ $(\sqrt{3}+i)^{-9}$ ⓒ $(-1+\sqrt{3}i)^{\frac{2}{5}}$

ⓓ -16 ⓔ $(-64i)^{\frac{1}{6}}$

Complex numbers — Homework 3

Q1 Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Q2 Comput the magnitud & phase for $e^{1.4-0.6i}$?

Q3 Find e^z in the form $u + iv$ ~~and~~ for

(A) $3+4i$ (B) $2\pi i(1+i)$ (C) $0.6-1.8i$ (d) $2+3\pi i$

Q4 write in exponential form.

(A) $4+3i$ (B) $1+i$ (C) $\dfrac{1}{1-z}$

Q5 Find the solution of

(A) $e^z = 1$ (B) $e^z = 4+3i$ (C) $e^z = 0$ (d) $e^z = -2$

Q6 Show that

(A) $\cosh z = \cosh x \cos y + i \sinh x \sin y$

(B) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

Q7 Find $\ln z$ when z equals

(A) -11 (B) $4+4i$ (C) $4-4i$ (d) $0.6+0.8i$

Complex numbers Homework 4

Q1 Find the Taylor series of

ⓐ $\dfrac{z+2}{1-z^2}$ ⓑ $\dfrac{1}{2+z^4}$ ⓒ $\dfrac{1}{(z+i)^2}$

Q2 Find the Laurent series of

ⓐ $\dfrac{1}{1-z^2}$, $z_0=0$ ⓑ $\dfrac{z^8}{1-z^4}$, $z_0=0$

ⓒ $\dfrac{1}{z}$, $z_0=1$. ⓓ $\dfrac{1}{z^2(z-i)}$, $z_0=i$

Q3 Find the residue of

ⓐ $\dfrac{8}{1+z^2}$ ⓑ $\dfrac{z^4}{z^2-iz+2}$

Q4 Evaluate

ⓐ $\oint_C \dfrac{dz}{(z^2+1)^3}$, $C: |z-i|=3$

ⓑ $\oint \dfrac{z+1}{z^4-2z^3}\,dz$, $C: |z-1|=2$