Power Series :-

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The "Power Series" is the standard method for solving linear ODEs with variable coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions, as we will see.

The power series of an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \cdots$$

or some text books is

$$\sum_{m=0}^{\infty} C_m (x-a)^m = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \cdots$$

where $C_0, C_1, C_2 \cdots$ are constants, called the "coefficients" of the series, a is a constant, called the "center" of the series and x is a variable. In particular, if $a=0$, we obtain a "power series in power of x" :-

$$\sum_{m=0}^{\infty} C_m x^m = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

Familiar Power Series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We assume a Solution in the form of the power Series;

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots = \sum_{m=0}^{\infty} C_m x^m.$$

we take the differential of the Series y;

$$y' = C_1 + 2C_2 x + 3C_3 x^2 + \cdots = \sum_{m=1}^{\infty} m C_m x^{m-1}$$

$$y'' = 2C_2 + 3 \cdot 2 C_3 x + \cdots = \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2}$$

Ex Solve $y' - y = 0$?

$$y = \sum_{m=0}^{\infty} C_m x^m = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y' = \sum_{m=1}^{\infty} m C_m x^{m-1} = C_1 + 2C_2 x + 3C_3 x^2 + \cdots$$

$$(C_1 + 2C_2 x + 3C_3 x^2 + \cdots) - (C_0 + C_1 x + C_2 x^2 + \cdots) = 0$$

Coefficient of each side :-

$$C_1 - C_0 = 0 \Rightarrow C_1 = C_0 \quad \text{—} (1)$$

$$2C_2 - C_1 = 0 \Rightarrow C_2 = \frac{C_1}{2} = \frac{C_0}{2} \quad \text{—} (2)$$

$$3C_3 - C_2 = 0 \Rightarrow C_3 = \frac{C_2}{3} = \frac{C_0}{3 \cdot 2} = \frac{C_0}{3!} \quad \text{—} (3)$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$= C_0 + C_0 x + \frac{C_0}{2} x^2 + \frac{C_0}{3!} x^3 + \cdots$$

$$\therefore y = C_0 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right)$$

$$\therefore y = C_0 e^x$$

<u>EX</u> Solve $y'' + y = 0$

$$y = \sum_{m=0}^{\infty} C_m x^m = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} = 2C_2 + 3 \cdot 2 C_3 x + 4 \cdot 3 C_4 x^2 + \cdots$$

By substitution two equations

$$(2C_2 + 3 \cdot 2 C_3 x + 4 \cdot 3 C_4 x^2 + \cdots) + (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots) = 0$$

By factors for each side

$$2C_2 + C_0 = 0 \Rightarrow C_2 = -\frac{C_0}{2}$$

$$3 \cdot 2 C_3 + C_1 = 0 \Rightarrow C_3 = -\frac{C_1}{3!}$$

$$4 \cdot 3 C_4 + C_2 = 0 \Rightarrow C_4 = -\frac{C_2}{4 \cdot 3} = \frac{C_0}{4!}$$

$$\therefore y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$\therefore y = C_0 + C_1 x - \frac{C_0}{2} x^2 - \frac{C_1}{3!} x^3 + \frac{C_0}{4!} x^4 + \frac{C_1}{5!} x^5$$

$$\therefore y = C_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) + C_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$\therefore y = C_0 \cos x + C_1 \sin x$$

<u>Ex</u> $y' - y = e^x$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$$

$$y' = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\therefore C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \cdots - [C_0 + C_1 x + C_2 x^2 + C_3 x^3$$
$$+ C_4 x^4 + \cdots] = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

By factors for each side

$$C_1 - C_0 = 1 \Rightarrow C_1 = C_0 + 1$$

$$2C_2 - C_1 = 1 \Rightarrow C_2 = \frac{1}{2} + \frac{C_1}{2} \Rightarrow C_2 = \frac{1}{2} + \frac{C_0}{2} + \frac{1}{2}$$

$$C_2 = 1 + \frac{1}{2} C_0$$

$$3C_3 - C_2 = \frac{1}{2} \Rightarrow C_3 = \frac{C_2}{3} + \frac{1}{3 \cdot 2}$$

$$= C_3 = \frac{1}{3} + \frac{1}{3 \cdot 2} C_0 + \frac{1}{3 \cdot 2}$$

$$= C_3 = \frac{1}{2} + \frac{1}{6} C_0$$

$$4C_4 - C_3 = \frac{1}{6} \Rightarrow C_4 = \frac{1}{24} + \frac{C_3}{4}$$

$$\Rightarrow C_4 = \frac{1}{24} + \frac{1}{8} + \frac{1}{24} C_0$$

$$= \frac{1}{6} + \frac{1}{24} C_0$$

$\therefore y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$

$$y = C_0 + C_0 x + x + x^2 + \frac{1}{2} C_0 x^2 + \frac{1}{2} x^3 + \frac{1}{6} C_0 x^3 + \frac{1}{24} C_0 x^4 + \frac{1}{6} x^4.$$

$$y = C_0 \left(1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \cdots\right) + x + x^2 + \frac{1}{2} x^3 + \frac{1}{6} x^4$$

$$y = C_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$\therefore y = C_0 e^x + x e^x$$

Ex Solve $(x+1)y' - (x+2)y = 0$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$y' = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \cdots$$

$$\therefore (x+1)\left[C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \cdots\right] - (x+2)\left[C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots\right] = 0$$

$$\therefore C_1 x + 2C_2 x^2 + 3C_3 x^3 + 4C_4 x^4 + \cdots + C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \cdots - C_0 x - C_1 x^2 - C_2 x^3 - C_3 x^4 - \cdots - 2C_0 - 2C_1 x - 2C_2 x^2 - 2C_3 x^3 - \cdots = 0$$

$$C_1 - 2C_0 = 0 \Rightarrow C_1 = 2C_0$$

$$C_1 + \cancel{2}C_2 - C_0 - \cancel{2}C_1 = 0 \Rightarrow C_2 = \frac{C_1}{2} + \frac{C_0}{2} \Rightarrow C_2 = \frac{2C_0}{2} + \frac{C_0}{2}$$

$$C_2 = \frac{3C_0}{2}$$

$$\cancel{2C_2} + 3C_3 - C_1 - \cancel{2C_2} = 0 \Rightarrow C_3 = \frac{C_1}{3} \Rightarrow C_3 = \frac{2}{3} C_0$$

$$3C_3 + 4C_4 - C_2 - 2C_3 = 0$$

$$C_4 = -\frac{C_3}{4} + \frac{C_2}{4} \Rightarrow C_4 = -\frac{2}{4\cdot 3}C_0 + \frac{3}{4\cdot 2}C_0$$

$$C_4 = \frac{5}{24}$$

$$\therefore y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$$

$$y = C_0 + 2C_0 x + \frac{3}{2}C_0 x^2 + \frac{2}{3}C_0 x^3 + \frac{5}{24}x^4 + \cdots$$

$$y = C_0(1+x)e^x$$

Legendre's Equation :-

Legendrés differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

The parameter "n" in eq (1) is given real number.

Legendre's equation is one of the most important ODEs in physics. It arises in numerous problems practicularly in boundary value problems for spheres. the equation involves a parameter "n", whose value depends on the physical or engineering problem.

Apply the power Series method.

Substitution :-

$$y = \sum_{m=0}^{\infty} C_m x^m \qquad (2)$$

and its derivatives in to " 1 " and denoting the Constant $n(n+1)$ by K, we obtain.

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} - 2x \sum_{m=1}^{\infty} m C_m x^{m-1} + K \sum_{m=0}^{\infty} C_m x^m = 0 \qquad (3)$$

By writing the first expansion as two seperate series we have.

$$\sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) C_m x^m - 2 \sum_{m=1}^{\infty} m C_m x^m + K \sum_{m=0}^{\infty} C_m x^m = 0 \qquad (4)$$

By subtitution values of m at equation (4)

$$2 \cdot 1 C_2 + 3 \cdot 2 C_3 x + 4 \cdot 3 C_4 x^2 + \cdots + (s+2)(s+1) C_{s+2} x^s + \cdots - 2 \cdot 1 C_2 x^2 - \cdots - s(s-1) C_s x^s - \cdots$$

$$- 2 \cdot 1 C_1 x_1 - 2 \cdot 2 C_2 x^2 - \cdots - 2 \cdot s C_s x^s + \cdots$$

$$+ K C_0 + K C_1 x + \cdots + K C_s x^s = 0$$

$$\sum_{s=0}^{\infty} (s+2)(s+1) C_{s+2} x^s - \sum_{s=2}^{\infty} s(s-1) C_s x^s - \sum_{s=1}^{\infty} 2 s C_s x^s + K \sum_{s=0}^{\infty} C_s x^s = 0$$

$2C_2 + KC_0 = 0$ ------ Coefficient of x^0

$6C_3 - 2.1\,C_1 + KC_1 = 0$ --- Coefficient of x^1

⋮

and in general, when $s = 2, 3,$ -----

$$(s+2)(s+1)\,C_{s+2} + [-s(s-1) - 2s + n(n+1)]\,C_s \quad \text{---} \ (5)$$

The expansion in brackets [-----] can be written

$$(n-s)(n+s+1)$$

we thus obtain from (5)

$$\boxed{C_{s+2} = \frac{-(n-s)(n+s+1)}{(s+1)(s+2)}\,C_s\,,\quad s = 0, 1, \text{----}}$$

This is called a "recursive relation" or "recursion form". It gives each coefficient in term of the second, except for C_0 and C_1. we find successively

$$C_2 = \frac{-n(n+1)}{2!}\,C_0\,,\quad C_3 = -\frac{(n-1)(n+2)}{3!}\,C_1$$

$$C_4 = -\frac{(n-2)(n+3)}{4\cdot3}\,C_2\,,\quad C_5 = -\frac{(n-3)(n+4)}{5\cdot4}$$

$$y(x) = C_0 y_1(x) + C_2 y_2(x)$$

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \cdots$$

$$y_2(x) = x - \frac{(n-1)(n-2)}{3!} x^3 + \frac{(n-3)(n-1)(n-1)}{5!} x^5 - \cdots$$

<u>EX</u> Solve $(1-x^2)y'' - 2xy' + 6y = 0$ by Legendre equation from the first prinesple.

$$n(n+1) = 6 \Rightarrow n^2 + n - 6 = 0 \Rightarrow n = -3,\ n = 2.$$

$$C_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} C_s$$

$$s=0 \Rightarrow C_2 = \frac{-n(n+1)}{2} C_0 \Rightarrow C_2 = -3C_0$$

$$s=1 \Rightarrow C_3 = \frac{-(n-1)(n+2)}{3\cdot 2} C_1 \Rightarrow C_3 = -\frac{2}{3} C_1$$

$$s=2 \Rightarrow C_4 = \frac{-(n-2)(n+3)}{4\cdot 3} C_2 \Rightarrow C_4 = 0$$

$$s=3 \Rightarrow C_5 = \frac{-(n-3)(n+4)}{5\cdot 4} C_3 \Rightarrow C_5 = -\frac{1}{5} C_1$$

$$\therefore y(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \cdots$$

$$\therefore y(x) = C_0 + C_1 x - 3C_0 x^2 - \frac{2}{3} C_1 x^3 - \frac{1}{5} C_1 x^5 + \cdots$$

$$\therefore y(x) = C_0(1 - 3x^2 + 0\cdot x^4 + \cdots) + C_1\left(x - \frac{2}{3}x^3 - \frac{1}{5}C_1 x^5 - \cdots\right)$$

Ex Solve $2(1-x^2)y'' - 6xy' + 6y = 0 \qquad \div 2$

$$(1-x^2)y'' - 3xy' + 3y = 0$$

$$y = \sum_{m=0}^{\infty} C_m x^m, \quad y' = \sum_{m=1}^{\infty} m C_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2}$$

$$\therefore (1-x^2)\sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} - 3x\sum_{m=1}^{\infty} m C_m x^{m-1} + 3\sum_{m=0}^{\infty} C_m x^m = 0$$

$$\sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) C_m x^m - 3\sum_{m=1}^{\infty} m C_m x^m + 3\sum_{m=0}^{\infty} C_m x^m = 0$$

$$2\cdot 1\, C_2 x^0 + 3\cdot 2\, C_3 x + \cdots (s+2)(s+1) C_{s+2} x^s$$

$$- 2\cdot 1\, C_2 x^2 - 3\cdot 2\, C_3 x^3 - \cdots - (s)(s-1) C_s x^s - 3\cdot 1\, C_1 x^1$$

$$- 3\cdot 2\, C_2 x^2 - \cdots - 3\cdot s\, C_s x^s + 3 C_0 x^0 + 3 C_1 x^1 + 3 C_2 x^2$$

$$+ \cdots + 3 C_s x^s = 0$$

$$(s+2)(s+1) C_{s+2} x^s - s(s-1) C_s x^s - 3\cdot s\, C_s x^s + 3 C_s x^s = 0$$

$$C_{s+2} = \frac{s(s-1)+(3s-3)}{(s+1)(s+2)} C_s$$ in general

$$s=0 \Rightarrow c_2 = \frac{-3}{2} c_0 \;, \quad s=1 \Rightarrow c_3 = c_1$$

$$s=2 \Rightarrow c_4 = -\frac{5}{8} c_0 \;, \quad s=3 \Rightarrow c_5 = \frac{3}{5} c_1$$

$$s=4 \Rightarrow c_6 = -\frac{7}{16} c_0 \;, \quad s=5 \Rightarrow c_7 = \frac{16}{35} c_1$$

$$\therefore\; y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$y = c_0 + c_1 x - \frac{3}{2} c_0 x^2 + c_1 x^3 - \frac{5}{8} c_0 x^4 + \frac{3}{5} c_1 x^5$$
$$- \frac{7}{16} c_0 x^6 + \frac{16}{35} c_1 x^7 + \cdots$$

$$\therefore\; y(x) = c_0 \left(1 - \frac{3}{2} x^2 - \frac{5}{8} x^4 - \frac{7}{16} x^6 - \cdots\right)$$
$$+ c_1 \left(x + x^3 + \frac{3}{5} x^5 + \frac{16}{35} x^7 + \cdots\right)$$

Extended power series method: "Frobenius method".

Several second-order ODE's of ~~and~~ considerable practical importance – the famous Bessel equation among them – have coefficients that are not analytic, so that these ODEs can still be solved by series (power series times a logarithm or times a fractional power of x, etc). Indeed, the following theorem permits an extension of the power series method the new method is called the "Frobenius method." Both the methods, that is, the power series method and the Frobenius method.

Any differential equation of the form

$$y'' + \frac{a(x)}{x} y' + \frac{b(x)}{x^2} y = 0 \quad \text{—— (1)}$$

To solve eq(1) is called the "Frobenius method".

$$\therefore\ x^2 y'' + x a(x) y' + b(x) y = 0 \quad \text{—— (2)}$$

we first expand $a(x)$ and $b(x)$ in power series:-

$$a(x) = a_0 + a_1 x + a_2 x^2 \text{----},\quad b(x) = b_0 + b_1 x + b_2 x^2 + \text{---}$$

Then we differential (2) term by term, finding

$$y(x) = x^r \sum_{m=0}^{\infty} C_m x^m = x^r \left[C_0 + C_1 x + C_2 x^2 + \text{-----} \right]$$

$$y'(x) = \sum_{m=0}^{\infty} (m+r) C_m x^{m+r-1} = x^{r-1}[r C_0 + (r+1) C_1 x + \cdots]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) C_m x^{m+r-2}$$

$$= x^{r-2}[r(r-1) C_0 + (r+1) r C_1 x + \cdots]$$

By inserting all these series into (2) we readily obtain:-

$$x^r [r(r-1) C_0 + \cdots] + (a_0 + a_1 x + \cdots) x^r (r C_0 + \cdots) +$$

$$(b_0 + b_1 x + \cdots) x^r (C_0 + C_1 x + \cdots) = 0$$

$$\therefore [r(r-1) + a_0 r + b_0] C_0 = 0$$

$$\therefore r^2 + (a_0 - 1) r + b_0 = 0$$

"Indicial equation"

This important quadratic equation is called the "Indicial equation" of the differential equation (1), there will be three cases depending on the roots of the indicial equation as follows:-

Case 1 :-

Distinct root which do not differ by an integer (1, 2, 3, ----) the solution is

$$y_1(x) = x^{r_1} (C_0 + C_1 x + C_2 x^2 + \cdots)$$

$$y_2(x) = x^{r_2} (C_0 + C_1 x + C_2 x^2 + \cdots).$$

$\underline{\underline{EX}}$ Solve the differential equation :-

$$x^2 y'' + (x^2 + \frac{5}{36}) y = 0$$

$$y(x) = \sum_{m=0}^{\infty} C_m x^{r+m} \quad , \quad y'(x) = (r+m) \sum_{m=0}^{\infty} C_m x^{r+m-1}$$

$$y''(x) = (r+m)(r+m-1) \sum_{m=0}^{\infty} C_m x^{r+m-2}$$

$$\therefore \quad \sum_{m=0}^{\infty} (m+r)(m+r-1) C_m x^{m+r} + \sum_{m=0}^{\infty} C_m x^{m+r+2}$$

$$+ \frac{5}{36} \sum_{m=0}^{\infty} C_m x^{m+r} = 0$$

By equating the sum of the coefficient of x^r to Zero we obtain the "indicial equation".

$$x^r [r(r-1) C_0 + \frac{5}{36} C_0] = 0$$

$$\therefore \quad r(r-1) + \frac{5}{36} = 0 \Rightarrow r^2 - r + \frac{5}{36} = 0$$

$$\therefore \quad r_1 = \frac{5}{6} \quad , \quad r_2 = \frac{1}{6}$$

By equating the sum of the coefficient x^{r+s} to zero, we find

$$[(r+1)r + \frac{5}{36}] C_1 = 0 \Rightarrow C_1 = 0 \, , \, C_3 = 0 \, , \, C_5 = 0 \, , \, \cdots$$

$$\therefore (s+r)(s+r-1)C_s + C_{s-2} + \frac{5}{36} C_s = 0$$

$$\text{at } r_1 = \frac{5}{6} \Rightarrow s(s+\frac{2}{3})C_s + C_{s-2} = 0$$

$$\therefore C_s = \frac{-1}{s(s+\frac{2}{3})} C_{s-2}$$

$$\text{at } s = 2 \Rightarrow C_2 = \frac{-1}{s(s+\frac{2}{3})} C_{s-2} = -\frac{3}{16} C_0$$

$$s = 4 \Rightarrow C_4 = \frac{9}{896} C_0$$

$$\therefore y_1(x) = C_0 x^{5/6} (1 - \frac{3}{36} x^2 + \frac{9}{896} x^4 - \cdots + \cdots)$$

at $r = \frac{1}{6}$ and sub.

$$[(r_2+1)r_2 + \frac{5}{36}] C_1 = 0 \quad , \quad C_1 = 0, C_3 = 0, C_5 = 0, \cdots$$

$$s(s-\frac{2}{3})C_s + C_{s-2} = 0$$

$$C_2 = \frac{-3}{8} C_0 , \quad C_4 = \frac{9}{320} C_0, \cdots$$

$$\therefore y_2 = C_0 x^{1/6} (1 - \frac{3}{8} x^2 + \frac{9}{320} x^4 + \cdots + \cdots).$$