Lecture ( )

Frequency Response

1) Introduction

In our sinusoidal circuit analysis, we have learned how to find voltages and currents in a circuit with a constant frequency source. If we let the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's ***frequency response***. The frequency response may be regarded as a complete description of the sinusoidal steady state behavior of a circuit as a function of frequency.

> The frequency response of a circuit is the variation in its behavior with change in signal frequency.

The sinusoidal steady-state frequency responses of circuits are of significance in many applications, especially in communications and control systems. A specific application is in electric filters that block out or eliminate signals with unwanted frequencies and pass signals of the desired frequencies. Filters are used in radio, TV, and telephone systems to separate one broadcast frequency from another.

2) Transfer Function

The transfer function $\boldsymbol{H(\omega)}$ (also called the ***network function***) is a useful analytical tool for finding the frequency response of a circuit. In fact, the frequency response of a circuit is the plot of the circuit's transfer function $\boldsymbol{H(\omega)}$ versus $\boldsymbol{\omega}$, with $\boldsymbol{\omega}$ varying from $\boldsymbol{\omega = 0}$ to $\boldsymbol{\omega = \infty}$.

A transfer function is the frequency-dependent ratio of a forced function to a forcing function (or of an output to an input). The idea of a transfer function was implicit when we used the concepts of impedance and admittance to relate voltage and current. In general, a linear network can be represented by the block diagram shown in Fig.2.1.

> The transfer function $\mathbf{H}(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\mathbf{Y}(\omega)$ (an element voltage or current) to a phasor input $\mathbf{X}(\omega)$ (source voltage or current).

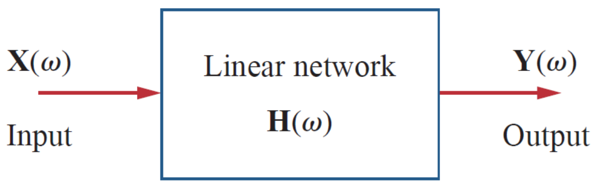


Fig.2.1. A block diagram representation of a linear network.

Thus,

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}} \qquad \textbf{(2.1)}$$

assuming zero initial conditions. Since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} \quad \textbf{(2.2a)}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} \quad \textbf{(2.2b)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)} \quad \textbf{(2.2c)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)} \quad \textbf{(2.2d)}$$

Being a complex quantity, $H(\omega)$ has a magnitude $\boldsymbol{H(\omega)}$ and a phase $\boldsymbol{\varphi}$; that is, $\boldsymbol{H(\omega) = H(\omega)\underline{/\varphi}}$.

To obtain the transfer function using **Eq. (2.2)**, we first obtain the frequency-domain equivalent of the circuit by replacing resistors, inductors, and capacitors with their impedances $\boldsymbol{R}$, $\boldsymbol{j\omega L}$, and $\boldsymbol{1/j\omega C}$. We then use any circuit technique (s) to obtain the appropriate quantity in **Eq. (2.2)**. We can obtain the frequency response of the circuit by plotting the magnitude and phase of the transfer function as the frequency varies. The transfer function $\boldsymbol{H(\omega)}$ can be expressed in terms of its numerator polynomial $\boldsymbol{N(\omega)}$ and denominator polynomial $\boldsymbol{D(\omega)}$ as

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}} \quad \textbf{(2.3)}$$

The roots of $\boldsymbol{N(\omega) = 0}$ are called the *zeros* of $\boldsymbol{H(\omega)}$ and are usually represented as $\boldsymbol{j\omega = z_1 . z_2 .}$ .... Similarly, the roots of $\boldsymbol{D(\omega) = 0}$ are the *poles* of $\boldsymbol{H(\omega)}$ and are represented as $\boldsymbol{j\omega = p_1 . p_2 .} \dots$

A zero, as a root of the numerator polynomial, is a value that results in a zero value of the function. A pole, as a root of the denominator polynomial, is a value for which the function is infinite.

To avoid complex algebra, it is expedient to replace $\boldsymbol{j\omega}$ temporarily with $\boldsymbol{s}$ when working with $\boldsymbol{H(\omega)}$ and replace $\boldsymbol{s}$ with $\boldsymbol{j\omega}$ at the end.

Example 1: For the RC circuit in Fig. 1.1(a), obtain the transfer function $\boldsymbol{V_o/V_s}$ and its frequency response. Let $v_s = V_m cos\omega t$.

Solution:

The frequency-domain equivalent of the circuit is in Fig. 1.1(b). By voltage division, the transfer function is given by

$$H(\omega) = \frac{V_o}{V_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

We obtain the magnitude and phase of $H(\omega)$ as

$$H = \frac{1}{\sqrt{1+(\omega/\omega_0)^2}}, \quad \varphi = -\tan^{-1}\frac{\omega}{\omega_0}$$

where $\boldsymbol{\omega_0 = 1/RC}$. To plot $\boldsymbol{H}$ and φ for $\boldsymbol{0 < \omega < \infty}$**,** we obtain their values at some critical points and then sketch.

At $\boldsymbol{\omega = 0}$. $\boldsymbol{H = 1}$ and $\boldsymbol{\varphi = 0}$. At $\boldsymbol{\omega = \infty}$. $\boldsymbol{H = 0}$ and $\boldsymbol{\varphi = -90^o}$. Also, at $\boldsymbol{\omega = \omega_0}$. $\boldsymbol{H = 1/\sqrt{2}}$ and $\boldsymbol{\varphi = -45^o}$. With these and a few more points as shown in **Table 1.1**, we find that the frequency response is as shown in Fig. 1.2.

TABLE 1.1

.For Example 1

ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	$-84°$
1	0.71	$-45°$	20	0.05	$-87°$
2	0.45	$-63°$	100	0.01	$-89°$
3	0.32	$-72°$	∞	0	$-90°$

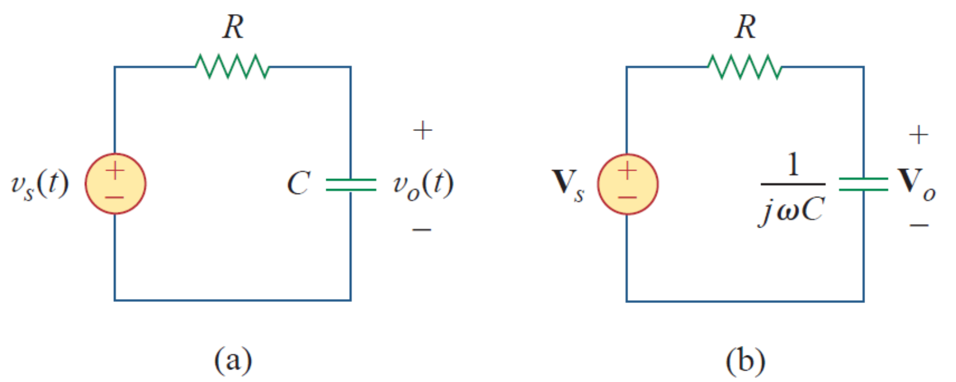


Fig. 1.1 (a) time-domain RC circuit, (b) frequency-domain RC circuit.

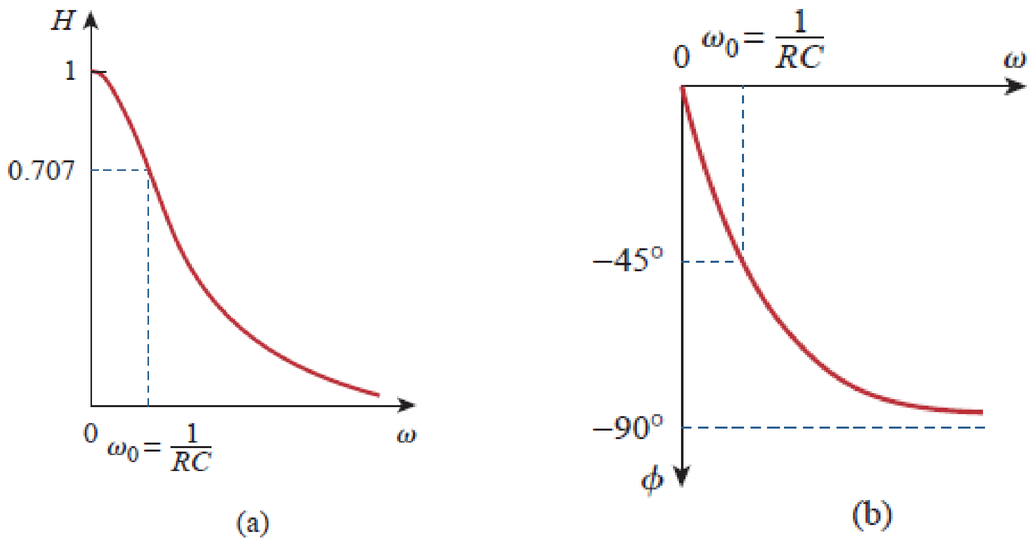
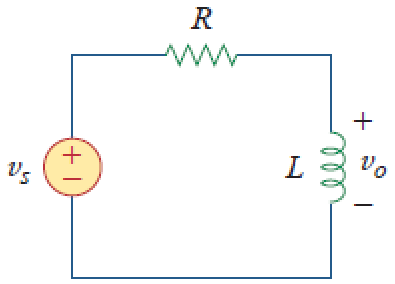


Fig. 1.2 Frequency response of the *RC* circuit: (a) amplitude response, (b) phase response.

H.W.1: Obtain the transfer function V_o/V_s of the *RL* circuit in Fig. 1.1, assuming $v_s = V_m cos\omega t$.. Sketch its frequency response.



Answer: $\boldsymbol{j\omega L/(R + j\omega L)}$**, see** Fig. 1.2 **for the response.**

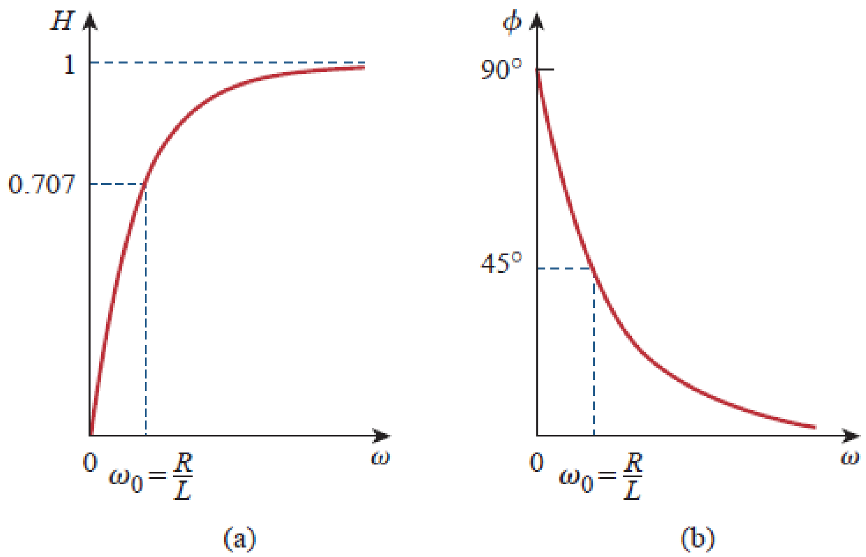


Fig. 1.2 Frequency response of the *RL* circuit in Fig. 1.1.

Example 2: For the circuit in Fig. 2.1, calculate the gain $\boldsymbol{I_o(\omega)/I_s(\omega)}$ and its poles and zeros.

Solution:

By current division,

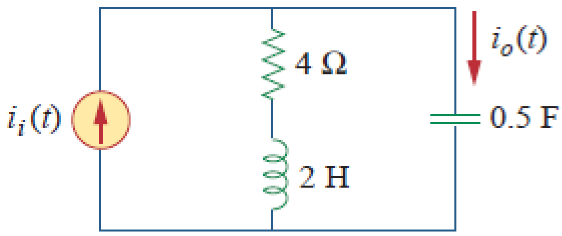


Fig. 2 · 1

$$I_o(\omega) = \frac{4 + j2\omega}{4 + j2\omega + 1/j0.5\omega} I_i(\omega)$$

or

$$\frac{I_o(\omega)}{I_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s+2)}{s^2 + 2s + 1}, \quad s = j\omega$$

The zeros are at

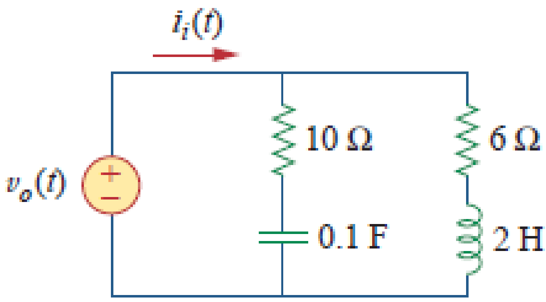
$$s(s+2) = 0 \;\Rightarrow\; z_1 = 0, z_2 = -2$$

The poles are at

$$s^2 + 2s + 1 = (s+1)^2 = 0$$

Thus, there is a repeated pole (or double pole) at $p = -1$.

H.W.2: Find the transfer function $V_o(\omega)/I_i(\omega)$ for the circuit shown below. Obtain its zeros and poles.



Answer: $\boldsymbol{\frac{10(s+1)(s+3)}{s^2+8s+5}, s = j\omega}$**; zeros:** $\boldsymbol{-1, -3}$**; poles:** $\boldsymbol{-0.683, -7.317}$

3) Decibel Scale

It is not always easy to get a quick plot of the magnitude and phase of the transfer function as did above. A more systematic way of obtaining the frequency response is to use **Bode plots**. Before begin to construct Bode plots, it should take care of two important issues: the use of logarithms and decibels in expressing gain.

Since Bode plots are based on logarithms, it is important to keep the following properties of logarithms in mind:

1. $\log P_1P_2 = \log P_1 + \log P_2$
2. $\log P_1/P_2 = \log P_1 - \log P_2$
3. $\log P^n = n \log P$
4. $\log 1 = 0$

In communications systems, gain is measured in ***bels***. Historically, the ***bel*** is used to measure the ratio of two levels of power or power gain $\boldsymbol{G}$; that is,

$$G = \text{Number of bels} = \log_{10}\frac{P_2}{P_1} \qquad \ldots(3.1)$$

The ***decibel*** $(\boldsymbol{dB})$ provides us with a unit of less magnitude. It is 1/l0th of a ***bel*** & given by

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1} \qquad \ldots(3.2)$$

When $P_1 = P_2$, there is no change in power and the gain is $0dB$. If $P_2 = 2P_1$, the gain is,

$$G_{dB} = 10\log_{10}2 = 3dB \qquad \ldots(3.3)$$

and when $P_2 = 0.5P_1$, the gain is

$$G_{dB} = 10\log_{10}0\cdot 5 = -3dB \qquad \ldots(3.4)$$

Equations (3.3) and (3.4) show another reason why logarithms are greatly used: The logarithm of the reciprocal of a quantity is simply negative the logarithm of that quantity. Alternatively, the gain G can be expressed in terms of voltage or current ratio. To do so, consider the network shown in Fig. 3.1. If P_1 is the input power, P_2 is the output (load) power, R_1 is the input resistance, and R_2 is the load resistance, then $P_1 = 0.5V_1^2/R_1$ and $P_2 = 0.5V_2^2/R_2$, and Eq. (3.2) becomes,

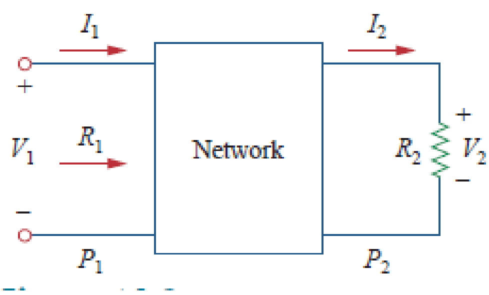


Fig. 3.1 Voltage-current relationships for a four terminal network.

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\frac{V_2^2/R_2}{V_1^2/R_1} = 10\log_{10}(\frac{V_2}{V_1})^2 + 10\log_{10}\frac{R_1}{R_2} \qquad \ldots(3.5)$$

$$G_{dB} = 20\log_{10}\frac{V_2}{V_1} - 10\log_{10}\frac{R_1}{R_1} \qquad \ldots(3.6)$$

For the case when $R_2 = R_1$, a condition that is often assumed when comparing voltage levels, Eq. (3.6) becomes

$$G_{dB} = 20\log_{10}\frac{V_2}{V_1} \qquad \ldots(3.7)$$

Instead, if $P_1 = I_1^2 R_1$ and $P_2 = I_2^2 R_2$, for $R_1 = R_2$, obtain

$$G_{dB} = 20\log_{10}\frac{I_2}{I_1} \qquad (3.8)$$

Three things are important to note from Eqs. (3.2), (3.7), and (3.8):

1. That $10\log_{10}$ is used for power, while $20\log_{10}$ is used for voltage or current, because of the square relationship between them
$(P = V^2/R = I^2R)$.

2. That the dB value is a logarithmic measurement ofthe *ratio* of one variable to another *of the same type*. Therefore, it applies in expressing the transfer function H in Eqs. (2.2a) and (2.2b), which are dimensionless quantities, but not in expressing H in Eqs. (2.2c) and (2.2d).

3. It is important to note that, only use voltage and current magnitudes in Eqs. (3.7) and (3.8). Negative signs and angles will be handled independently.

4) Bode Plots

Obtaining the frequency response from the transfer function is an uphill task. The frequency range required in frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis. Also, there is a more systematic way of locating the important features of the magnitude and phase plots of the transfer function. For these reasons, it has become standard practice to plot the transfer function on a pair of semilogarithmic plots: the magnitude in decibels is plotted against the logarithm of the frequency; on a separate plot, the phase in degrees is plotted against the logarithm of the frequency. Such semilogarithmic plots of the transfer function—known as ***Bode plots***—have become the industry standard.

Bode plots are semilog plots of the magnitude (in decibels) and phase (in degrees) of a transfer function versus frequency.

Bode plots contain the same information as the nonlogarithmic plots discussed in the previous section.

The transfer function can be written as

$$H = H\underline{/\varphi} = He^{j\varphi} \qquad \ldots(4.1)$$

Taking the natural logarithm of both sides,

$$ln\ H = \text{In}\ H + lne^{j\varphi} = lnH + j\varphi \qquad \ldots(4.2)$$

Thus, the real part of lnH is a function of the magnitude while the imaginary part is the phase. In a Bode magnitude plot, the gain

$$H_{dB} = 20\log_{10} H \qquad \ldots(4.3)$$

is plotted in decibels (dB) versus frequency.

In a Bode phase plot, φ is plotted in degrees versus frequency. Both magnitude and phase plots are made on semilog graph paper.

A transfer function in the form of Eq. (2.3) may be written in terms of factors that have real and imaginary parts. One such representation might be,

$$H(\omega) = \frac{K(j\omega)^{\pm 1}(1+j\omega/z_1)[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2]}{(1+j\omega/p_1)[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]} \qquad \ldots(4.4)$$

which is obtained by dividing out the poles and zeros in $H(\omega)$. The representation of $H(\omega)$ as in Eq. (4.4) is called the ***standard form***. $H(\omega)$ may include up to seven types of different factors that can appear in various combinations in a transfer function. These are:

1. A gain K.

2. A pole $(j\omega)^{-1}$ or zero $(j\omega)$ at the origin.

3. A simple pole $1/(1 + j\omega/p_1)$ or zero $(1 + j\omega/z_1)$

4. A quadratic pole $1/[1 + J2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$ or zero $[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]$

In constructing a Bode plot, we plot each factor separately and then add them graphically. The factors can be considered one at a time and then combined additively because of the logarithms involved. It is this mathematical convenience of the logarithm that makes Bode plots a powerful engineering tool. We will now make straight-line plots of the factors listed above. We shall find that these straight-line plots known as Bode plots approximate the actual plots to a reasonable degree of accuracy.

Notes:

1) The origin is where $\omega = 1$ or $\log\omega = 0$ and the gain is zero.

2) A decade is an interval between two frequencies with a ratio of 10; e.g., between ω_0 and $10\omega_0$, or between 10 and 100 Hz. Thus, 20 dB/decade means that the magnitude changes 20 dB whenever the frequency changes tenfold or one decade.

Constant term: For the gain K, the magnitude is $20\log_{10}K$ and the phase is 0^o; both are constant with frequency. Thus, the magnitude and phase plots of the gain are shown in Fig. 4.1. If K is negative, the magnitude remains $20\log_{10}|K|$ but the phase is $\pm180^o$.

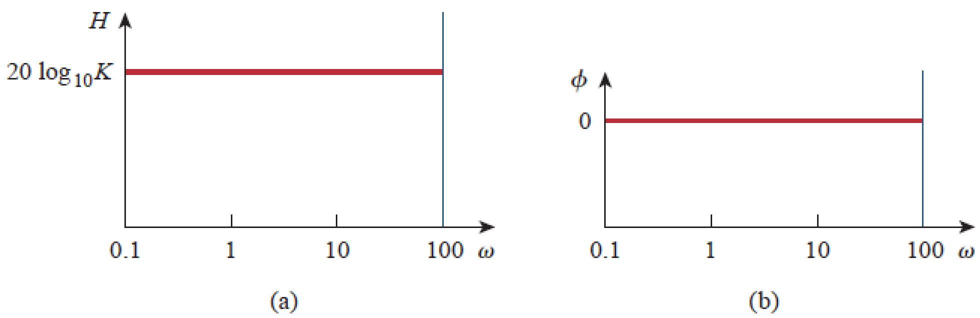


Fig. 4.1. Bode plots for gain K: (a) magnitude plot, (b) phase plot.

Pole/zero at the origin: For the zero $(j\omega)$ at the origin, the magnitude is $20\log_{10}\omega$ and the phase is is 90^0. These are plotted in Fig. 4.2, where we notice that the slope of the magnitude plot is 20 dB/decade, while the phase is constant with frequency. The Bode plots for the pole $(j\omega)^{-1}$ are similar except that the slope of the magnitude plot is $-20dB/$decade while the phase is -90^o. In general, for $(j\omega)^N$, where N is an integer, the magnitude plot will have a slope of $20NdB/$decade, while the phase is $90N$ degrees.

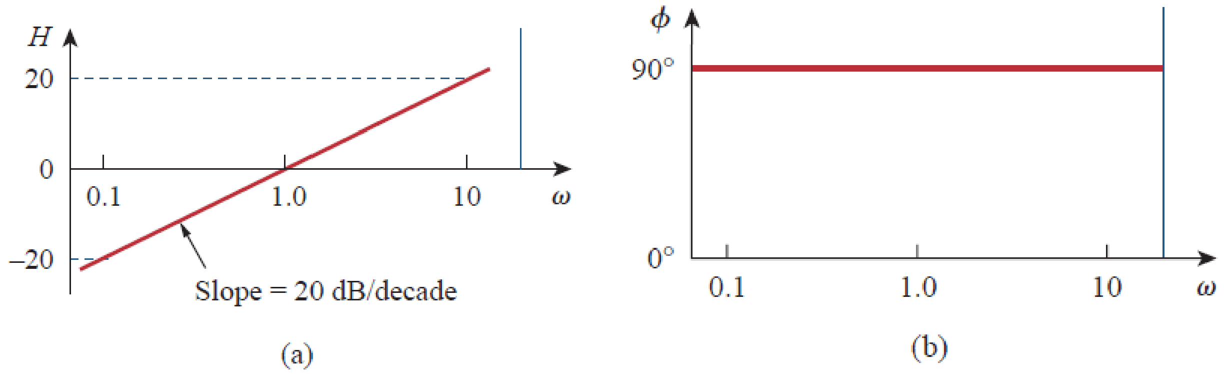


Fig. 4.2. Bode plots for a zero $(j\omega)$ at the origin: (a) magnitude plot, (b) phase plot.

Simple *pole/zero*: For the simple zero $(1+j\omega/z_1)$, the magnitude is $20\log_{10}|1+j\omega/z_1|$ and the phase is $\tan^{-1}\omega/z_1$. We notice that ,

$$H_{dB} = 20\log_{10}|1+\frac{j\omega}{z_1}| \Rightarrow 20\log_{10}1 = 0 \quad \text{as } \omega \to 0 \qquad \ldots(4.5)$$

$$H_{dB} = 20\log_{10}|1+\frac{j\omega}{z_1}| \Rightarrow 20\log_{10}\frac{\omega}{z_1} \quad \text{as } \omega \to \infty \qquad \ldots(4.6)$$

showing that we can approximate the magnitude as zero (a straight line with zero slope) for small values of ω and by a straight line with slope 20 $dB/$decade for large values of ω. The frequency $\omega = z_1$ where the two asymptotic lines meet is called the *corner frequency* or *break frequency*. Thus the approximate magnitude plot is shown in Fig. 4.3(a), where the actual plot is also shown. Notice that the approximate plot is close to the actual plot except at the break frequency, where $\omega = z_1$ and the deviation is $20\log_{10}|(1+j1)| = 20\log_{10}\sqrt{2} \cong 3dB$.

The phase $\tan^{-1}(\omega/z_1)$ can be expressed as

$$\varphi = \tan^{-1}(\frac{\omega}{z_1}) = \begin{cases} 0 & \omega = 0 \\ 45^o & \omega = z_1 \\ 90^o & \omega \to \infty \end{cases} \qquad \ldots(4.7)$$

As a straight-line approximation, let $\varphi \cong 0$ for $\omega \le z_1/10$, $\varphi \cong 45^o$ for $\omega = z_1$, and $\varphi = 90^o$ for $\omega \ge 10z_1$. As shown in Fig. 4.3 (b) along with the actual plot, the straight-line plot has a slope of 45^o per decade.

The Bode plots for the pole $1/(1 + j\omega/p_1)$ are similar to those in Fig. 4.3 except that the corner frequency is at $\omega = p_1$, the magnitude has a slope of $-20\ dB/$decade, and the phase has a slope $of - 45^o$ per decade.

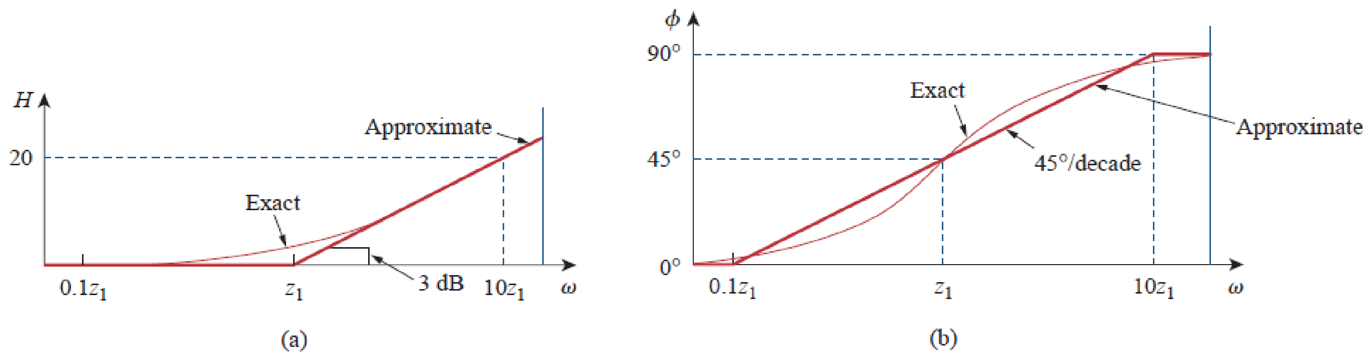


Fig. 4.3. Bode plots for a zero $(1 + \frac{j\omega}{z_1})$: (a) magnitude plot, (b) phase plot.

Quadratic $pole/zero$: The magnitude of the quadratic pole $1/[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$ is $-20\log_{10}|1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2|$ and the phase is $-\tan^{-1}(2\zeta_2\omega/\omega_n)/(1 - \omega^2/\omega_n^2)$. But

$$H_{dB} = -20\log_{10}|1 + \frac{j2\zeta_2\omega}{\omega_n} + (\frac{j\omega}{\omega_n})^2| \Rightarrow 0 \quad \text{as } \omega \to 0 \qquad \ldots(4.8)$$

and

$$H_{dB} = -20\log_{10}|1 + \frac{j2\zeta_2\omega}{\omega_n} + (\frac{j\omega}{\omega_n})^2| \ \Rightarrow\ -40\log_{10}\frac{\omega}{\omega_n} \quad \text{as } \omega \to \infty \qquad \ldots(4.9)$$

Thus, the amplitude plot consists of two straight asymptotic lines: one with zero slope for $\omega < \omega_n$ and the other with slope $-40dB/$ decade for $\omega > \omega_n$, with ω_n as the corner frequency. Fig. 4.4(a) shows the approximate and actual amplitude plots. Note that the actual plot depends on the damping factor ζ_2 as well as the corner frequency ω_n. The significant peaking in the neighborhood of the corner frequency should be added to the straight-line approximation if a high level of accuracy is desired. However, we will use the straight-line approximation for the sake of simplicity.

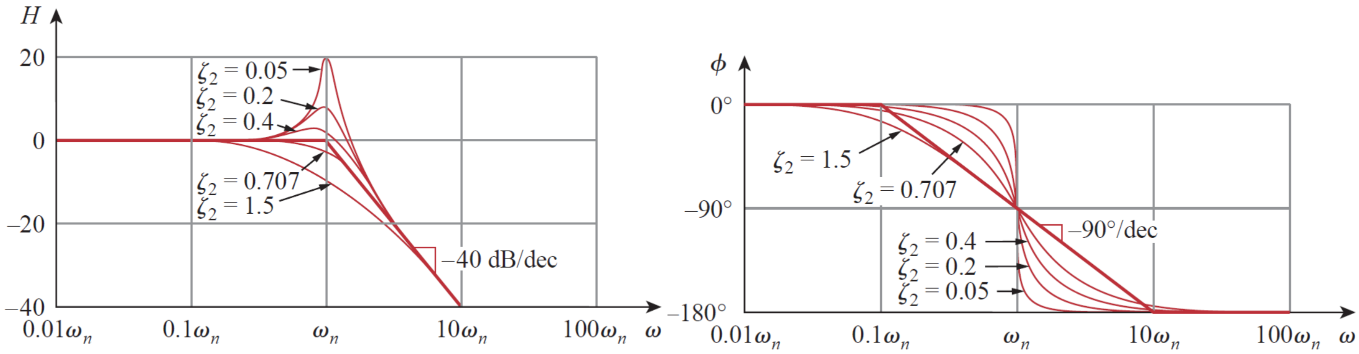


Fig. 4.4. Bode plots of quadratic pole $[1 + j2\zeta\omega/\omega_n - \omega^2/\omega_n^2]^{-1}$: (a) magnitude plot, (b) phase plot.

The phase can be expressed as

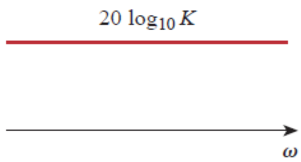
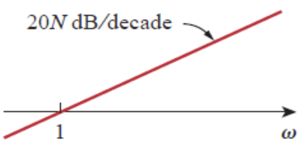
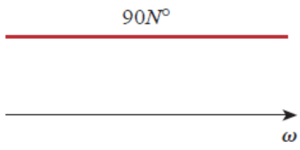
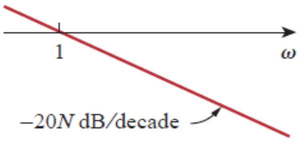
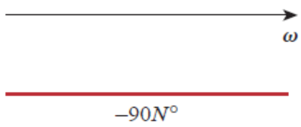
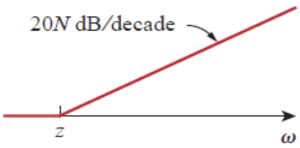
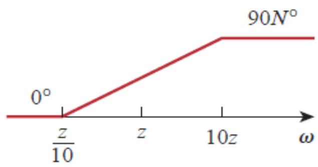
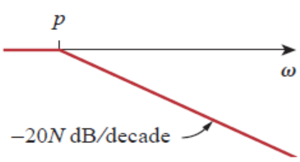
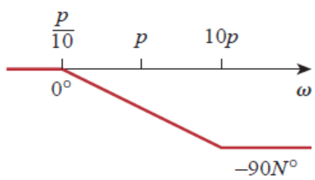
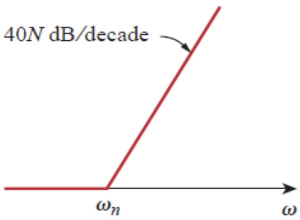
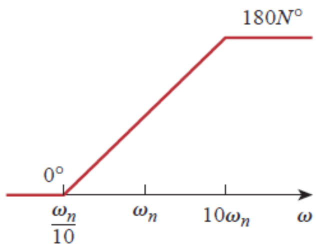
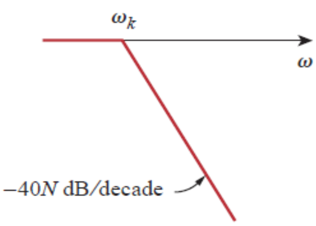
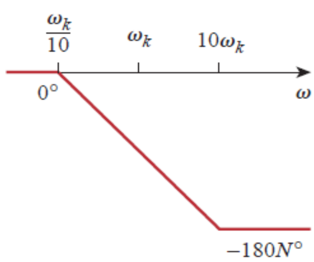
$$\varphi = -\tan^{-1}\frac{2\zeta_2\omega/\omega_n}{1-\omega^2/\omega_n^2} = \begin{cases} 0. & \omega = 0 \\ -90^o. & \omega = \omega_n \\ -180^o. & \omega \to \infty \end{cases} \qquad \dots(4.10)$$

The phase plot is a straight line with a slope $of - 90^o$ per decade starting at $\omega_n/10$ and ending at $10\omega_n$, as shown in Fig. 4.4. (b). We see again that the difference between the actual plot and the straight-line plot is due to the damping factor. Notice that the straight-line approximations for both magnitude and phase plots for the quadratic pole are the same as those for a double pole, i.e. $(1 + j\omega/\omega_n)^{-2}$. We should expect this because the double pole $(1 + j\omega/\omega_n)^{-2}$ equals the quadratic pole $1/[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$ when $\zeta_2 = 1$. Thus, the quadratic pole can be treated as a double pole as far as straight-line approximation is concerned. For the quadratic zero $[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]$, the plots in Fig. 4.4. are inverted because the magnitude plot has a slope of 40 $dB/$decade while the phase plot has a slope of 90^o per decade.

Table 4.1 presents a summary of Bode plots for the seven factors. Of course, not every transfer function has all seven factors. To sketch the Bode plots for a function $H(\omega)$ in the form of Eq. (4.4), for example, we first record the corner frequencies on the semilog graph paper, sketch the factors one at a time as discussed above, and then combine

TABLE 4.1

Summary of Bode straight-line magnitude and phase plots.

Factor	Magnitude	Phase
K		
$(j\omega)^N$		
$\dfrac{1}{(j\omega)^N}$		
$\left(1+\dfrac{j\omega}{z}\right)^N$		
$\dfrac{1}{(1+j\omega/p)^N}$		
$\left[1+\dfrac{2j\omega\zeta}{\omega_n}+\left(\dfrac{j\omega}{\omega_n}\right)^2\right]^N$		
$\dfrac{1}{[1+2j\omega\zeta/\omega_k+(j\omega/\omega_k)^2]^N}$		

additively the graphs of the factors. The combined graph is often drawn from left to right, changing slopes appropriately each time a corner frequency is encountered. The following examples illustrate this procedure.

Example 3: Construct the Bode plots for the transfer function

$$H(\omega) = \frac{200j\omega}{(j\omega + 2)(j\omega + 10)}$$

Solution:

We first put $H(\omega)$ in the standard form by dividing out the poles and zeros. Thus,

$$H(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)} = \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|}\angle 90^o - \tan^{-1}\omega/2 - \tan^{-1}\omega/10$$

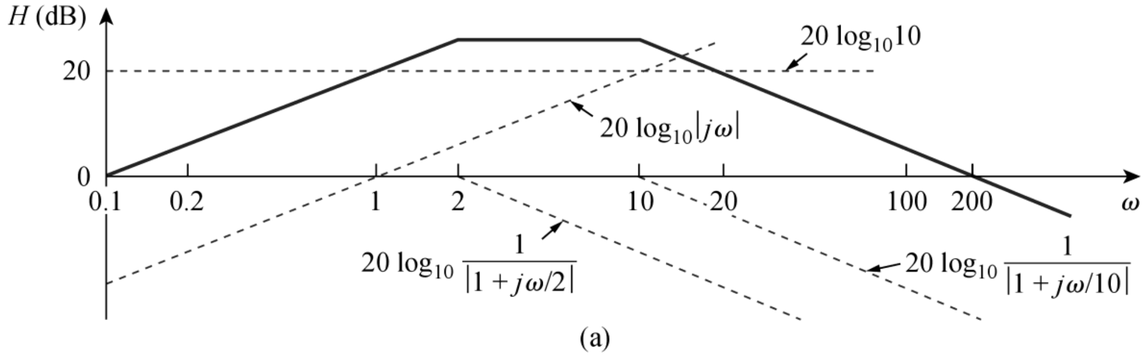
Hence, the magnitude and phase are

$$H_{dB} = 20\log_{10}10 + 20\log_{10}|j\omega| - 20\log_{10}|1 + \frac{j\omega}{2}|$$

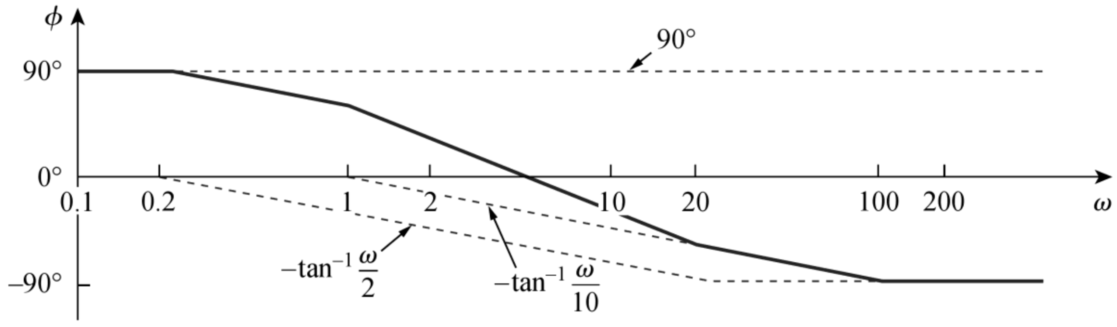
$$-20\log_{10}|1 + \frac{j\omega}{10}|$$

$$\varphi = 90^o - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$

We notice that there are two corner frequencies at $\omega = 2.10$. For both the magnitude and phase plots, we sketch each term as shown by the dotted lines in Fig. below. We add them up graphically to obtain the overall plots shown by the solid curves.



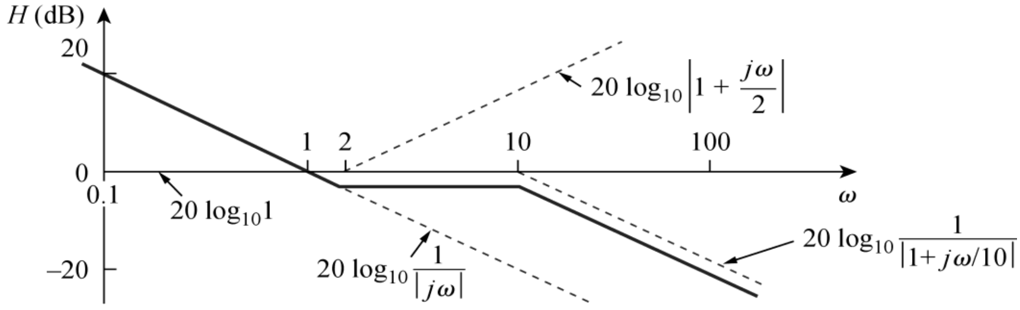
(a) magnitude plot,



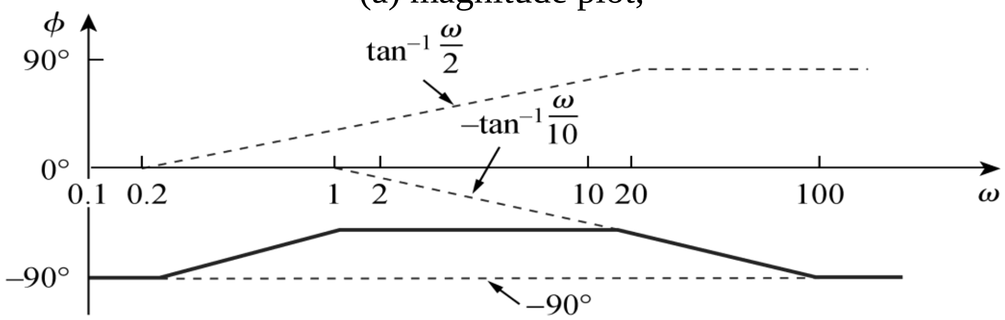
(b) phase plot.

H.W.3: Draw the Bode plots for the transfer function $H(\omega)=\frac{5(j\omega+2)}{j\omega(j\omega+10)}$

Answer:



(a) magnitude plot,



(b) phase plot.

Example 4: Obtain the Bode plots for $H(\omega)=\frac{j\omega+10}{j\omega(j\omega+5)^2}$

Solution:

Putting $H(\omega)$ in the standard form, we get

$$H(\omega)=\frac{0.4(1+j\omega/10)}{j\omega(1+j\omega/5)^2}$$

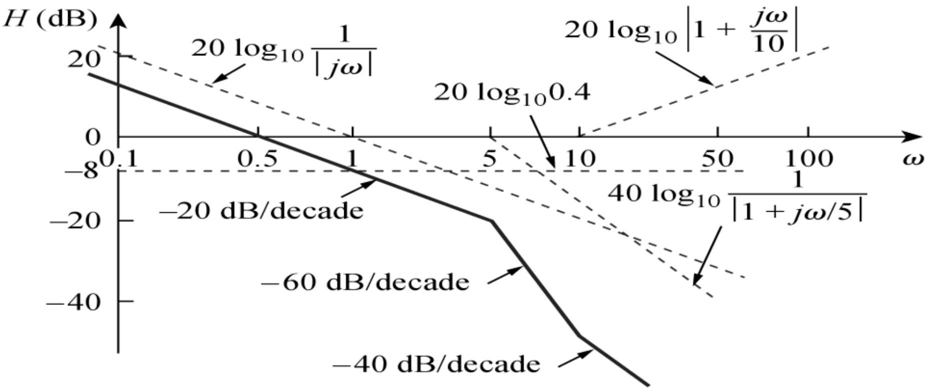
From this, we obtain the magnitude and phase as

$$H_{dB}=20\log_{10}0.4+20\log_{10}|1+\frac{j\omega}{10}|-20\log_{10}|j\omega|$$

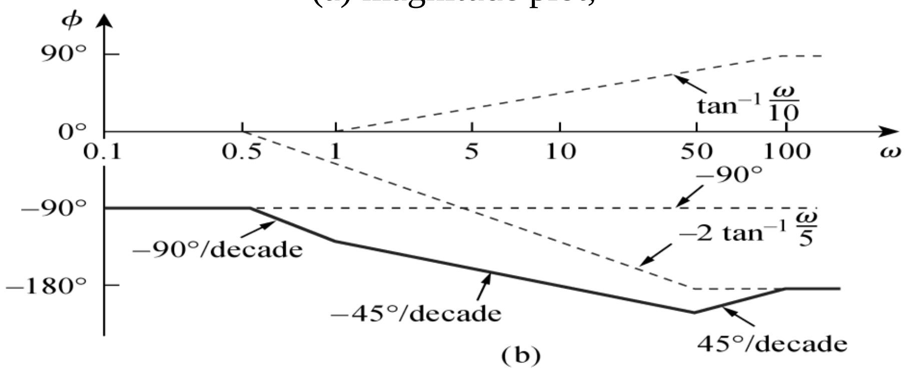
$$-40\log_{10}|1+\frac{j\omega}{5}|$$

$$\varphi=0^o+\tan^{-1}\frac{\omega}{10}-90^0-2\tan^{-1}\frac{\omega}{5}$$

There are two comer frequencies at $\omega=5.10 rad/s$. For the pole with corner frequency at $\omega=5$, the slope ofthe magnitude plot is–40 $dB/$decade and that of the phase plot is -90^0 per decade due to the power of 2. The magnitude and the phase plots for the individual terms (in dotted lines) and the entire $H(j\omega)$ (in solid lines) are in Fig. below.



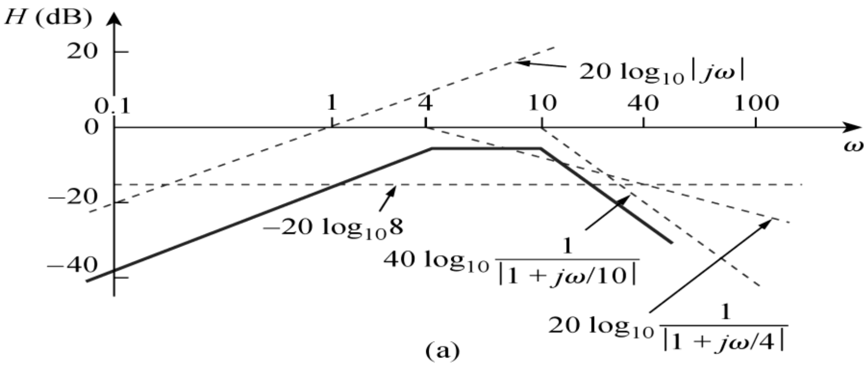
(a) magnitude plot,



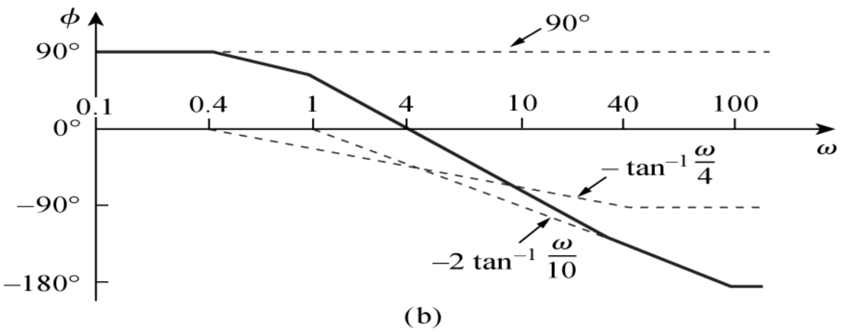
(b) phase plot.

H.W.4: Sketch the Bode plots for $H(\omega) = \frac{50j\omega}{(j\omega+4)(j\omega+10)^2}$

Answer:



(a) magnitude plot,



(b) phase plot.

Example 5: Draw the Bode plots for $H(s) = \frac{s+1}{s^2+12s+100}$

Solution:

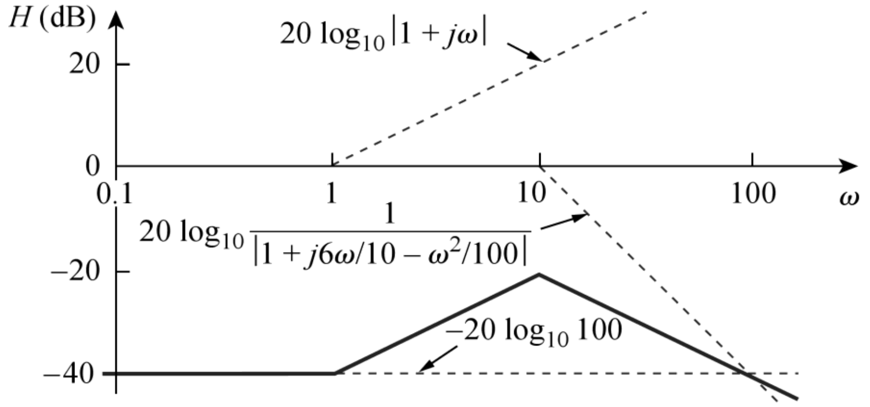
We express $H(s)$ as

$$H(\omega) = \frac{1/100(1+j\omega)}{1 + j\omega 1.2/10 + (j\omega/10)^2}$$

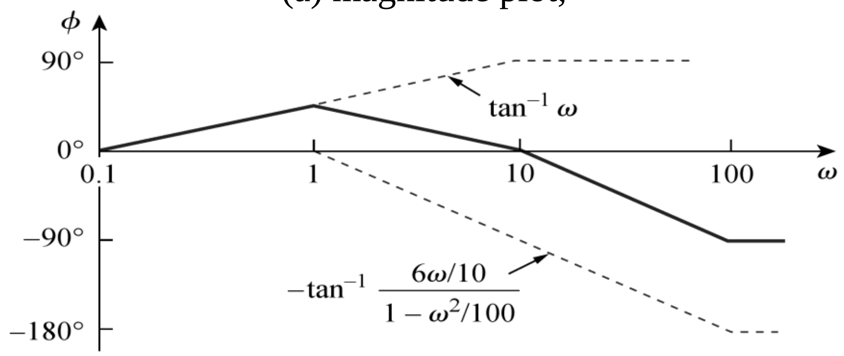
For the quadratic pole, $\omega_n = 10rad/s$, which serves as the corner frequency. The magnitude and phase are

$$H_{dB} = -20\log_{10}100 + 20\log_{10}|1+j\omega|$$
$$-20\log_{10}|1 + \frac{j\omega 1.2}{10} - \frac{\omega^2}{100}|$$
$$\varphi = 0^o + \tan^{-1}\omega - \tan^{-1}[\frac{\omega 1.2/10}{1-\omega^2/100}]$$

Notice that the quadratic pole is treated as a repeated pole at ω_k, that is, $(1 + j\omega/\omega_k)^2$. which is an approximation.



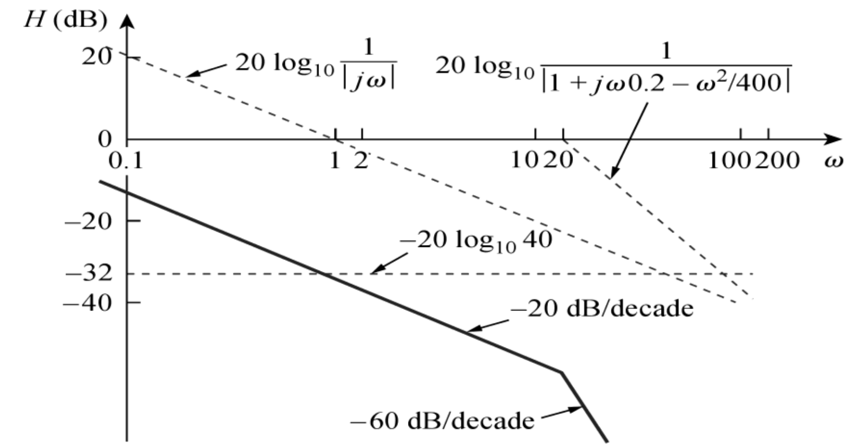
(a) magnitude plot,



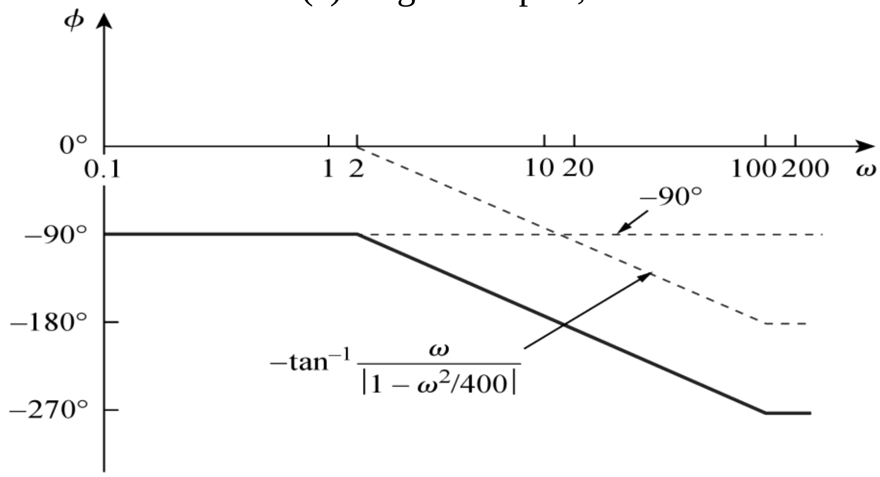
(b) phase plot.

H.W.5: Construct the Bode plots for $H(s) = \frac{10}{s(s^2+80s+400)}$

Answer:



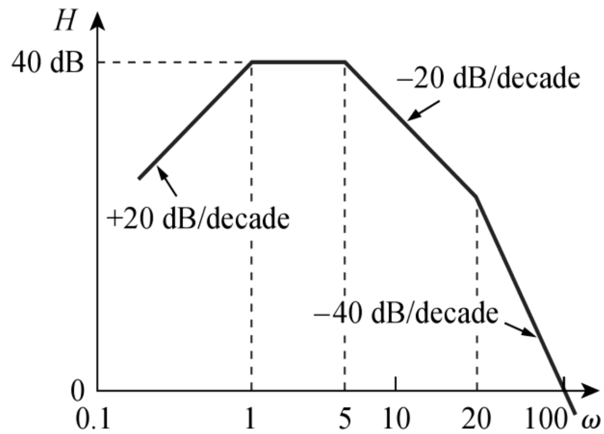
(a) magnitude plot,



(b) phase plot.

Example 6: Given the Bode plot in Fig. below, obtain the transfer function $H(\omega)$.

Solution:



To obtain $H(\omega)$ ffom the Bode plot, we keep in mind that a zero always causes an upward turn at a corner frequency, while a pole causes a downward turn. We notice from Fig. that there is a zero $j\omega$ at the origin which should have intersected the frequency axis at $\omega = 1$. This is indicated by the straight line with slope $+20dB/$ decade. The fact that this straight line is shifted by 40 dB indicates that there is a40-dB gain; that is,

$$40 = 20\log_{10}K \;\Rightarrow\; \log_{10}K = 2$$

or

$$K = 10^2 = 100$$

In addition to the zero $j\omega$ at the origin, we notice that there are three factors with corner frequencies at $\omega = 1 \cdot 5$, and 20 rad/s. Thus, we have:

1. A pole at $p = 1$ with slope $-20dB/$decade to cause a down-ward turn and counteract the zero at the origin. The pole at $p = 1$ is determined as $1/(1 + j\omega/1)$.

2. Another pole at $p = 5$ with slope $-20dB/$decade causing a downward turn. The pole is $1/(1 + j\omega/5)$.

3. A third pole at $p = 20$ with slope $-20dB/$decade causing a further downward turn. The pole is $1/(1 + j\omega/20)$.

Putting all these together gives the corresponding transfer function as

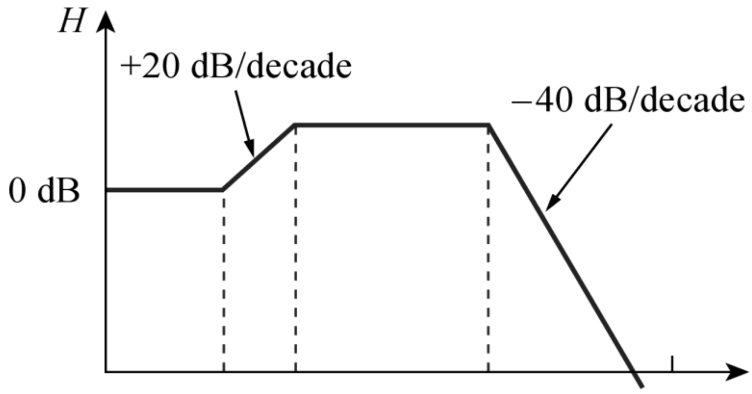
$$H(\omega) = \frac{100j\omega}{(1 + j\omega/1)(1 + j\omega/5)(1 + j\omega/20)} = \frac{j\omega 10^4}{(j\omega + 1)(j\omega + 5)(j\omega + 20)}$$

or

$$H(s) = \frac{10^{4s}}{(s + 1)(s + 5)(s + 20)}, \quad s = j\omega$$

H.W.6: Obtain the transfer function $H(\omega)$ corresponding to the Bode plot in Fig. below

Answer: $\boldsymbol{H(\omega) = \frac{4.000(s+5)}{(s+10)(s+100)^2}}$.



5) Series Resonance

The most prominent feature of the frequency response of a circuit may be the ***sharp peak*** (or ***resonant peak***) exhibited in its amplitude characteristic. The concept of resonance applies in several areas of science and engineering. Resonance occurs in any system that has a complex conjugate pair of poles; it is the cause of oscillations of stored energy from one form to another. It is the phenomenon that allows frequency discrimination in communications networks. Resonance occurs in any circuit that has at least one inductor and one capacitor.

Resonance is a condition in an RLC circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.

Resonant circuits (series or parallel) are useful for constructing filters, as their transfer functions can be highly frequency selective. They are used in many applications such as selecting the desired stations in radio and *TV* receivers.

Consider the series *RLC* circuit shown in Fig. 5.1 in the frequency domain. The input impedance is

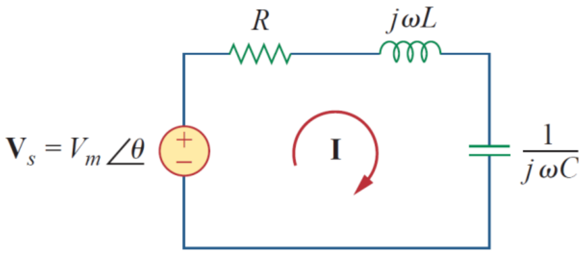


Fig. 5.1 The series resonant circuit.

$$Z = H(\omega) = \frac{V_s}{I} = R + j\omega L + \frac{1}{j\omega C} \qquad \ldots(5.1)$$

$$\therefore Z = R + j(\omega L - \frac{1}{\omega C}) \qquad \ldots(5.2)$$

Resonance results when the imaginary part of the transfer function is zero, or

$$\omega L - \frac{1}{\omega C} = 0 \qquad \ldots(5.3)$$

The value of ω that satisfies this condition is called the ***resonant frequency*** $\boldsymbol{\omega_0}$. Thus, the resonance condition is

$$\omega_0 L = \frac{1}{\omega_0 C} \qquad \ldots(5.4)$$

$$\boxed{\therefore \omega_0 = \frac{1}{\sqrt{LC}} rad/s} \qquad \ldots(5.5)$$

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}} \text{ Hz} \qquad \ldots(5.6)$$

Notes:

$$|V_L| = \frac{V_m}{R}\,\omega_0 L = QV_m$$

$|V_C| = \frac{V_m}{R}\frac{1}{\omega_0 C} = QV_m$, where Q is the quality factor.

Note that at resonance:

1. The impedance is purely resistive, thus, $Z = R$. In other words, the LC series combination acts like a short circuit, and the entire voltage is across R.

2. The voltage V_s and the current I are in phase, so that the power factor is unity.

3. The magnitude of the transfer function $H(\omega) = Z(\omega)$ is minimum.

4. The inductor voltage and capacitor voltage can be much more than the source voltage.

$$I = |I| = \frac{V_m}{\sqrt{R^2+(\omega L-1/\omega C)^2}} \qquad \ldots(5.7)$$

The frequency response of the circuit's current magnitude is shown in Fig. 5.2. The average power dissipated by the *RLC* circuit is

$$P(\omega) = \frac{1}{2}I^2R \qquad \ldots(5.8)$$

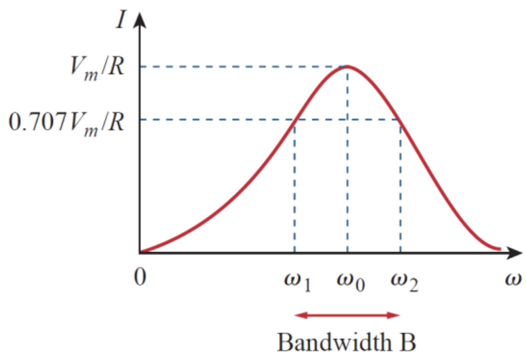


Fig. 5.2 The current amplitude versus frequency for the series resonant circuit of Fig. 5.1

The highest power dissipated occurs at resonance, when $I = V_m/R$, so that

$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R} \qquad \ldots(5.9)$$

At certain frequencies $\omega = \omega_1 . \omega_2$, the dissipated power is half the maximum value; that is,

$$P(\omega_1) = P(\omega_2) = \frac{(V_m/\sqrt{2})^2}{2R} = \frac{V_m^2}{4R} \qquad \ldots(5.10)$$

Hence, ω_1 and ω_2 are called the ***half-power frequencies.***

The half-power frequencies are obtained by setting Z equal to $\sqrt{2}R$, and writing

$$\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} = \sqrt{2}R \qquad \ldots(5.11)$$

Solving for ω, we obtain

$$\boxed{\begin{aligned} \omega_1 &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \\ \omega_2 &= \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \end{aligned}} \qquad \ldots(5.12)$$

From Eqs. (5.5) and (5.12),

$$\omega_0 = \sqrt{\omega_1 \omega_2} \qquad \ldots(5.13)$$

showing that the resonant frequency is the geometric mean of the half- power frequencies. Notice that ω_1 and ω_2 are in general not symmetrical around the resonant frequency ω_0, because the frequency response is not generally symmetrical.

Although the height of the curve in Fig. 5.2 is determined by R, the width of the curve depends on other factors. The width of the response curve depends on the ***bandwidth B***, which is defined as the difference between the two half-power frequencies,

$$B = \omega_2 - \omega_1 \qquad \ldots(5.14)$$

The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the ***quality factor*** Q (dimensionless). At resonance, the reactive energy in the circuit oscillates between the inductor and the capacitor. The quality factor relates the maximum or peak energy stored to the energy dissipated in the circuit per cycle of oscillation:

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} \qquad \ldots(5.15)$$

$$\therefore Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2 R(1/f_0)} = \frac{2\pi f_0 L}{R} \qquad \ldots(5.16)$$

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}} \qquad \ldots(5.17)$$

The relationship between the bandwidth B and the quality factor Q is obtained by substituting Eq. (5.12) into Eq. (5.14) and utilizing Eq. (5.17).

$\boxed{B = \frac{R}{L} = \frac{\omega_0}{Q}}$ or $\boxed{B = \omega_0^2 CR}$ ...(5.18)

The quality factor of a resonant circuit is the ratio of its resonant frequency to its bandwidth. The quality factor is a measure of the selectivity (or "sharpness" of resonance) of the circuit.

Keep in mind that Eqs. (5.12), (5.17), and (5.18) only apply to a series *RLC* circuit. As illustrated in Fig. 5.3, the higher the value of *Q*, the more selective the circuit is but the smaller the bandwidth.

The *selectivity* of an RLC circuit is the ability of the circuit to respond to a certain frequency and discriminate against all other frequencies.

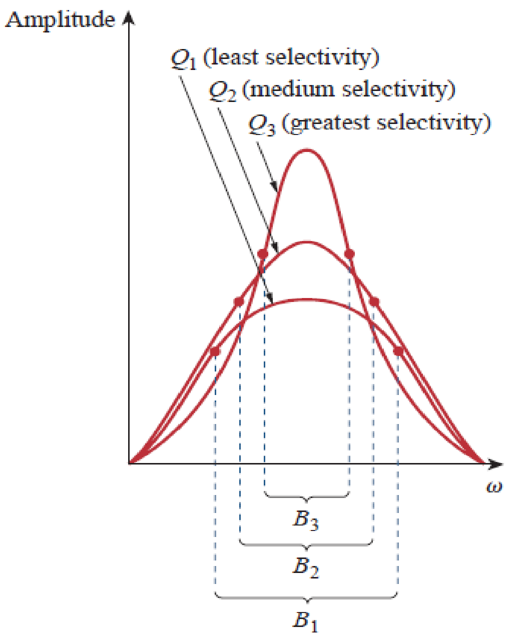


Fig. 5.3 The higher the circuit Q, the smaller the bandwidth.

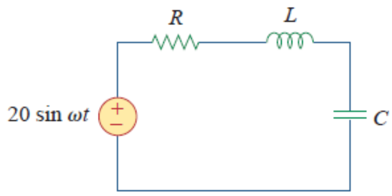
A resonant circuit is designed to operate at or near its resonant frequency. It is said to be a ***high-Q circuit*** when its quality factor is equal to or greater than 10. For high-Q circuits ($Q \geq 10$) the half-power frequencies are, for all practical purposes, symmetrical around the resonant frequency and can be approximated as

$$\omega_1 \approx \omega_0 - \frac{B}{2}, \qquad \omega_2 \approx \omega_0 + \frac{B}{2} \qquad ...(5.19)$$

High-Q circuits are used often in communications networks.

We see that a resonant circuit is characterized by five related parameters: ***the two half-power frequencies ω_1 and ω_2, the resonant frequency ω_0, the bandwidth B,*** and ***the quality factor Q.***

Example 7: In the circuit of Fig. below, $R = 2\Omega$, $L = 1mH$, and $C = 0 \cdot 4\mu F$. (a) Find the resonant frequency and the half-power frequencies. (b) Calculate the quality factor and bandwidth. (c) Determine the amplitude of the current at ω_0, ω_1, and ω_2.



Solution:

(a) $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10\times04\times1036}} = 50\ krad/s$

METHOD 1 The lower & upper half-power frequencies are

$$\omega_1 = -\frac{R}{2L} + \sqrt{(\frac{R}{2L})^2 + \frac{1}{LC}} = -\frac{2}{2\times10^{-3}} + \sqrt{(10^3)^2 + (50\times10^3)^2} = -1 + \sqrt{1+2500}krad/s = 49krad/s$$

$$\omega_2 = 1 + \sqrt{1+2500}krad/s = 51\ krad/s$$

(b) The bandwidth is $B = \omega_2 - \omega_1 = 2krad/s$ or $B = \frac{R}{L} = \frac{2}{10^{-3}} = 2krad/s$

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

METHOD 2 Alternatively, we could find

$$Q = \frac{\omega_0 L}{R} = \frac{50\times10^3\times10^{-3}}{2} = 25$$

$$B = \frac{\omega_0}{Q} = \frac{50\times10^3}{25} = 2\ krad|ss$$

Since $Q > 10$, this is a high-Q circuit and we can obtain the half- power frequencies as

$\omega_1 = \omega_0 - \frac{B}{2} = 50 - 1 = 49krad/s$ & $\omega_2 = \omega_0 + \frac{B}{2} = 50 + 1 = 51\ krad/s$

(c) At $\omega = \omega_0$, $I = \frac{V_m}{R} = \frac{20}{2} = 10\ A$

At $\omega = \omega_1 . \omega_2$ $\rightarrow I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7\cdot071\ A$

H.W.7: A series-connected circuit has $R = 4\Omega$ and $L = 25mH$. (a) Calculate the value of C that will produce a quality factor of 50. (b) Find $\omega_1 . \omega_2 .$ and B. (c) Determine the average power dissipated at $\omega = \omega_0 . \omega_1 . \omega_2$. Take $V_m = 100V$.

Answer: (a) 0.625 μF. (b) 7920 rad/s, 8080 $\frac{rad}{s}$. 160 $\frac{rad}{s}$. (c)1 · 25kW. 0.625 kW, 0.625 kW.

6) Parallel Resonance

The parallel RLC circuit in Fig. 6.1 is the dual of the series RLC circuit. So we will avoid needless repetition. The admittance is

$$Y = H(\omega) = \frac{I}{V} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} \qquad \ldots(6.1)$$

$$\therefore Y = \frac{1}{R} + j(\omega C - \frac{1}{\omega L}) \qquad \ldots(6.2)$$

Resonance occurs when the imaginary part of Y is zero,

$$\omega C - \frac{1}{\omega L} = 0 \qquad \ldots(6.3)$$

$$\boxed{\therefore \omega_0 = \frac{1}{\sqrt{LC}} rad/s} \qquad \ldots(6.4)$$

which is the same as Eq. (5.5) for the series resonant circuit. The voltage $|V|$ is sketched in Fig. 6.2 as a function of frequency. Notice that at resonance, the parallel LC combination acts like an open circuit, so that the entire current flows through R. Also, the inductor and capacitor current can be much more than the source current at resonance.

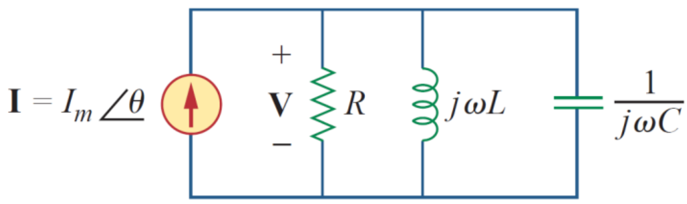


Fig. 6.1 The parallel resonant circuit.

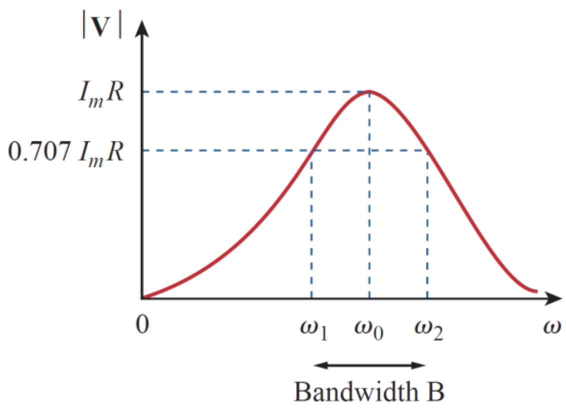


Fig. 6.2 The current amplitude versus frequency for the series resonant circuit of Fig. 6.1.

Notes:

$$|I_L| = \frac{I_m R}{\omega_0 L} = Q I_m$$

$|I_C| = \omega_0 C I_m R = Q I_m$, where Q is the quality factor.

We exploit the duality between Fig. 5.1 and Fig. 6.1 by comparing Eq. (6.2) with Eq. (5.2). By replacing R, L, and C in the expressions for the series circuit with $1/R$, C, and L respectively, we obtain for the parallel circuit

$$\boxed{\begin{aligned}\omega_1 &= -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \\ \omega_2 &= \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\end{aligned}} \quad \ldots(6.5)$$

$$\boxed{B = \omega_2 - \omega_1 = \frac{1}{RC}} \quad \ldots(6.6)$$

$$\boxed{Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}} \quad \ldots(6.7)$$

It should be noted that Eqs. (6.5) to (6.7) apply only to a parallel RLC circuit. Using Eqs. (6.5) and (6.7) we can express the half- power frequencies in terms of the quality factor. The result is

$$\omega_1 = \omega_o\sqrt{1 + (\frac{1}{2Q})^2} - \frac{\omega_0}{2Q}., \quad \omega_2 = \omega_o\sqrt{1 + (\frac{1}{2Q})^2} + \frac{\omega_0}{2Q} \quad \ldots(6.8)$$

Again, for high-Q circuits ($Q \geq 10$)

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \quad \omega_2 \simeq \omega_0 + \frac{B}{2} \quad \ldots(6.9)$$

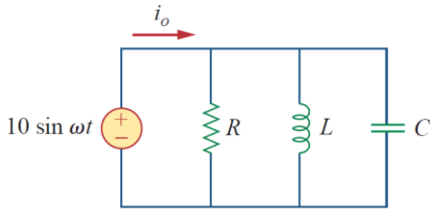
Table 6.1 presents a summary of the characteristics of the series and parallel resonant circuits.

TABLE 6.1

Summary of the characteristics of resonant RLC circuits.

Characteristic	Series circuit	Parallel circuit
Resonant frequency, ω_0	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$
Quality factor, Q	$\frac{\omega_0 L}{R}$ or $\frac{1}{\omega_0 RC}$	$\frac{R}{\omega_0 L}$ or $\omega_0 RC$
Bandwidth, B	$\frac{\omega_0}{Q}$	$\frac{\omega_0}{Q}$
Half-power frequencies, ω_1, ω_2	$\omega_0\sqrt{1+\left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$	$\omega_0\sqrt{1+\left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$
For $Q \geq 10$, ω_1, ω_2	$\omega_0 \pm \frac{B}{2}$	$\omega_0 \pm \frac{B}{2}$

Example 8: In the parallel RLC circuit of Fig. shown, let $R = 8k\Omega$, $L = 0 \cdot 2mH$, and $C = 8\mu F$. (a) Calculate ω_0, Q, and B. (b) Find ω_1 and ω_2.
(c) Determine the power dissipated at ω_0 ,ω_1, and ω_2.



Solution:

(a) $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{02\times10^{-3}\times8\times10^{-6}}} = \frac{10^5}{4} = 25 \text{ krad }/s$

$$Q = \frac{R}{\omega_0 L} = \frac{8 \times 10^3}{25 \times 10^3 \times 0.2 \times 10^{-3}} = 1600$$

$B = \frac{\omega_0}{Q} = 15 \cdot 625 \text{ rad }/s$

(b) Due to the high value of Q, we can regard this as a high-Q circuit, Hence,

$$\omega_1 = \omega_0 - \frac{B}{2} = 25000 - 7.812 = 24992 rad/s$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25000 + 7.812 = 25008 rad/s$$

(c) At $\omega = \omega_0$, $Y = 1/R$ or $Z = R = 8k\Omega$. Then

$$I_o = \frac{V}{Z} = \frac{10\underline{/-90^o}}{8000} = 1 \cdot 25\underline{/-90^o mA}$$

Since the entire current flows through R at resonance, the average power dissipated at $\omega = \omega_0$ is

$$P = \frac{1}{2}|I_o|^2 R = \frac{1}{2}(1 \cdot 25 \times 10^{-3})^2 (8 \times 10^3) = 6 \cdot 25 mW$$

or $P = \frac{V_m^2}{2R} = \frac{100}{2\times8\times10^3} = 6 \cdot 25 mW$

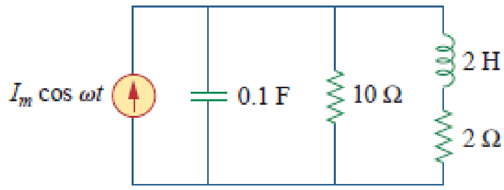
At $\omega = \omega_1, \omega_2, \rightarrow P = \frac{V_m^2}{4R} = 3 \cdot 125 mW$

H.W.8: A parallel resonant circuit has $R = 100k\Omega$, $L = 20mH$, and $C = 5nF$. Calculate ω_0, ω_1, ω_2, Q, and B.
Answer: 100 $krad/s$, 99$krad/s$, 101$krad/s$, 50.2$krad/s$ ·

Example 9: Determine the resonant frequency of the circuit in Fig. shown.

Solution:

The input admittance is



$$Y = j\omega 0 \cdot 1 + \frac{1}{10} + \frac{1}{2 + j\omega 2}$$

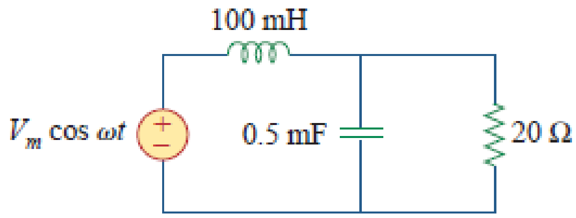
$$= 0 \cdot 1 + j\omega 0 \cdot 1 + \frac{2 - j\omega 2}{4 + 4\omega^2}$$

At resonance, $\text{Im}\,(Y) = 0$ and

$$\omega_0 0 \cdot 1 - \frac{2\omega_0}{4+4\omega_0^2} = 0 \Rightarrow \omega_0 = 2 \text{ rad}/s$$

H.W.9: Calculate the resonant frequency of the circuit in Fig. shown.

Answer: 100 rad/s.

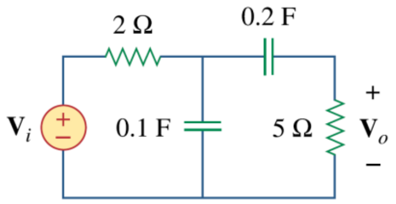


Lecture ()

Frequency Response

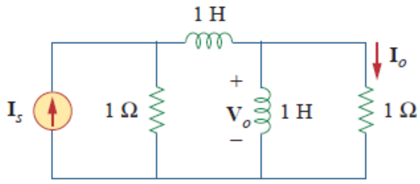
Problems

H.W.(1): For the circuit shown in Fig., find **H**(s) = **V**$_o$/**V**$_i$(s).



[Answer: $H(s) = \frac{V_0(s)}{V_i(s)} = \frac{5s}{s^2+8s+5}$ **]**

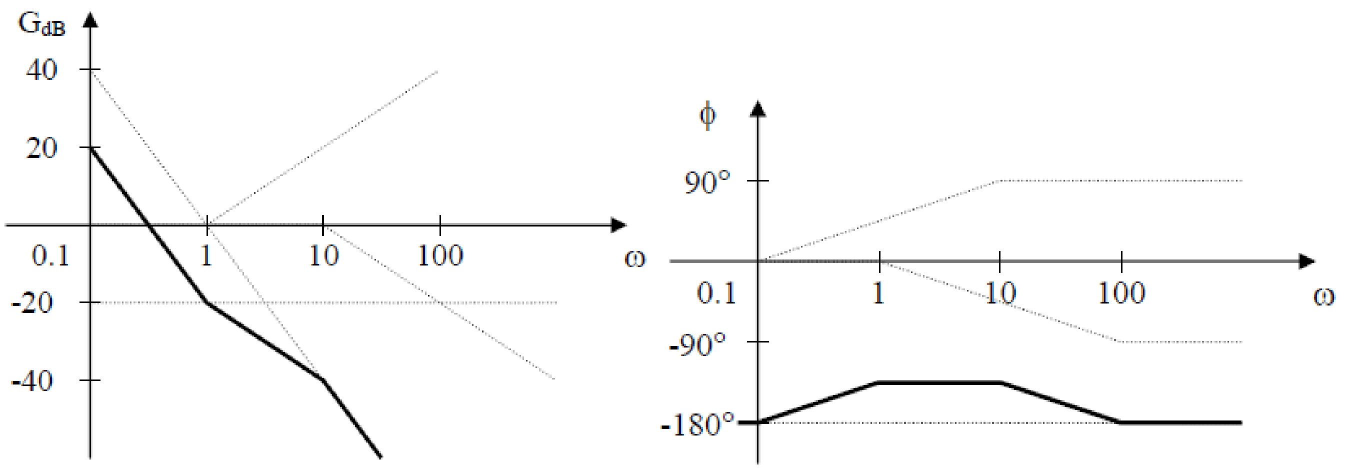
H.W.(2): For the circuit shown in Fig.,find **H**(s) = **I**$_o$(s)/**I**$_s$(s).



[Answer: $H(s) = \frac{I_0(s)}{I_s(s)} = \frac{s}{s^2+3s+1}$ **]**

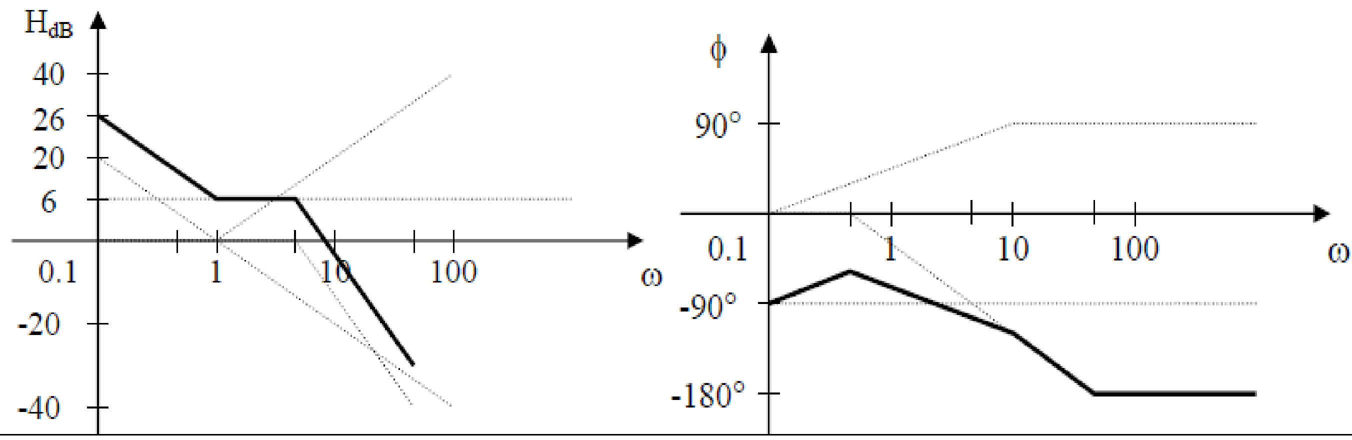
H.W.(3): Construct the Bode plots for $G(s) = \frac{s+1}{s^2(s+10)}$

[Answer:]



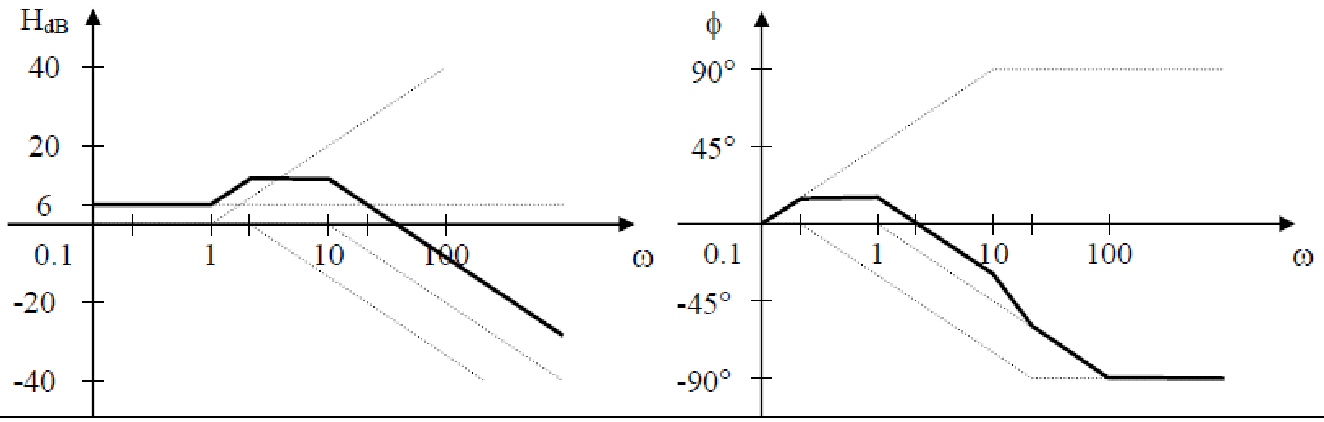
H.W.(4): Draw the Bode plots for $G(j\omega) = \frac{50(j\omega+1)}{j\omega(-\omega^2+10j\omega+25)}$

[Answer:]



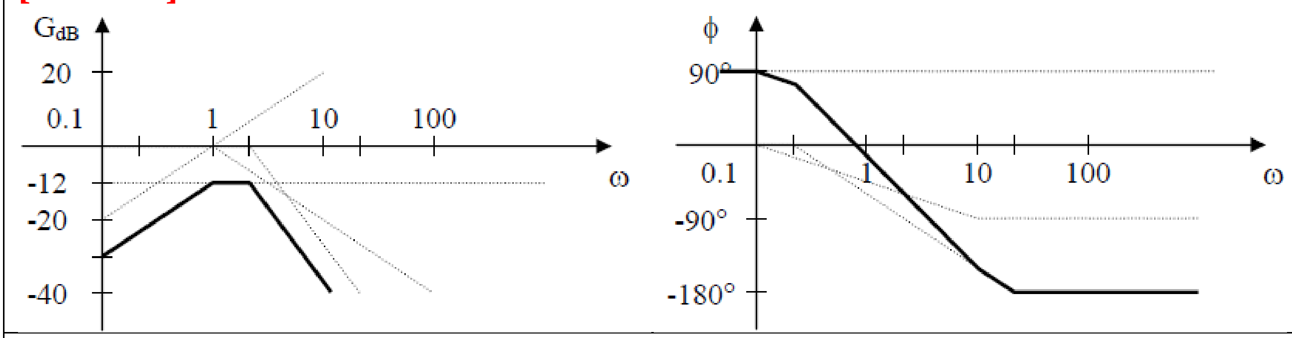
H.W.(5): Construct the Bode plots for $G(s) = \frac{40(s+1)}{(s+2)(s+10)}$

[Answer:]



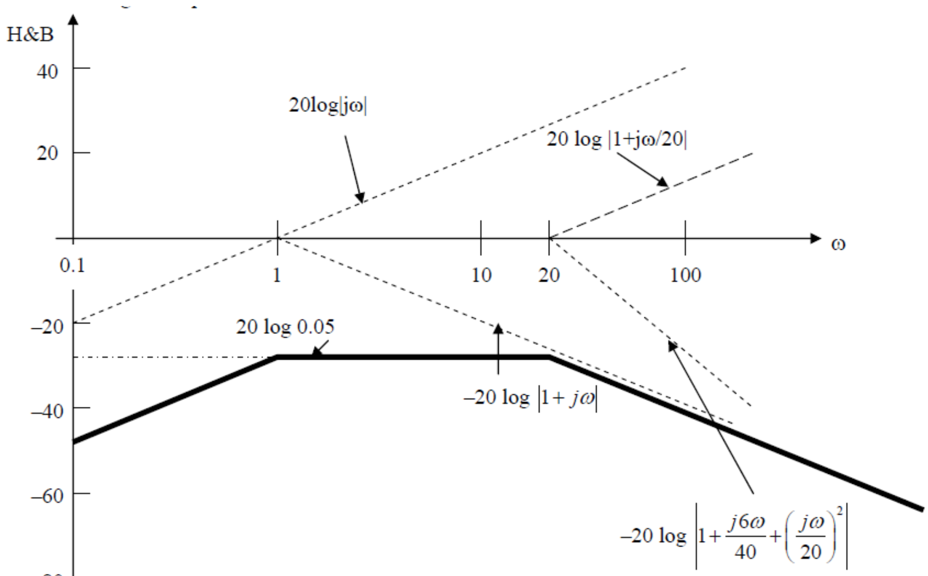
H.W.(6): Construct the Bode plots for $G(s) = \frac{s}{(s+2)^2+(s+1)}$

[Answer:]



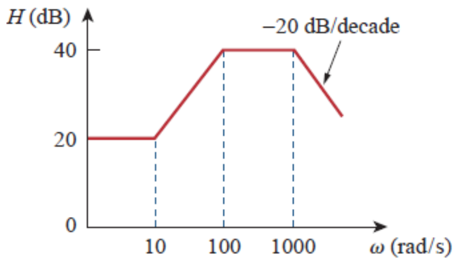
H.W.(7): Construct the Bode plots for $G(s) = \frac{s(s+20)}{(s+1)(s^2+60s+400)}$

[Answer:]



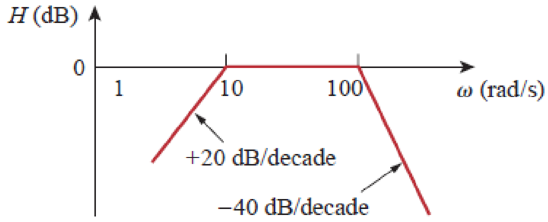
H.W.(8): Find the transfer function **H**(ω) with the Bode magnitude plot shown in Fig.

[Answer: $\mathbf{H(\omega) = \dfrac{10^4(2+j\omega)}{(20+j\omega)(100+j\omega)}}$**]**



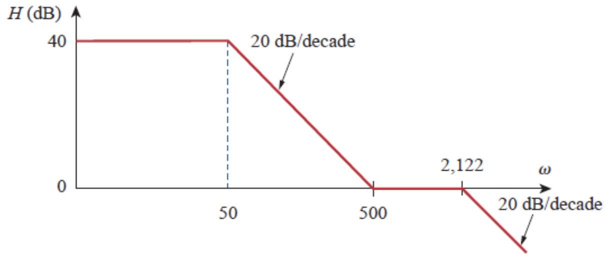
H.W.(9): The Bode magnitude plot of **H**(ω) is shown in Fig. Find **H**(ω).

[Answer: $\mathbf{H(\omega) = \dfrac{100j\omega}{(1+j\omega)(10+j\omega)^2}}$**]**



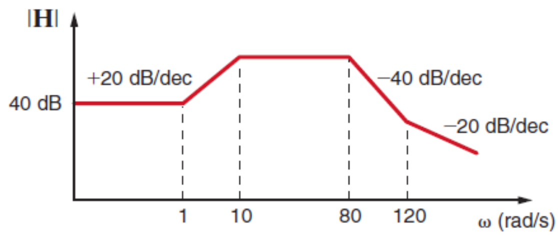
H.W.(10): The magnitude plot in Fig. represents the transfer function of a preamplifier. Find *H*(*s*).

[Answer: $\mathbf{H(s) = \dfrac{8488(s+500)}{(s+50)(s+2122)}}$**]**



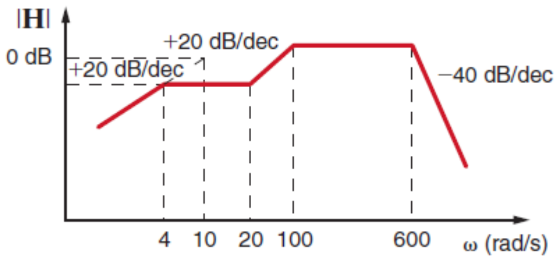
H.W.(11): Find the transfer function **H**(ω) with the Bode magnitude plot shown in Fig.

[Answer: $\mathbf{H(\omega) = \dfrac{5.33\times10^4(1+j\omega)(120+j\omega)}{(10+j\omega)(80+j\omega)^2}}$**]**



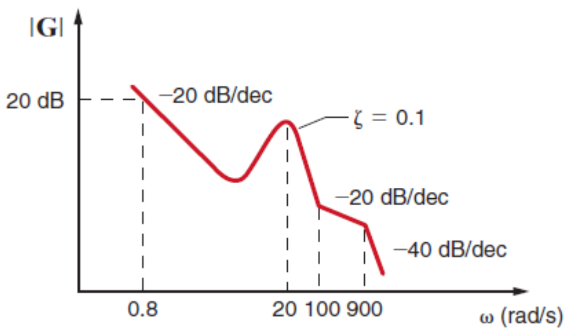
H.W.(12): Find the transfer function **H**(ω) with the Bode magnitude plot shown in Fig.

[Answer: $\mathbf{H(\omega) = \dfrac{7.2\times10^5(j\omega)(20+j\omega)}{(4+j\omega)(100+j\omega)(600+j\omega)^2}}$**]**



H.W.(13): Find the transfer function **H**(ω) with the Bode magnitude plot shown in Fig.

[Answer: $\mathbf{H(\omega) = \dfrac{288(100+j\omega)^2}{j\omega(900+j\omega)[(j\omega)^2+4j\omega+400]}}$**]**



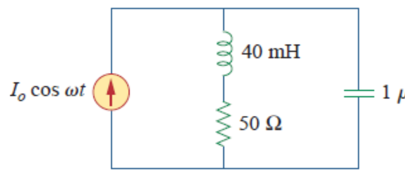
H.W.(14): Design a parallel resonant *RLC* circuit with $\omega_0 = 10$ rad/s and Q = 20. Calculate the bandwidth of the circuit. Let *R* = 10 Ω .

[Answer: $8.796\text{x}10^6$ rad/s]

H.W.(15): A parallel *RLC* circuit has R = 5kΩ, L = 8 mH, and C = μF. Determine:
(a) the resonant frequency (b) the bandwidth (c) the quality factor
[Answer: 40Ω, 10 μF, 2.5 μH, 2.5 krad / s, 198.75 krad/s, 201.25 krad/s]

H.W.(16): It is expected that a parallel *RLC* resonant circuit has a midband admittance of 25×110^{-3} S, quality factor of 80, and a resonant frequency of 200 krad/s. Calculate the values of R, L, and C. Find the bandwidth and the half-power frequencies.
[Answer: 40Ω, 10 μF, 2.5 μH, 2.5 krad/ s, 198.75 krad/s, 201.25 krad/s]

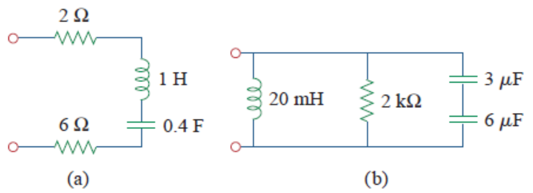
H.W.(17): For the circuit in Fig, find the resonant frequency.



[Answer: 4.841 krad/s]

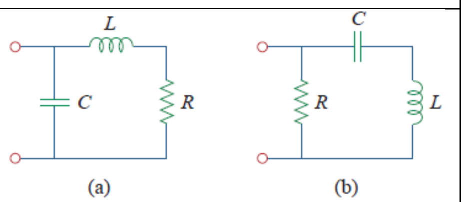
H.W.(18): A parallel resonance circuit has a resistance of 2 k Ω and half-power frequencies of 86 kHz and 90 kHz. Determine:
(a) the capacitance (b) the inductance (c) the resonant frequency (d) the bandwidth (e) the quality factor
[Answer: (a)19.89Nf (b) 164.45 μH (c) 552.9krad / s (d) 25.13krad / s (e)Q = 22]

H.W.(19): For the circuits in Fig, find the resonant frequency ω_o , the quality factor Q, and the bandwidth B.



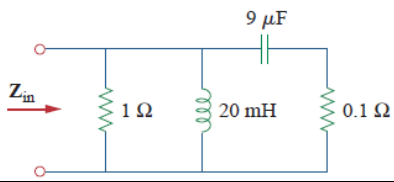
[Answer: (a) 1.5811 rad / s, Q = 0.1976, B = 8 rad / s (b) 5 krad / s, Q = 20, B = 250 rad / s]

H.W.(20): Calculate the resonant frequency of each of the circuits in Fig.



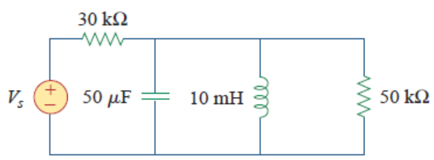
[Answer:(a) $\boldsymbol{\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}}$ **(b)** $\boldsymbol{\omega_0 = \frac{1}{\sqrt{LC}}}$**]**

H.W.(21): For the circuit in Fig., find:
(a) the resonant frequency ω_o
(b) $\mathbf{Z}_{in}(\omega_0)$



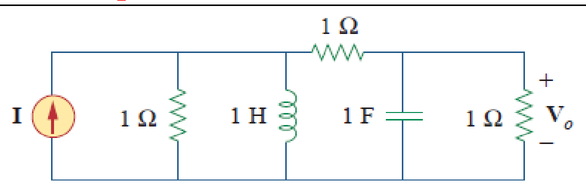
[Answer (a) 2.357 krad/ s (b) 1Ω

H.W.(22): For the circuit shown in Fig., find ω_0 , B, and Q, as seen by the voltage across the inductor.



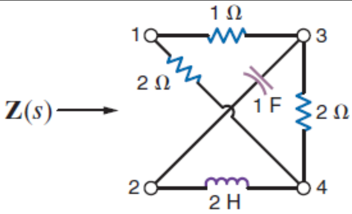
[Answer: ω_0 = 447.21 rad/s, B = 1.067 rad/s, Q = 419.13]

H.W.(23): For the network shown in Fig., find
(a) the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o(\omega)/\mathbf{I}(\omega)$,
(b) the magnitude of $\mathbf{H}$ at ω_o = 1 rad/s.



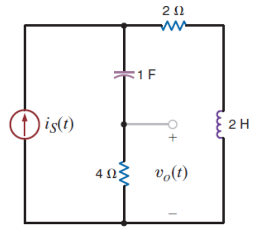
[Answer: (a) $\boldsymbol{H(\omega) = \frac{j\omega}{2(1+j\omega)^2}}$ **(b) 0.25]**

H.W.(24): Determine the driving point impedance at the input terminals of the network shown.



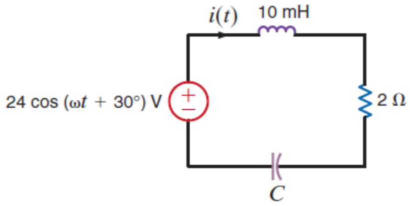
[Answer:]

H.W.(25): Find the transfer impedance or transfer function $V_o(s)/I_s(s)$ for the network shown in Fig.



[Answer: $H(s) = \frac{V_0(s)}{I_s(s)} = \frac{8s}{2s^2+6s+1}$]

H.W.(26): The series *RLC* circuit in Fig. is driven by a variable-frequency source. If the resonant frequency of the network is selected as ω_0 = 1600 rad/s, find the value of C. In addition, compute the current at resonance and at $\omega_0/4$ and $4\omega_0$.

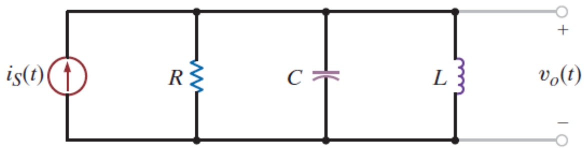


[Answer: C = 39 μF, $I(\omega_0) = 12\angle 30^0 . I(\omega_0/4) = 0\cdot 4\angle 118^0 . I(4\omega_0) = 0\cdot 4\angle -58^0$]

H.W.(27): A parallel *RLC* circuit, which is driven by a variable frequency 2-A current source, has the following values: R = 1 kΩ, L = 100mH and C = 10 μF. Find the bandwidth of the network, the half-power frequencies, and the voltage across the network at the half-power frequencies.

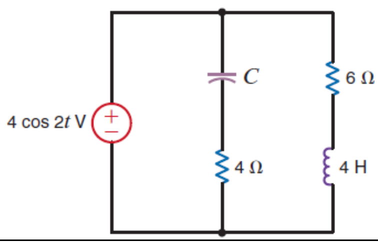
[Answer: B = 100 rad/s, $\omega_1 = 951\cdot 25\frac{rad}{s} . \omega_2 = 1051\cdot 25\frac{rad}{s}$.
$V = \sqrt{2}\angle 45^o\ kV\ at\ \omega_1 . V = \sqrt{2}\angle -45^o\ kV\ at\ \omega_2$]

H.W.(28): The source in the network in Fig. is $i_s(t) = \cos 1000t + \cos 1500t\ A$. R = 200 Ω and C = 500 μF . If ω_0 = 1000 rad/s, find *L*, *Q*, and the BW. Compute the output voltage $v_0(t)$ and discuss the magnitude of the output voltage at the two input frequencies.



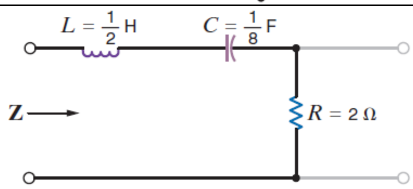
[Answer: L = 2 mH, Q = 100, B = 10 rad/s, $v_0(t) = 200cos\,1000t + 2\cdot 4cos(1500t - 89\cdot 31^0)V$]

H.W.(29): Determine the value of *C* in the network shown in Fig. for the circuit to be in resonance.



[Answer: either C = 345 mF or C = 45 mF]

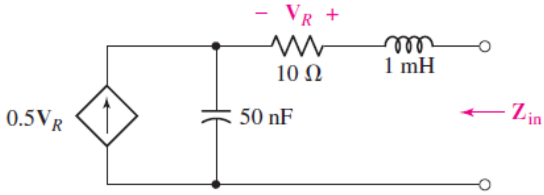
H.W.(30): Determine the new parameters of the network shown in Fig. if $Z_{new} = 10^4 Z_{old}$



[Answer: L_{new} = 5 kH, C_{new} = 12.5 μF, R_{new} = 20 kΩ]

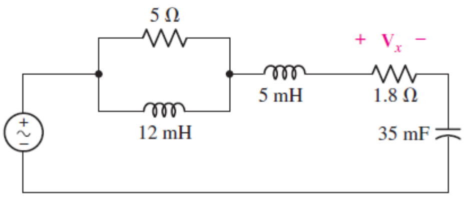
H.W.(31): A parallel *RLC* network is constructed using $R = 5\ \Omega$, $L = 100$ mH, and $C = 1$ mF. (*a*) Compute Q_0. (*b*) Determine at which frequencies the impedance magnitude drops to 90% of its maximum value.
[Answer: (a) 0.5 (b) $\omega_1 = 62.7$ rad/s; $\omega_1 = 159.5$ rad/s]

H.W.(32): After deriving $\mathbf{Z}_{in}(\mathbf{s})$ in Fig., find *(a)* ω_0; *(b)* Q_0.



[Answer: Z = R + j(ωL − 1/ωC), ω_0=1.4*10^5; Q_0=14.1]

H.W.(33): For the circuit shown in Fig., the voltage source has magnitude 1 V and phase angle 0◦. Determine the resonant frequency ω_0 and the value of $\mathbf{V}_x$ at $0.95\omega_0$.



[Answer: At ω_0 set imaginary part of input impedance, Z(ω), to zero to obtain ω_0 = 41 rad/s. Then, Z(0.95ω_0) = Z(39) = 1.845 ∠-2.2^0 Ω, I_s(39) = 0.542 ∠2.2° A and V_x(39) = 0.98 ∠2.2^0 V]

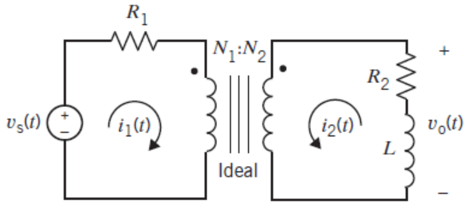
H.W.(34): The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{3}{\left(1 + j\frac{\omega}{2}\right)\left(1 + j\frac{\omega}{5}\right)}$$

Determine the value of the inductance L and of the gain A of the voltage-controlled voltage source (VCVS).
[Answer: A =3 and L = 2H.]

H.W.(35): The network function of this circuit is

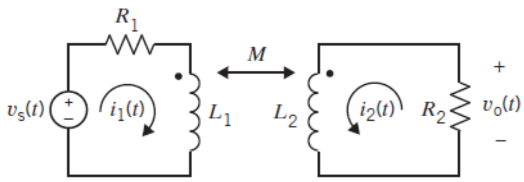
$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_s(\omega)} = k\frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$$



Determine expressions that relate the network function parameters k, z, and p to the circuit parameters R_1, R_2, L, N_1, and N_2.
[Answer:]

H.W.(36): The network function of this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_s(\omega)} = k\frac{j\omega}{1 + j\dfrac{\omega}{p}}$$



Determine expressions that relate the network function parameters k and p to the circuit parameters R_1, R_2, M, L_1, and L_2.
[Answer:]

Lecture ( )

Passive Filters

1) Introduction

The reactances of inductors and capacitors depend on the frequency of the a.c. signal applied to them. That is why these devices are known as ***frequency-selective***. By using various combinations of resistors, inductors and capacitors, we can make circuits that have the property of passing or rejecting either low or high frequencies or bands of frequencies. These frequency selective networks, which alter the amplitude and phase characteristics of the input a.c. signal, are called ***filters***. Their performance is usually expressed in terms of how much attenuation a band of frequencies experiences by passing through them. Attenuation is commonly expressed in terms of decibels (dB).

As a frequency-selective device, a filter can be used to limit the frequency spectrum of a signal to some specified band of frequencies. Filters are the circuits used in radio and TV receivers to allow us to select one desired signal out of a multitude of broadcast signals in the environment.

A filter is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.

2) Types of Electrical Filters

1) A ***passive filter*** is a filter consists of only passive elements *R*, *L*, and *C*.
2) An ***active filter*** is a filter consists of active elements (such as transistors and op amps) in addition to passive elements *R*, *L*, and *C*.

We consider passive filters in this section. *LC* filters have been used in practical applications for more than eight decades. *LC* filter technology feeds related areas such as equalizers, impedance-matching networks, transformers, shaping networks, power dividers, attenuators, and directional couplers, and is continuously providing practicing engineers with oppurtunities to innovate and experiment.

There are four types of filters whether passive or active:

1. A ***lowpass filter*** passes low frequencies and stops high frequencies, as shown ideally in Fig.2.1 (a).
2. A ***highpass filter*** passes high frequencies and rejects low frequencies, as shown ideally in Fig.2.1 (b).
3. A ***bandpass filter*** passes frequencies within a frequency band and blocks or attenuates frequencies outside the band, as shown ideally in Fig.2.1 (c).
4. A ***bandstop filter (or band-rejection)*** passes frequencies outside a frequency band and blocks or attenuates frequencies within the band, as shown ideally in Fig.2.1 (d).

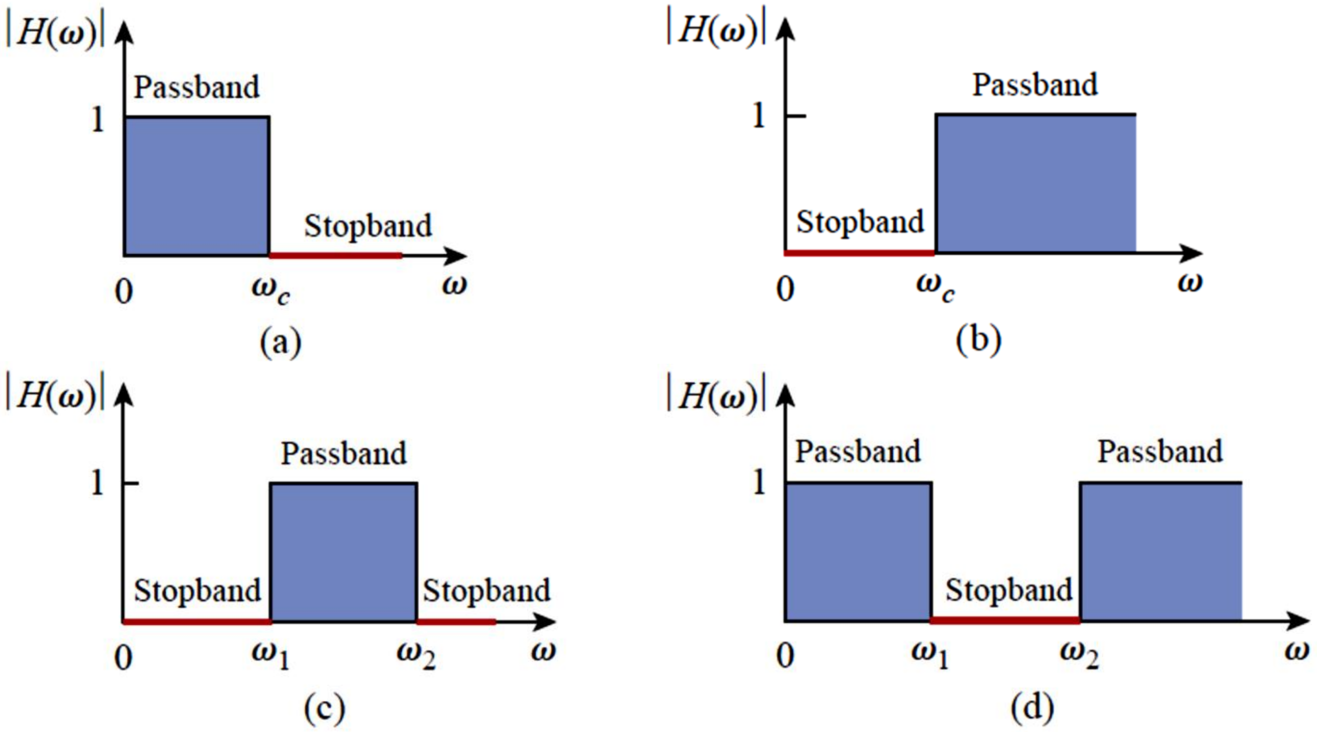


Fig.2.1 Ideal frequency response of four types of filter:
(a) lowpass filter, (b) highpass filter, (c) bandpass filter, (d) bandstop filter.

Table 2.1 presents a summary of the characteristics of these filters. Be aware that the characteristics in Table 2.1 are only valid for first- or second-order filters—but one should not have the impression that only these kinds of filter exist. We now consider typical circuits for realizing the filters shown in Table 2.1.

TABLE 2.1

Summary of the characteristics of ideal filters.

Type of Filter	$H(0)$	$H(\infty)$	$H(\omega_c)$ or $H(\omega_0)$
Lowpass	1	0	$1/\sqrt{2}$
Highpass	0	1	$1/\sqrt{2}$
Bandpass	0	0	1
Bandstop	1	1	0

ω_c is the cutoff frequency for lowpass and highpass filters; ω_0 is the center frequency for bandpass and bandstop filters.

2.1) Lowpass Filter

2.1.1) RC LowPass Filter: A typical ***lowpass filter*** is formed when the output of an *RC* circuit is taken off the capacitor as shown in Fig.2.2. At low frequencies, the capacitor has a very large reactance. Consequently, at low frequencies the capacitor is essentially an open circuit resulting in the voltage across the capacitor, $\mathbf{V}_{out}$, to be essentially equal to the applied voltage, $\mathbf{V}_{in}$. At high frequencies, the capacitor has a very small reactance, which essentially short-circuits the output terminals. The voltage at the output will therefore approach zero as the frequency increases. The transfer function (see also **Example1**) is

$$\boxed{H(\omega) = \frac{V_o}{V_i} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{1+j\omega RC}} \qquad \ldots(2.1)$$

Note that $\mathbf{H}(0) = 1$. $\mathbf{H}(\infty) = 0$. Fig.2.3 shows the plot of $|\mathbf{H}(\omega)|$ along with the ideal characteristic. The range of frequencies passed by a filter within specified limits is called the ***passband*** of the filter. The half-power frequency, which is equivalent to the corner frequency on the Bode plots but in the context of filters is usually known as the ***cutoff frequency (or roll off or break or critical frequency)*** is obtained by setting the magnitude of $\mathbf{H}(\omega)$ equal to $\frac{1}{\sqrt{2}}$ thus,

$$\boxed{H(\omega_c) = \frac{1}{\sqrt{1+\omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}}} \qquad \ldots(2.2)$$

For the RC circuit, the cutoff frequency occurs at

$$\boxed{\omega_c = \frac{1}{\tau} = \frac{1}{RC}} \qquad \ldots(2.3)$$

$$\boxed{\therefore H(\omega) = \frac{1}{1+j\frac{\omega}{\omega_c}}} \qquad \ldots(2.4)$$

Note: ***cutoff frequency sometimes called -3db frequency*** because the output voltage is down 3 dB from its maximum at this frequency. The term *dB* (***decibel***) is a commonly used unit in filter measurements.

A lowpass filter is designed to pass only frequencies from dc up to the cutoff frequency $\boldsymbol{\omega_c}$.

Note: The cutoff frequency is the frequency at which the transfer function **H** drops in magnitude to 70.71% of its maximum value. It is also regarded as the frequency at which the power dissipated in a circuit is half of its maximum value.

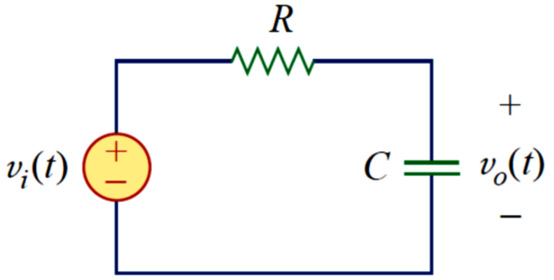


Fig.2.2 A lowpass filter

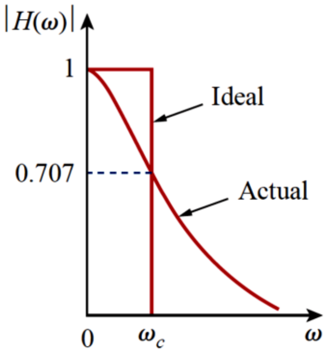


Fig.2.3 Ideal and actual frequency response of a lowpass filter

2.1.2) The RL LowPass Filter: A low-pass filter circuit may be made up of a resistor and an inductor as illustrated in Fig.2.4. In a manner similar to that used for the *RC* low-pass filter, we may write the transfer function for the circuit of Fig.2.4 as follows:

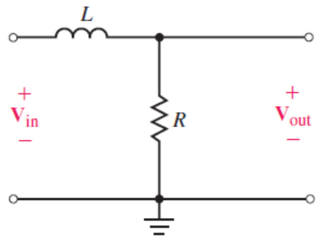


Fig.2.4 RL lowpass filter

$$\boxed{H(\omega)=\frac{V_o}{V_i}=\frac{R}{R+j\omega L}=\frac{1}{1+j\omega\frac{L}{R}}} \quad \ldots(2.5)$$

Since the cutoff frequency is found as $\omega_c=\frac{1}{\tau}\,.so$

$$\boxed{\omega_c=\frac{1}{\tau}=\frac{1}{\frac{L}{R}}=\frac{R}{L}} \quad \ldots(2.6)$$

$$\boxed{H(\omega)=\frac{1}{1+j\frac{\omega}{\omega_c}}} \quad \ldots(2.7)$$

2.2) Highpass Filter

2.2.1) RC HighPass Filter: A ***highpass filter*** is formed when the output of an *RC* circuit is taken off the resistor as shown in Fig.2.5. At low frequencies, the reactance of the capacitor will be very large, effectively preventing any input signal from passing through to the output. At high frequencies, the capacitive reactance will approach a short-circuit condition, providing a very low impedance path for the signal from the input to the output. The transfer function is

$$\boxed{H(\omega)=\frac{V_o}{V_i}=\frac{R}{R+1/j\omega C}=\frac{j\omega RC}{1+j\omega RC}} \quad \ldots(2.8)$$

Note that $\mathbf{H}(0)=0.\,\mathbf{H}(\infty)=1.$ Fig.2.6 shows the plot of $|\mathbf{H}(\omega)|$. Again, the corner or cutoff frequency is

$$\boxed{\omega_c=\frac{1}{\tau}=\frac{1}{RC}} \quad \ldots(2.9)$$

$$\boxed{H(\omega)=\frac{j\frac{\omega}{\omega_c}}{1+j\frac{\omega}{\omega_c}}} \quad \ldots(2.10)$$

A highpass filter is designed to pass all frequencies above its cutoff frequency $\boldsymbol{\omega_c}$.

A ***highpass filter*** can also be formed when the output of an *RL* circuit is taken off the inductor.

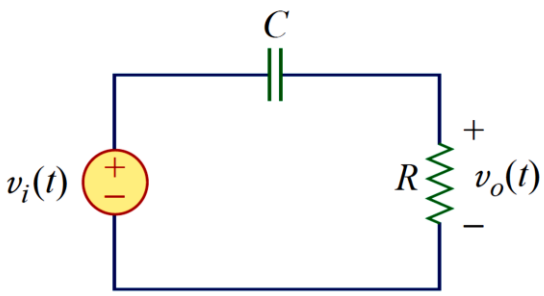


Fig.2.5 A highpass filter

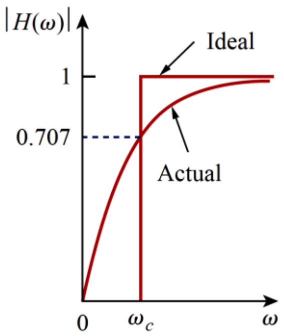


Fig.2.6 Ideal and actual frequency response of a highpass filter

2.2.2) RL HighPass Filter

A typical *RL* high-pass filter circuit is shown in Fig.2.7. At low frequencies, the inductor is effectively a short circuit, which means that the output of the circuit is essentially zero at low frequencies. Inversely, at high frequencies, the reactance of the inductor approaches infinity and greatly exceeds the resistance, effectively preventing current. The voltage across the inductor is therefore very nearly equal to the applied input voltage signal. The transfer function for the high-pass RL circuit is derived as follows:

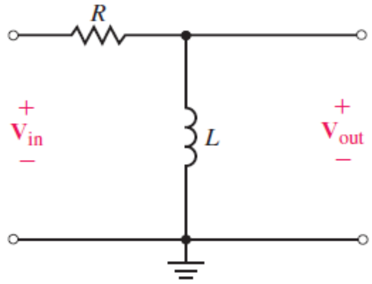


Fig.2.7. RL high-pass filter

$$\boxed{H(\omega)=\frac{V_o}{V_i}=\frac{j\omega L}{R+j\omega L}=\frac{j\omega\frac{L}{R}}{1+j\omega\frac{L}{R}}} \quad \ldots(2.11)$$

Since the cutoff frequency is found as $\omega_c=\frac{1}{\tau}\ .so$

$$\boxed{\omega_c=\frac{1}{\tau}=\frac{1}{\frac{L}{R}}=\frac{R}{L}} \quad \ldots(2.12)$$

$$\boxed{H(\omega)=\frac{j\frac{\omega}{\omega_c}}{1+j\frac{\omega}{\omega_c}}} \quad \ldots(2.13)$$

The preceding expression is identical to the transfer function for a high-pass RC filter, with the exception that in this case we have $\tau=\frac{L}{R}$.

2.3) Bandpass Filter

The *RLC* series resonant circuit provides a ***bandpass filter*** when the output is taken off the resistor as shown in Fig.2.7. The transfer function is

$$\boxed{H(\omega)=\frac{V_o}{V_i}=\frac{R}{R+j(\omega L-1/\omega C)}} \quad \ldots(2.14)$$

We observe that $\mathbf{H}(0)=0.\ \mathbf{H}(\infty)=0.$ Fig.2.8 shows the plot of $|\mathbf{H}(\omega)|$. The ***bandpass filter*** passes a band of frequencies $(\boldsymbol{\omega_1}<\boldsymbol{\omega}<\boldsymbol{\omega_2})$ centered on $\boldsymbol{\omega_0}$ the center frequency, which is given by

$$\boxed{\omega_c=\frac{1}{\sqrt{LC}}} \quad \ldots(2.15)$$

A bandpass filter is designed to pass all frequencies within a band of frequencies, $\boldsymbol{\omega_1}<\boldsymbol{\omega}<\boldsymbol{\omega_2}$

Since the ***bandpass filter*** in Fig.2.7 is a series resonant circuit, the half-power frequencies, the bandwidth, and the quality factor are determined as in series resonant circuit. A ***bandpass filter*** can also be formed by cascading the ***lowpass filter*** (where $\boldsymbol{\omega_2}=\boldsymbol{\omega_c}$) in Fig.2.2 with the ***highpass filter*** (where $\boldsymbol{\omega_1}=\boldsymbol{\omega_c}$) in Fig.2.5. However, the result would not be the same as just adding the output of the ***lowpass filter*** to the input of the ***highpass filter***, because one circuit loads the other and alters the desired transfer function.

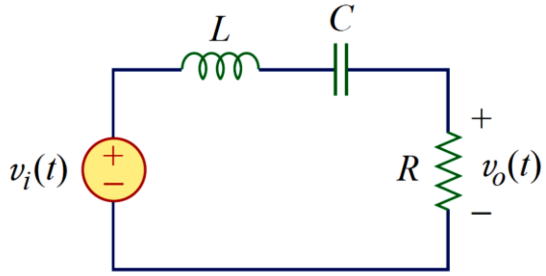


Fig.2.7 A bandpass filter

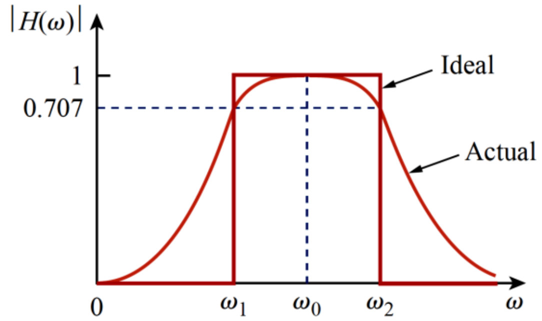


Fig.2.8 Ideal and actual frequency response of a bandpass filter

2.4) Bandstop Filter

A filter that prevents a band of frequencies between two designated values ($\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$) from passing is variably known as a ***bandstop***, ***bandreject***, or ***notch filter***. A ***bandstop filter*** is formed when the output *RLC* series resonant circuit is taken off the *LC* series combination as shown in Fig.2.9. The transfer function is

$$\boxed{H(\omega)=\frac{V_o}{V_i}=\frac{j(\omega L-1/\omega C)}{R+j(\omega L-1/\omega C)}} \qquad \ldots(2.16)$$

Note that $\mathbf{H}(0)=1.\,\mathbf{H}(\infty)=1$. Fig.2.10 shows the plot of $|\mathbf{H}(\omega)|$. Again, the corner or cutoff frequency is

$$\boxed{\omega_c=\frac{1}{\sqrt{LC}}} \qquad \ldots(2.17)$$

while the half-power frequencies, the bandwidth, and the quality factor are calculated using the formulas for a series resonant circuit. Here, $\boldsymbol{\omega_0}$ is called the ***frequency of rejection***, while the corresponding bandwidth ($\boldsymbol{B}=\boldsymbol{\omega_2}-\boldsymbol{\omega_1}$) is known as the ***bandwidth of rejection***. Thus,

A bandstop filter is designed to stop or eliminate all frequencies within a band of frequencies, $\boldsymbol{\omega_1}<\boldsymbol{\omega}<\boldsymbol{\omega_2}$.

Notice that adding the transfer functions of the ***bandpass*** and the ***bandstop*** gives unity at any frequency for the same values of *R*, *L*, and *C*. Of course, this is not true in general but true for the circuits treated here. This is due to the fact that the characteristic of one is the inverse of the other.

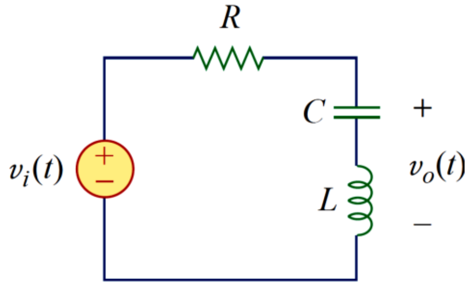


Fig.2.9 A bandstop filter

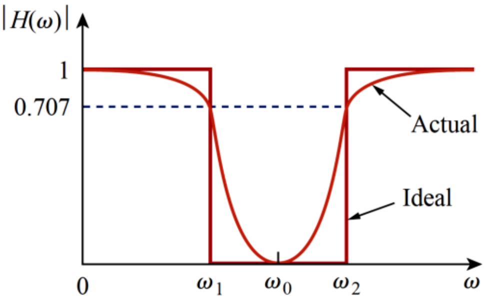


Fig.2.10 Ideal and actual frequency response of a bandstop filter

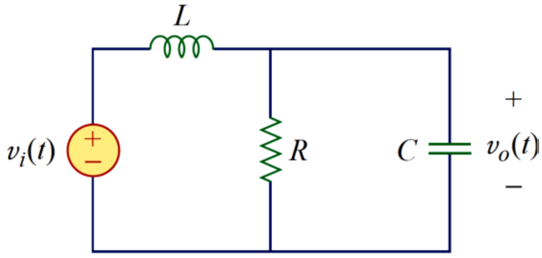
We should note that:

1. From Eqs. (2.1), (2.5), (2.8), (2.11), (2.14), and (2.16), the maximum gain of a ***passive filter*** is unity. To generate a gain greater than unity, one should use an active filter.
2. There are other ways to get the types of filters.
3. The filters treated here are the simple types. Many other filters have sharper and complex frequency responses.

There are three major limitations to the passive filters considered in the previous section. First, they cannot generate gain greater than 1; passive elements cannot add energy to the network. Second, they may require bulky and expensive inductors. Third, they perform poorly at frequencies below the audio frequency range (300 Hz < f < 3000 Hz) Nevertheless, passive filters are useful at high frequencies.

Example 1: Determine what type of filter is shown in Fig. Calculate the corner or cutoff frequency. Take R = 2 kΩ, L = 2 H, C = 2 μF

Solution:



$$H(s) = \frac{V_o}{V_i} = \frac{R\|\frac{1}{sC}}{sL+R\|\frac{1}{sC}}, s = j\omega \qquad \ldots(1)$$

$$R\|\frac{1}{sC} = \frac{R/sC}{R+1/sC} = \frac{R}{1+sRC}$$

Substituting this into Eq. (1) gives

$$H(s) = \frac{\frac{R}{1+sRC}}{sL+\frac{R}{1+sRC}} = \frac{R}{s^2RLC+sL+R}, s = j\omega$$

or $$H(\omega) = \frac{R}{-\omega^2RLC+j\omega L+R} \qquad \ldots(2)$$

Since $H(0) = 1$ and $H(\infty) = 0$, we conclude from Table 2.1 that the circuit of this example is a second-order lowpass filter. The magnitude of H is

$$H = \frac{R}{\sqrt{(R-\omega^2RLC)^2+\omega^2L^2}} \qquad \ldots(3)$$

The corner frequency is the same as the half-power frequency, i.e., where H is reduced by a factor of $1/\sqrt{2}$. Since the dc value of $H(\omega)$ is 1, at the corner frequency, Eq. (3) becomes after squaring

$$H^2 = \frac{1}{2} = \frac{R^2}{(R-\omega_c^2RLC)^2+\omega_c^2L^2} \qquad \text{or} \qquad 2 = (1-\omega_c^2LC)^2 + \left(\frac{\omega_cL}{R}\right)^2$$

Substituting the values of R, L, and C, we obtain

$$2 = (1 - \omega_c^2 4 \times 10^{-6})^2 + (\omega_c 10^{-3})^2$$

Assuming that ω_c is in $krad/s$,

$$2 = (1 - 4\omega_c^2)^2 + \omega_c^2 \qquad \text{or} \qquad 16\omega_c^4 - 7\omega_c^2 - 1 = 0$$

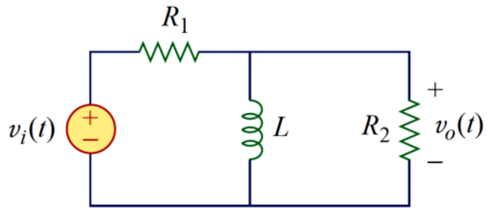
Solving the quadratic equation in ω_c^2, we get $\omega_c^2 = 0 \cdot 5509 and - 0 \cdot 1134$. Since ω_c is real,

$$\omega_c = 0 \cdot 742 \text{ krad}/s = 742 \text{ rad}/s$$

H.W.1: For the circuit shown, obtain the transfer function $\frac{V_O(\omega)}{V_i(\omega)}$. Identify the type of filter and determine the corner frequency. Take $R_1 = 100\ \Omega = R_2$, $L = 2$ mH.

Answer: $\frac{R_2}{R_1+R_2}\left(\frac{j\omega}{j\omega+\omega_c}\right)$ **, highpass filter**

$$\omega_c = \frac{R_1 R_2}{(R_1+R_2)L} = 25 \text{ krad}/s.$$



Example 2: If the bandstop filter in Fig.2.8 is to reject a 200-Hz sinusoid while passing other frequencies, calculate the values of L and C. Take R = 150 Ω and the bandwidth as 100 Hz.

Solution:

We use the formulas for a series resonant circuit

$$B = 2\pi(100) = 200\pi rad/s$$

But $B = \frac{R}{L} \Rightarrow L = \frac{R}{B} = \frac{150}{200\pi} = 0 \cdot 2387H$

Rejection of the 200-Hz sinusoid means that f_0 is 200 Hz, so that ω_0 in is

$$\omega_0 = 2\pi f_0 = 2\pi(200) = 400\pi$$

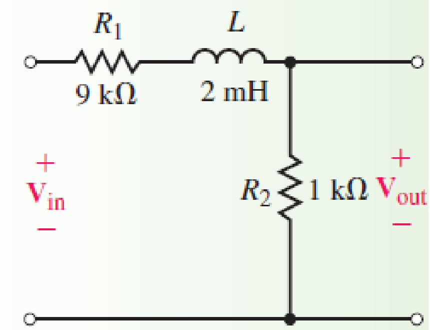
Since $\omega_0 = 1/\sqrt{LC}$,

$$\therefore C = \frac{1}{\omega_0^2 L} = \frac{1}{(400\pi)^2(0.2387)} = 2 \cdot 653\mu F$$

H.W.2: Design a bandpass filter of the form in Fig.2.6 with a lower cutoff frequency of 20.1 kHz and an upper cutoff frequency of 20.3 kHz. Take R = 20 kΩ Calculate L, C & Q.

Answer: 15.92 H, 3.9 pF, 101.

Example : Consider the low-pass circuit of Figure below.
a. Write the transfer function for the circuit.
b. Sketch the frequency response.



Solution:

a. $H(\omega) = \frac{V_o}{V_s} = \frac{R_2}{R_2+R_1+j\omega L} = \frac{R_2}{R_2+R_1}\left(\frac{1}{1+j\omega\frac{L}{R_2+R_1}}\right)$

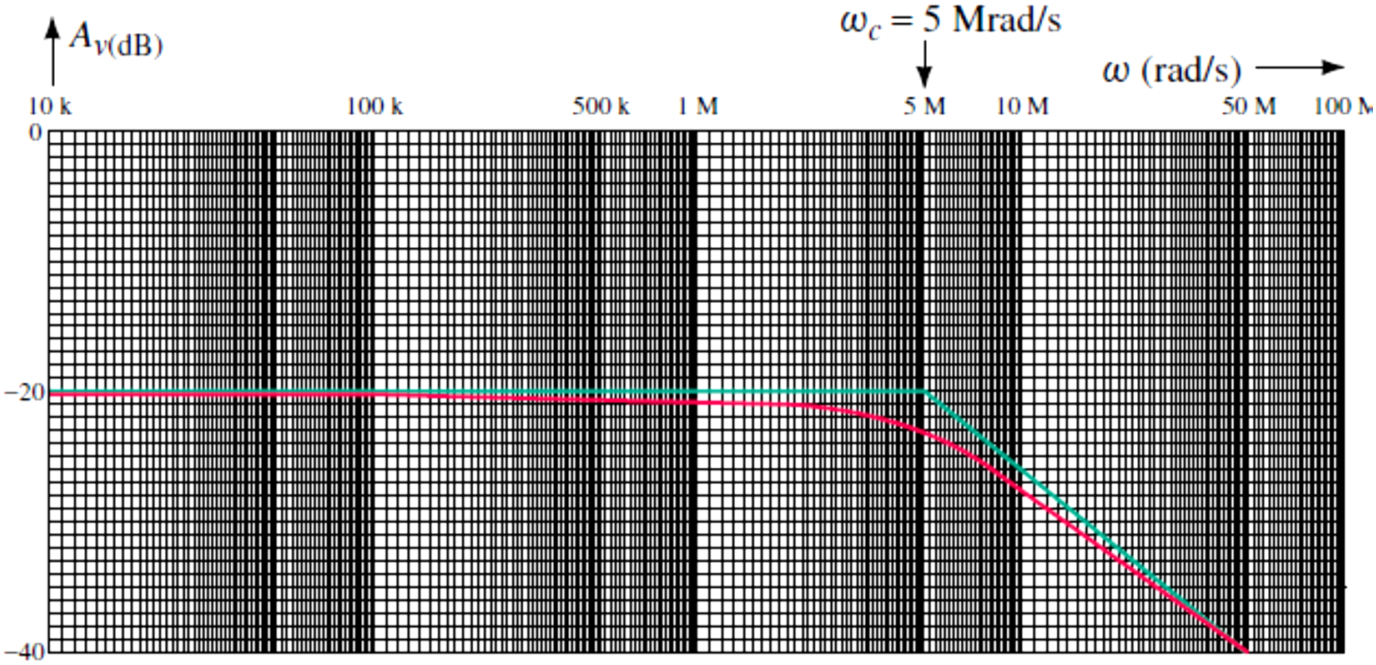
b. From the transfer function of part (a), we see that the dc gain will no longer be 1 $(0dB)$ but rather is found as

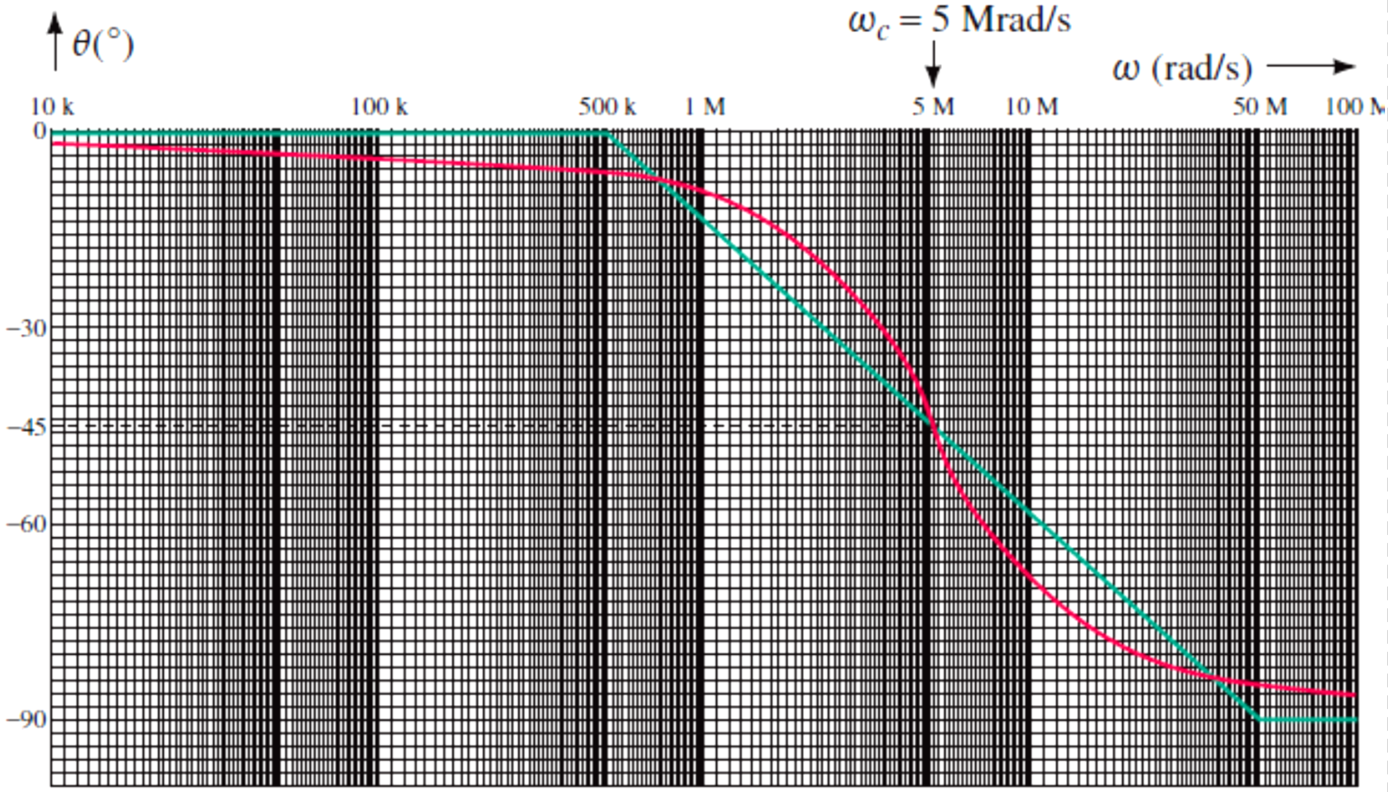
$$A_{v(dc)} = 20 \log \left(\frac{R_2}{R_1+R_2}\right) = 20 \log \left(\frac{1}{10}\right) = -20dB$$

The cutoff frequency occurs at

$$\omega_c = \frac{1}{\tau} = \frac{1}{\frac{L}{R_1+R_2}} = \frac{R_1+R_2}{L} = \frac{10k\Omega}{2mH} = 5.0Mrad/s$$

The resulting Bode plot is shown in Figure 22-24. Notice that the frequency response of the phase shift is precisely the same as for other low-pass filters. However, the response of the voltage gain now starts at−20 dB and then drops at a rate of−20 dB/decade above the cutoff frequency, $\omega_c = 5Mrad/s$.





H.W. : Design a low-pass *RC* filter to have a cutoff frequency of 30 krad/s. Use a 0.01-μF capacitor.

Answer:

$R = 3333\ \Omega$ in series with $C = 0.01\ \mu F$ (output across C)

H.W. : Design a low-pass *RL* filter to have a cutoff frequency of 20 kHz and a dc gain of -6 dB. Use a 10-mH inductor. (Assume that the inductor has no internal resistance.)

Answer:

$R_1 = 630\ \Omega$ in series with $L = 10$ mH and $R_2 = 630\ \Omega$ (output across R_2)

H.W. : A series *RL* low-pass filter with a cutoff frequency of 2 kHz is needed. Using R = 5 kΩ, compute (a) *L*; (b) $|H(j\omega)|$ at 50 kHz; and (c) $\theta(j\omega)$ at 50 kHz.

Answer: (a) 0.4 H, (b) 0.04, (c) -87.71^0

H.W. : Compute the transfer function of a series *RC* low-pass filter that has a load resistor R_L in parallel with its capacitor.

Answer: $H(s) = \dfrac{1/RC}{s+1/KRC}$ where $K = \dfrac{R_L}{R+R_L}$

H.W. : Use a 25-mH inductor to design a high-pass filter circuit having a cutoff frequency of 80 krad/s and a high-frequency gain of -12 dB. Sketch the frequency response of the filter.

Answer:

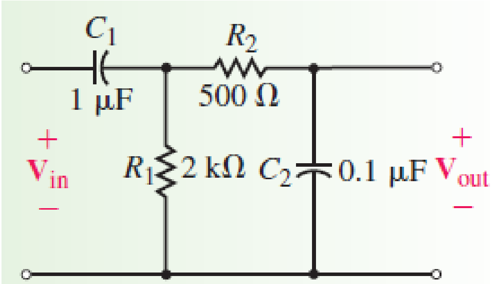
2. $$\mathrm{TF} = \frac{j\omega(3.125 \times 10^{-6})}{1 + j\omega(12.5 \times 10^{-6})}$$

$R_1 = 8\ \mathrm{k\Omega}$ is in series with $L = 25\ \mathrm{mH} \| R_2 = 2.67\ \mathrm{k\Omega}$ (output across the parallel combination)

Example : Write the transfer function for the circuit of Figure below. Sketch the resulting Bode plot and determine the expected bandwidth for the band-pass filter.

Solution:

It is easier to recognize that the circuit consists of two stages: one a low-pass stage and the other a high-pass stage. If the cutoff frequencies of each stage are separated by more than one decade, then we may assume that the impedance of one stage will not adversely affect the operation of the other stage. Based on the previous assumption, the transfer function of the first stage is determined as



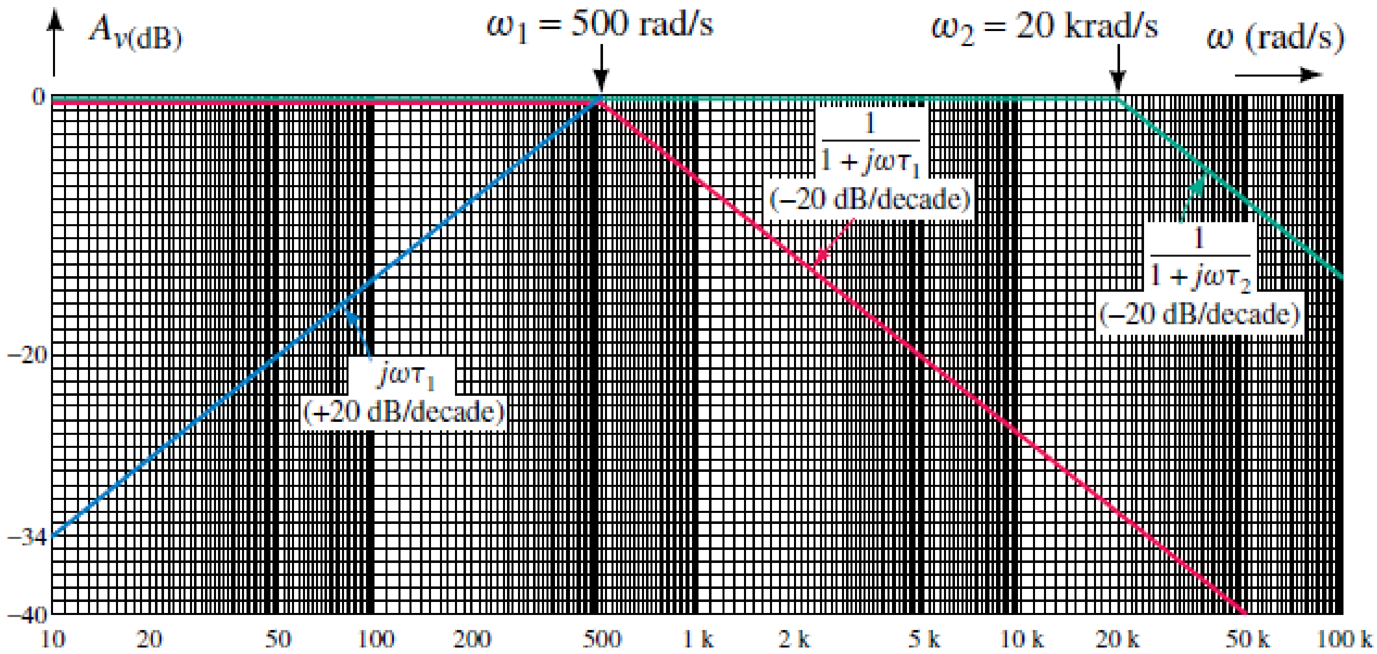
$$H_1(\omega) = \frac{V_1}{V_{in}} = \frac{j\omega R_1 C_1}{1 + j\omega R_1 C_1}$$

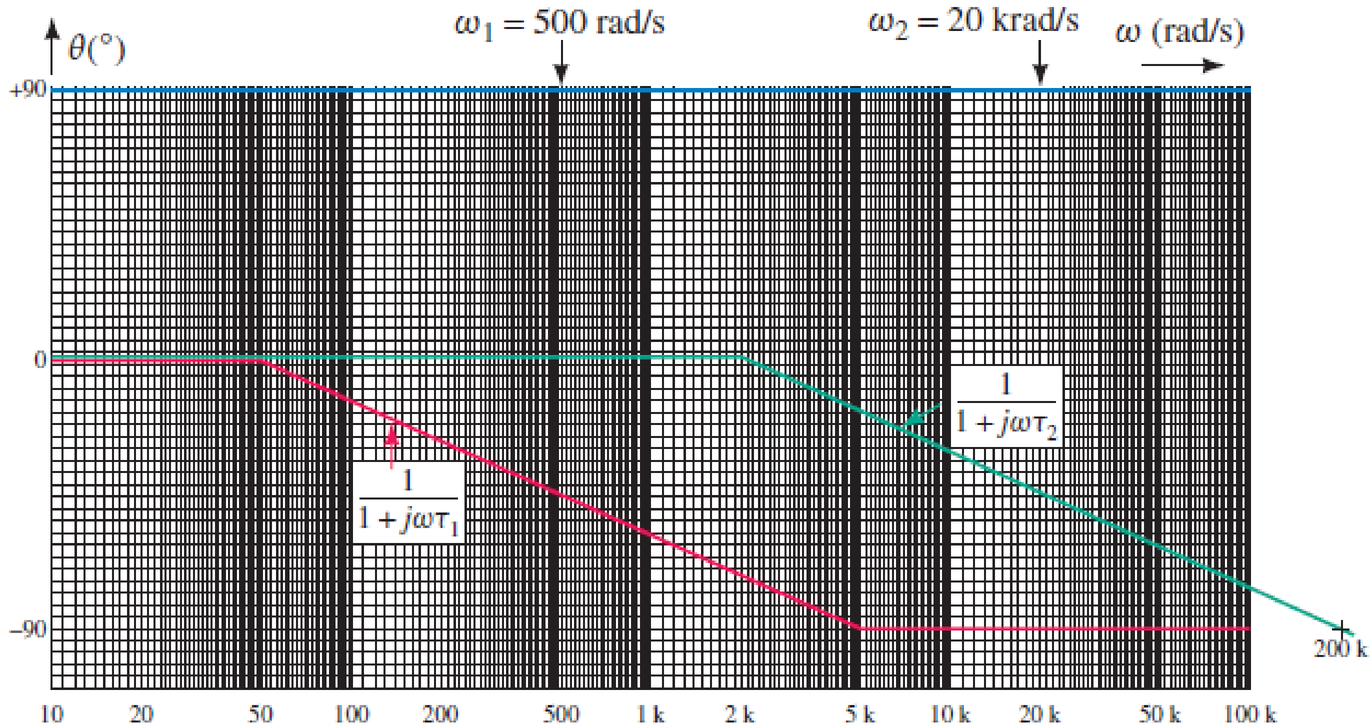
And for the second stage,

$$H_2(\omega) = \frac{V_{out}}{V_1} = \frac{1}{1 + j\omega R_2 C_2}$$

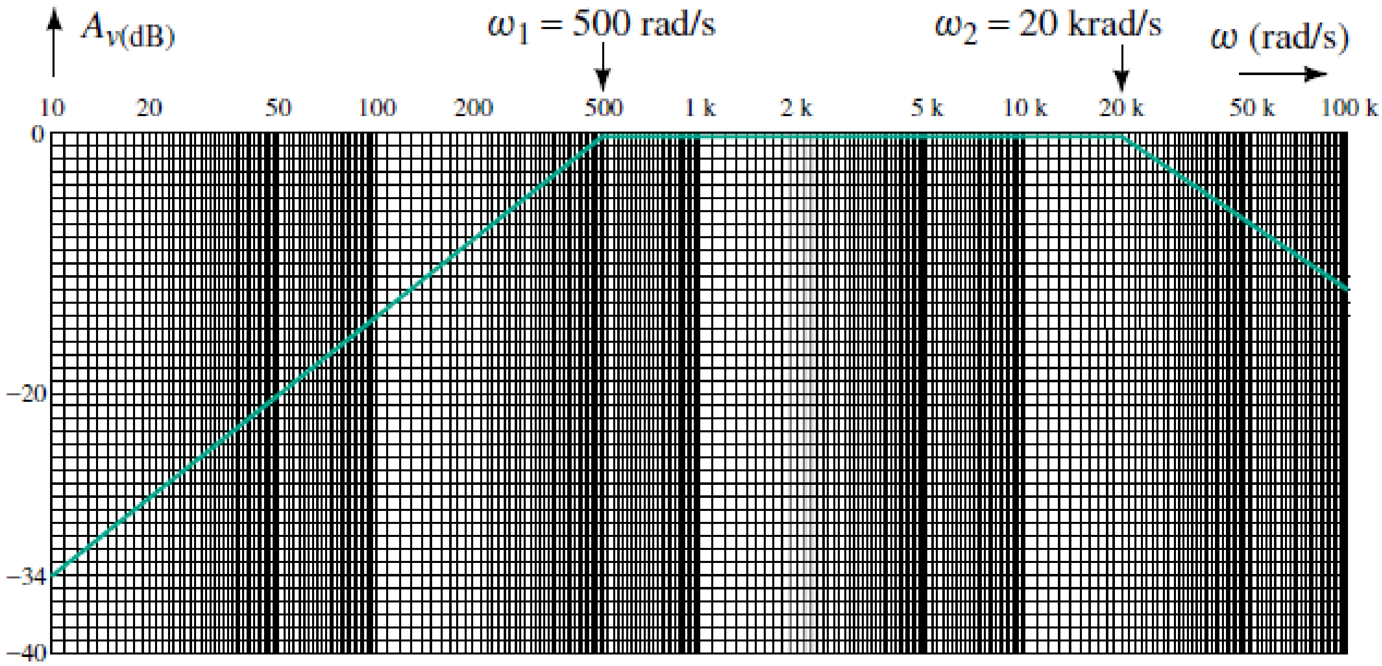
Combining the preceding results, we have

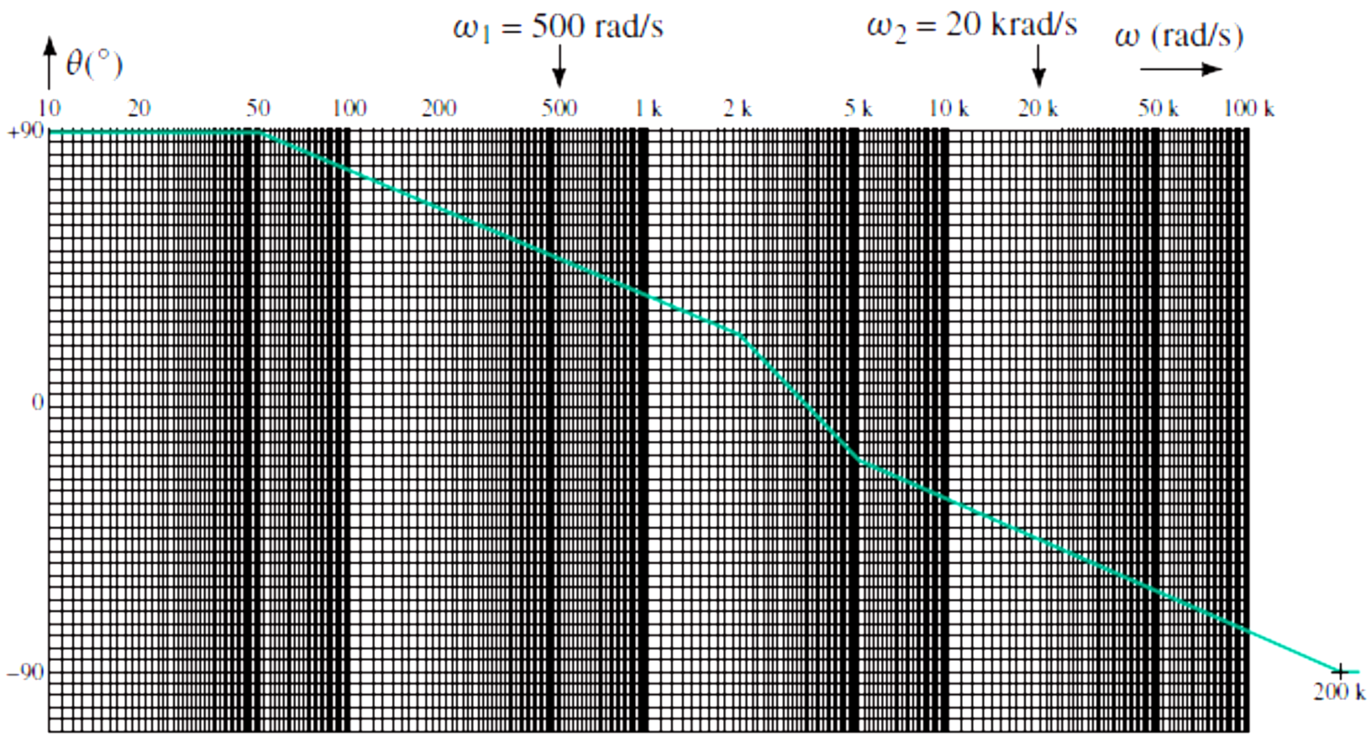
$$H(\omega) = \frac{V_{out}}{V_{in}} = H_1(\omega) H_2(\omega) = \frac{j\omega R_1 C_1}{(1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2)} = \frac{j\omega\tau_1}{(1 + j\omega\tau_1)(1 + j\omega\tau_2)}$$





The resulting frequency response is determined by the summation of the individual responses as shown in Figure below.





From the Bode plot, we determine that the bandwidth of the resulting filter is

$$BW(\text{rad/s}) = \omega_2 - \omega_1 = 20\ \text{krad/s} - 0.5\ \text{krad/s} = 19.5\ \text{krad/s}$$

Example : Determine the transfer function of a third-order Butterworth low-pass filter having a cutoff frequency equal to 500 rad/s, if the transfer function al cutoff frequency equal 1 rad/s is given by,

$$H_n(s) = \frac{1}{(s+1)(s^2+s+1)}$$

Solution:

A technique called ***frequency scaling*** is used to adjust the cutoff frequency to $\omega_c = 500\,rad/s$. Frequency scaling can be accomplished by replacing each s in $H_n(s)$ by s/ω_c. That is,

$$H(s) = \frac{1}{(\frac{s}{\omega_c}+1)((\frac{s}{\omega_c})^2+\frac{s}{\omega_c}+1)}$$

In this case, $\omega_c = 500\,rad/s$, so

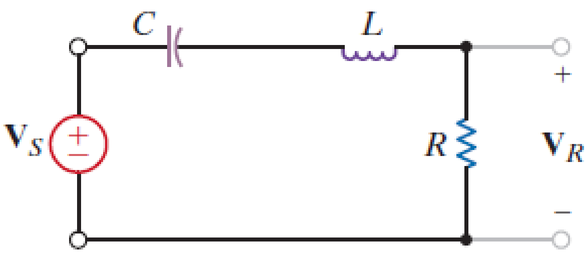
$$H(s) = \frac{1}{(\frac{s}{500}+1)((\frac{s}{500})^2+\frac{s}{500}+1)} = \frac{500^3}{(s+500)(s^2+500s+500^2)} = \frac{125000000}{(s+500)(s^2+500s+250000)}$$

$H(s)$ is the transfer function of a third-order Butterworth low-pass filter having a cutoff frequency equal to 500 rad/s.

Example : We wish to determine the parameters R, L, and C so that the circuit shown in Fig. operates as a band-pass filter with an ω_0 of 1000 rad/s and a bandwidth of 100 rad/s.

Solution:

The voltage gain for the network is



$$\mathrm{H}(\omega) = \frac{(R/L)j\omega}{(j\omega)^2 + (R/L)j\omega + 1/LC}$$

$$\therefore \omega_0 = \frac{1}{\sqrt{LC}}$$

and since $\omega_0 = 10^3$, $\quad \frac{1}{LC} = 10^6$

$$Q = \frac{\omega_0}{BW} = \frac{1000}{100} = 10$$

However, $Q = \frac{\omega_0 L}{R} = \frac{1000L}{R} = 10$

Note that we have two equations in the three unknown circuit parameters R, L, and C. Hence, if we select $C = 1\mu F$, then

$L = \frac{1}{10^6 C} = 1H \quad$ and $\quad \frac{1000(1)}{R} = 10$

$\therefore R = 100\Omega$

Therefore, the parameters $R = 100\Omega$, $L = 1H$, and $C = 1\mu F$ will produce the proper filter characteristics.

3) Scalling

In designing and analyzing filters and resonant circuits or in circuit analysis in general, it is sometimes convenient to work with element values of 1 Ω, 1 H, or 1 F, and then transform the values to realistic values by ***scaling***. We have taken advantage of this idea by not using realistic element values in most of our examples and problems; mastering circuit analysis is made easy by using convenient component values. We have thus eased calculations, knowing that we could use scaling to then make the values realistic.

There are two ways of scaling a circuit: ***magnitude*** or ***impedance scaling***, and ***frequency scaling***. Both are useful in scaling responses and circuit elements to values within the practical ranges. While magnitude scaling leaves the frequency response of a circuit unaltered, frequency scaling shifts the frequency response up or down the frequency spectrum.

3.1) Magnitude Scalling

Magnitude scaling is the process of increasing all impedances in a network by a factor, the frequency response remaining unchanged.

Recall that impedances of individual elements R, L, and C are given by

$$Z_R = R. \qquad Z_L = j\omega L. \qquad z_c = \frac{1}{j\omega C} \qquad \ldots(3.1)$$

In magnitude scaling, we multiply the impedance of each circuit element by a factor K_m and let the frequency remain constant. This gives the new impedances as

$$Z'_R = K_m Z_R = K_m R. \qquad Z'_L = K_m Z_L = j\omega K_m L, \qquad Z'_C = K_m Z_C = \frac{1}{j\omega C/K_m} \qquad \ldots(3.2)$$

Comparing Eq. (3.2) with Eq. (3.1), we notice the following changes in the element values: $R \rightarrow K_m R$, $L \rightarrow K_m L$, and $C \rightarrow C/K_m$. Thus, in magnitude scaling, the new values of the elements and frequency are

$$R' = K_m R. \qquad L' = K_m L, \qquad C' = \frac{C}{K_m}, \qquad \omega' = \omega \qquad \ldots(3.3)$$

The primed variables are the new values and the unprimed variables are the old values. Consider the series or parallel *RLC* circuit. We now have

$$\omega'_0 = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{K_m LC/K_m}} = \frac{1}{\sqrt{LC}} = \omega_0 \qquad \ldots(3.4)$$

showing that the resonant frequency, as expected, has not changed. Similarly, the quality factor and the bandwidth are not affected by magnitude scaling. Also, magnitude scaling does not affect transfer functions in the forms of Eqs. (2.2a) and (2.2b) in previous lecture, which are dimensionless quantities.

3.2) Frequency Scalling

Frequency scaling is the process of shifting the frequency response of a network up or down the frequency axis while leaving the impedance the same.

We achieve frequency scaling by multiplying the frequency by a factor K_f while keeping the impedance the same.

From Eq. (3.1), we see that the impedances of L and C are frequency-dependent. If we apply frequency scaling to $\boldsymbol{Z_L}\ (\boldsymbol{\omega})$ and $\boldsymbol{Z_C}\ (\boldsymbol{\omega})$in Eq. (3.1), we obtain

$$\boxed{Z_L = j(\omega K_f)L' = j\omega L \ \Rightarrow \ L' = \frac{L}{K_f}} \qquad \ldots(3.5a)$$

$$\boxed{Z_C = \frac{1}{j(\omega K_f)C}{}' = \frac{1}{j\omega C} \ \Rightarrow \ C' = \frac{C}{K_f}} \qquad \ldots(3.5b)$$

since the impedance of the inductor and capacitor must remain the same after frequency scaling. We notice the following changes in the element values: $L \rightarrow L/K_f$ and $C \rightarrow C/K_f$. The value of R is not affected, since its impedance does not depend on frequency. Thus, in frequency scaling, the new values of the elements and frequency are

$$\boxed{R' = R. \qquad L' = \frac{L}{K_f}, \qquad C' = \frac{C}{K_f}, \qquad \omega' = K_f\omega} \qquad \ldots(3.6)$$

Again, if we consider the series or parallel RLC circuit, for the resonant frequency

$$\omega_0' = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{(L/K_f)(C/K_f)}} = \frac{K_f}{\sqrt{LC}} = K_f\omega_0 \qquad \ldots(3.7)$$

and for the bandwidth

$$B' = K_f B \qquad \ldots(3.8)$$

but the quality factor remains the same $(Q' = Q)$.

3.3) Magnitude and Frequency Scaling

If a circuit is scaled in magnitude and frequency at the same time, then

$$R' = K_m R. \qquad L' = \frac{K_m}{K_f}L, \qquad C' = \frac{1}{K_m K_f}C. \qquad \omega' = K_f\omega \qquad \ldots(3.9)$$

These are more general formulas than those in Eqs. (3.3) and (3.6). We set $K_m = 1$ in Eq. (3.9) when there is no magnitude scaling or $K_f = 1$ when there is no frequency scaling.

Example 3: A fourth-order Butterworth lowpass filter is shown in Fig.(a). The filter is designed such that the cutoff frequency $\boldsymbol{\omega_c = 1}$ **rad/s.** Scale the circuit for a cutoff frequency of 50 kHz using 10- kΩ resistors.

Solution:

If the cutoff frequency is to shift from $\omega_c = 1 rad/s$ to $\omega_c' = 2\pi(50)\ krad/s$, then the frequency scale factor is

$$K_f = \frac{\omega_c'}{\omega_c} = \frac{100\pi \times 10^3}{1} = \pi \times 10^5$$

Also, if each 1 Ω resistor is to be replaced by a 10 $k\Omega$ resistor, then the magnitude scale factor must be

$$K_m = \frac{R'}{R} = \frac{10 \times 10^3}{1} = 10^4$$

Using Eq. (3.9),

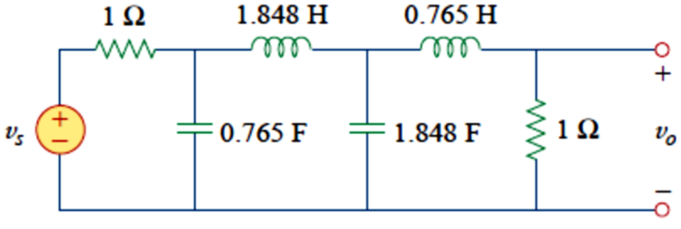
$$L_1' = \frac{K_m}{K_f} L_1 = \frac{10^4}{\pi \times 10^5}(1 \cdot 848) = 58.82 mH$$

$$L_2' = \frac{K_m}{K_f} L_2 = \frac{10^4}{\pi \times 10^5}(0 \cdot 765) = 24 \cdot 35 mH$$

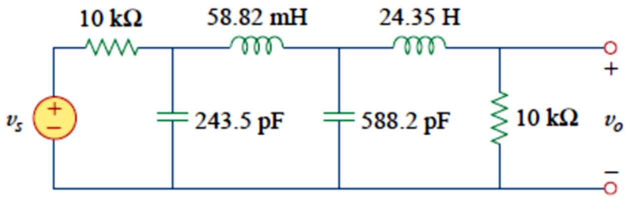
$$C\acute{i} = \frac{C_1}{K_m K_f} = \frac{0.765}{\pi \times 10^9} = 243 \cdot 5 pF$$

$$C_2' = \frac{C_2}{K_m K_f} = \frac{1.848}{\pi \times 10^9} = 588 \cdot 2 pF$$

The scaled circuit is shown in Fig.(b). This circuit uses practical values and will provide the same transfer function as the prototype in Fig.(a), but shifted in frequency.



(a)

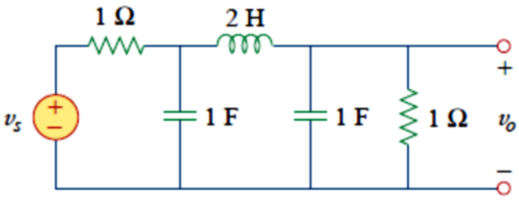


(b)

H.W.3: A third-order Butterworth filter normalized to $\boldsymbol{\omega_c = 1}$ **rad/s.** is shown in Fig. Scale the circuit to a cutoff frequency of 10 kHz. Use 15-nF capacitors.

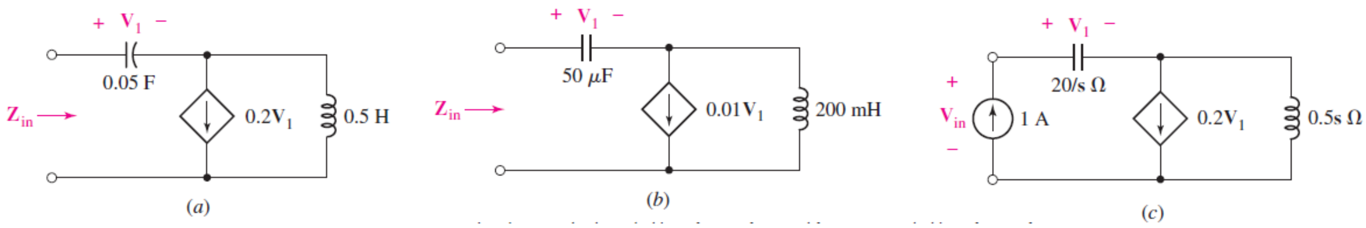
Answer: $\boldsymbol{R_1' = R_2' = 1\cdot 061k\Omega,}$

$\boldsymbol{C\acute{\imath} = C_2' = 15nF,\ L' = 33\cdot 77mH}$.



Example : Scale the network shown in Fig.(a) by $K_m = 20$ and $K_f = 50$, and then find $Z_{in}(s)$ for the scaled network.

Solution:



$$C_{scaled} = \frac{0.05}{(20)(50)} = 50\mu F\ , \qquad L_{scaled} = \frac{(20)(0.5)}{50} = 200mH$$

In scaling the dependent source, only magnitude scaling need be considered, as frequency scaling does not affect dependent sources. Since this is a voltage-controlled *current* source, the multiplying constant 0.2 has units $of\, A/V$, or S. Since the factor has units of admittance, we divide by K_m, so that the new term is $0.01V_1$. The resulting (scaled) network is shown in Fig.(b).

To find the impedance of the new network, we need to apply a 1 A test source at the input terminals. We may work with either circuit; however, let's proceed by first finding the impedance of the *unscaled* network in Fig.(a), and then scaling the result.

Referring to Fig.(c)

$$V_{in} = V_1 + 0.5s(1 - 0.2V_1)$$

$$V_1 = \frac{20}{s}(1)$$

$$Z_{in} = \frac{V_{in}}{1} = \frac{s^2 - 4s + 40}{2s}$$

To scale this quantity to correspond to the circuit of Fig.(b) we multiply by $K_m = 20$, and replace s with $s/K_f = s/50$. Thus,

$$Z_{in_{scaled}} = \frac{0.2s^2 - 40s + 20000}{s}\ \Omega$$

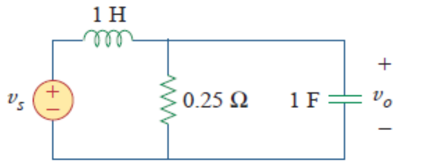
Lecture ()

Passive Filters

Problems

H.W.(1): Find the transfer function $\mathbf{V}_o/\mathbf{V}_s$ of the circuit in Fig. Show that the circuit is a lowpass filter.

[Answer: $\mathbf{H}(\boldsymbol{\omega}) = \frac{R}{R+j\omega L-\omega^2 RLC}$, $\mathbf{H}(0) = 1$ & $\mathbf{H}(\infty) = 0$, this circuit is a lowpass filter]



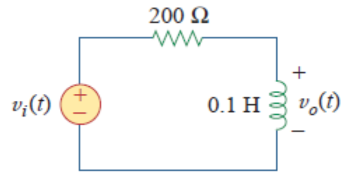
H.W.(2): Determine the cutoff frequency of the lowpass filter described by

$$\mathbf{H}(\boldsymbol{\omega}) = \frac{\mathbf{4}}{\mathbf{2 + j\omega 10}}$$

Find the gain in dB and phase of $\mathbf{H}(\omega)$ at $\omega = 2$ rad/s.

[Answer: 0.2 Hz, - 14.023, - 84.3°]

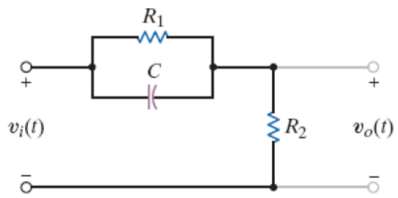
H.W.(3): Determine what type of filter is in Fig. Calculate the corner frequency f_c .



[Answer: H(0) = 0 & H(∞) = 1, this circuit is a highpass filter, 318.3 Hz]

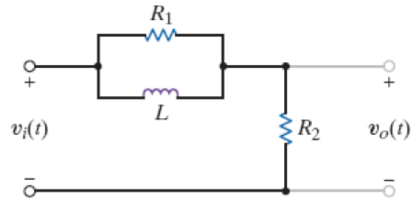
H.W.(4): Determine what type of filter the network shown in Fig. represents by determining the voltage transfer function.

[Answer: the filter is highpass filter]



H.W.(5): Determine what type of filter the network shown in Fig. represents by determining the voltage transfer function.

[Answer: the filter is lowpass filter]



H.W.(6): Determine the range of frequencies that will be passed by a series RLC bandpass filter with R = 10 Ω , L = 25mH, and C = 0.4 μ F. Find the quality factor.

[Answer: 1.56 kHz < f < 1.62 kHz, Q = 25]

H.W.(7): (a) Show that for a bandpass filter

$$\mathbf{H}(s) = \frac{sB}{\mathbf{s^2 + sB + \omega_0^2}}$$

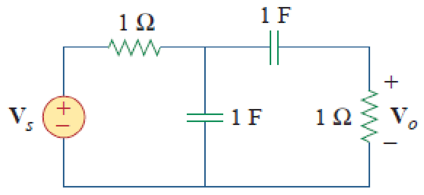
where B = bandwidth of the filter and ω_0 is the center frequency.

(b) Similarly, show that for a bandstop filter,

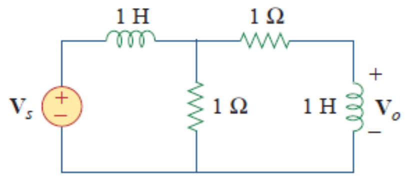
$$\mathbf{H}(s) = \frac{\mathbf{s^2 + \omega_0^2}}{\mathbf{s^2 + sB + \omega_0^2}}$$

[Answer]

H.W.(8): Determine the center frequency and bandwidth of the bandpass filters in Fig.



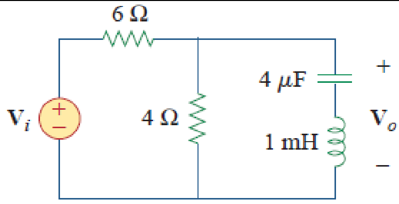
(a)



(b)

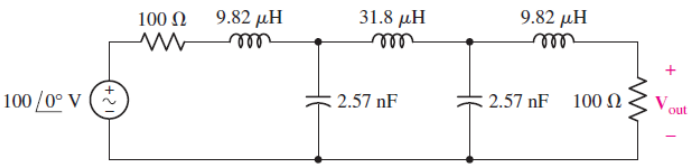
[Answer: (a) $\omega_0 = 1$ rad / s, B = 3 rad / s (b) $\omega_0 = 1$ rad / s, B = 3 rad / s]

H.W.(9): Find the bandwidth and center frequency of the bandstop filter of Fig.

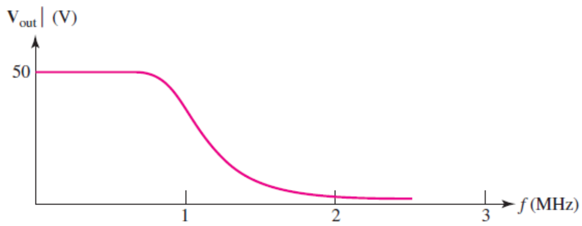


[Answer: B = 2.408 krad/ s, 15.811 krad/ s]

H.W.(10): The filter shown in Fig.a has the response curve shown in Fig.b *(a)* Scale the filter so that it operates between a 50 Ω source and a 50 Ω load and has a cutoff frequency of 20 kHz. *(b)* Draw the new response curve.

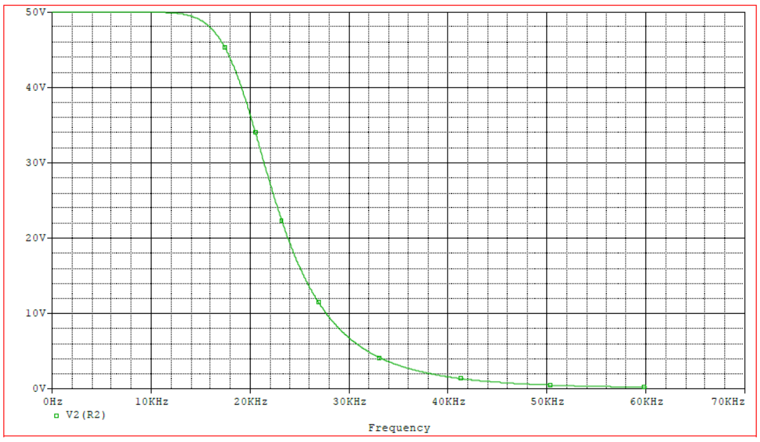


(a)



(b)

[Answer: $k_m = 0.5$; $k_f = 20*10^{-3}$, 100 Ω → 50 Ω, 9.82μH → 245.5 μH, 31.8 μH → 780 μH, 2.57 nF → 257 nF]



H.W.(11): Design a low-pass filter using one resistor and one capacitor that will produce a 4.24-volt output at 159 Hz when 6 volts at 159 Hz are applied at the input.

[Answer: select R = 1 kΩ then C = 1 μF]

H.W.(12): Design a band-pass filter with a low cutoff frequency of approximately 4535 Hz and a high cutoff frequency of approximately 5535 Hz.

[Answer: select C = 100Nf, then L = 10 mH and R = 62.83 Ω]