Lecture ( )

Transient
First- Order Circuits

1) Introduction

The analysis of *RC* and *RL* circuits is carried out by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to *RC* and *RL* circuits produces differential equations, which are more difficult to solve than algebraic equations. The differential equations resulting from analyzing *RC* and *RL* circuits are of the first order. Hence, the circuits are collectively known as ***first-order*** circuits.

A first-order circuit is characterized by a first-order differential equation.

In addition to there being two types of first-order circuits (*RC* and *RL*), there are two ways to excite the circuits.

1) The first way is by initial conditions of the storage elements in the circuits. In these so-called ***source-free circuits***, we assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors. Although source free circuits are by definition free of independent sources, they may have dependent sources.
2) The second way of exciting first-order circuits is by independent sources (dc and ac sources).

2) The Source-Free RC Circuit

A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig.2.1. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Now to determine the circuit response. Since the capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is

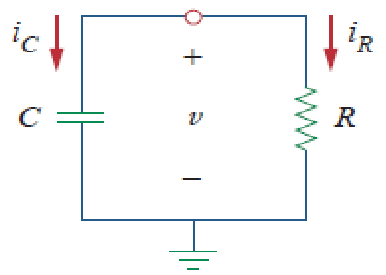


Fig.2.1.source-free RC circuit

$$v(0) = V_0 \qquad \ldots(2.1)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2 \qquad \ldots(2.2)$$

Applying KCL at the top node of the circuit in Fig. 2.1 yields

$$i_C + i_R = 0 \qquad \ldots(2.3)$$

By definition, $i_C = Cdv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad \ldots(2.4a)$$

$$\therefore \frac{dv}{dt} + \frac{v}{RC} = 0 \qquad \ldots(2.4b)$$

This is a ***first-order differential equation***, since only the first derivative of u is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt \qquad \ldots(2.5)$$

Integrating both sides, we get

$$ln\ v = -\frac{t}{RC} + ln\ A$$

where $ln\ A$ is the integration constant. Thus,

$$ln\frac{v}{A} = -\frac{t}{RC} \qquad \ldots(2.6)$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(O) = A = V_0$. Hence,

$$v(t) = V_0e^{-t/RC} \qquad \ldots(2.7)$$

This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the ***natural response*** of the circuit.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in Fig.2.2. Note that at $t = 0$, we have the correct initial condition as in Eq. (2.1). As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the ***time constant***, denoted by $\boldsymbol{\tau}$, the lowercase Greek letter tau.

The time constant of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

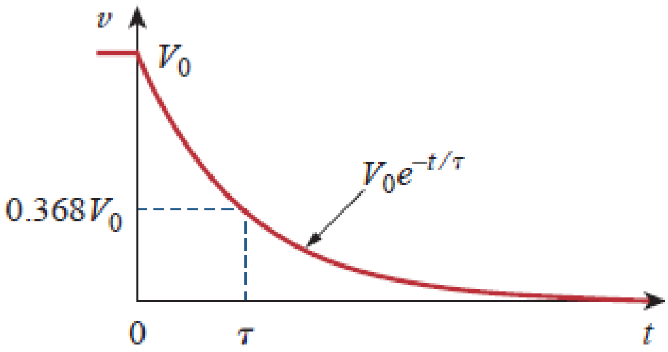


Fig.2.2. The voltage response of the RC circuit.

This implies that at $t = \tau$, Eq. (2.7) becomes

$$V_0e^{-\tau/RC} = V_0e^{-1} = 0.368V_0$$

or

$$\boxed{\tau = RC} \quad \ldots(2.8)$$

In terms of the time constant, Eq. (2.7) can be written as

$$\boxed{v(t) = V_0e^{-t/\tau}} \quad \ldots(2.9)$$

It is evident from **Table 7.1** that the voltage $v(t)$ is less than 1 percent of V_0 after $\mathbf{5\tau}$ (five time constants). ***Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants.*** In other words, it takes $\mathbf{5\tau}$ for the circuit to reach its final state or steady state when no changes take place with time. ***Notice that for every time interval of*** $\boldsymbol{\tau}$***, the voltage is reduced by 36.8 percent of its previous value,*** $\boldsymbol{v(t+\tau) = v(t)/e = 0.368v(t)}$ ***, regardless of the value of*** $\boldsymbol{t}$.

TABLE 2.1

Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state (this is illustrated in Fig. 2.3***). At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.***

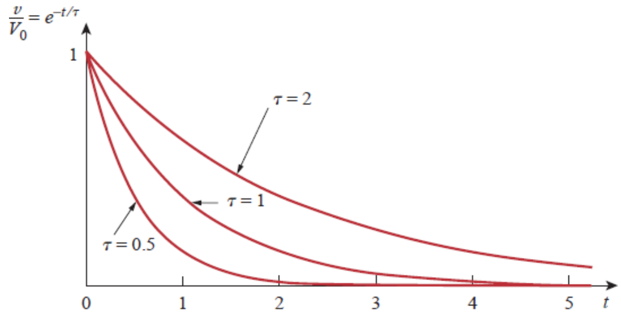


Figure 2.3. Plot of $v(t)/V_0 = e^{-t/\tau}$ for various values of the time constant.

With the voltage $v(t)$ in Eq. (2.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \quad \ldots(2.10)$$

The power dissipated in the resistor is

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau} \quad \ldots(2.11)$$

The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt = -\frac{\tau V_0^2}{2R} e^{-2t/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}),\ \tau = RC \quad .(2.12)$$

Notice that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} C V_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

With these two items, we obtain the response as the capacitor voltage $v_C(t) = v(t) = v(0)e^{-t/\tau}$ other variables (capacitor current i_C, resistor voltage v_R, and resistor current i_R) can be determined. ***In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor C and find $R = R_{Th}$ at its terminals.***

Example 1: In Fig. Fig.1, let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

Solution:

We first need to make the circuit in Fig.1 conform with the standard RC circuit in Fig.2.1. We find the equivalent resistance or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage v_C. From this, we can determine v_x and i_x.

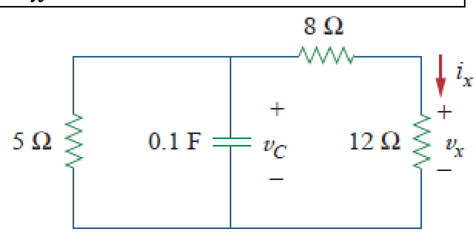


Fig.1

$$R_{eq} = (8 + 12) \parallel 5 \ \Rightarrow\ \therefore R_{eq} = \frac{20 \times 5}{20 + 5} = 4\Omega$$

$\therefore$ the equivalent circuit is as shown in Fig.2.

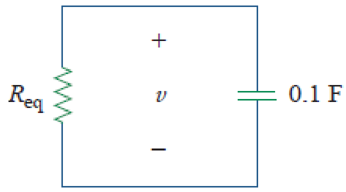


Fig.2

$\tau = R_{eq}C = 4(0.1) = 0.4s$

$v = v(0)e^{-t/\tau} = 15e^{-t/04}V. v_C = v = 15e^{-2.5t}V$

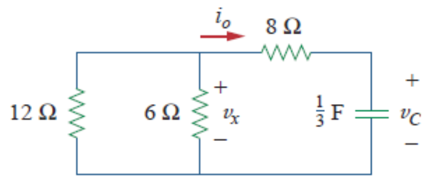
use voltage division to get v_x so,

$v_x = \frac{12}{12+8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}V$

$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}A$

H.W.1: Let $v_C(0) = 45$ V. Determine v_C, v_x. and i_o for $f \geq 0$.

Answer:



$\mathbf{45e^{-0.25t}V. 15e^{-025t}V. -3.75e^{-0.25t}A}$.

Example 2: The switch in the circuit in Fig.1 has been closed fora long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

Solution:

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc , as represented in Fig.2(a). Using voltage division

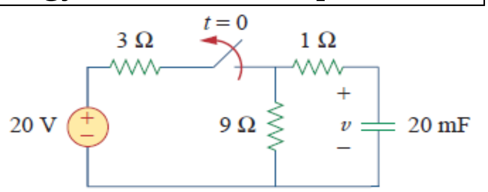


Fig. 1

$v_C(t) = \frac{9}{9+3}(20) = 15V. t < 0$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$v_C(0) = V_0 = 15V$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig.2 (b). [Notice that the RC circuit in Fig.2 (b) is source free; the independent source in Fig.1 is needed to provide V_0 or the initial energy in the capacitor.]

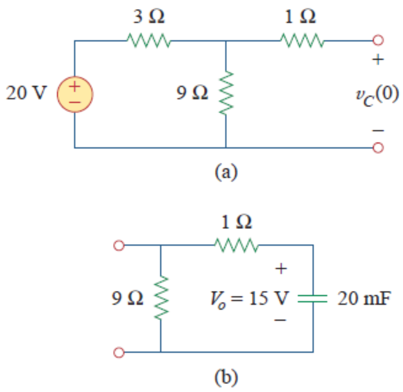


Fig.2

$R_{eq} = 1 + 9 = 10\Omega$, $\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0 \cdot 2s$

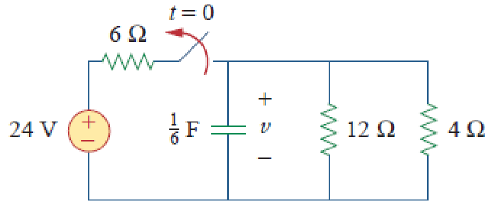
Thus, the voltage across the capacitor for $t \geq 0$ is

$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0\cdot 2}V = 15e^{-5t}V$

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25J$$

H.W.2: If the switch in Fig. shown opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

Answer: $\mathbf{8e^{-2t}V.\ 5.33J}$



3) The Source-Free RL Circuit

Our goal is to determine the circuit response (current $i(t)$ through the inductor). We select the inductor current as the response in order to take advantage of the idea $\underline{i}$ that the inductor current cannot change instantaneously. At $t = 0$, we assume that the inductor has an initial current I_0, or with the corresponding energy stored in the inductor as

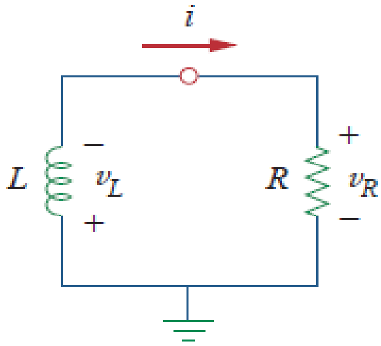


Fig.3.1 A source-free RL circuit

$$i(0) = I_o \tag{3.1}$$

$$w(0) = \frac{1}{2}LI_0^2 \tag{3.2}$$

Applying KVL around the loop in Fig.3.1,

$$v_L + v_R = 0 \tag{3.3}$$

But $v_L = Ldi/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0 \quad \Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = 0 \tag{3.4}$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt \quad \Rightarrow \quad \ln i\,\Big|_{I_o}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \Rightarrow \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

$$\therefore \ln\frac{i(t)}{I_0} = -\frac{Rt}{L} \tag{3.5}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \tag{3.6}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig.3.2.

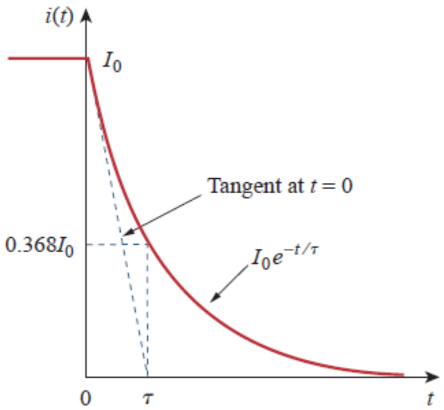


Fig.3.2. The current response of the RL circuit

It is evident from Eq. (3.6) that the time constant for the RL circuit is

$$\boxed{\tau = \frac{L}{R} \quad \text{in (s)}} \tag{3.7}$$

$$\therefore i(t) = I_0 e^{-t/\tau} \tag{3.8}$$

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{3.9}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{3.10}$$

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2\, R e^{-2t/\tau} dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}) \tag{3.11}$$

Note that as $t \to \infty$,$w_R(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in Eq. (3.2). Again, energy initially stored in the inductor is eventually dissipated in the resistor.

With the two items, we obtain the response as the inductor current $i_L(t) = i(t) = i(0)e^{-t/\tau}$. Once we determine the inductor current i_L, other variables (inductor voltage v_L, resistor voltage v_R, and resistor current i_R) can be obtained. *Note that in general, **R** in Eq. (3.7) is the Thevenin resistance at the terminals of the inductor.*

Example 3: Assuming that $i(O) = 10A$, calculate $i(t)$ and $i_x(t)$ in the circuit of Fig.1.

Solution:

There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductor terminals and then use Eq. (3.8). The other way is to start from scratch by using Kirchhoff's voltage law. Whichever approach is taken, it is always better to first obtain the inductor current.

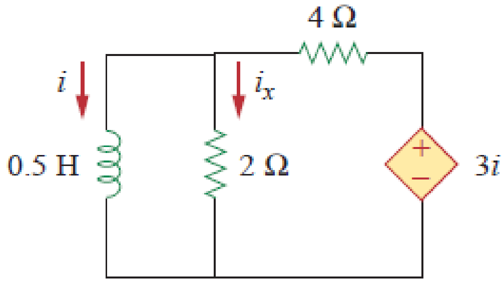


Fig.1

METHOD 1 The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1V$ at the inductor terminals a-b, as in Fig. 2(a). (We could also insert a 1-A current source at the terminals.) Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \Rightarrow i_1 - i_2 = -\frac{1}{2} \tag{1}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \Rightarrow i_2 = \frac{5}{6}i_1 \tag{2}$$

Substituting Eq. (2) into Eq. (1) gives

$$i_1 = -3A \quad i_o = -i_1 = 3A$$

$$\therefore R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}s$$

$$i(t) = i(0)e^{-\frac{t}{\tau}} = 10e^{-\left(\frac{2}{3}\right)t}A \quad t > 0$$

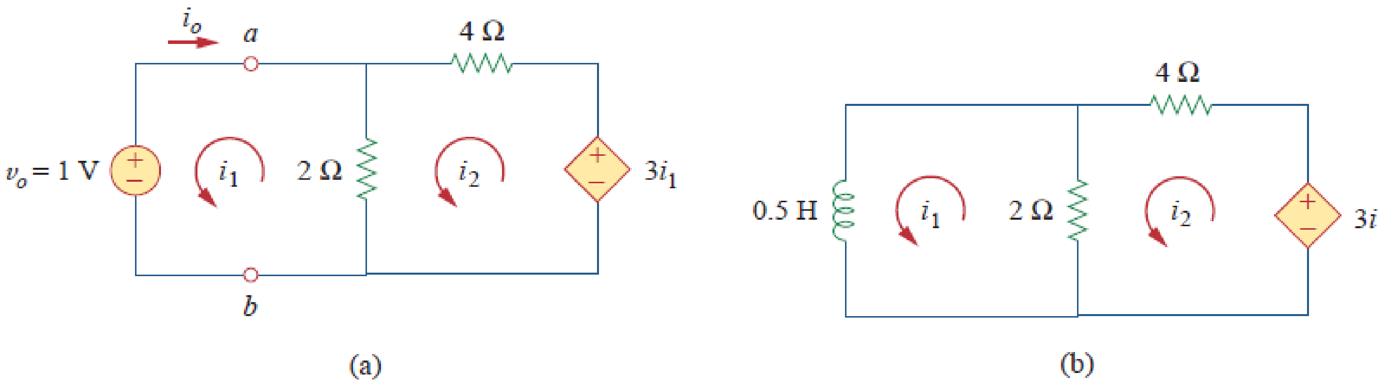


Fig. 2

METHOD 2 We may directly apply KVL to the circuit as in Fig. 2(b).

For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

$$\therefore \ \frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \qquad (1)$$

For loop 2,

$$6\,i_2 - 2i_1 - 3i_1 = 0 \ \Rightarrow \ i_2 = \frac{5}{6}i_1 \qquad (2)$$

Substituting Eq. (2) into Eq. (1) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0 \qquad \Rightarrow \qquad \frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i|_{i(0)}^{i(t)} = -\frac{2}{3}t|_0^t \qquad \Rightarrow \qquad \ln\frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{-\left(\frac{2}{3}\right)t} = 10e^{-\left(\frac{2}{3}\right)t}A \qquad t > 0$$

which is the same as by **Method 1**.

$$v_L = L\frac{di}{dt} = 0.5(10)(-\frac{2}{3})e^{-(2/3)t} = -\frac{10}{3}e^{-\left(\frac{2}{3}\right)t} \quad V$$

Since the inductor and the $2-\Omega$ resistor are in parallel, ($v_L = v_R$)

$$i_x(t) = \frac{v_R}{2} = -1.6667e^{-\left(\frac{2}{3}\right)t}A \qquad t > 0$$

H.W.3: Find i and v_x in the circuit of Fig. 1. Let $i(O) = 5$ A.

Answer: $5e^{-4t}V \quad -20e^{-4t}V$.

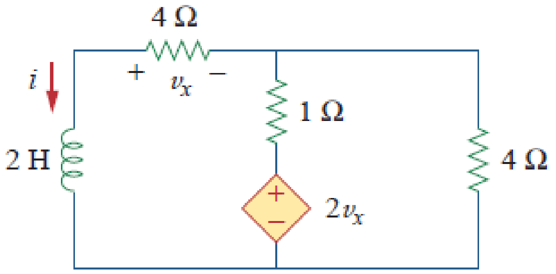


Fig. 1.

Example 4: The switch in the circuit of Fig. 1. has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

Solution:

When $t < 0$, the switch is closed, and the inductor acts as a short circuit to dc. The $16-\Omega$ resistor is short-circuited; the resulting circuit is shown in Fig. 2(a). To get i_1 in Fig. 2(a), we combine the $4-\Omega$ and $12-\Omega$ resistors in parallel to get

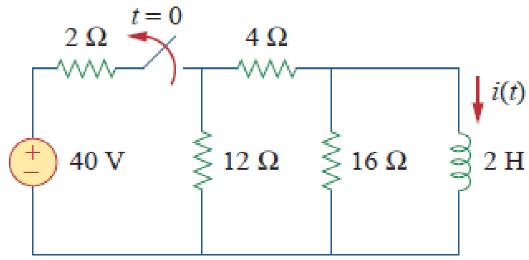


Fig. 1.

$$\frac{4\times 12}{4+12} = 3\Omega$$

$$i_1 = \frac{40}{2+3} = 8A$$

We obtain $i(t)$ from i_1 in Fig. 2(a) using current division, by writing

$$i(t) = \frac{12}{12+4} i_1 = 6A, \quad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(O) = i(0^-) = 6A$$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the source-free RL circuit in Fig. 2(b).

$$R_{eq} = (12+4)\|16 = 8\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4}s$$

$$\therefore i(t) = i(0)e^{-t/\tau} = 6e^{-4t}A$$

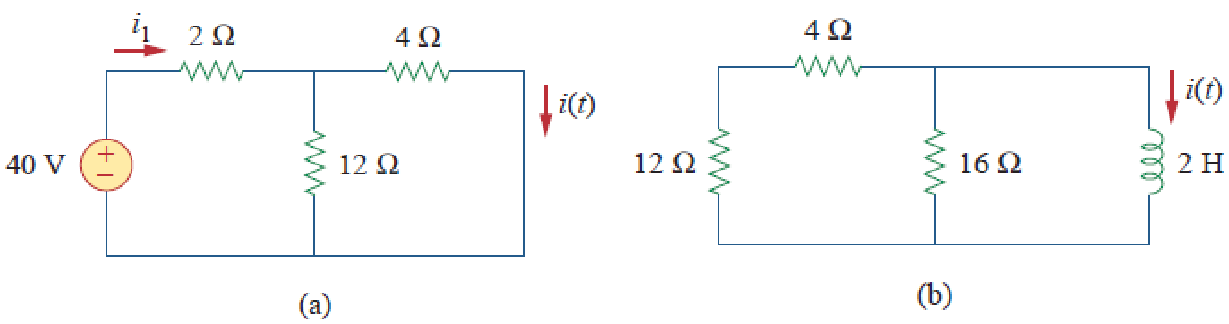


Fig. 2 Solving the circuit of Fig.1: (a) for t < 0, (b) for t > 0.

H.W.4: For the circuit in Fig.1, find $i(t)$ for $t > 0$.

Answer: $2e^{-2t}$ A, t > 0

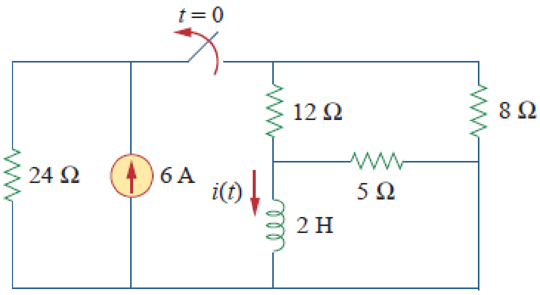


Fig.1

Example 5: In the circuit shown in Fig.1, find i_o, v_o, and i for all time, assuming that the switch was open for a long time.

Solution: It is better to first find the inductor current i and then obtain other quantities from it. For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the $6-\Omega$ resistor is short-circuited, so that we have the circuit shown in Fig.2 (a). Hence,

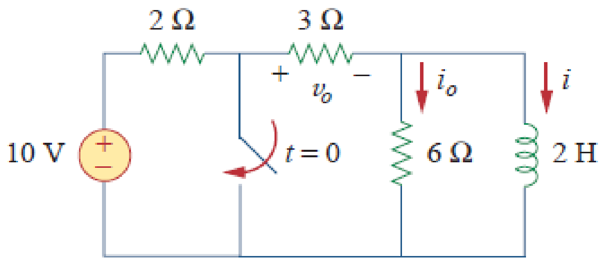


Fig.1

$$i_o = 0, \text{ and } \quad i(t) = \frac{10}{2+3} = 2A. \qquad t < 0$$

$$v_o(t) = 3i(t) = 6V. \quad t < 0$$

Thus, $i(O) = 2$.

For $t > 0$, the switch is closed, so that the voltage source is short- circuited. We now have a source-free RL circuit as shown in Fig.2 (b). At the inductor terminals,

$$R_{Th} = 3||6 = 2\Omega$$

$$\tau = \frac{L}{R_{Th}} = 1s$$

$$i(t) = i(0)e^{-\frac{t}{\tau}} = 2e^{-t}A. \qquad t > 0$$

Since the inductor is in parallel with the $6-\Omega$ and $3-\Omega$ resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}V. \qquad t > 0$$

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}A. \qquad t > 0$$

Thus, for all time,

$$i_o(t) = \begin{cases} 0A & t < 0 \\ -\frac{2}{3}e^{-t}\ A & t > 0 \end{cases} \quad , \quad v_o(t) = \begin{cases} 6\ V & t < 0 \\ 4e^{-t}\ V & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2A & t < 0 \\ 2e^{-t}A & t \geq 0 \end{cases}$$

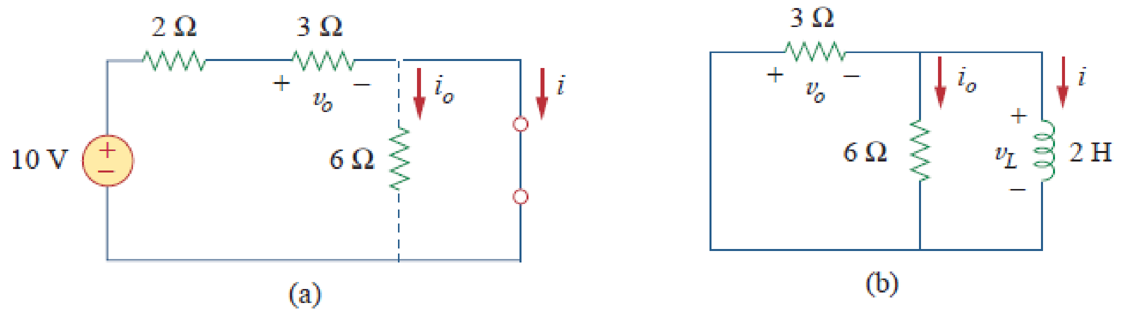


Fig.2. The circuit in Fig.1 for: (a) $t < 0$ (b) $t > 0$

We notice that the inductor current is continuous at $t = 0$, while the current through the $6 - \Omega$ resistor drops from 0 to $-2/3$ at $t = 0$, and the voltage across the $3 - \Omega$ resistor drops from 6 to 4 at $t = 0$. We also notice that the time constant is the same regardless of what the output is defined to be. Fig.3. plots i and i_o.

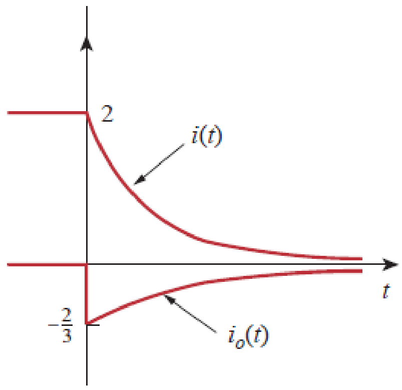


Fig.3. A plot of i and i_o

H.W.5: Determine i, i_o, and v_o for all t in the circuit shown in Fig.1. Assume that the switch was closed for a long time. It should be noted that opening a switch in series with an ideal current source creates an infinite voltage at the current source terminals. Clearly this is impossible. For the purposes of problem solving, we can place a shunt resistor in parallel with the source (which now makes it a voltage source in series with a resistor). In more practical circuits, devices that act like current sources are, for the most part, electronic circuits. These circuits will allow the source to act like an ideal current source over its operating range but voltage-limit it when the load resistor becomes too large (as in an open circuit).

Answer: $i = \begin{cases} 12A & t < 0 \\ 12e^{-2t}A & t \geq 0 \end{cases}$, $i_o = \begin{cases} 6A & t < 0 \\ -4e^{-2t}A & t > 0 \end{cases}$, $v_o = \begin{cases} 24V & t < 0 \\ 8e^{-2t}V & t > 0 \end{cases}$

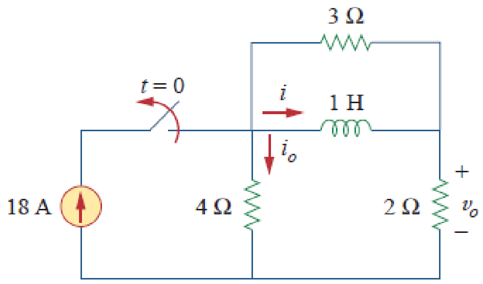


Fig.1

4) Singularity Functions

A basic understanding of singularity functions will help us make sense of the response of first-order circuits to a sudden application of an independent dc voltage or current source. Singularity functions (also called ***switching functions***) are very useful in circuit analysis. They serve as good approximations to the switching signals that arise in circuits with switching operations. They are helpful in the neat, compact description of some circuit phenomena, especially the step response of RC or RL circuits .

Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

The three most widely used singularity functions in circuit analysis are:

1. the ***unit step*** function.
2. the ***unit impulse*** function.
3. the ***unit ramp*** function.

4.1) Unit step function

The unit step function ***u(t)*** is 0 for negative values of t and 1 for positive values of t.

The unit step function is undefined at $t = 0$, where it changes abruptly from 0 to 1 (shown in Fig.4.1). It is dimensionless, like other mathematical functions such as sine and cosine. In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{4.1}$$

If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, which is the same as saying that $u(t)$ is delayed by t_0 seconds, as shown in Fig.4.2(a), the unit step function becomes

$$u(t - t_o) = \begin{cases} 0, & t < t_o \\ 1, & t > t_o \end{cases} \tag{4.2}$$

If the change is at $t = -t_0$, meaning that $u(t)$ is advanced by t_0 seconds, Fig.4.2(b), the unit step function becomes

$$u(t + t_o) = \begin{cases} 0, & t < -t_o \\ 1, & t > -t_o \end{cases} \tag{4.3}$$

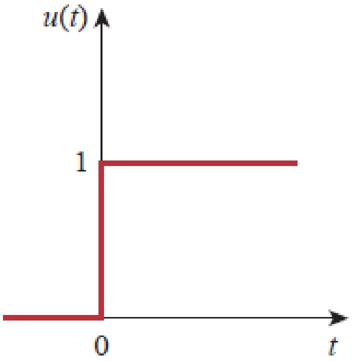


Fig.4.1The unit step function.

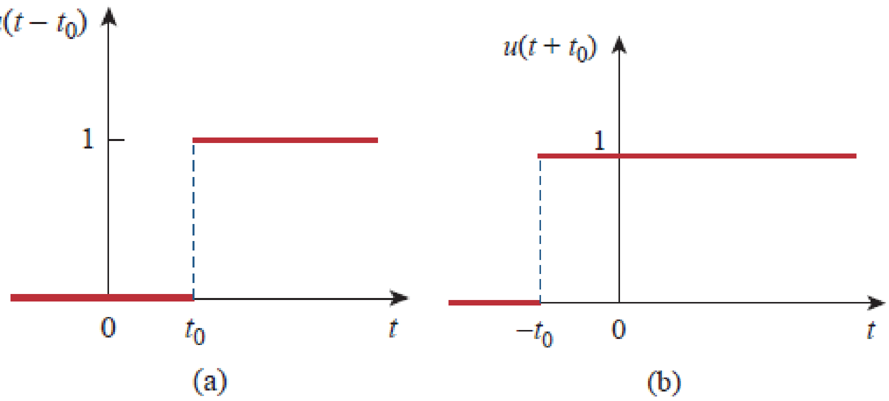


Fig.4.2 The unit step function
(a) delayed by t_0 (b) advanced by t_0.

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \tag{4.4}$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \tag{4.5}$$

If we let $t_0 = 0$ then $v(t)$is simply the step voltage $V_0 u(t)$. A voltage source of $V_0 u(t)$ is shown in Fig.4.3 (a); its equivalent circuit is shown in Fig.4.3 (b). It is evident in Fig.4.3 (b) that terminals *a*-*b* are short circuited ($v = 0$) for $t < 0$ and that $v = V_0$ appears at the terminals

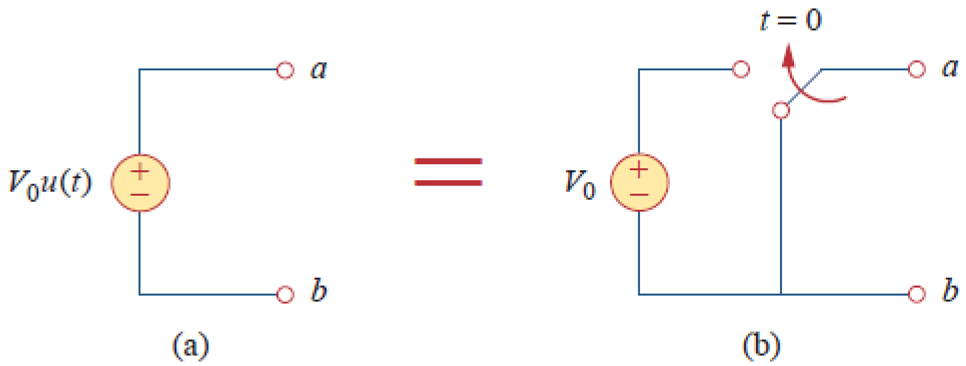


Fig.4.3 (a) A voltage source of $V_0u(t)$ (b) its equivalent circuit.

for $t > 0$. Similarly, a current source of $I_0u(t)$ is shown in Fig.4.4 (a), while its equivalent circuit is in Fig.4.4 (b). Notice that for $t < 0$, there is an open circuit $(i = 0)$, and that $i = I_0$ flows for $t > 0$.

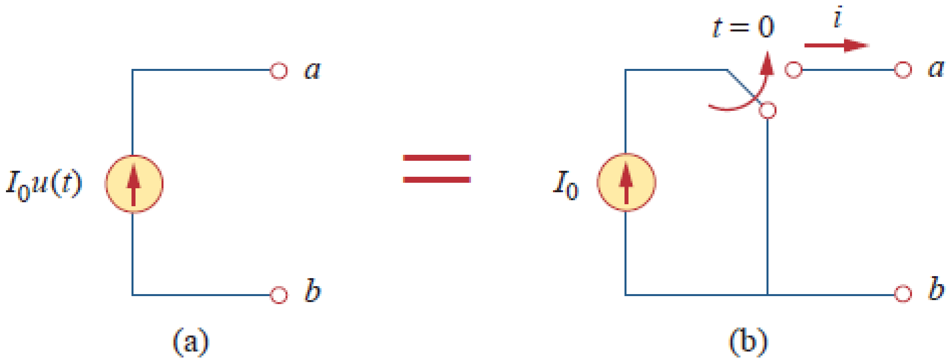


Fig.4.4 (a) A current source of $I_0u(t)$ (b) its equivalent circuit.

4.2) Unit impulse function

The derivative of the unit step function $u(t)$ is the ***unit impulse function*** (*or delta function*) $\boldsymbol{\delta(t)}$, which we write as

$$\delta(t) = \frac{d}{dt}\,u(t) = \begin{cases} 0 & t < 0 \\ Undefined & t = 0 \\ 0 & t > 0 \end{cases} \qquad (4.6)$$

The unit impulse function $\delta(t)$ is zero everywhere except at t = 0, where it is undefined.

Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources. Although the unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematical tool.

The unit impulse may be regarded as an applied or resulting shock. It may be visualized as a very short duration pulse of unit area. This may be expressed mathematically as

$$\int_{0^-}^{0^+} \delta(t)dt = 1 \tag{4.7}$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$. For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function, as in Fig.4.5. The unit area is known as the *strength* of the impulse function. When an impulse function has a strength other than unity, the area of the impulse is equal to its strength. For example, an impulse function $10\delta(t)$ has an area of 10. Fig.4.6 shows the impulse functions $5\delta(t+2)$, $10\delta(t)$, and $-4\delta(t-3)$.

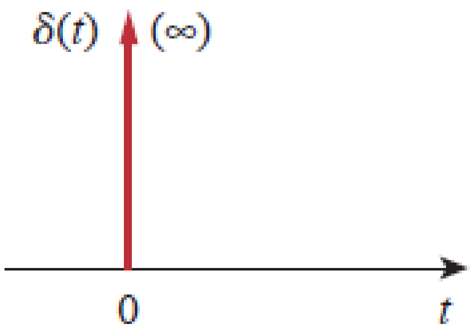


Fig.4.5 The unit impulse function.

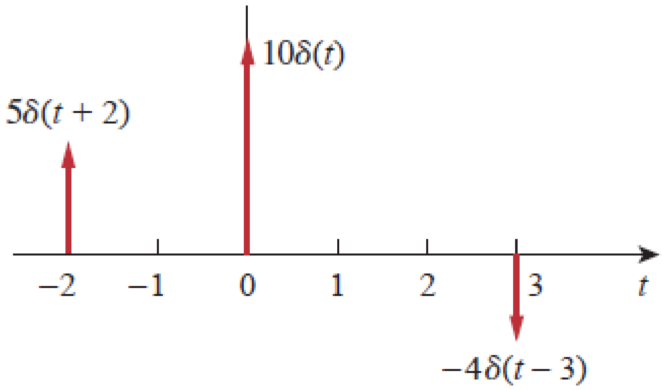


Fig.4.6 Three impulse functions.

To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t-t_0)dt \tag{4.8}$$

where $a < t_0 < b$. Since $\delta(t-t_0) = 0$ except at $t = t_0$, the integrand is zero except at t_0. Thus,

$$\int_a^b f(t)\delta(t-t_0)dt = \int_a^b f(t_0)\delta(t-t_0)dt = f(t_0)\int_a^b \delta(t-t_0)dt = f(t_0)$$

$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0) \tag{4.9}$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This is a highly useful property of the impulse function known as the ***sampling*** or ***sifting*** property. The special case of Eq. (4.8) is for $t_0 = 0$. Then Eq. (4.9) becomes

$$\int_{0^-}^{0^+} f(t)\delta(t)dt = f(0) \tag{4.10}$$

4.3) Unit ramp function

Integrating the unit step function $u(t)$ results in the ***unit ramp function*** $\boldsymbol{r(t)}$;

$$r(t) = \int_{-\infty}^{t} u(t)dt = tu(t) \tag{4.11}$$

or

$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases} \tag{4.12}$$

The unit ramp function is zero for negative values of t and has a unit slope for positive values of t.

Fig.4.7 shows the unit ramp function. In general, a ramp is a function that changes at a constant rate.

The unit ramp function may be delayed or advanced as shown in Fig.4.8. For the delayed unit ramp function,

$$r(t - t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & r \ge t_0 \end{cases} \tag{4.13}$$

and for the advanced unit ramp function,

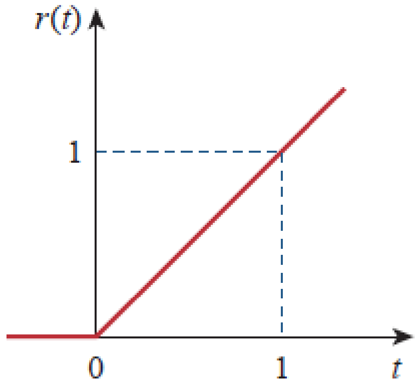
$$r(t + t_0) = \begin{cases} 0, & t \le -t_0 \\ t + t_0, & t \ge -t_0 \end{cases} \tag{4.14}$$

We should keep in mind that the three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt} \tag{4.15}$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt, \qquad r(t) = \int_{-\infty}^{t} u(t)dt \tag{4.16}$$



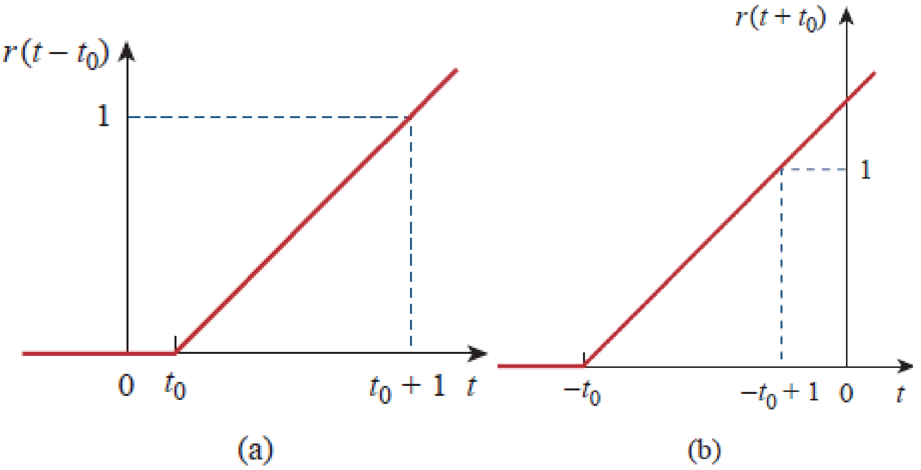


Fig.4.7 The unit ramp function.

Fig.4.8 The unit ramp function
(a) delayed by t_0 (b) advanced by t_0.

Example 6: Express the voltage pulse in Fig.1 in terms of the unit step. Calculate its derivative and sketch it.

Solution: The type of pulse in Fig.1 is called the ***gate function***. It may be regarded as a step function that switches on at one value of t and switches off at another value of t.

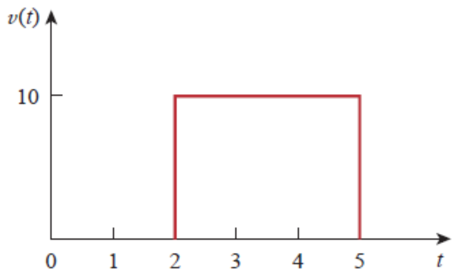


Fig.1

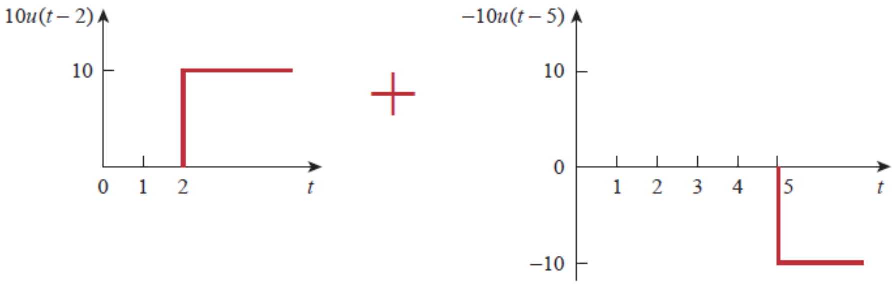
This gate function switches on at $t = 2s$ and switches off at $t = 5s$. It consists of the sum of two unit step functions as shown in Fig.2 (a). From the figure, it is evident that

$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

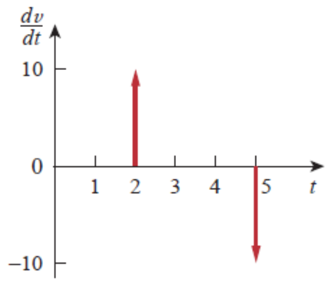
Taking the derivative of this gives

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

which is shown in Fig.2 (b). We can obtain Fig.2 (b). directly from Fig.1. by simply observing that there is a sudden increase by 10 V at $t = 2s$ leading to $10\delta(t-2)$. At $t = 5s$, there is a sudden decrease by 10 V leading to $-10V\ \delta(t-5)$.



(a)



(b)

Fig.2 (a) Decomposition of the pulse in Fig.1, (b) derivative of the pulse in Fig.1.

H.W. 6: Express the current pulse in Fig.1in terms of the unit step. Find its integral and sketch it.

Answer: $10[u(t) - 2u(t-2) + u(t-4)]$, $10[r(t) - 2r(t-2) + r(t-4)]$. See Fig.2.

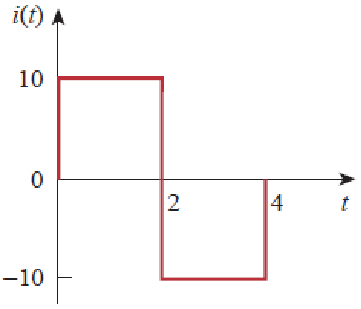


Fig.1

Fig.2

Example 7: Express the ***sawtooth*** function shown in Fig.1in terms of singularity functions.

Solution:

There are three ways of solving this problem. The first method is by mere observation of the given function, while the other methods involve some graphical manipulations of the function.

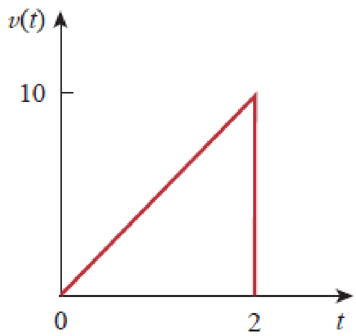


Fig.1

METHOD 1 By looking at the sketch of $v(t)$ in Fig.1, it is not hard to notice that the given function $v(t)$ is a combination of singularity functions. So we let

$$v(t) = v_1(t) + v_2(t) + \cdots \tag{1}$$

The function $v_1(t)$ is the ramp function of slope 5, shown in Fig.2 (a); that is,

$$v_1(t) = 5r(t) \tag{2}$$

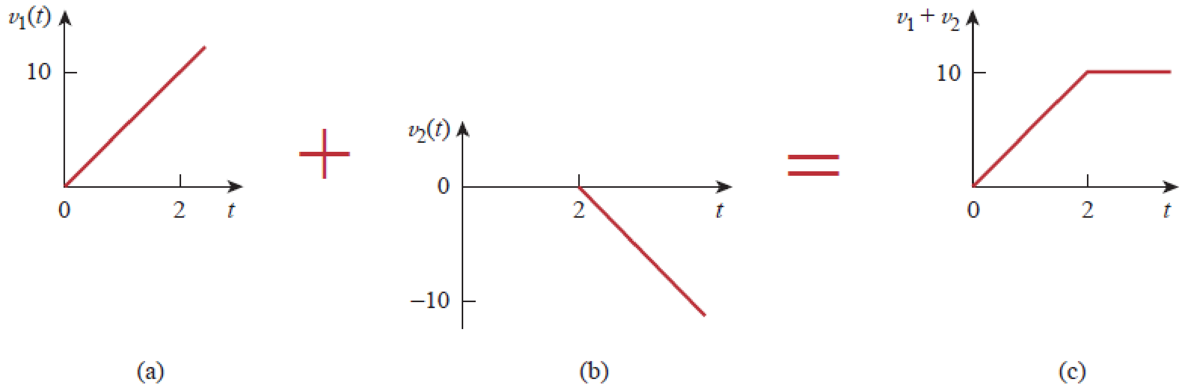


Fig.2 Partial decomposition of $v(t)$ in Fig.1.

Since $v(t)$ goes to infinity, we need another function at $t = 2s$ in order to get $v(t)$. We let this function be v_2, which is a ramp function of slope −5, as shown in Fig.2 (b); that is,

$$v_2(t) = -5r(t-2) \tag{3}$$

Adding v_1 and v_2 gives us the signal in Fig.2 (c). Obviously, this is not the same as $v(t)$ in Fig.1. But the difference is simply a constant 10 units for $t > 2s$. By adding a third signal v_3, where

$$v_3 = -10u(t-2) \tag{4}$$

we get $v(t)$, as shown in Fig.3. Substituting Eqs. (2) through (4) into Eq. (1) gives

$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

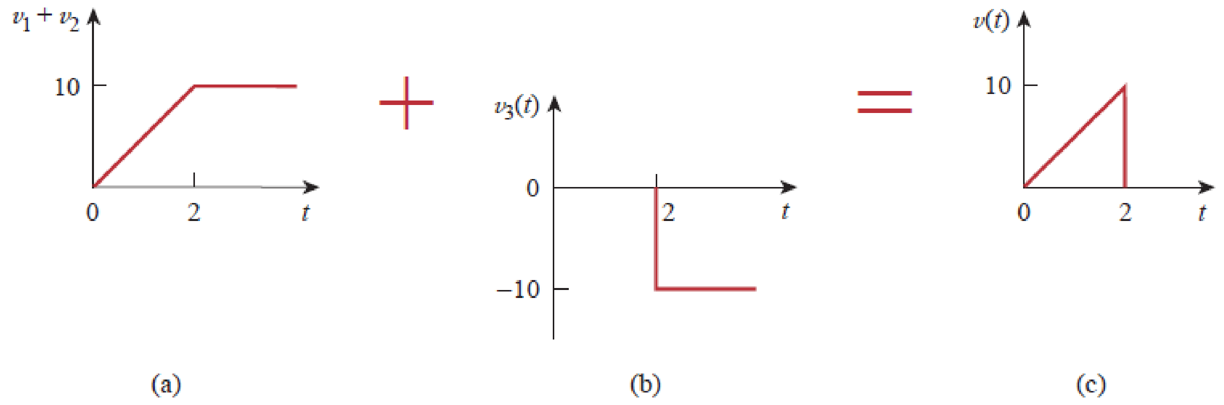


Fig.3 Complete decomposition of $v(t)$ in Fig.1.

METHOD 2 A close observation of Fig.1 reveals that $v(t)$ is a multiplication of two functions: a ramp function and a gate function. Thus,

$$v(t) = 5t[u(t) - u(t-2)] = 5tu(t) - 5tu(t-2)$$

$$= 5r(t) - 5(t-2+2)u(t-2) = 5r(t) - 5(t-2)u(t-2) - 10u(t-2)$$

$$= 5r(t) - 5r(t-2) - 10u(t-2)$$

the same as before.

METHOD 3 This method is similar to **Method 2**. We observe from Fig.1 that $v(t)$ is a multiplication of a ramp function and a unit step function, as shown in Fig.4. Thus,

$$v(t) = 5r(t)u(-t+2)$$

If we replace $u(-t)$ by $1 - u(t)$, then we can replace $u(-t+2)$ by $1 - u(t-2)$. Hence,

$$v(t) = 5r(t)[1 - u(t-2)]$$

which can be simplified as in **Method 2** to get the same result.

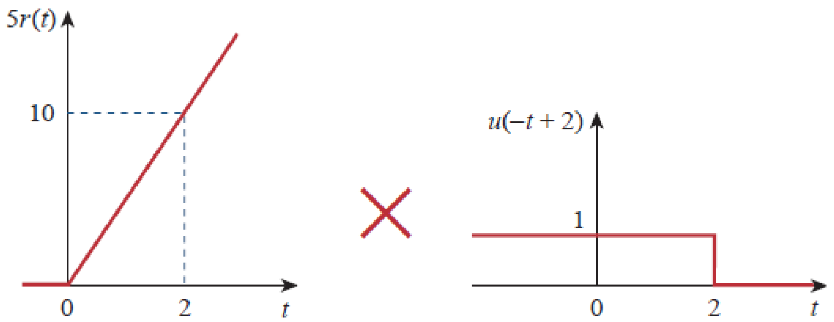


Fig.4. Decomposition of $v(t)$ in Fig.1.

H.W. 7: Refer to Fig.1. Express $i(t)$ in terms of singularity functions.

Answer: $\mathbf{2u(t) - 2r(t) + 4r(t-2) - 2r(t-3)}$ **.**

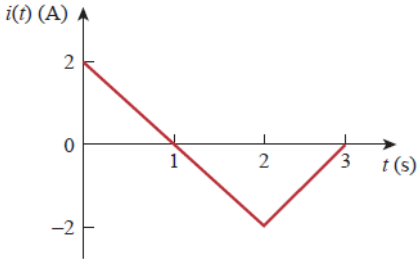


Fig.1

Example 8: Given the signal

$$g(t) = \begin{cases} 3, & t < 0 \\ -2, & 0 < t < 1 \\ 2t - 4, & t > 1 \end{cases}$$

express $g(t)$ in terms of step and ramp functions.

Solution:

The signal $g(t)$ may be regarded as the sum of three functions specified within the three intervals $t < 0, 0 < t < 1$, and $t > 1$.

For $t < 0$, $g(t)$ may be regarded as 3 multiplied by $u(-t)$, where $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Within the time interval $0 < t < 1$, the function may be considered as -2 multiplied by a gated function $[u(t) - u(t-1)]$. For $t > 1$, the function may be regarded as $2t - 4$ multiplied by the unit step function $u(t-1)$. Thus,

$$g(t) = 3u(-t) - 2[u(t) - u(t-1)] + (2t - 4)u(t-1)$$

$$= 3u(-t) - 2u(t) + (2t - 4 + 2)u(t-1)$$

$$= 3u(-t) - 2u(t) + 2(t-1)u(t-1) = 3u(-t) - 2u(t) + 2r(t-1)$$

One may avoid the trouble of using $u(-t)$ by replacing it with $1 - u(t)$. Then

$$g(t) = 3[1 - u(t)] - 2u(t) + 2r(t-1) = 3 - 5u(t) + 2r(t-1)$$

Alternatively, we may plot $g(t)$ and apply **Method 1** from **Example 7**.

H.W. 8: If

$$h(t) = \begin{cases} 0, & t < 0 \\ 8, & 0 < t < 2 \\ 2t + 6, & 2 < t < 6 \\ 0, & t > 6 \end{cases}$$

express $h(t)$ in terms of the singularity functions.

Answer: $\mathbf{8u(t) + 2u(t-2) + 2r(t-2) - 18u(t-6) - 2r(t-6)}$.

Example 9: Evaluate the following integrals involving the impulse function:

$$\int_{0}^{10} (t^2 + 4t - 2)\delta(t - 2)dt$$

$$\int_{-\infty}^{\infty} [\delta(t - 1)e^{-t} \cos t + \delta(t + 1)e^{-t} \sin t]dt$$

Solution:

For the first integral, we apply the sifting property in Eq. (4.9).

$$\int_{0}^{10}(t^2 + 4t - 2)\delta(t - 2)dt = (t^2 + 4t - 2)|_{t=2} = 4 + 8 - 2 = 10$$

Similarly, for the second integral,

$$\int_{-\infty}^{\infty} [\delta(t - 1)e^{-t} \cos t + \delta(t + 1)e^{-t} \sin t]dt$$

$$= e^{-t} \cos t|_{t=1} + e^{-t} \sin t|_{t=-1} = e^{-1} \cos 1 + e^{1} \sin (-1) = 0.1988 - 2.2873 = -2.0885$$

H.W. 9: Evaluate the following integrals:

$$\int_{-\infty}^{\infty} (t^3 + 5t^2 + 10)\delta(t + 3)dt, \qquad \int_{0}^{10} \delta\,(t - \pi) \cos 3t dt$$

Answer: 28, $-$1.

5) Step Response of an RC Circuit

When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a ***step response.***

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig.5.1 (a) which can be replaced by the circuit in Fig.5.1 (b), where V_s is a constant dc voltage source. We assume an initial voltage V_0 on the capacitor, although this is not necessary for the step response.

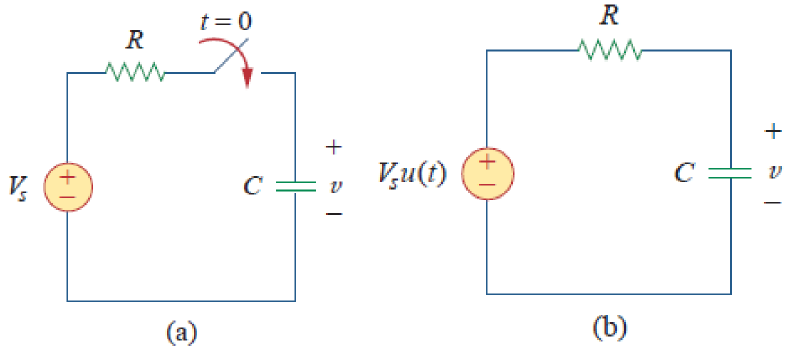


Fig.5.1 An RC circuit with voltage step input.

Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \qquad (5.1)$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

$$\therefore \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \qquad (5.2)$$

where v is the voltage across the capacitor. For $t > 0$, Eq. (5.2) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad (5.3)$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

$$\therefore \frac{dv}{v - V_s} = -\frac{dt}{RC} \tag{5.4}$$

Integrating both sides and introducing the initial conditions,

$$ln\,(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$ln\,(v(t) - V_s) - ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$ln\,\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \tag{5.5}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \quad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$\therefore v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, \quad t > 0 \tag{5.6}$$

Thus,

$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \tag{5.7}$$

This is known as the ***complete response*** (or ***total response***) of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Fig.5.2.

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (5.7) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{5.8}$$

which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})u(t) \tag{5.9}$$

This is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (5.8) using $i(t) = \frac{Cdv}{dt}$. We get

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-\frac{t}{\tau}}, \qquad \tau = RC, t > 0$$

$$\therefore\ i(t) = \frac{V_s}{R}e^{-t/\tau}u(t) \tag{5.10}$$

Fig.5.3 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.

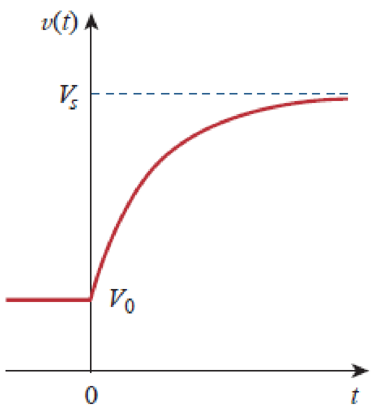


Fig.5.2 Response of an *RC* circuit with initially charged capacitor.

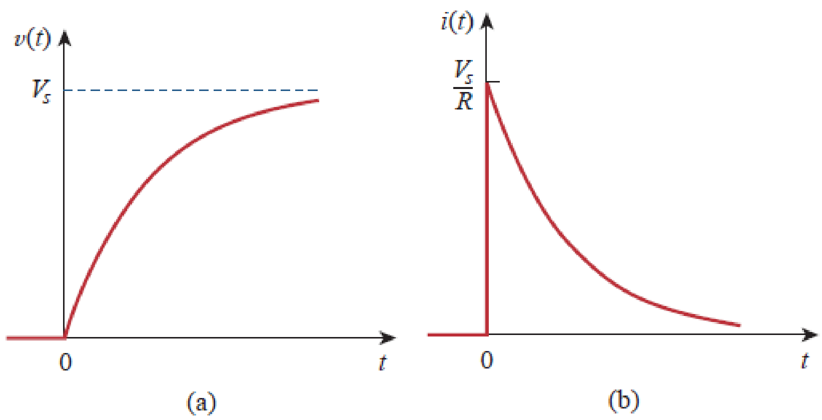


Fig.5.3 Step response of an *RC* circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an *RC* or *RL* circuit. Let us reexamine Eq. (5.6), which is more general than Eq. (5.9). It is evident that $v(t)$ has two components.

Classically there are two ways of decomposing this into two components. The first is to break it into a "***natural response and a forced response***" and the second is to break it into a "***transient response and a steady-state response.***" Starting with the natural response and forced (a) response, we write the total or complete response as

$$\boxed{\text{Complete response} = \underbrace{\text{natural response}}_{\text{stored energy}} + \underbrace{\text{forced response}}_{\text{independent source}}}$$

or

$$v = v_n - v_f \tag{5.11}$$

where $v_n = V_o e^{-t/\tau}$ and $v_f = V_s(1 - e^{-t/\tau})$

We are familiar with the natural response v_n of the circuit, as discussed response, (b) current response. in Section 2. v_f is known as the ***forced response*** because it is produced by the circuit when an external "***force***" (a voltage source in this case) is applied. It represents what the circuit is forced to do by the input excitation. The natural response eventually dies out along with the transient component of the forced response, leaving only the steady- state component of the forced response.

Another way of looking at the complete response is to break into two components—one temporary and the other permanent, i.e.,

$$\text{Complete response} = \underset{\text{temporary part}}{\text{transient response}} + \underset{\text{permanent part}}{\text{steady-state response}}$$

or

$$v = v_t - v_{ss} \tag{5.12}$$

where $v_t = (V_o - V_s)e^{-t/\tau}$ and $v_{ss} = V_s$

The ***transient response*** v_t is temporary; it is the portion of the complete response that decays to zero as time approaches infinity. Thus,

The transient response is the circuit's temporary response that will die out with time.

The ***steady-state response*** v_{ss} is the portion of the complete response that remains after the transient reponse has died out. Thus,

The steady-state response is the behavior of the circuit a long time after an external excitation is applied.

The first decomposition of the complete response is in terms of the source of the responses, while the second decomposition is in terms of the permanency of the responses. Under certain conditions, the natural response and transient response are the same. The same can be said about the forced response and steady-state response.

Whichever way we look at it, the complete response in Eq. (5.6) may be written as

$$v(t) = v(\infty) + [v(O) - v(\infty)]e^{-t/\tau} \tag{5.13}$$

where $v(O)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady- state value. Thus, to find the step response of an RC circuit requires three things:

1. The initial capacitor voltage $v(O)$.

2. The final capacitor voltage $v(\infty)$.

3. The time constant τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (5.13). This technique equally applies to RL circuits, as we shall see in the next section.

Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$, there is a time delay in the response so that Eq. (5.13) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau} \qquad (5.14)$$

where $v(t_0)$ is the initial value at $t = t_0^+$. Keep in mind that Eq. (5.13) or (5.14) applies only to step responses, that is, when the input excitation is constant.

Example 10: The switch in Fig.1 has been in position A for along time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1s$ and $4\ s$.

Solution: For $t < 0$, the switch is at position A. The capacitor acts like an open circuit to dc, but v is the same as the voltage across the $5 - k\Omega$ resistor. Hence, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

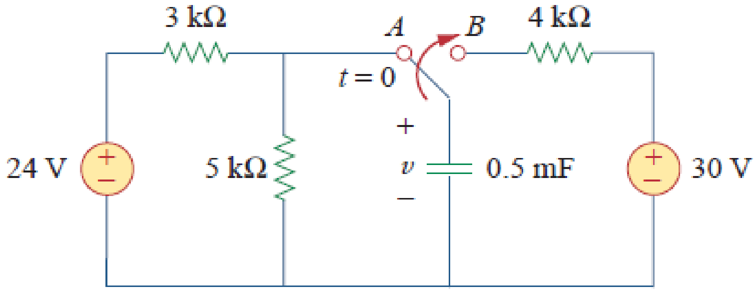


Fig.1

$$v(0^-) = \frac{5}{5+3}(24) = 15V$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15V$$

For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{Th} = 4k\Omega$, and the time constant is

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2s$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

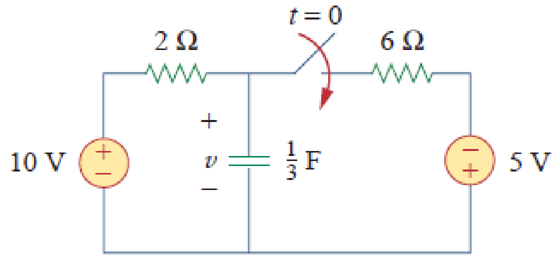
$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} = 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})V$$

At $t = 1, \Rightarrow\ v(1) = 30 - 15e^{-0.5} = 20.9V$

At $t = 4, \Rightarrow\ v(4) = 30 - 15e^{-2} = 27.97V$

H.W. 10: Find $v(t)$ for $t > 0$ in the circuit of Fig.1. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$

Answer: $(6.25 + 3.75e^{-2t})V$ **for all** $t > 0, 7.63V$.



Example 11: In Fig.1, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

Solution: The resistor current i can be discontinuous at $t = 0$, while the capacitor voltage v cannot. Hence, it is always better to find v and then obtain i from v.

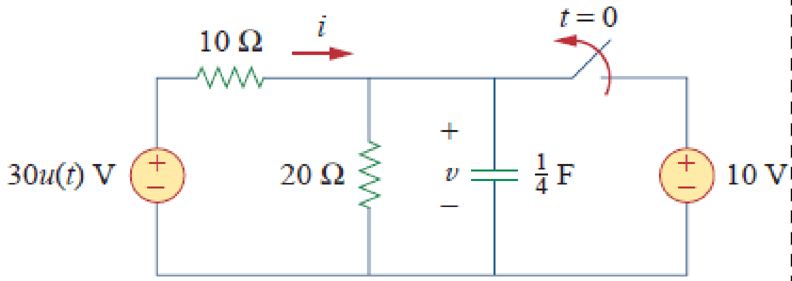


Fig.1

By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as contributing nothing to v. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit. Hence, the circuit becomes that shown in Fig.2 (a) for $t < 0$. From this circuit we obtain

$$v = 10V, \qquad i = -\frac{v}{10} = -1A$$

Since the capacitor voltage cannot change instantaneously,

$$v(O) = v(0^-) = 10V$$

For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig.2 (b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit (b) again. We obtain $v(\infty)$ by using voltage division, writing

$$v(\infty)=\frac{20}{20+10}(30)=20V$$

The Thevenin resistance at the capacitor terminals is

$$R_{Th}=10\|20=\frac{10\times 20}{30}=\frac{20}{3}\Omega$$

and the time constant is

$$\tau=R_{Th}C=\frac{20}{3}\cdot\frac{1}{4}=\frac{5}{3}s$$

Thus,

$$v(t)=v(\infty)+[v(0)-v(\infty)]e^{-t/\tau}$$

$$=20+(10-20)e^{-\left(\frac{3}{5}\right)t}$$

$$=(20-10e^{-0.6t})V$$

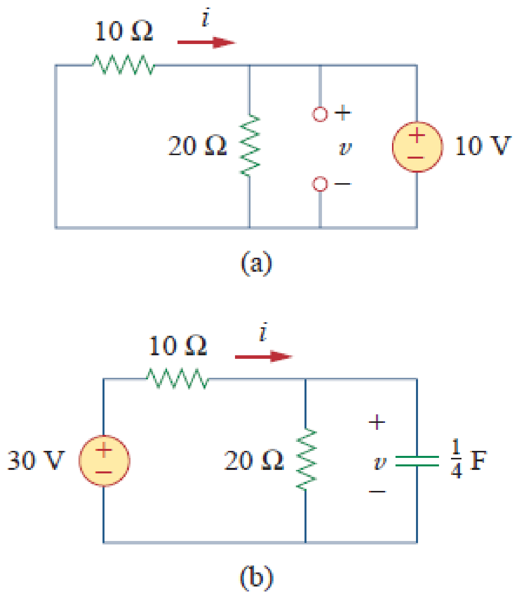


Fig.2 (a) for t<0, for t>0.

To obtain i, we notice from Fig.2 (b) that i is the sum of the currents through the $20-\Omega$ resistor and the capacitor; that is,

$$i=\frac{v}{20}+C\frac{dv}{dt}=1-0.5e^{-0.6t}+0.25(-0.6)(-10)e^{-0.6t}=(1+e^{-0.6t})A$$

Notice from Fig.2 (b) that $v+10i=30$ is satisfied, as expected. Hence,

$$v=\begin{cases}10V, & t<0\\ (20-10e^{-0.6t})V, & t\geq 0\end{cases}\quad i=\begin{cases}\text{-}1A, & t<0\\ (1+e^{-0.6t})A, & t>0\end{cases}$$

Notice that the capacitor voltage is continuous while the resistor current is not.

H.W. 11: The switch in Fig.1 is closed at $t=0$. Find $i(t)$ and $u(t)$ for all time. Note that $u(-t)=1$ for $t<0$ and 0 for $t>0$. Also, $u(-t)=1-u(t)$.

Answer:

$$i(t)=\begin{cases}0, & t<0\\ -2(1+e^{-15t})A, & t>0'\end{cases}$$

$$v=\begin{cases}20V, & t<0\\ 10(1+e^{-15t})V, & t>0\end{cases}$$

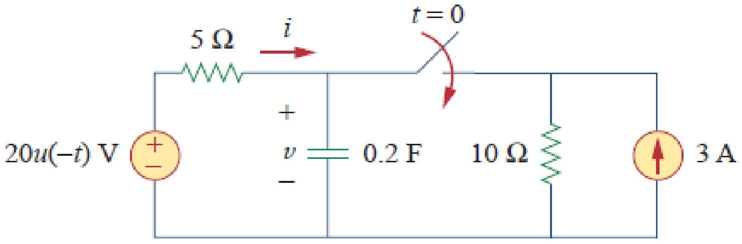


Fig.1

6) Step Response of an RL Circuit

Consider the RL circuit in Fig.6.1 (a), which may be replaced by the circuit in Fig.6.1 (b). Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (5.11) through (5.14). Let the response be the sum of the transient response and the steady-state response,

$$i = i_t + i_{ss} \tag{6.1}$$

We know that the transient response is always a decaying exponential, that is,

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \tag{6.2}$$

where A is a constant to be determined.

The steady-state response is the value of the current a long time after the switch in Fig.6.1 (a) is closed. We know that the transient response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_s appears across R. Thus, the steady-state response is

$$i_{ss} = \frac{V_s}{R} \tag{6.3}$$

Substituting Eqs. (6.2) and (6.3) into Eq. (6.1) gives

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \tag{6.4}$$

We now determine the constant A ffom the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than V_s. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \tag{6.5}$$

Thus, at $t = 0$, Eq. (6.4) becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_s}{R}$$

Substituting for A in Eq. (6.4), we get

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau} \tag{6.6}$$

This is the complete response of the RL circuit. It is illustrated in Fig.6.2. The response in Eq. (6.6) may be written as

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \tag{6.7}$$

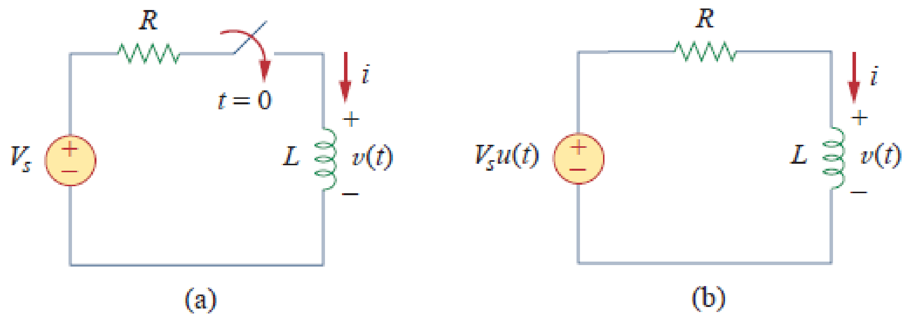


Fig.6.1 An RL circuit with a step input voltage.

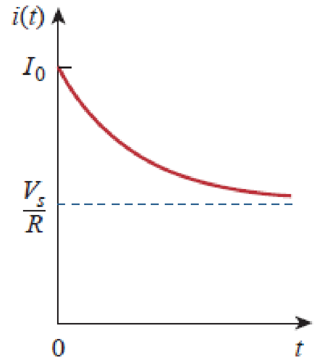


Fig.6.2 Total response of the RL circuit with initial inductor current I_0.

where $i(0)$ and $i(\infty)$ are the initial and final values of i, respectively. Thus, to find the step response of an RL circuit requires three things:

1. The initial inductor current $i(0)$ at $t = 0$.

2. The final inductor current $i(\infty)$.

3. The time constant τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (6.7). Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$, Eq. (6.7) becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau} \tag{6.8}$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{6.9a}$$

or

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t) \tag{6.9b}$$

This is the step response of the RL circuit with no initial inductor current. The voltage across the inductor is obtained from Eq. (6.9) using $v = Ldi/dt$. We get

$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \tau = \frac{L}{R}, t > 0$$

or

$$v(t) = V_s e^{-t/\tau}u(t) \tag{6.10}$$

Fig.6.3 shows the step responses in Eqs. (6.9) and (6.10).

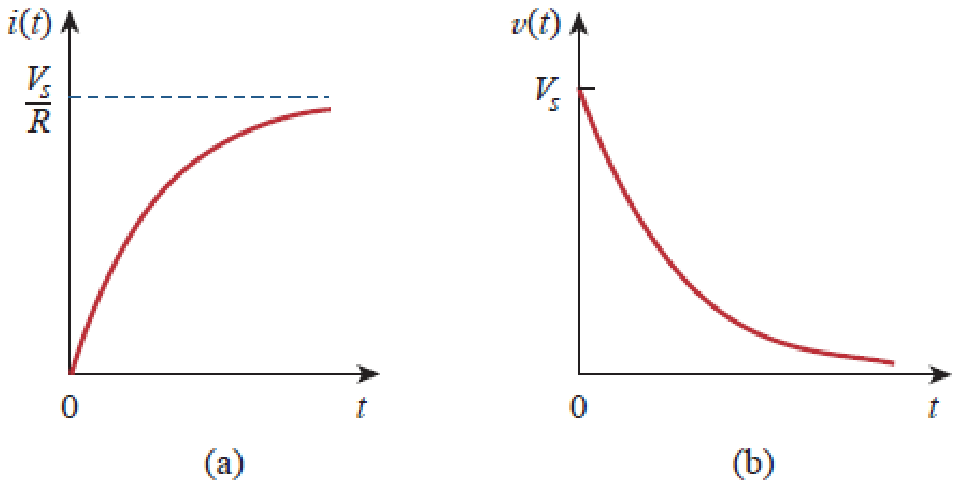


Fig.6.3 Step responses of an RL circuit with no initial inductor current: (a) current response, (b) voltage response.

Example 12: Find $i(t)$ in the circuit of Fig.1 for $t > 0$. Assume that the switch has been closed for a long time.

Solution: When $t < 0$, the $3-\Omega$ resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ ($i.e.$, just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5A$$

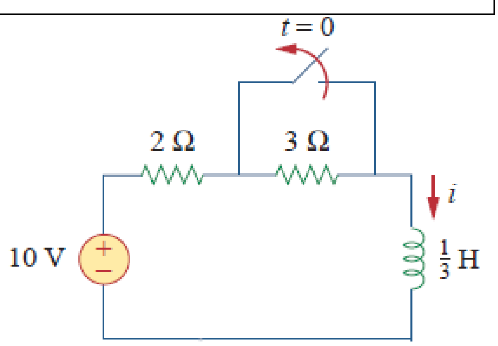


Fig.1

Since the inductor current cannot change instantaneously,

$i(O) = i(0^+) = i(0^-) = 5A$

When $t > 0$, the switch is open. The $2 - \Omega$ and $3 - \Omega$ resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2A$$

The Thevenin resistance across the inductor terminals is

$R_{Th} = 2 + 3 = 5\Omega$

For the time constant,

$$\tau = \frac{L}{R_{Th}} = \frac{\frac{1}{3}}{5} = \frac{1}{15}s$$

Thus,

$i(t) = i(\infty) + [i(O) - i(\infty)]e^{-t/\tau}$

$= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t}A, t > 0$

Check.· In Fig.1, for $t > 0$, KVL must be satisfied; that is,

$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + [\frac{1}{3}(3)(-15)e^{-15t}] = 10$$

This confirms the result.

H.W. 12: The switch in Fig.1 has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$.

Answer: $(6 + 3e^{-10t})$ A for all $t > 0$.

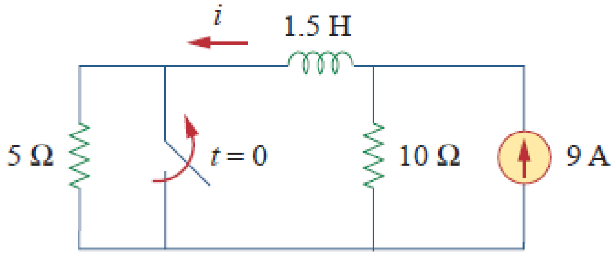


Fig.1

Example 13: At $t = 0$, switch 1 in Fig.1is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2s$ and $t = 5s$.

Solution: We need to consider the three time intervals $t \leq 0, 0 \leq t \leq 4$, and $t \geq 4$ separately. For $t < 0$, switches S_1 and S_2 are open so that $i = 0$. Since the inductor current cannot change instantly,

$$i(0^-) = i(O) = i(0^+) = 0$$

For $0 \leq t \leq 4$, S_1 is closed so that the $4 - \Omega$ and $6 - \Omega$ resistors are in series. (Remember, *at* this time, S_2 *is* still open.) Hence, assuming for now that S_1 is closed forever,

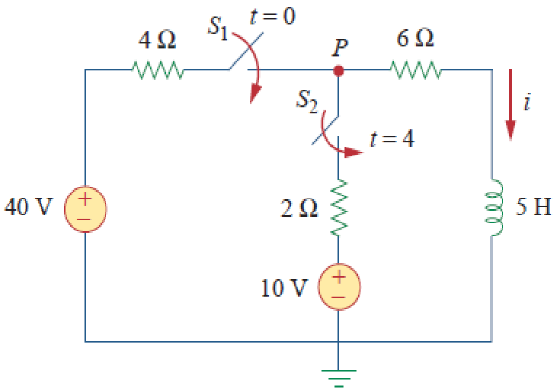


Fig.1

$$i(\infty) = \frac{40}{4+6} = 4A, \quad R_{Th} = 4 + 6 = 10\Omega$$

$$\tau = \frac{L}{R_{Th}} = \frac{5}{10} = \frac{1}{2}s$$

Thus,

$$i(t) = i(\infty) + [i(O) - i(\infty)]e^{-t/\tau} = 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t})A, 0 \leq t \leq 4$$

For $t \geq 4$, S_2 is closed; the 10-V voltage source is connected, and the circuit changes. This sudden change does not affect the inductor current because the current cannot change abruptly. Thus, the initial current is

$$i(4) = i(4^-) = 4(1 - e^{-8}) = 4A$$

To find $i(\infty)$, let v be the voltage at node P in Fig.1. Using KCL,

$$\frac{40 - v}{4} + \frac{10 - v}{2} = \frac{U}{6} \Rightarrow v = \frac{180}{11}V$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727A$$

The Thevenin resistance at the inductor terminals is

$$R_{Th} = 4 \| 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3}\Omega$$

and

$$\tau = \frac{L}{R_{Th}} = \frac{5}{\frac{22}{3}} = \frac{15}{22} s$$

Hence,

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, t \geq 4$$

We need $(t-4)$ in the exponential because of the time delay. Thus,

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \tau = \frac{15}{22}$$

$$= 2.727 + 1.273e^{-14667(t-4)}, t \geq 4$$

Putting all this together,

$$i(t) = \begin{cases} 0, & t \leq 0 \\ 4(1 - e^{-2t}), & 0 \leq t \leq 4 \\ 2.727 \; + 1.273e^{-14667(t-4)}, & t \geq 4 \end{cases}$$

At $t = 2$,

$$i(2) = 4(1 - e^{-4}) = 3.93A$$

At $t = 5$,

$$i(5) = 2.727 + 1.273e^{-14667} = 3.02A$$

H.W. 13: Switch S_1 in Fig.1 is closed at $t = 0$, and switch S_2 is closed at $t = 2s$. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.

Answer:

$$\boldsymbol{i(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-9t}), & 0 < t < 2 \\ 3.6 \; - 1.6e^{-5(t-2)}, & t > 2 \end{cases}}$$

$\boldsymbol{i(1) = 1.9997A, i(3) = 3.589A.}$

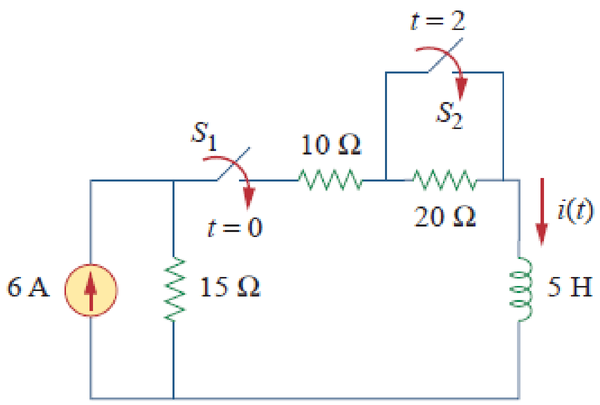


Fig.1

Lecture ( )

Transient
Second- Order Circuits

1) Introduction

In the previous lecture we considered circuits with a single storage element (a capacitor or an inductor). *Such circuits are first-order because the differential equations describing them are first-order.* In this lecture we will consider circuits containing two storage elements. *These are known as second-order circuits because their responses are described by differential equations that contain second derivatives.*

Typical examples of second-order circuits are RLC circuits, in which the three kinds of passive elements are present. Examples of such circuits are shown in Fig. 1.1(a) and (b). Other examples are RL and RC circuits, as shown in Fig. 1.1(c) and (d). It is apparent from Fig. 1.1 that *a second-order circuit may have two storage elements of different type or the same type (provided elements of the same type cannot be represented by an equivalent single element).*

A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.

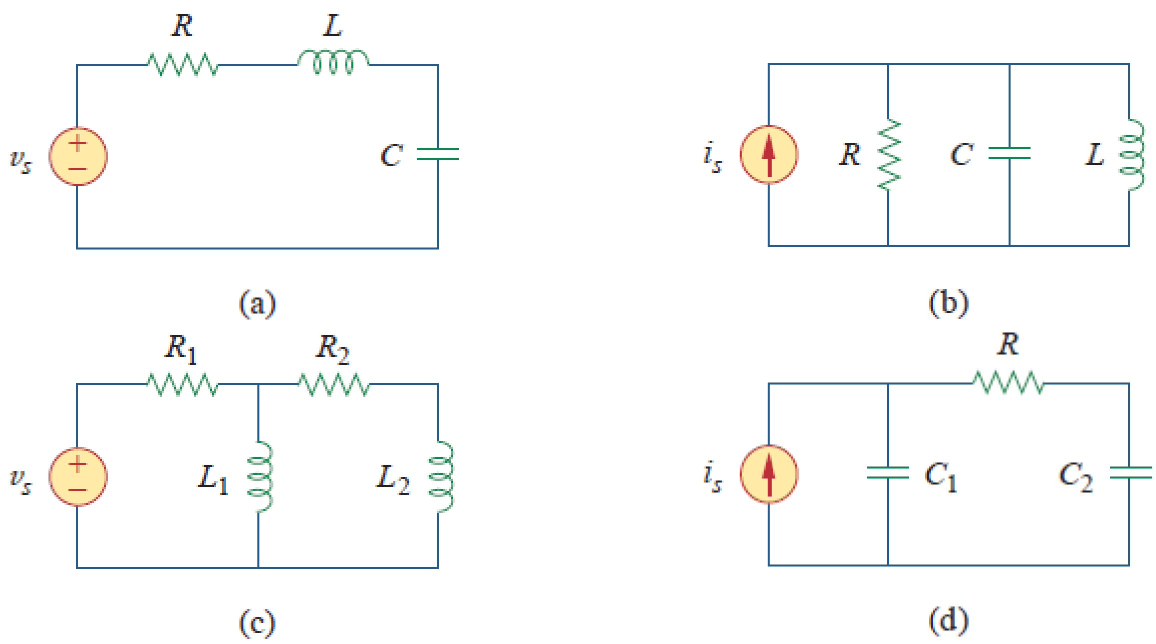


Fig. 1.1 Typical examples of second-order circuits: (a) series *RLC* circuit, (b) parallel *RLC* circuit, (c) *RL* circuit, (d) *RC* circuit.

2) *Finding Initial and Final Values*

There are two key points to keep in mind in determining the initial conditions. **First**—as always in circuit analysis—we must carefully handle the polarity of voltage $v(t)$ across the capacitor and the direction of the current $i(t)$ through the inductor. Keep in mind that v and i are defined strictly according to the passive sign convention. **Second**, keep in mind that the capacitor voltage is always continuous so that

$$v(0^+) = v(0^-) \tag{2.1a}$$

and the inductor current is always continuous so that

$$i(0^+) = i(0^-) \tag{2.1b}$$

where $t = 0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event, assuming that the switching event takes place at $t = 0$.

Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly, capacitor voltage and inductor current, by applying Eq. (2.1). The following examples illustrate these ideas.

Example 1: The switch in Fig.1 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+), v(0^+), (b) di(0^+)/dt, dv(0^+)/dt, (c) i(\infty), v(\infty)$.

Solution:

(a) If the switch is closed a long time before $t = 0$, it means that the circuit has reached dc steady state at $t = 0$. At dc steady state, the inductor acts like a short circuit, while the capacitor acts like an open circuit, so we have the circuit in Fig.2 (a) at $t = 0^-$. Thus,

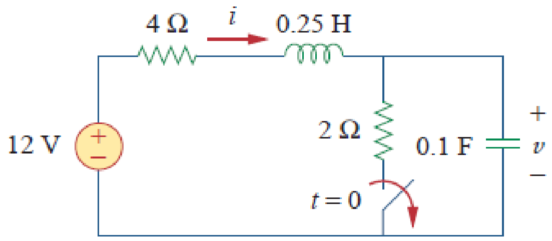


Fig.1

$$i(0^-) = \frac{12}{4+2} = 2A, \qquad v(0^-) = 2i(0^-) = 4V$$

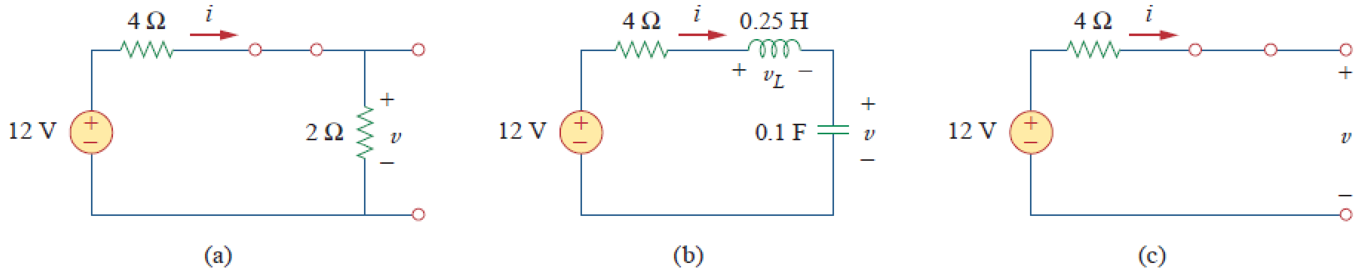


Fig.2 Equivalent circuit of that in Fig.1 for: (a) $t = 0^-$, $(b) t = 0^+$, $(c) t \to \infty$.

As the inductor current and the capacitor voltage cannot change abruptly,

$i(0^+) = i(0^-) = 2A, v(0^+) = v(0^-) = 4V$

(b) At $t = 0^+$, the switch is open; the equivalent circuit is as shown in Fig. 2(b). The same current flows through both the inductor and capacitor. Hence,

$i_C(0^+) = i(0^+) = 2A$

Since $Cdv/dt = i_C$, $dv/dt = i_C/C$, and

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20V/s$$

Similarly, since $Ldi/dt = v_L$, $di/dt = v_L/L$. We now obtain v_L by applying KVL to the loop in Fig. 2(b). The result is

$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$

or

$v_L(0^+) = 12 - 8 - 4 = 0$

Thus,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0A/s$$

(c) For $t > 0$, the circuit undergoes transience. But as $t \to \infty$, the circuit reaches steady state again. The inductor acts like a short circuit and the capacitor like an open circuit, so that the circuit in Fig. 2(b) becomes that shown in Fig. 2(c), from which we have

$i(\infty) = OA, v(\infty) = 12V$

H.W.1: The switch in Fig. 1 was open for a long time but closed at $t = 0$. Determine: (a) $i(0^+), v(0^+), (b) di(0^+)/dt, dv(0^+)/dt, (c) i(\infty), v(\infty)$.

Answer: (a) 1 A, 2V, (b)25A/s, 0V/s, (c)6A, 12V.

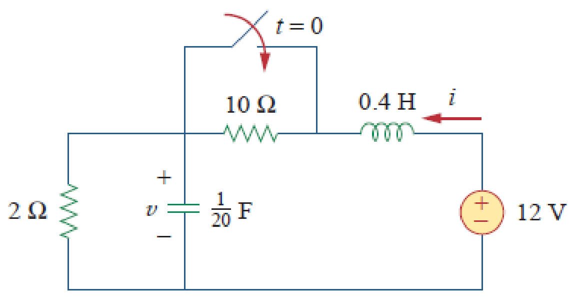


Fig. 1

Example 2: In the circuit of Fig. 1 , calculate: (a) $i_L(0^+), v_C(0^+), v_R(0^+)$, (b)$di_L(0^+)/dt, dv_c(0^+)/dt, dv_R(0^+)/dt$, $(c) i_L(\infty), v_c(\infty), v_R(\infty)$.

Solution:

(a) For $t < 0,\ 3u(t) = 0$. At $t = 0^-$ since the circuit has reached steady state, the inductor can be replaced by a short circuit, while the capacitor is replaced by an open circuit as shown in Fig. 2(a). From this figure we obtain

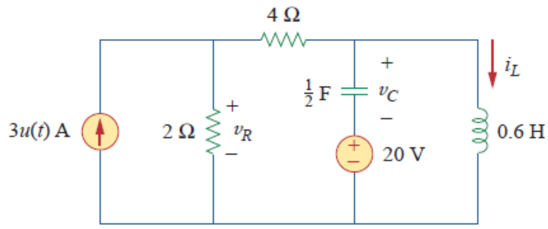


Fig. 1

$$i_L(0^-) = 0, v_R(0^-) = 0, v_C(0^-) = -20V \tag{1.1}$$

Although the derivatives of these quantities at $t = 0^-$ are not required, it is evident that they are all zero, since the circuit has reached steady state and nothing changes.

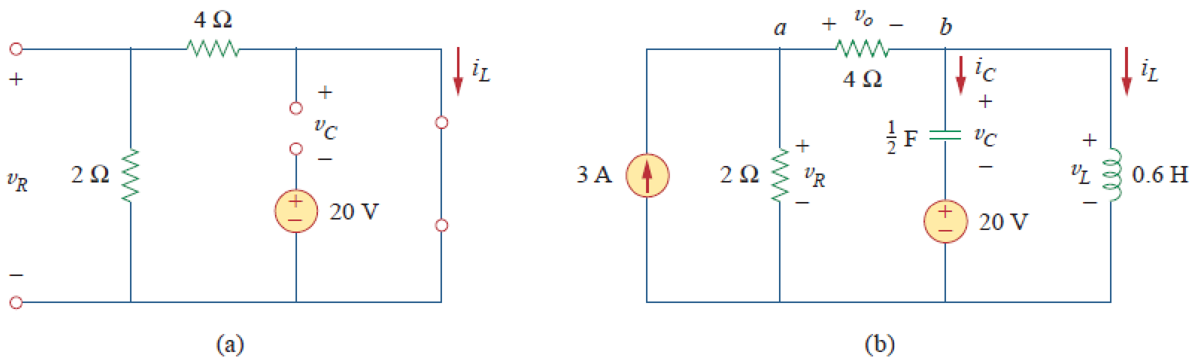


Fig. 2 The circuit in Fig. 1 for: (a) $t = 0^-$, $(b) t = 0^+$

For $t > 0,\ \ 3u(t) = 3$, so that the circuit is now equivalent to that in Fig. 2(b). Since the inductor current and capacitor voltage cannot change abruptly,

$$i_L(0^+) = i_L(0^-) = 0, \quad v_C(0^+) = v_C(0^-) = -20V \tag{1.2}$$

Although the voltage across the $4 - \Omega$ resistor is not required, we will use it to apply KVL and KCL; let it be called v_o. Applying KCL at node a in Fig. 2(b) gives

$$3 = \frac{v_R(0^+)}{2} + \frac{v_o(0^+)}{4} \tag{1.3}$$

Applying KVL to the middle mesh in Fig. 2(b) yields

$$-v_R(0^+) + v_o(0^+) + v_C(0^+) + 20 = 0 \tag{1.4}$$

Since $v_C(0^+) = -20V$ from Eq. (1.2), Eq. (1.4) implies that

$$v_R(0^+) = v_o(0^+) \tag{1.5}$$

From Eqs. (1.3) and (1.5), we obtain

$$v_R(0^+) = v_o(0^+) = 4V \tag{1.6}$$

(b) Since $Ldi_L/dt = v_L$,

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

But applying KVL to the right mesh in Fig. 2(b) gives

$$v_L(0^+) = v_c(0^+) + 20 = 0$$

Hence,

$$\frac{di_L(0^+)}{dt} = 0 \tag{1.7}$$

Similarly, since $Cdv_C/dt = i_C$, then $dv_C/dt = i_C/C$. We apply KCL at node b in Fig. 2(b) to get i_C:

$$\frac{v_o(0^+)}{4} = i_C(0^+) + i_L(0^+) \tag{1.8}$$

Since $v_o(0^+) = 4$ and $i_L(0^+) = 0$, $i_C(0^+) = 4/4 = 1$ A. Then

$$\frac{dv_c(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{1}{0.5} = 2V/s \tag{1.9}$$

To get $dv_R(0^+)/dt$, we apply KCL to node a and obtain

$$3 = \frac{v_R}{2} + \frac{v_o}{4}$$

Taking the derivative of each term and setting $t = 0^+$ gives

$$0 = 2\frac{dv_R(0^+)}{dt} + \frac{dv_o(0^+)}{dt} \tag{1.10}$$

We also apply KVL to the middle mesh in Fig. 2(b) and obtain

$$-v_R + v_c + 20 + v_o = 0$$

Again, taking the derivative of each term and setting $t = 0^+$ yields

$$-\frac{dv_R(0^+)}{dt}+\frac{dv_c(0^+)}{dt}+\frac{dv_o(0^+)}{dt}=0$$

Substituting for $dv_C(0^+)/dt = 2$ gives

$$\frac{dv_R(0^+)}{dt}=2+\frac{dv_o(0^+)}{dt} \tag{1.11}$$

From Eqs. (1.10) and (1.11), we get

$$\frac{dv_R(0^+)}{dt}=\frac{2}{3}V/s$$

We can find $di_R(0^+)/dt$ although it is not required. Since $v_R = 5i_R$,

$$\frac{di_R(0^+)}{dt}=\frac{1}{5}\frac{dv_R(0^+)}{dt}=\frac{1}{5}\frac{2}{3}=\frac{2}{15}A/s$$

(c) As $t \to \infty$, the circuit reaches steady state. We have the equivalent circuit in Fig. 2(a) except that the 3-A current source is now operative. By current division principle,

$$i_L(\infty)=\frac{2}{2+4}3A=1A \tag{1.12}$$

$$v_R(\infty)=\frac{4}{2+4}3A\times 2=4V,\quad v_C(\infty)=-20V$$

H.W.2: For the circuit in Fig. 1, find: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, $(c) i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

Answer: (a) $-6A$, $0, 0$, $(b) 0, 20V/s, 0$, $(c) -2A, 20V, 20V$.

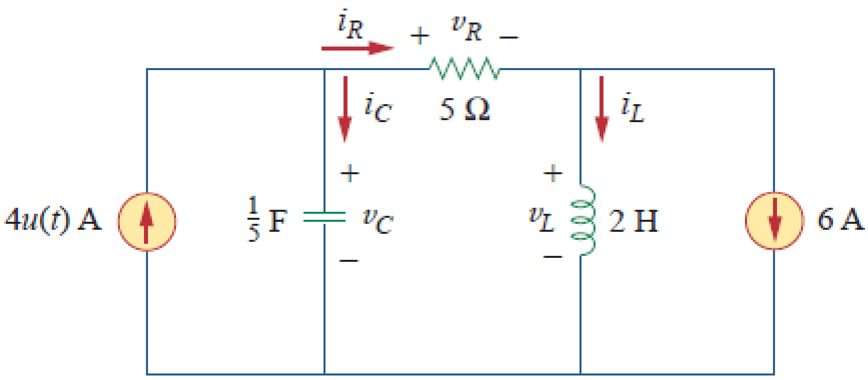


Fig. 1

3) The Source-Free Series RLC Circuit

Consider the series *RLC* circuit shown in Fig. 3.1. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0. Thus, at $t = 0$,

$$v(O) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \tag{3.1a}$$

$$i(0) = I_0 \tag{3.1b}$$

Applying KVL around the loop in Fig. 3.1,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0 \tag{3.2}$$

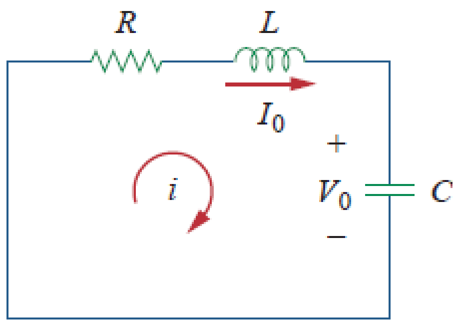


Fig. 3.1. A source-free series *RLC* circuit.

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{3.3}$$

This is a ***second-order differential equation*** and is the reason for calling the *RLC* circuits in this lecture ***second-order circuits***. To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of i and its first derivative or initial values of some i and v. The initial value of i is given in Eq. (8.2b). We get the initial value of the derivative of i from Eqs. (3.1a) and (3.2); that is,

$$Ri(O) + L\frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \tag{3.4}$$

With the two initial conditions in Eqs. (3.1b) and (3.4), we can now solve Eq. (3.4). Our experience in the preceding lecture on first-order circuits suggests that the solution is of exponential form. So we let

$$i = Ae^{st} \tag{3.5}$$

where A and s are constants to be determined. Substituting Eq. (3.5) into Eq. (3.3) and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}(s^2 + \frac{R}{L}s + \frac{1}{LC}) = 0 \tag{3.6}$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{3.7}$$

This quadratic equation is known as the ***characteristic equation*** of the formula to differential Eq. (3.3), since the roots of the equation dictate the character of i. The two roots of Eq. (3.7) are

$$s_1 = -\frac{R}{2L} + \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}} \tag{3.8a}$$

$$s_2 = -\frac{R}{2L} - \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}} \tag{3.8b}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}} \tag{3.9}$$

where

$$\alpha = \frac{R}{2L}, \omega_0 = \frac{1}{\sqrt{LC}} \tag{3.10}$$

The roots s_1 and s_2 are called ***natural frequencies***, measured in **nepers per second** ($\boldsymbol{Np/s}$), because they are associated with the natural response of the circuit; ω_0 is known as the ***resonant frequency*** or strictly as the ***undamped natural frequency***, expressed in **radians per second** ($\boldsymbol{rad/s}$) ;and α is the ***neper frequency*** or the ***damping factor***, expressed in nepers per second. In terms of α and ω_0, Eq. (3.7) can be written as

$$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad (3.7a)$$

Notes:

1) The neper (Np) is a dimensionless unit named after John Napier (1550–1617), a Scottish mathematician.
2) The ratio α/ω_0 is known as the damping ratio ζ.

The variables s and ω_0 are important quantities we will be discussing throughout the rest of the lecture.

The two values of s in Eq. (3.9) indicate that there are two possible solutions for i, each of which is of the form of the assumed solution in Eq. (3.5); that is,

$$i_1 = A_1 e^{s_1 t},\ i_2 = A_2 e^{s_2 t} \qquad (3.11)$$

Since Eq. (3.3) is a linear equation, any linear combination of the two distinct solutions i_1 and i_2 is also a solution of Eq. (3.3). A complete or total solution of Eq. (3.3) would therefore require a linear combination of i_1 and i_2. Thus, the natural response ofthe series *RLC* circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (3.12)$$

where the constants A_1 and A_2 are determined ffom the initial values $i(O)$ and $di(O)/dt$ in Eqs. (3.1b) and (3.4).

From Eq. (3.9), we can infer that there are three types of solutions

1. If $\alpha > \omega_0$, we have the ***overdamped*** case.

2. If $\alpha = \omega_0$, we have the ***critically damped*** case.

3. If $\alpha < \omega_0$, we have the ***underdamped*** case.

We will consider each of these cases separately.

Note: The response is overdamped when the roots of the circuit's characteristic equation are unequal and real, critically damped when the roots are equal and real, and underdamped when the roots are complex.

Overdamped Case ($\alpha > \omega_0$)

From Eqs. (3.8) and (3.9), $\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{3.13}$$

which decays and approaches zero as t increases. Fig. 3.2(a) illustrates a typical overdamped response.

Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{3.14}$$

For this case, Eq. (3.12) yields

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

where $A_3 = A_1 + A_2$. This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant A_3. What then could be wrong? Our assumption of an exponential solution is incorrect for the special case of critical damping. Let us go back to Eq. (3.3). When $\alpha = \omega_0 = R/2L$, Eq. (3.3) becomes

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \alpha^2 i = 0$$

or

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0 \tag{3.15}$$

If we let

$$f = \frac{di}{dt} + \alpha i \tag{3.16}$$

then Eq. (8. 15) becomes

$$\frac{df}{dt} + \alpha f = 0$$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where A_1 is a constant. Equation (3.16) then becomes

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

or

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1 \tag{3.17}$$

This can be written as

$$\frac{d}{dt}(e^{\alpha t} i) = A_1 \tag{3.18}$$

Integrating both sides yields

$$e^{\alpha t} i = A_1 t + A_2$$

or

$$i = (A_1 t + A_2)e^{-\alpha t} \tag{3.19}$$

where A_2 is another constant. Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}} \tag{3.20}$$

A typical critically damped response is shown in Fig. 3.2 (b). In fact, Fig. 3.2 (b) is a sketch of $i(t) = te^{-\alpha t}$, which reaches a maximum value of e^{-1}/α at $t = 1/\alpha$, one time constant, and then decays all the way to zero.

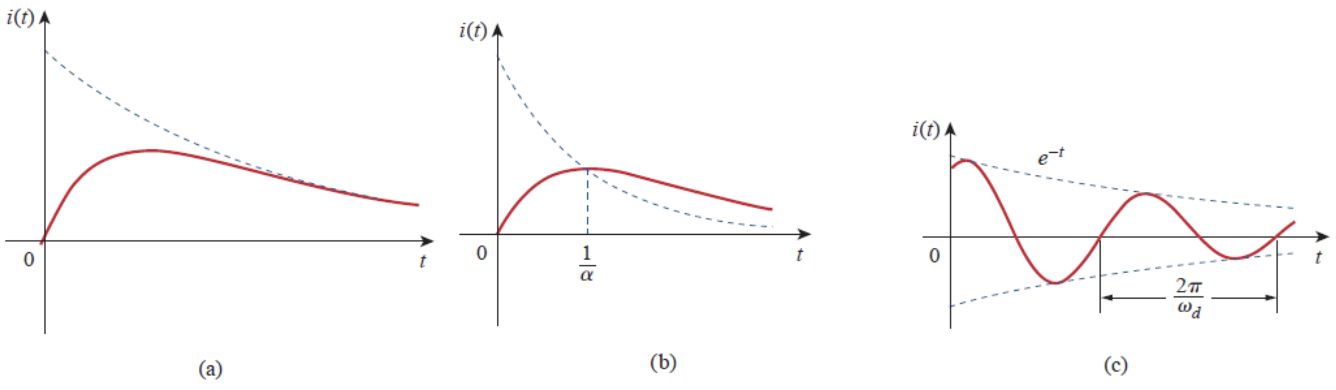


Fig. 3.2 (a) Overdamped response, (b) critically damped response, (c) underdamped response.

Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0, C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \quad (3.21a)$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \quad (3.21b)$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the ***damping frequency***. Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is often called the ***undamped natural frequency***, ω_d is called the ***damped natural frequency***. The natural response is

$$i(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \quad (3.22)$$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, e^{-j\theta} = \cos\theta - j\sin\theta \quad (3.23)$$

we get

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned} \quad (3.24)$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \quad (3.25)$$

With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature. The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$. Fig. 3.2(c) depicts a typical underdamped response. [Fig. 3.2 assumes for each case that $i(O) = 0$.].

Once the inductor current $i(t)$ is found for the *RLC* series circuit as shown above, other circuit quantities such as individual element voltages can easily be found. For example, the resistor voltage is $v_R = Ri$, and the inductor voltage is $v_L = Ldi/dt$. The inductor current $i(t)$ is selected as the key variable to be determined first in order to take advantage of Eq. (2.lb).

We conclude this section by noting the following interesting, peculiar properties of an *RLC* network:

1. The behavior of such a network is captured by the idea of ***damping***, which is the gradual loss of the initial stored energy, as evidenced by because of the inherent losses in them. the continuous decrease in the amplitude of the response. The damping effect is due to the presence of resistance R. The damping factor α determines the rate at which the response is damped. If $R = 0$, then $\alpha = 0$, and we have an LC circuit with $1/\sqrt{LC}$ as the undamped natural frequency. Since $\alpha < \omega_0$ in this case, the response is not only undamped but also oscillatory. The circuit is said to be ***loss-less***, because the dissipating or damping element (R) is absent. By adjusting the value of R, the response may be made ***undamped, overdamped, critically damped*** or ***undamped.***

2. Oscillatory response is possible due to the presence of the two types of storage elements. Having both L and C allows the flow of energy back and forth between the two. The damped oscillation exhibited by the underdamped response is known as ***ringing***. It stems from the ability of the storage elements L and C to transfer energy back and forth between them.

3. Observe from Fig. 3.2 that the waveforms of the responses differ. In general, it is difficult to tell from the waveforms the difference between the overdamped and critically damped responses. The critically damped case is the borderline between the underdamped and overdamped cases and it decays the fastest. With the same initial conditions, the overdamped case has the longest settling time, because it takes the longest time to dissipate the initial stored energy. If we desire the response that approaches the final value most rapidly without oscillation or ringing, the critically damped critically damped circuit. circuit is the right choice.

Example 3: In Fig.3.1, $R = 40\Omega, L = 4H$, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, under- damped, or critically damped?

Solution: We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \quad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped. This is also evident from the fact that the roots are real and negative.

H.W. 3: If $R = 10\Omega$, $L = 5H$, and $C = 2mF$ in Fig.3.1, find α, ω_0, s_1, and s_2. What type of natural response will the circuit have?

Answer:

Answer: 1, 10, $-1 \pm j9.95$, underdamped.

Example 4: Find $i(t)$in the circuit of Fig.1. Assume that the circuit has reached steady state at $t = 0^-$.

Solution:

For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in Fig.2(a). Thus, at $t = 0$,

$$i(O) = \frac{10}{4+6} = 1A, \qquad v(O) = 6i(O) = 6V$$

where $i(O)$ is the initial current through the inductor and $v(O)$ is the initial voltage across the capacitor.

For $t > 0$, the switch is opened and the voltage source is disconnected. The equivalent circuit is shown in Fig.2(b), which is a source- free series RLC circuit. Notice that the 3Ω and 6Ω resistors, which are in series in Fig.1 when the switch is opened, have been combined to give $R = 9\Omega$ in Fig.2(b). The roots are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \;\; \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100} = -9 \pm j4.359$$

Hence, the response is underdamped $(\alpha < \omega)$; that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \tag{1}$$

We now obtain A_1 and A_2 using the initial conditions. At $t = 0$,

$$i(O) = 1 = A_1 \tag{2}$$

From Eq. (3.4),

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(O) + v(O)] = -2[9(1) - 6] = -6A/s \tag{3}$$

Note that $v(O) = V_0 = -6V$ is used, because the polarity of v in Fig.2 (b) is opposite that in Fig. 3.1. Taking the derivative of $i(t)$ in Eq. (1),

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

Imposing the condition in Eq. (3) at $t = 0$ gives

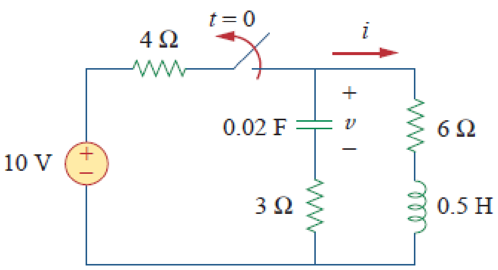
$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

But $A_1 = 1$ from Eq. (2). Then

$$-6 = -9 + 4.359A_2 \;\Rightarrow\; A_2 = 0.6882$$

Substituting the values of A_1 and A_2 in Eq. (1) yields the complete solution as

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t)A$$



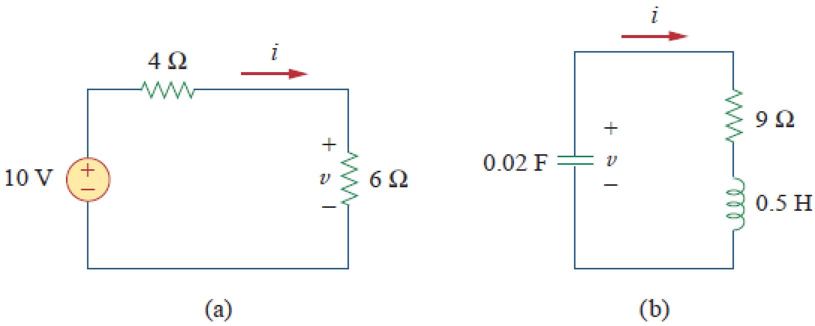


Fig.1

Fig.2 The circuit in Fig.1: (a) for $t < 0$, (b) for $t > 0$.

H.W. 4: The circuit in Fig.1 has reached steady state at $t = 0^-$. If the make before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$

Answer: $\boldsymbol{e^{-2.5t}(5\cos 1.6583t - 7.5378 \sin 1.6583t)A}$

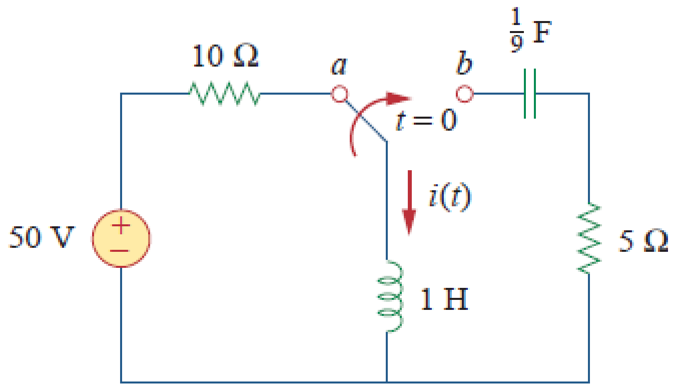


Fig.1

4) The Source-Free Parallel RLC Circuit

Parallel *RLC* circuits find many practical applications, notably in communications networks and filter designs.
Consider the parallel *RLC* circuit shown in Fig.4.1. Assume initial inductor current I_o and initial capacitor voltage V_o,

$$i(0) = I_o = \frac{1}{L}\int_{\infty}^{0} v(t)dt \tag{4.1a}$$

$$v(0) = V_0 \tag{4.1b}$$

Since the three elements are in parallel, they have the same voltage *v* across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0 \tag{4.2}$$

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{4.3}$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2. By following the same reasoning used in establishing Eqs. (3.3) through (3.7), the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \tag{4.4}$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \tag{4.5}$$

or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \tag{4.6}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{4.7}$$

The names of these terms remain the same as in the preceding section, as they play the same role in the solution. Again, there are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$, or $\alpha < \omega_0$. Let us consider these cases separately.

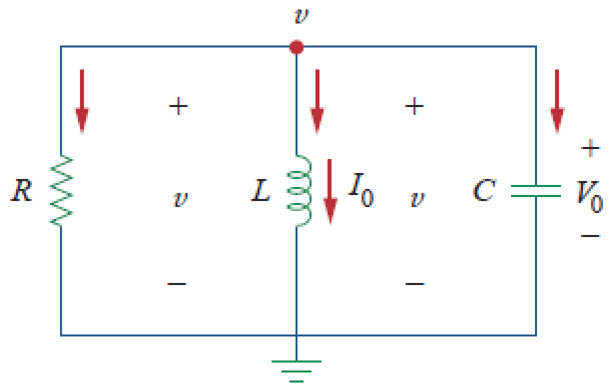


Fig.4.1 A source-free parallel *RLC* circuit.

Overdamped Case $(\alpha > \omega_0)$

From Eq. (4.7), $\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{4.8}$$

Critically Damped Case $(\alpha = \omega_0)$

For $\alpha = \omega_0$, $L = 4R^2C$. The roots are real and equal so that the response is

$$\boxed{v(t) = (A_1 + A_2 t)e^{-\alpha t}} \tag{4.9}$$

Underdamped Case $(\alpha < \omega_0)$

When $\alpha < \omega_0$, $L < 4R^2C$. In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \tag{4.10}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{4.11}$$

The response is

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)} \tag{4.12}$$

The constants A_1 and A_2 in each case can be determined ffom the initial conditions. We need $v(O)$ and $dv(O)/dt$. The first term is known from Eq. (4.1b). We find the second term by combining Eqs. (4.1) and (4.2), as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \qquad (4.13)$$

The voltage waveforms are similar to those shown in Fig. 3.2 and will depend on whether the circuit is overdamped, underdamped, or critically damped.

Having found the capacitor voltage $v(t)$ for the parallel *RLC* circuit as shown above, we can readily obtain other circuit quantities such as individual element currents. For example, the resistor current is $i_R = v/R$ and the capacitor voltage is $v_C = Cdv/dt$. We have selected the capacitor voltage $v(t)$ as the key variable to be determined first in order to take advantage of Eq. (2.la). Notice that we first found the inductor current $i(t)$ for the *RLC* series circuit, whereas we first found the capacitor voltage $v(t)$ for the parallel *RLC* circuit.

Example 5: In the parallel circuit of Fig. 4.1, find $v(t)$ for $t > 0$, assuming $v(O) = 5V$, $i(O) = 0$, $L = 1H$, and $C = 10mF$. Consider these cases: $R = 1.923\Omega$, $R = 5\Omega$, and $R = 6.25\Omega$.

Solution:

CASE 1 If $R = 1.923\Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2{\times}1.923{\times}10{\times}10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1{\times}10{\times}10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

and the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t} \qquad (1)$$

We now apply the initial conditions to get A_1 and A_2.

$$v(O) = 5 = A_1 + A_2 \qquad (2)$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

But differentiating Eq. (1),

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t = 0$,

$$-260 = -2A_1 - 50A_2 \qquad (3)$$

From Eqs. (2) and (3), we obtain $A_1 = -0.2083$ and $A_2 = 5.208$. Substituting A_1 and A_2 in Eq. (1) yields

$$v(t) = -0.2083e^{-2t} + 5.208e^{-50t} \qquad (4)$$

CASE 2 When $R = 5\Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

while $\omega_0 = 10$ remains the same. Since $\alpha = \omega_0 = 10$, the response is critically damped. Hence, $s_1 = s_2 = -10$, and

$$v(t) = (A_1 + A_2 t)e^{-10t} \qquad (5)$$

To get A_1 and A_2, we apply the initial conditions

$$u(0) = 5 = A_1 \qquad (6)$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

But differentiating Eq. (5),

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$$-100 = -10A_1 + A_2 \tag{7}$$

From Eqs. (6) and (7), $A_1 = 5$ and $A_2 = -50$. Thus,

$$v(t) = (5 - 50t)e^{-10t}V \tag{8}$$

CASE 3 When $R = 6.25\Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times6.25\times10\times10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same. As $\alpha < \omega_0$ in this case, the response is underdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Hence,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \tag{9}$$

We now obtain A_1 and A_2, as

$$v(O) = 5 = A_1 \tag{10}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25\times10\times10^{-3}} = -80$$

But differentiating Eq. (9),

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$$-80 = -8A_1 + 6A_2 \tag{11}$$

From Eqs. (10) and (11), $A_1 = 5$ and $A_2 = -6.667$. Thus,

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)e^{-8t} \tag{12}$$

Notice that by increasing the value of R, the degree of damping decreases and the responses differ. Fig.1 plots the three cases.

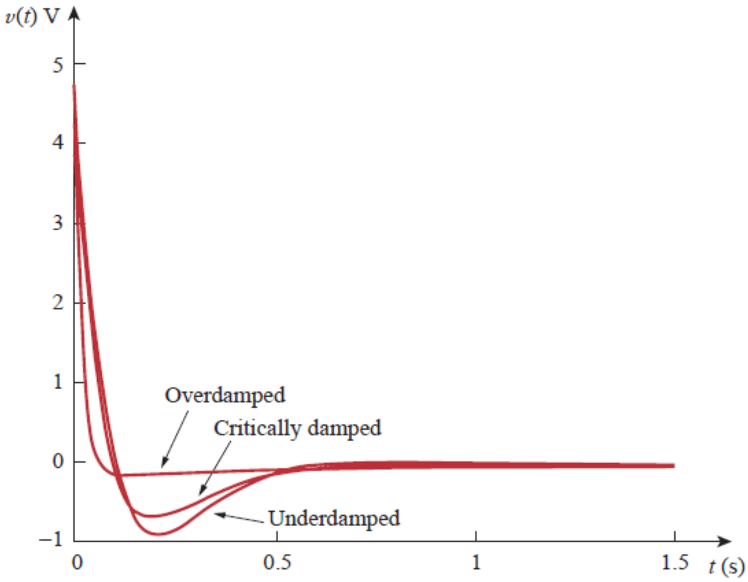


Fig.1: responses for three degrees of damping.

H.W. 5: In Fig.4.1, let $R = 2\Omega$, $L = 0.4H$, $C = 25mF$, $v(O) = 0$, $i(O) = 10mA$. Find $v(t)$ for $t > 0$.

Answer: $-400te^{-10t}v(t)mV$.

Example 6: Find $v(t)$ for $t > 0$ in the RLC circuit of Fig.1.

Solution:

When $t < 0$, the switch is open; the inductor acts like a short circuit while the capacitor behaves like an open circuit. The initial voltage across the capacitor is the same as the voltage across the $50-\Omega$ resistor; that is,

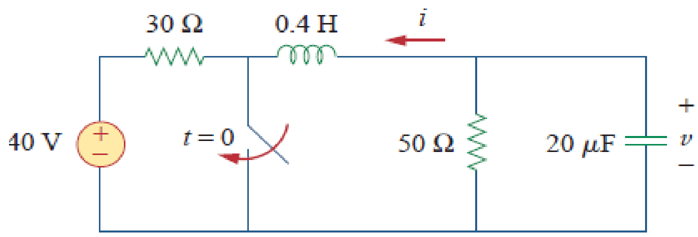


Fig.1.

$$v(O) = \frac{50}{30+50}(40) = \frac{5}{8}\times 40 = 25V \quad (1)$$

The initial current through the inductor is

$$i(0) = -\frac{40}{30+50} = -0.5A$$

The direction of i is as indicated in Fig.1. to conform with the direction of I_0 in Fig.4.1, which is in agreement with the convention that current flows into the positive terminal of an inductor. We need to express this in terms of dv/dt, since we are looking for v.

$$\frac{dv(0)}{dt} = -\frac{v(0)+Ri(0)}{RC} = -\frac{25-50\times 0.5}{50\times 20\times 10^{-6}} = 0 \qquad (2)$$

When $t > 0$, the switch is closed. The voltage source along with the $30-\Omega$ resistor is separated ffom the rest of the circuit. The parallel RLC circuit acts independently of the voltage source, as illustrated in Fig.2. Next, we determine that the roots of the characteristic equation are

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times 50\times 20\times 10^{-6}} = 500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{04\times 20\times 10^{-6}}} = 354$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -500 \pm \sqrt{250{,}000 - 124{,}9976} = -500 \pm 354$$

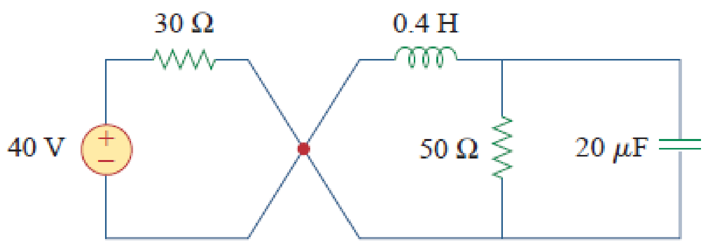


Fig.2 The circuit in Fig.1 when t >0. The parallel *RLC* circuit on the right-hand side acts independently of the circuit on the left-hand side of the junction.

or

$$s_1 = -854, s_2 = -146$$

Since $\alpha > \omega_0$, we have the overdamped response

$$v(t) = A_1 e^{-854t} + A_2 e^{-146t} \qquad (3)$$

At $t = 0$, we impose the condition in Eq. (1),

$$v(O) = 25 = A_1 + A_2 \ \Rightarrow\ A_2 = 25 - A_1 \qquad (4)$$

Taking the derivative of $v(t)$ in Eq. (3),

$$\frac{dv}{dt} = -854A_1 e^{-854t} - 146A_2 e^{-146t}$$

Imposing the condition in Eq. (2),

$$\frac{dv(0)}{dt} = 0 = -854A_1 - 146A_2$$

or

$$0 = 854A_1 + 146A_2 \tag{5}$$

Solving Eqs. (4) and (5) gives

$A_1 = -5.156, A_2 = 30.16$

Thus, the complete solution in Eq. (3) becomes

$v(t) = -5.156e^{-854t} + 30.16e^{-146t}V$

H.W. 6: Refer to the circuit in Fig.1. Find $v(t)$ for $t > 0$.

Answer: $\boldsymbol{100(e^{-10t} - e^{-2.5t})V}$.

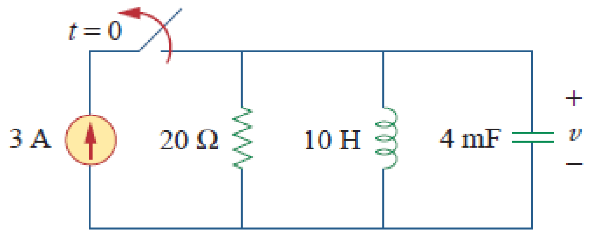


Fig.1

5) Step Response of a Series RLC Circuit

Consider the series *RLC* circuit shown in Fig.5.1. Applying KVL around the loop for $t > 0$,

$$L\frac{di}{dt} + Ri + v = V_s \tag{5.1}$$

But

$$i = C\frac{dv}{dt}$$

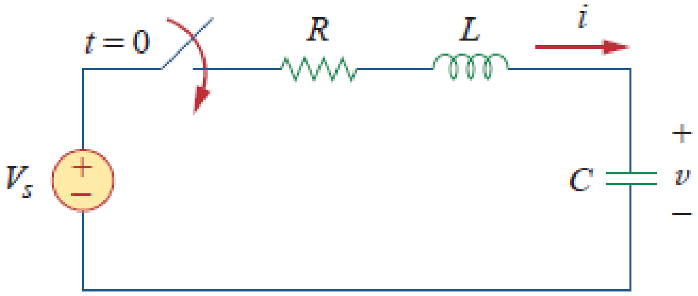


Fig.5.1Step voltage applied to a series *RLC* circuit

Substituting for i in Eq. (8.39) and rearranging terms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \tag{5.2}$$

which has the same form as Eq. (3.3). More specifically, the coefficients are the same (and that is important in determining the frequency parameters) but the variable is different.

(Likewise, see Eq. (5.9).) Hence, the characteristic equation for the series *RLC* circuit is not affected by the presence of the dc source.

The solution to Eq. (5.2) has two components: the transient response $v_t(t)$ and the steady-state response $v_{ss}(t)$; that is,

$$v(t) = v_t(t) + v_{ss}(t) \tag{5.3}$$

The transient response $v_t(t)$ is the component of the total response that dies out with time. The form of the transient response is the same as the form of the solution obtained in **Section 3** for the source-free circuit, given by Eqs. (3.13), (3.20), and (3.25). Therefore, the transient repsonse $v_t(t)$ for the overdamped, underdamped, and critically damped cases are:

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)} \tag{5.4a}$$

$$v_t(t) = (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)} \tag{5.4b}$$

$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)} \tag{5.4c}$$

The steady-state response is the final value of $v(t)$. In the circuit in Fig.5.1, the final value of the capacitor voltage is the same as the source voltage V_s. Hence,

$$v_{ss}(t) = v(\infty) = V_s \tag{5.5}$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)} \tag{5.6a}$$

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)} \tag{5.6b}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)} \tag{5.6c}$$

The values of the constants A_1 and A_2 are obtained from the initial conditions: $v(O)$ and $dv(O)/dt$. Keep in mind that v and i are, respectively, the voltage across the capacitor and the current through the inductor. Therefore, Eq. (5.6) only applies for finding v. But once the capacitor voltage $v_C = v$ is known, we can determine $i = Cdv/dt$, which is the same current through the capacitor, inductor, and resistor. Hence, the voltage across the resistor is $v_R = iR$, while the inductor voltage is $v_L = Ldi/dt$.

Alternatively, the complete response for any variable $x(t)$ can be found directly, because it has the general form

$$x(t) = x_{ss}(t) + x_t(t) \tag{5.7}$$

where the $x_{ss} = x(\infty)$ is the final value and $x_t(t)$ is the transient response. The final value is found as in Section 8.2. The transient response has the same form as in Eq. (5.4), and the associated constants are deter- mined from Eq. (5.6) based on the values of $x(O)$ and $dx(O)/dt$.

Example 7: For the circuit in Fig. 1, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\Omega$, $R = 4\Omega$, and $R = 1\Omega$

Solution:

CASE 1 When $R = 5\Omega$. For $t < 0$, the switch is closed for a long time. The capacitor behaves like an open circuit while the inductor acts like a short circuit. The initial current through the Figure 8.19 inductor is

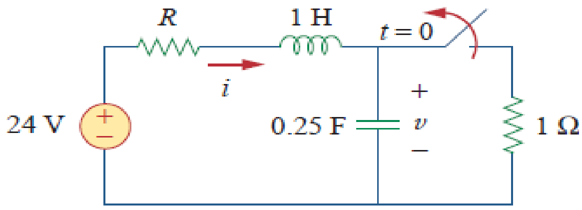


Fig. 1

$$i(0) = \frac{24}{5+1} = 4A$$

and the initial voltage across the capacitor is the same as the voltage across the $1-\Omega$ resistor; that is,

$$v(O) = 1i(O) = 4V$$

For $t > 0$, the switch is opened, so that we have the $1-\Omega$ resistor disconnected. What remains is the series *RLC* circuit with the voltage source. The characteristic roots are determined as follows:

$$\alpha = \frac{R}{2L} = \frac{5}{2\times 1} = 2.5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 025}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

Since $\alpha > \omega_0$, we have the overdamped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t})$$

where v_{ss} is the steady-state response. It is the final value of the capacitor voltage. In Fig. 1, $v_f = 24$ V. Thus,

$$v(t) = 24 + (A_1 e^{-t} + A_2 e^{-4t}) \tag{1}$$

We now need to find A_1 and A_2 using the initial conditions.

$$v(0) = 4 = 24 + A_1 + A_2$$

or

$$-20 = A_1 + A_2 \tag{2}$$

The current through the inductor cannot change abruptly and is the same current through the capacitor at $t = 0^+$ because the inductor and capacitor are now in series. Hence,

$$i(0) = C\frac{dv(0)}{dt} = 4 \Rightarrow \frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16$$

Before we use this condition, we need to take the derivative of v in Eq. (1).

$$\frac{dv}{dt} = -A_1e^{-t} - 4A_2e^{-4t} \tag{3}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2 \tag{4}$$

From Eqs. (2) and (4), $A_1 = -64/3$ and $A_2 = 4/3$. Substituting A_1 and A_2 in Eq. (1), we get

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t})V \tag{5}$$

Since the inductor and capacitor are in series for $t > 0$, the inductor current is the same as the capacitor current. Hence,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (3) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t})A \tag{6}$$

Note that $i(O) = 4A$, as expected.

CASE 2 When $R = 4\Omega$. Again, the initial current through the inductor is

$$i(0) = \frac{24}{4+1} = 4.8A$$

and the initial capacitor voltage is

$$v(O) = 1i(O) = 4.8V$$

For the characteristic roots,

$$\alpha = \frac{R}{2L} = \frac{4}{2\times 1} = 2$$

while $\omega_0 = 2$ remains the same. In this case, $s_1 = s_2 = -\alpha = -2$, and we have the critically damped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1 + A_2 t)e^{-2t}$$

and, as before $v_{ss} = 24V$,

$$v(t) = 24 + (A_1 + A_2 t)e^{-2t} \qquad (7)$$

To find A_1 and A_2, we use the initial conditions. We write

$$v(O) = 4.8 = 24 + A_1 \;\Rightarrow\; A_1 = -19.2 \qquad (8)$$

Since $i(O) = Cdv(O)/dt = 4.8$

or

$$\frac{dv(0)}{dt} = \frac{4.8}{C} = 19.2$$

From Eq. (7),

$$\frac{dv}{dt} = (-2A_1 - 2tA_2 + A_2)e^{-2t} \qquad (9)$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 19.2 = -2A_1 + A_2 \qquad (10)$$

From Eqs. (8) and (10), $A_1 = -19.2$ and $A_2 = -19.2$. Thus, Eq. (7) becomes

$$v(t) = 24 - 19.2(1 + t)e^{-2t}V \qquad (11)$$

The inductor current is the same as the capacitor current; that is,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (9) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = (4.8 + 9.6t)e^{-2t}A \qquad (12)$$

Note that $i(O) = 4.8A$, as expected.

CASE 3 When $R = 1\Omega$. The initial inductor current is

$$i(0) = \frac{24}{1+1} = 12A$$

and the initial voltage across the capacitor is the same as the voltage across the $1-\Omega$ resistor,

$$v(O) = 1i(O) = 12V$$

$$\alpha = \frac{R}{2L} = \frac{1}{2\times1} = 0.5$$

Since $\alpha = 0.5 < \omega_0 = 2$, we have the underdamped response

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936$$

The total response is therefore

$$v(t) = 24 + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t} \qquad (13)$$

We now determine A_1 and A_2. We write

$$v(O) = 12 = 24 + A_1 \;\Rightarrow\; A_1 = -12 \qquad (14)$$

Since $i(O) = Cdv(O)/dt = 12$,

$$\frac{dv(0)}{dt} = \frac{12}{C} = 48 \qquad (15)$$

But

$$\frac{dv}{dt} = e^{-0.5t}(-1.936A_1 \sin 1.936t + 1.936A_2 \cos 1.936t) \\ -0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t) \qquad (16)$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 48 = (-0 + 1.936A_2) - 0.5(A_1 + 0)$$

Substituting $A_1 = -12$ gives $A_2 = 21.694$, and Eq. (13) becomes

$$v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t}V \qquad (17)$$

The inductor current is

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (16) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t}A \quad (18)$$

Note that $i(O) = 12A$, as expected.

Fig.2 plots the responses for the three cases. From this figure, we observe that the critically damped response approaches the step input of 24 V the fastest.

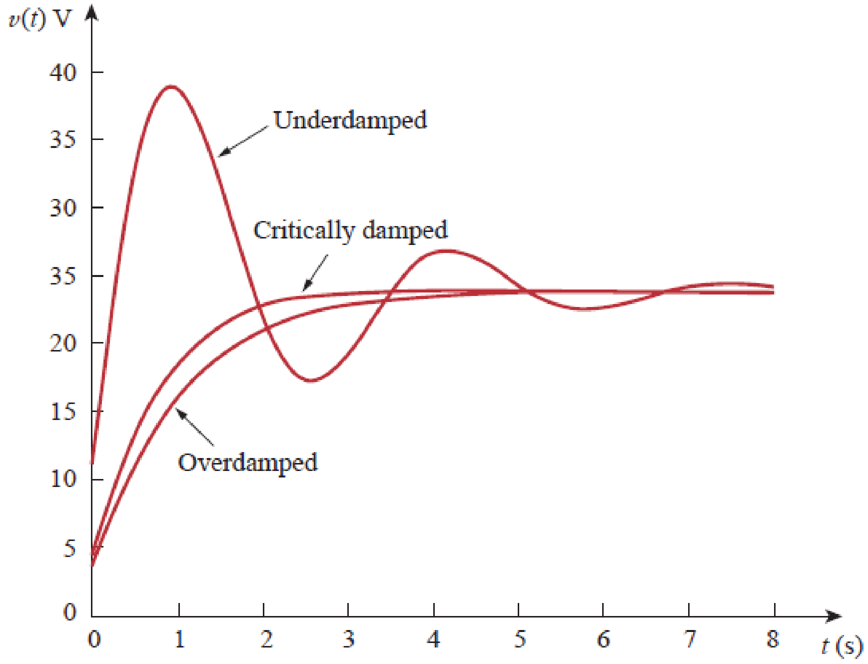


Fig.2 response for three degrees of damping.

H.W. 7: Having been in position a for a long time, the switch in Fig.1 is moved to position b at $t = 0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.

Answer: 10—$(1.1547\ \sin 3.464t + 2\cos 3.464t)e^{-2t}V, 2.31e^{-2t} \sin 3.464tV$.

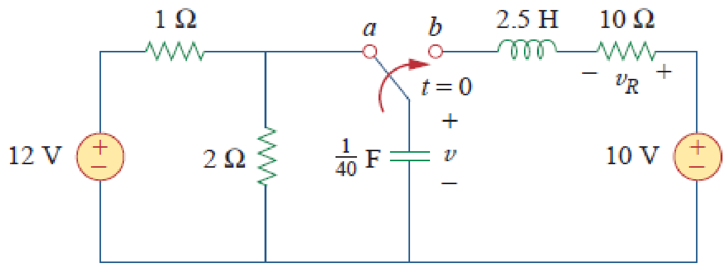


Fig.1

6) Step Response of a Parallel RLC Circuit

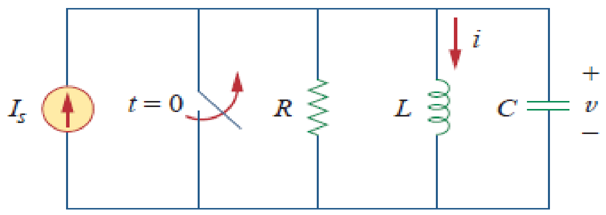


Fig.6.1 Parallel *RLC* circuit with an applied current.

Consider the parallel *RLC* circuit shown in Fig.6.1. We want to find i due to a sudden application of a dc current. Applying KCL at the top node for $t > 0$,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \tag{6.1}$$

But

$$v = L\frac{di}{dt}$$

Substituting for v in Eq. (6.1) and dividing by LC, we get

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \tag{6.2}$$

which has the same characteristic equation as Eq. (4.3).

The complete solution to Eq. (6.2) consists of the transient response $i_t(t)$ and the steady-state response i_{ss}; that is,

$$i(t) = i_t(t) + i_{ss}(t) \tag{6.3}$$

The transient response is the same as what we had in ***Section 4***. The steady-state response is the final value of i. In the circuit in Fig.6.1, the final value of the current through the inductor is the same as the source current I_s. Thus,

$$i(t) = I_s + A_1e^{s_1t} + A_2e^{s_2t} \quad \text{(Overdamped)} \tag{6.4a}$$

$$i(t) = I_s + (A_1 + A_2t)e^{-\alpha t} \quad \text{(Critically damped)} \tag{6.4b}$$

$$i(t) = I_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)} \tag{6.4c}$$

The constants A_1 and A_2 in each case can be determined from the initial conditions for i and di/dt. Again, we should keep in mind that Eq. (6.4) only applies for finding the inductor current i. But once the inductor current $i_L = i$ is known, we can find $v = Ldi/dt$, which is the same voltage across inductor, capacitor, and resistor. Hence, the current through the resistor is $i_R = v/R$, while the capacitor current is $i_C = Cdv/dt$. Alternatively, the complete response for any variable $x(t)$ may be found directly, using

$$x(t) = x_{ss}(t) + x_t(t) \tag{6.5}$$

where x_{ss} and x_t are its final value and transient response, respectively.

Example 8: In the circuit of Fig.1, find $i(t)$ and $i_R(t)$ for $t > 0$.

Solution:

For $t < 0$, the switch is open, and the circuit is partitioned into two independent subcircuits. The 4-A current flows through the inductor, so that

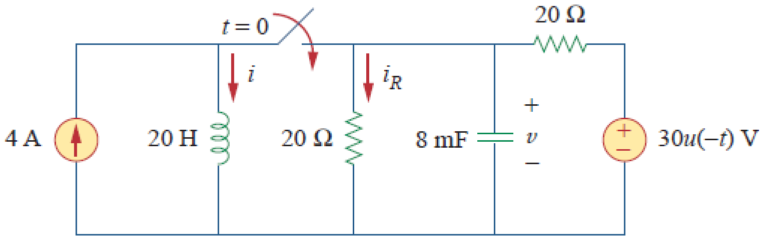


Fig.1

$$i(0) = 4A$$

Since $30u(-t) = 30$ when $t < 0$ and 0 when $t > 0$, the voltage source is operative for $t < 0$. The capacitor acts like an open circuit and the voltage across it is the same as the voltage across the $20 - \Omega$ resistor connected in parallel with it. By voltage division, the initial capacitor voltage is

$$v(O) = \frac{20}{20+20}(30) = 15V$$

For $t > 0$, the switch is closed, and we have a parallel RLC circuit with a current source. The voltage source is zero which means it acts like a short-circuit. The two $20 - \Omega$ resistors are now in parallel. They are combined to give $R = 20||20 = 10\Omega$. The characteristic roots are determined as follows:

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times10\times8\times10^{-3}} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20\times8\times10^{-3}}} = 2.5$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -6.25 \pm \sqrt{390625 - 625} = -6.25 \pm 5.7282$$

or

$$s_1 = -11.978, s_2 = -0.5218$$

Since $\alpha > \omega_0$, we have the overdamped case. Hence,

$$i(t) = I_s + A_1 e^{-11978t} + A_2 e^{-0.5218t} \tag{1}$$

where $I_s = 4$ is the final value of $i(t)$. We now use the initial conditions to determine A_1 and A_2. At $t = 0$,

$$i(0) = 4 = 4 + A_1 + A_2 \;\Rightarrow\; A_2 = -A_1 \tag{2}$$

Taking the derivative of $i(t)$ in Eq. (1),

$$\frac{di}{dt} = -11.978 A_1 e^{-11978t} - 0.5218 A_2 e^{-0.5218t}$$

so that at $t = 0$,

$$\frac{di(0)}{dt} = -11.978 A_1 - 0.5218 A_2 \tag{3}$$

But

$$L\frac{di(0)}{dt} = v(0) = 15 \Rightarrow \frac{di(0)}{dt} = \frac{15}{L} = \frac{15}{20} = 0.75$$

Substituting this into Eq. (3) and incorporating Eq. (2), we get

$$0.75 = (11.978 - 0.5218)A_2 \;\Rightarrow\; A_2 = 0.0655$$

Thus, $A_1 = -0.0655$ and $A_2 = 0.0655$. Inserting A_1 and A_2 in Eq. (1) gives the complete solution as

$$i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11978t})A$$

From $i(t)$, we obtain $v(t) = Ldi/dt$ and

$$i_R(t) = \frac{v(t)}{20} = \frac{L}{20}\frac{di}{dt} = 0.785e^{-11978t} - 0.0342e^{-0.5218t}A$$

H.W. 8: Find $i(t)$ and $u(t)$ for $t > 0$ in the circuit of Fig.1.

Answer: $12(1 - \cos t)A$, $60 \sin tV$.

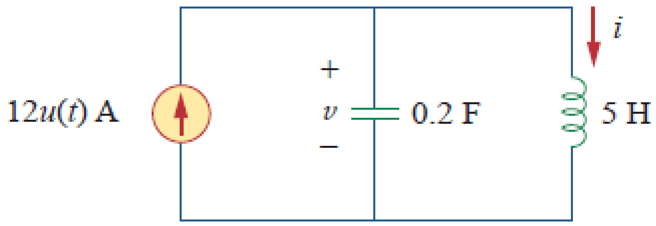


Fig.1

7) General Second-Order Circuits

Now that we have mastered series and parallel *RLC* circuits, we are prepared to apply the ideas to any second-order circuit having one or more independent sources with constant values. Although the series and parallel *RLC* circuits are the second-order circuits of greatest interest, other second-order circuits including op amps are also useful. Given a second-order circuit, we determine its step response *x(t)* (which may be voltage or current) by taking the following four steps:

1. We first determine the initial conditions $x(O)$ and $dx(O)/dt$ and the final value $x(\infty)$, as discussed in Section 2.

2. We turn off the independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL. Once a second-order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_t(t)$ with two unknown constants as we did in the previous sections.

3. We obtain the steady-state response as

$$x_{ss}(t) = x(\infty) \tag{7.1}$$

where $x(\infty)$ is the final value of x, obtained in step 1.

4. The total response is now found as the sum of the transient response and steady-state response

$$x(t) = x_t(t) + x_{ss}(t) \tag{7.2}$$

We finally determine t*hec*onstants associated with the transients response by imposing the initial conditions x(0) and dx(0)/dt, determined in step 1.

We can apply this general procedure to find the step response of any second-order circuit, including those with op amps. The following examples illustrate the four steps.

Example 9: Find the complete response v and then i for $t > 0$ in the circuit of Fig.1

Solution:

We first find the initial and final values. At $t = 0^-$, the circuit is at steady state. The switch is open; the equivalent circuit is shown in Fig.2(a). It is evident from the figure that

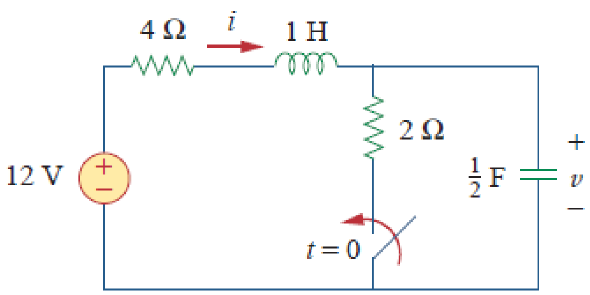


Fig.1

$$v(0^-) = 12V, \qquad i(0^-) = 0$$

At $t = 0^+$, the switch is closed; the equivalent circuit is in Fig.2(b). By the continuity of capacitor voltage and inductor current, we know that

$$v(0^+) = v(0^-) = 12V, i(0^+) = i(0^-) = 0 \quad (1)$$

To get $dv(0^+)/dt$, we use $Cdu/dt = i_C$ or $dv/dt = i_C/C$. Applying KCL at node a in Fig.2(b),

$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2}$$

$$0 = i_C(0^+) + \frac{12}{2} \Rightarrow i_C(0^+) = -6A$$

Hence,

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12V/s \quad (2)$$

The final values are obtained when the inductor is replaced by a short circuit and the capacitor by an open circuit in Fig.2(b), giving

$$i(\infty) = \frac{12}{4+2} = 2A, v(\infty) = 2i(\infty) = 4V \quad (3)$$

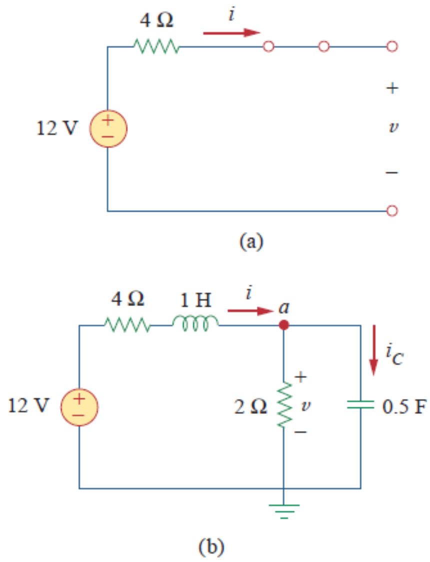


Fig.2 Equivalent circuit of the circuit in Fig.1 for: (a) $t < 0$, $(b) t > 0$.

Next, we obtain the form of the transient response for $t > 0$. By turning off the 12-V voltage source, we have the circuit in Fig.3. Applying KCL at node a in Fig.3 gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \quad (4)$$

Applying KVL to the left mesh results in

$$4i + 1\frac{di}{dt} + v = 0 \quad (5)$$

Since we are interested in u for the moment, we substitute i from Eq. (4) into Eq. (5). We obtain

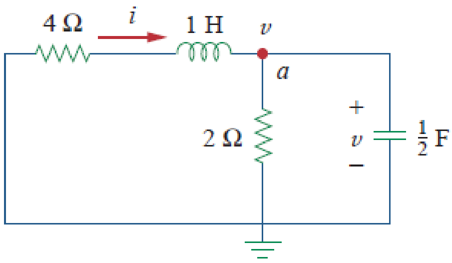


Fig.3 Obtaining the form of the transient response

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0$$

or

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0$$

From this, we obtain the characteristic equation as

$s^2 + 5s + 6 = 0$

with roots $s = -2$ and $s = -3$. Thus, the natural response is

$$v_n(t) = Ae^{-2t} + Be^{-3t} \tag{6}$$

where A and B are unknown constants to be determined later. The steady-state response is

$$v_{ss}(t) = v(\infty) = 4 \tag{7}$$

The complete response is

$$v(t) = v_t + v_{ss} = 4 + Ae^{-2t} + Be^{-3t} \tag{8}$$

We now determine A and B using the initial values. From Eq. (1), $v(O) = 12$. Substituting this into Eq. (8) at $t = 0$ gives

$$12 = 4 + A + B \;\Rightarrow\; A + B = 8 \tag{9}$$

Taking the derivative of v in Eq. (8),

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t} \tag{10}$$

Substituting Eq.(2) into Eq.(10), at t=0 gives

$$-12 = -2A - 3B \Rightarrow 2A + 3B = 12 \tag{11}$$

From Eqs. (9) and (11), we obtain

$A = 12, \qquad B = -4$

so that Eq. (8) becomes

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t}V, \qquad t > 0 \tag{12}$$

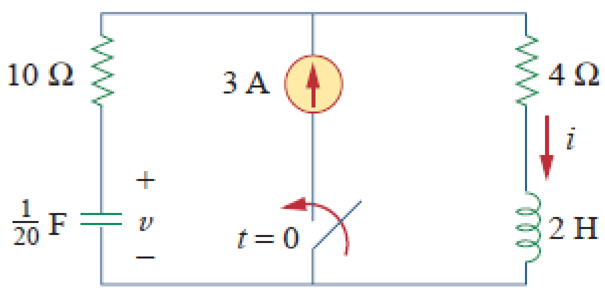
From v, we can obtain other quantities of interest by referring to Fig.2(b). To obtain i, for example,

$$\begin{aligned} i &= \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t} \\ &= 2 - 6e^{-2t} + 4e^{-3t}A, \qquad t > 0 \end{aligned} \tag{13}$$

Notice that $i(O) = 0$, in agreement with Eq. (1).

H.W. 9: Determine v and i for $t > 0$ in the circuit of Fig.1.

Answer: $\mathbf{12(1 - e^{-5t})V,\ 3(1 - e^{-5t})A}$.



Example 10: Find $v_o(t)$ for $t > 0$ in the circuit of Fig.1.

Solution:

This is an example of a second-order circuit with two inductors. We first obtain the mesh currents i_1 and i_2, which happen to be the currents through the inductors. We need to obtain the initial and final values of these currents.

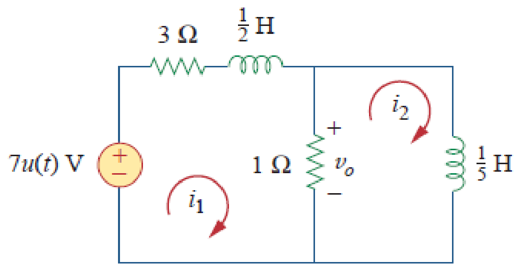


Fig.1

For $t < 0$, $7u(t) = 0$, so that $i_1(0^-) = 0 = i_2(0^-)$.

For $t > 0$, $7u(t) = 7$, so that the equivalent circuit is as shown in Fig.2(a). Due to the continuity of inductor current,

$$i_1(0^+) = i_1(0^-) = 0,\ i_2(0^+) = i_2(0^-) = 0 \tag{1}$$

$$v_{L_2}(0^+) = v_o(0^+) = 1[(i_1(0^+) - i_2(0^+)] = 0 \tag{2}$$

Applying KVL to the left loop in Fig.2(a) at $t = 0^+$,

$$7 = 3i_1(0^+) + v_{L_1}(0^+) + v_o(0^+)$$

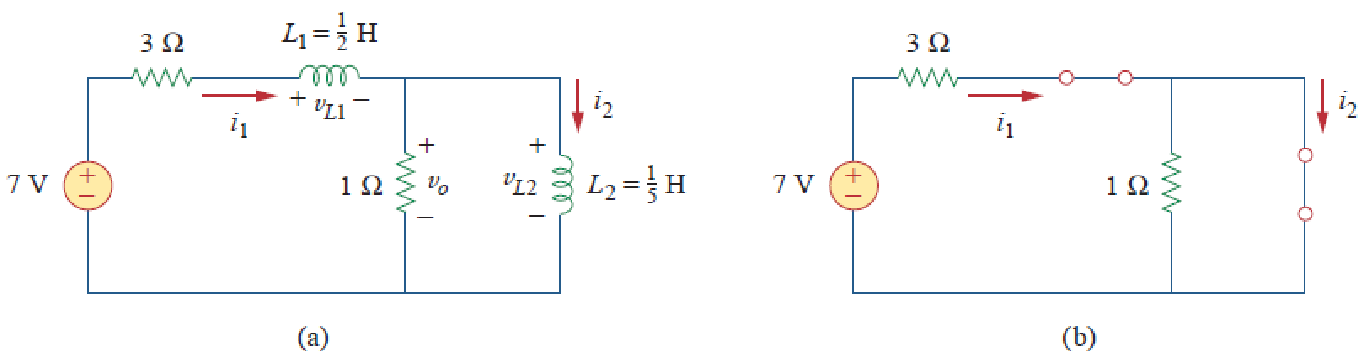


Fig.2 Equivalent circuit of that in Fig.1 for: (a) $t > 0$, $(b) t \to \infty$.

or

$$v_{L_1}(0^+) = 7V$$

Since $L_1 di_1/dt = v_{L_1}$,

$$\frac{di_1(0^+)}{dt} = \frac{v_{L_1}}{L_1} = \frac{7}{\frac{1}{2}} = 14V/s \tag{3}$$

Similarly, since $L_2 di_2/dt = v_{L_2}$,

$$\frac{di_2(0^+)}{dt} = \frac{v_{L_2}}{L_2} = 0 \tag{4}$$

As $t \to \infty$, the circuit reaches steady state, and the inductors can be replaced by short circuits, as shown in Fig.2 (b). From this figure,

$$i_1(\infty) = i_2(\infty) = \frac{7}{3}A \tag{5}$$

Next, we obtain the form of the transient responses by removing the voltage source, as shown in Fig.3. Applying KVL to the two meshes yields

$$4i_1 - i_2 + \frac{1}{2}\frac{di_1}{dt} = 0 \tag{6}$$

and

$$i_2 + \frac{1}{5}\frac{di_2}{dt} - i_1 = 0 \tag{7}$$

From Eq. (6),

$$i_2 = 4i_1 + \frac{1}{2}\frac{di_1}{dt} \tag{8}$$

Substituting Eq. (8) into Eq. (7) gives

$$4i_1 + \frac{1}{2}\frac{di_1}{dt} + \frac{4}{5}\frac{di_1}{dt} + \frac{1}{10}\frac{d^2i_1}{dt^2} - i_1 = 0$$

$$\frac{d^2i_1}{dt^2} + 13\frac{di_1}{dt} + 30i_1 = 0$$

From this we obtain the characteristic equation as

$$s^2 + 13s + 30 = 0$$

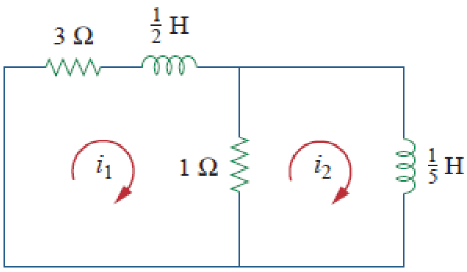


Fig.3 Obtaining the form of the transient response

which has roots $s = -3$ and $s = -10$. Hence, the form of the transient response is

$$i_{1n} = Ae^{-3t} + Be^{-10t} \tag{9}$$

where A and B are constants. The steady-state response is

$$i_{1ss} = i_1(\infty) = \frac{7}{3}A \tag{10}$$

From Eqs. (9) and (10), we obtain the complete response as

$$i_1(t) = \frac{7}{3} + Ae^{-3t} + Be^{-10t} \tag{11}$$

We finally obtain A and B ffom the initial values. From Eqs. (1) and (11),

$$0 = \frac{7}{3} + A + B \tag{12}$$

Taking the derivative of Eq. (11), setting $t = 0$ in the derivative, and enforcing Eq. (3), we obtain

$$14 = -3A - 10B \tag{13}$$

From Eqs. (12) and (13), $A = -4/3$ and $B = -1$. Thus,

$$i_1(t) = \frac{7}{3} - \frac{4}{3}e^{-3t} - e^{-10t} \qquad (14)$$

We now obtain i_2 from i_1. Applying KVL to the left loop in Fig.2 (a) gives

$$7 = 4i_1 - i_2 + \frac{1}{2}\frac{di_1}{dt} \Rightarrow i_2 = -7 + 4i_1 + \frac{1}{2}\frac{di_1}{dt}$$

Substituting for i_1 in Eq. (14) gives

Substituting for i_1 in Eq. (8.10.14) gives

$$i_2(t) = -7 + \frac{28}{3} - \frac{16}{3}e^{-3t} - 4e^{-10t} + 2e^{-3t} + 5e^{-10t}$$
$$= \frac{7}{3} - \frac{10}{3}e^{-3t} + e^{-10t} \qquad \textbf{(8.10.15)}$$

From Fig. 8.29,

$$v_o(t) = 1[i_1(t) - i_2(t)] \qquad \textbf{(8.10.16)}$$

Substituting Eqs. (8.10.14) and (8.10.15) into Eq. (8.10.16) yields

$$v_o(t) = 2(e^{-3t} - e^{-10t}) \qquad \textbf{(8.10.17)}$$

Note that $v_o(0) = 0$, as expected from Eq. (8.10.2).

$$i_2(t) = -7 + \frac{28}{3} - \frac{16}{3}e^{-3t} - 4e^{-10t} + 2e^{-3t} + 5e^{-10t}$$
$$= \frac{7}{3} - \frac{10}{3}e^{-3t} + e^{-10t} \qquad (15)$$

From Fig.1,

$$v_o(t) = 1[i_1(t) - i_2(t)] \qquad (16)$$

Substituting Eqs. (14) and (15) into Eq. (16) yields

$$v_o(t) = 2(e^{-3t} - e^{-10t}) \qquad (17)$$

Note that $v_o(0) = 0$, as expected from Eq. (2).

H.W. 10: For $t > 0$, obtain $v_o(t)$ in the circuit of Fig.1. (*Hint*: First find v_1 and v_2.)

Answer: $8(e^{-t} - e^{-6t})V, t > 0$.