6.1 INTRODUCTION

The procedure for determining the electric field $\mathbf{E}$ in the preceding chapters has generally been using either Coulomb's law or Gauss's law when the charge distribution is known, or using ($\mathbf{E} = -\nabla V$) when the potential V is known throughout the region.

In this chapter, we shall consider practical electrostatic problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find ($\mathbf{E}$ and V) throughout the region. Such problems are usually tackled using Poisson's or Laplace's equation or the method of images, and they are usually referred to as boundary value problems. We shall use Laplace's equation in deriving the resistance of an object and the capacitance of a capacitor.

6.2 POISSON'S AND LAPLACE'S EQUATIONS

Poisson's and Laplace's equations are easily derived from Gauss's law (for a linear material medium)

$$\nabla . \mathbf{D} = \nabla . \epsilon \, \mathbf{E} = \rho_v \tag{6.1}$$

and

$$\mathbf{E} = -\nabla V \tag{6.2}$$

Substituting equation (6.2) into equation (6.1) gives

$$\nabla . (-\epsilon \, \nabla V) = \rho_v \tag{6.3}$$

for an inhomogeneous medium. For a homogeneous medium, equation (6.3) becomes

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \tag{6.4}$$

This is known as Poisson's equation. A special case of this equation occurs when $\rho_v = 0$ (i.e., for a charge-free region). Equation (6.4) then becomes

$$\nabla^2 V = 0 \tag{6.5}$$

which is known as Laplace's equation. Note that in taking (ϵ) out of the left-hand side of equation (6.3) to obtain equation (6.4), we have assumed that (ϵ) is constant throughout the region in which V is defined; for an inhomogeneous region, (ϵ) is not constant and equation (6.4) does not follow equation (6.3). Equation (6.3) is Poisson's equation for an inhomogeneous medium; it becomes Laplace's equation for an inhomogeneous medium when $\rho_v = 0$.

Where ∇^2 is called the Laplacian operator.

Thus Laplace's equation in Cartesian, cylindrical, or spherical coordinates respectively is given by

$$\nabla^2 . V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (Cartesian\ coordinate) \qquad (6.6)$$

$$\nabla^2 . V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (cylindrical\ coordinate) \qquad (6.7)$$

$$\nabla^2 . V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$(spherical\ coordinate) \qquad (6.8)$$

depending on whether the potential is $V(x, y, z)$, $V(\rho, \phi, z)$, or $V(r, \theta, \phi)$. Poisson's equation in those coordinate systems may be obtained by simply replacing zero on the right-hand side of equations (6.6), (6.7), and (6.8) with $-\frac{\rho_v}{\epsilon}$.

Laplace's equation is of primary importance in solving electrostatic problems involving a set of conductors maintained at different potentials.

6.3 UNIQUENESS THEOREM

If a solution of Laplace's equation satisfies a given set of boundary conditions, there is only one solution. We say that the solution is unique. Thus any solution of Laplace's equation which satisfies the same boundary conditions must be the only solution regardless of the method used. This is known as the uniqueness theorem.

We assume that there are two solutions V_1 and V_2 of Laplace's equation both of which satisfy the prescribed boundary conditions. Thus

$$\nabla^2 V_1 = 0; \quad and \qquad \nabla^2 V_2 = 0 \tag{6.9a}$$

$$V_1 = V_2 \qquad \text{on the boundary} \tag{6.9b}$$

We consider their difference

$$V_d = V_2 - V_1 \tag{6.10}$$

which obeys

$$\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \tag{6.11a}$$

$$V_d = 0 \qquad \text{on the boundary} \tag{6.11b}$$

according to equation (6.9). From the divergence theorem.

$$\int_v \nabla .\ \mathbf{A}\, dv = \oint_S \mathbf{A} \,.\, dS \tag{6.12}$$

We let $\mathbf{A} = V_d\, \nabla V_d$ and use a vector identity

$$\nabla . \mathbf{A} = \nabla . [V_d \nabla V_d] = V_d [\nabla^2 V_d] + \nabla V_d . \nabla V_d$$

But $\nabla^2 V_d = 0$ according to equation (6.11), so

$$\nabla . \mathbf{A} = \nabla V_d . \nabla V_d \tag{6.13}$$

Substituting equation (6.13) into eq. (6.12) gives

$$\int_v \nabla V_d . \nabla V_d\, dv = \oint_S V_d \nabla V_d . dS \tag{6.14}$$

From equations (6.9) and (6.11), it is evident that the right-hand side of eq. (6.14) vanishes.

Hence:

$$\int_v [\nabla V_d]^2\, dv = 0$$

Since the integration is always positive.

$$[\nabla V_d]^2 = 0 \tag{6.15a}$$

$$\nabla V_d = 0 \tag{6.15b}$$

$$or \qquad V_d = V_2 - V_1 = \text{constant everywhere in } v \tag{6.15c}$$

But equation (6.15) must be consistent with equation (6.9b). Hence, $V_d = 0$ or $V_1 = V_2$ everywhere, showing that V_1 and V_2 cannot be different solutions of the same problem.

This is the uniqueness theorem: If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

Similar steps can be taken to show that the theorem applies to Poisson's equation and to prove the theorem for the case where the electric field (potential gradient) is specified on the boundary.

Before we begin to solve boundary-value problems, we should bear in mind the three things that uniquely describe a problem:

- The appropriate differential equation (Laplace's or Poisson's equation)
- The solution region
- The prescribed boundary conditions

A problem does not have a unique solution and cannot be solved completely if any of the three items is missing.

6.4 GENERAL PROCEDURE FOR SOLVING POISSON'S OR LAPLACE'S EQUATION

The following general procedure may be taken in solving a given boundary-value problem involving Poisson's or Laplace's equation:

1. Solve Laplace's (if $\rho_v = 0$) or Poisson's (if $\rho_v \neq 0$) equation using either (a) direct integration when V is a function of one variable, or (b) separation of variables if V is a function of more than one variable. The solution at this point is

not unique but expressed in terms of unknown integration constants to be determined.

2. Apply the boundary conditions to determine a unique solution for V. Imposing the given boundary conditions makes the solution unique.

3. Having obtained V, find **E** using $\mathbf{E} = -\nabla V$ and **D** from $\mathbf{D} = \epsilon\,\mathbf{E}$.

4. If desired, find the charge Q induced on a conductor using $Q = \int_S \rho_s\, dS$ where $\rho_s = D_n$ and D_n is the component of **D** normal to the conductor. If necessary, the capacitance between two conductors can be found using $C = Q/V$.

Solving Laplace's (or Poisson's) equation, as in step 1, is not always as complicated as it may seem. In some cases, the solution may be obtained by mere inspection of the problem. Also a solution may be checked by going backward and finding out if it satisfies both Laplace's (or Poisson's) equation and the prescribed boundary conditions.

Example 7.1:

Let us assume that V is a function only of x and worry later about which physical problem we are solving when we have a need for boundary conditions.

Applying Laplace’s equation to V

$$\nabla^2 . V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

V is not a function of y or z, then Laplace’s equation reduces to

$$\nabla^2 . V = \frac{\partial^2 V}{\partial x^2} = 0$$

and the partial derivative may be replaced by an ordinary derivative, since V is not a function of y or z,

$$\frac{d^2 V}{dx^2} = 0$$

We integrate twice, obtaining

$$\frac{dV}{dx} = \mathrm{A}$$

And

$$V = Ax + B$$

where A and B are constants of integration.

Since the field varies only with x and is not a function of y and z, then V is a constant if x is a constant, in other words, the equipotential surfaces are described by setting x constant. These surfaces are parallel planes normal to the x axis. The field is thus that of a parallel-plate capacitor, and as soon as we specify the potential on any two planes, we may evaluate our constants of integration.

In general, let $V = V_1$ at $x = x_1$ and $V = V_2$ at $x = x_2$. This is boundary conditions

$V_1 = Ax_1 + B$ and $V_2 = Ax_2 + B$

$A = \dfrac{V_1 - V_2}{x_1 - x_2}$ $\quad B = \dfrac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$

and

$$V = \frac{V_1(x - x_2) - V_2(x - x_1)}{x_1 - x_2}$$

A simpler answer would have been obtained by choosing simpler boundary conditions. If we had fixed $V = 0$ at $x = 0$ and $V = V_o$ at $x = d$, then

$$A = \frac{V_o}{d} \qquad and \qquad B = 0$$

and

$$V = \frac{V_o}{d}x$$

Here, we have

$$V = \frac{V_o}{d}x$$

$$\mathbf{E} = -\frac{V_o}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon\frac{V_o}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}|_{x=0} = -\epsilon\frac{V_o}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon\frac{V_o}{d} = \rho_S$$

$$Q = \int_S -\epsilon \frac{V_o}{d} dS = -\,\epsilon \frac{V_o S}{d}$$

and the capacitance is

$$C = \frac{|Q|}{V_o} = \frac{\epsilon\, S}{d}$$

Example 7.2:

Let us assume that V is a function only of ρ, i.e we assume that the variation with respect to ρ only.

Applying Laplace's equation to V

$$\nabla^2 . V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace's equation becomes

$$\nabla^2 . V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

or the partial derivative may be replaced by an ordinary derivative

$$\nabla^2 . V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

Noting the ρ in the denominator, we exclude $\rho = 0$ from our solution and then multiply by ρ and integrate,

$$\rho \frac{dV}{d\rho} = A$$

rearrange, and integrate again,

$$V = A \ln \rho + B$$

The equipotential surfaces are given by $\rho = constant$ and are cylinders, and the problem is that of the coaxial capacitor or coaxial transmission line. We choose a potential difference of V_o by letting $V = V_o$ at $\rho = a$, $V = 0$ at $\rho = b$, $b > a$, and obtain

$$V = V_o \frac{\ln(\frac{b}{\rho})}{\ln(\frac{b}{a})}$$

from which

$$\mathbf{E} = \frac{V_o}{\rho} \frac{1}{\ln(\frac{b}{a})} \mathbf{a}_\rho$$

$$D_{N(\rho=a)} = \frac{\epsilon\, V_o}{a \ln(\frac{b}{a})}$$

$$Q = \frac{\epsilon\, V_o\, 2\pi a L}{a \ln(\frac{b}{a})}$$

$$C = \frac{2\pi\epsilon L}{\ln(\frac{b}{a})}$$

Example 7.3:

Let us assume that V is a function only of ϕ in cylindrical coordinates. We see that equipotential surfaces are given by $\phi = constant$. These are radial planes. Boundary conditions might be $V = 0$ at $\phi = 0$ and $V = V_o$ at $\phi = \alpha$, leading to the physical problem detailed in Fig. 6.1.

Applying Laplace's equation to V

$$\nabla^2 . V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial\phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace's equation becomes

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad \text{or} \qquad \frac{d^2 V}{d\phi^2} = 0$$

The solution is

$$V = A\phi + B$$

The boundary conditions determine A and B, and

$$V = V_o \frac{\phi}{\alpha}$$

Taking the gradient of equation above produces the electric field intensity,

$$\mathbf{E} = \frac{V_o}{\alpha\rho}\mathbf{a}_\phi$$

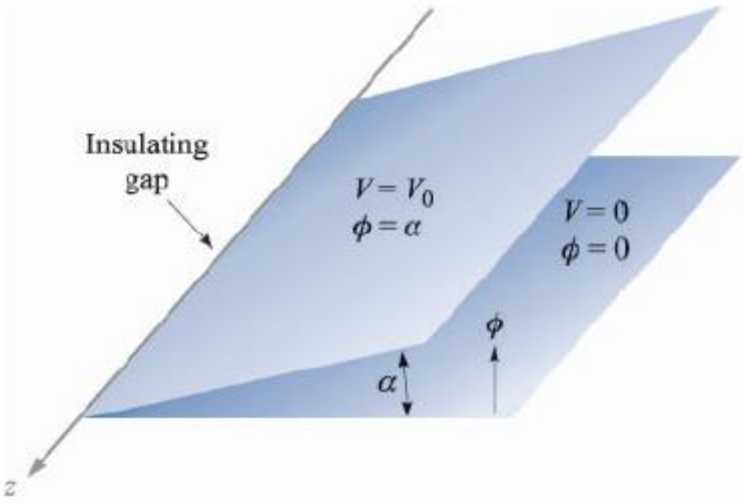

Fig 6.1 Two infinite radial planes with an interior angle α. An infinitesimal insulating gap exists at $\rho = 0$.

Note that **E** is a function of ρ and not of ϕ. This does not contradict our original assumptions, which were restrictions only on the potential field. Note, however, that the vector field **E** is a function of ϕ.