8.1 INTRODUCTION

8.2 FORCES DUE TO MAGNETIC FIELDS

There are at least three ways in which force due to magnetic fields can be experienced. The force can be [2].

- due to a moving charged particle in a **B** field,
- on a current element in an external **B** field, or
- between two current elements.

8.2.1 Force on a Charged Particle (Moving Charge)

In chapter two, we discussed that the electric force $\overrightarrow{F_e}$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's law and is related to the electric field intensity **E** as [2]:

$$\overrightarrow{F_e} = Q\,\mathbf{E} \tag{8.1}$$

This shows that if Q is positive, $\overrightarrow{F_e}$ and **E** have the same direction and $\overrightarrow{F_e}$ is directly proportional to both **E** and Q [1].

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\overrightarrow{F_m}$ experienced by a charge Q moving with a velocity **u** in a magnetic field **B** is [2]:

$$\overrightarrow{F_m} = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

The force $\overrightarrow{F_m}$ has direction perpendicular to both **u** and **B** and whose magnitude is proportional to the product of the magnitudes of the charge Q, its velocity **u**, the flux density **B**, and to the sine of the angle between the vectors **u** and **B** [1].

A comparison between the electric force and the magnetic force can be made [2].

- The electric force $\overrightarrow{F_e}$ is independent of the velocity of the charge.
- The electric force $\overrightarrow{F_e}$ can perform work on the charge and change its kinetic energy.
- The magnetic force $\overrightarrow{F_m}$ depends on the charge velocity and is normal to it.
- The magnetic force $\overrightarrow{F_m}$ cannot perform work because it is at right angles to the direction of motion of the charge ($\overrightarrow{F_m} \,.\, d\mathbf{L} = 0$); it does not cause an increase in kinetic energy of the charge.
- The magnitude of $\overrightarrow{F_m}$ is generally small compared to $\overrightarrow{F_e}$ except at high velocities.

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by superposition [2].

$$\vec{F} = \overrightarrow{F_e} + \overrightarrow{F_m} = Q\,\mathbf{E} + Q\mathbf{u} \times \mathbf{B}$$

$$\vec{F} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{8.3}$$

This equation is known as the Lorentz force equation, and its solution is required in determining electron orbits in the magnetron, proton paths in the cyclotron, plasma characteristics in a magneto-hydrodynamic (MHD) generator, or, in general, charged-particle motion in combined electric and magnetic fields [1].

Lorentz force equation relates mechanical force to electrical force. If the mass of the charged particle moving in **E** and **B** fields is (m), by Newton's second law of motion [2].

$$\vec{F} = \mathrm{m}\frac{d\mathbf{u}}{dt} = \mathrm{m}\mathbf{a} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{8.4}$$

Table (8.1) Force on a Charged Particle

No.	State of Particle	Electric Field	Magnetic Field	Combined **E** and **B** Fields
1	Stationary	QE	– – – –	QE
2	Moving	QE	$Q\mathbf{u} \times \mathbf{B}$	$Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

8.2.2 Force on a Differential Current Element.

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge [1].

$$d\mathbf{F} = dQ\mathbf{u} \times \mathbf{B} \tag{8.5}$$

The force on a current element $(Id\mathbf{L})$ of a current-carrying conductor due to the magnetic field, can be determined by using the fact that for convection current density [2]:

$$\vec{\mathrm{J}} = \rho_v\,\mathbf{u} \tag{8.6}$$

The differential element of charge in equation (8.5) may also be expressed in terms volume charge density [1].

$$dQ = \rho_v dv$$

Thus

$$d\mathbf{F} = \rho_v dv\mathbf{u} \times \mathbf{B}$$

or

$$d\mathbf{F} = \vec{\mathrm{J}} \times \mathbf{B}\,dv \tag{8.7}$$

The relationship between differential current elements is [2]:

$$I\mathrm{d}\mathbf{L} = \mathbf{K}dS = \vec{\mathrm{J}}dv \tag{8.8}$$

From equations (8.6) and (8.8) yields

$$I\mathrm{d}\mathbf{L} = \vec{\mathrm{J}}dv = \rho_v\,\mathbf{u}\,dv = \rho_v dv\,\mathbf{u} = dQ\,\mathbf{u}$$

or

$$I\mathrm{d}\mathbf{L} = \frac{dQ}{dt}\,d\mathbf{L} = dQ\,\frac{d\mathbf{L}}{dt} = dQ\,\mathbf{u}$$

Hence,

$$Id\mathbf{L} = dQ\,\mathbf{u} \tag{8.9}$$

An elemental charge dQ moving with velocity $\mathbf{u}$ is equivalent to a conduction current element $Id\mathbf{L}$. The force on a current element $Id\mathbf{L}$ in a magnetic field $\mathbf{B}$ is found by using equation (8.2) with replace $Q\mathbf{u}$ by $Id\mathbf{L}$; that is [2].

$$\overrightarrow{F_m} = Q\mathbf{u} \times \mathbf{B}$$

$$d\mathbf{F} = Id\mathbf{L} \times \mathbf{B} \tag{8.10}$$

If the current I is through a closed path L or circuit, the force on the circuit is given by

$$\mathbf{F} = \oint Id\mathbf{L} \times \mathbf{B} \tag{8.11}$$

The magnetic field produced by the current element $Id\mathbf{L}$ does not exert force on the element itself just as a point charge does not exert force on itself. The $\mathbf{B}$ field that exerts force on $Id\mathbf{L}$ must be due to another element. In other words, The $\mathbf{B}$ field in equation (8.10) or (8.11) is external to the current element $Id\mathbf{L}$. If we have surface current elements $\mathbf{K}dS$ or a volume current element $\vec{\mathrm{J}}dv$, thus the Lorentz force equation become [1,2]:

$$d\mathbf{F} = \mathbf{K}dS \times \mathbf{B} \tag{8.12}$$

$$d\mathbf{F} = \vec{\mathrm{J}}dv \times \mathbf{B} \tag{8.13}$$

or

$$\mathbf{F} = \int_S \mathbf{K}dS \times \mathbf{B} \tag{8.14}$$

$$\mathbf{F} = \int_v \vec{\mathrm{J}}dv \times \mathbf{B} \tag{8.15}$$

and

$$\mathbf{F} = \oint I d\mathbf{L} \times \mathbf{B} = -I \oint d\mathbf{L} \times \mathbf{B} \tag{8.16}$$

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B} \tag{8.17}$$

The magnitude of the force is given by the familiar equation

$$F = BIL \sin\theta \tag{8.18}$$

Where, θ is the angle between the vectors representing the direction of the current flow and the direction of the magnetic flux density. Equation (17) or (18) applies only to a portion of the closed circuit [1].

The magnetic field **B**: is defined as the force per unit current element. The magnetic field **B**: may be defined from equation (8.2) as the vector which satisfies $\overrightarrow{F_m}/Q = \mathbf{u} \times \mathbf{B}$. Both of these definitions of **B** show that **B** describes the force properties of a magnetic field [2].

8.2.3 Force between Two Current Elements

Consider the force between two elements $I_1 d\mathbf{L}_1$ and $I_2 d\mathbf{L}_2$. According to Biot-Savart's law, both current elements produce magnetic fields. So we may find the force $d(d\mathbf{F}_1)$ on element $I_1 d\mathbf{L}_1$ due to the field $d\mathbf{B}_2$ produced by element $I_2 d\mathbf{L}_2$ as shown in Fig. 8.1.

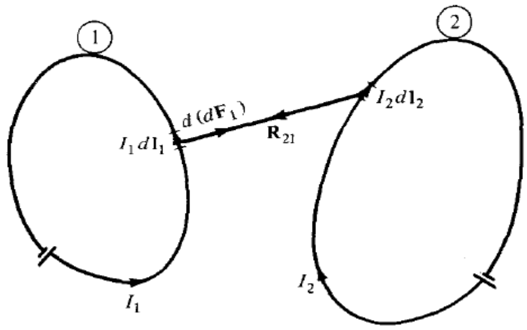


Fig. 8.1 Force between two current loops

From equation (8.7) $\quad d\mathbf{F} = I d\mathbf{L} \times \mathbf{B}$

$$d(d\mathbf{F}_1) = I_1 d\mathbf{L}_1 \times d\mathbf{B}_2 \tag{8.12}$$

But from Biot-Savart's law,

$$d\mathbf{B}_2 = \frac{\mu_o I_2 d\mathbf{L}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2} \tag{8.13}$$

Hence,

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\, d\mathbf{L}_1 \times \left(I_2\, d\mathbf{L}_2 \times \mathbf{a}_{R_{21}}\right)}{4\pi\, R_{21}^2} \qquad (8.14)$$

The law of force between two current elements. The total force $\mathbf{F}_1$, on current loop 1 due to current loop 2 shown in Fig. 8.1 as

$$\mathbf{F}_1 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{L}_1 \times \left(d\mathbf{L}_2 \times \mathbf{a}_{R_{21}}\right)}{R_{21}^2} \qquad (8.15)$$

The force $\mathbf{F}_2$ on loop 2 due to the magnetic field $\mathbf{B}_1$ from loop 1 is obtained from equation (8.15) by interchanging subscripts 1 and 2. It can be shown that $\mathbf{F}_2 = -\mathbf{F}_1$ thus $\mathbf{F}_1$ and $\mathbf{F}_2$ obey Newton's third law that action and reaction are equal and opposite.

Example 8.1 [3]: Find the force on a straight conductor of length 0.30 m carrying a current of 5.0 A in the $-\mathbf{a}_z$, direction, where the field is

$$\mathbf{B} = 3.50 * 10^{-3}(\mathbf{a}_x - \mathbf{a}_y)\ T$$

Solution:

$$\mathbf{F} = I(\mathbf{L} \times \mathbf{B})$$

$$\mathbf{F} = (5.0)\left[(0.30)(-\mathbf{a}_z) \times \left(3.50 * 10^{-3}(\mathbf{a}_x - \mathbf{a}_y)\right)\right]$$

$$\mathbf{F} = 7.42 * 10^{-3} \left(\frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}\right)\ \text{N}$$

The force, of magnitude 7.42 mN, is at right angles to both the field **B** and the current direction, as shown in Fig. 8.2.

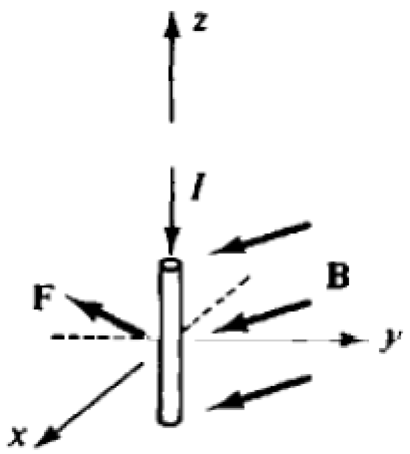


Fig. 8.2

Example 8.2 [3]: Find the force on a particle of mass $1.70 * 10^{-27}$ kg and charge $1.60 * 10^{-19}$ C, if it enters a field $B = 5$ mT with an initial speed of 83.5 km/s.

Solution: Unless directions are known for **B** and $\mathbf{U}_o$, the particle's initial velocity, the force cannot be calculated. Assuming that $\mathbf{U}_o$ and **B** are perpendicular, as shown in Fig. 8.3.

$$F = |Q|UB = (1.60 \times 10^{-19})(83.5 \times 10^3)(5 \times 10^{-3}) = 6.68 \times 10^{-17}\ \text{N}$$

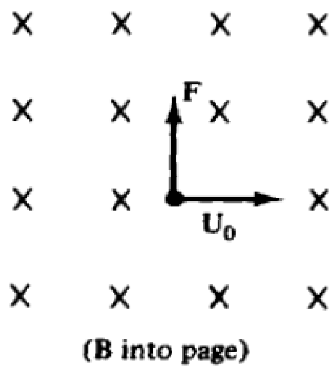


Fig. 8.3

Example 8.3 [2]: A charged particle of mass 2kg and charge 3C starts at point $(1, -2, 0)$ with velocity $4\mathbf{a}_x + 3\mathbf{a}_z$ m/s in an electric field ($12\mathbf{a}_x + 10\mathbf{a}_y$ V/m). At time $t = 1$ s, determine

(a) The acceleration of the particle, (b) Its velocity, (c) Its kinetic energy, (d) Its position

Solution: (a) According to Newton's second law of motion,

$$F = \mathrm{m}\mathbf{a} = Q\mathbf{E}$$

Where, **a** is the acceleration of the particle. Hence,

$$\mathbf{a} = \frac{Q\mathbf{E}}{\mathrm{m}} = \frac{3}{2}(12\mathbf{a}_x + 10\mathbf{a}_y) = 18\mathbf{a}_x + 15\mathbf{a}_y \text{ m/s}^2$$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(u_x, u_y, u_z) = 18\mathbf{a}_x + 15\mathbf{a}_y$$

(b) Equating components gives

$$\frac{du_x}{dt} = 18 \quad \Rightarrow \quad u_x = 18t + A$$

$$\frac{du_y}{dt} = 15 \quad \Rightarrow \quad u_x = 15t + B$$

$$\frac{du_z}{dt} = 0 \quad \Rightarrow \quad u_z = C$$

Where A, B, and C are integration constants. But at $t = 0$, $\mathbf{u} = 4\mathbf{a}_x + 3\mathbf{a}_z$. Hence,

$$u_x(t = 0) = 4 \;\Rightarrow\; 4 = 0 + A \;\Rightarrow\; A = 4$$

$$u_y(t = 0) = 0 \;\Rightarrow\; 0 = 0 + B \;\Rightarrow\; B = 0$$

$$u_z(t = 0) = 3 \;\Rightarrow\; 3 = C \;\Rightarrow\; C = 3$$

Substituting the values of A, B, and C, then

$$\mathbf{u}_{(t)} = (u_x, u_y, u_z) = (18t + 4, 15t, 3)$$

Hence

$$\mathbf{u}_{(t=1s)} = (u_x, u_y, u_z) = 22\mathbf{a}_x + 15\mathbf{a}_y + 3\mathbf{a}_z \text{ m/s}$$

(c) Kinetic energy

$$K.E. = \frac{1}{2}m|\mathbf{u}|^2 = \frac{1}{2}(2)[22^2 + 15^2 + 3^2] = 118 \text{ J}$$

(d)

$$\mathbf{u} = \frac{d\mathbf{L}}{dt} = \frac{d}{dt}(x, y, z) = (18t + 4, 15t, 3)$$

Equating components yields

$$\frac{dx}{dt} = u_x = 18t + 4 \quad \Rightarrow \quad x = 9t^2 + 4t + A_1$$

$$\frac{dy}{dt} = u_y = 15t \quad \Rightarrow \quad y = 7.5t^2 + B_1$$

$$\frac{dz}{dt} = u_z = 3 \quad \Rightarrow \quad z = 3t + C_1$$

At $t = 0$, $(x, y, z) = (1, -2, 0)$; hence, $\quad A_1 = 1, \quad B_1 = -2, \quad C_1 = 0$

Substituting the values of A_1, B_1 and C_1, we get

$$(x, y, z) = (9t^2 + 4t + 1, 7.5t^2 - 2, 3t)$$

Hence, at $t = 1$, $(x, y, z) = (14, 5.5, 3)$.

Example 8.4 [2]: A charged particle of mass 1kg and charge 2C starts at the origin with zero initial velocity in a region where $\mathbf{E} = 3\mathbf{a}_z$ V/m. Find

(a) The force on the particle, (b) The time it takes to reach point $P(0, 0, 12\ m)$,

(c) Its velocity and acceleration at P, (d) Its K.E. at P.

Solution: (a)

$$F = m\frac{\partial \vec{u}}{\partial t} = Q\mathbf{E} = 2(3\mathbf{a}_z) = 6\mathbf{a}_z\ N$$

(b)

$$\frac{\partial \vec{u}}{\partial t} = 6\mathbf{a}_z = \frac{\partial}{\partial t}(u_x, u_y, u_z)$$

$$\frac{\partial u_x}{\partial t} = 0 \quad \Rightarrow \quad u_x = A$$

$$\frac{\partial u_y}{\partial t} = 0 \quad \Rightarrow \quad u_y = B$$

$$\frac{\partial u_z}{\partial t} = 6 \quad \Rightarrow \quad u_z = 6t + C$$

Since, $\quad \vec{u}(t = 0) = 0 \qquad \Rightarrow \quad A = B = C = 0$

$$u_x = u_y = 0 \quad , \quad u_z = 6t$$

$$u_x = \frac{\partial x}{\partial t} = 0 \quad \Rightarrow \quad x = A_1$$

$$u_y = \frac{\partial y}{\partial t} = 0 \quad \Rightarrow \quad y = B_1$$

$$u_z = \frac{\partial z}{\partial t} = 6t \quad \Rightarrow \quad z = 3t^2 + C_1$$

At, $t = 0$; $(x, y, z) = (0,0,0)$ $\Rightarrow$ $A_1 = B_1 = C_1 = 0$

Hence, $(x, y, z) = (0,0,3t^2)$ $\Rightarrow$ $\vec{u} = 6t\mathbf{a}_z$

At any time, at $P(0, 0, 12 \text{ m})$ $\Rightarrow$ $z = 12 = 3t^2 \Rightarrow t = 2$ sec

(c) $$\vec{u} = 6t\mathbf{a}_z = 12\mathbf{a}_z \text{ m/s}$$

$$\mathbf{a} = \frac{\partial \vec{u}}{\partial t} = 6\mathbf{a}_z \text{ m/s}^2$$

(d) Kinetic energy

$$K.E. = \frac{1}{2}m|\mathbf{u}|^2 = \frac{1}{2}(1)(12^2) = 72 \text{ J}$$

Example 8.5 [2]: A charged particle of mass 2kg and 1C starts at the origin with velocity $3\mathbf{a}_y$ m/s and travels in a region of uniform magnetic field $\mathbf{B} = 10\,\mathbf{a}_z$ Wb/m^2. At t = 4 s. Calculate (a) The velocity and acceleration of the particle (b) The magnetic force on it (c) Its K.E. and location

Solution: (a)

$$F = m\frac{d\vec{u}}{dt} = Q\vec{u} \times \vec{B}$$

$$\mathbf{a} = \frac{d\vec{u}}{dt} = \frac{Q}{m}(\vec{u} \times \vec{B})$$

Hence

$$\frac{d}{dt}(u_x\mathbf{a}_x + u_y\mathbf{a}_y + u_z\mathbf{a}_z) = \frac{1}{2}\begin{pmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ u_x & u_y & u_z \\ 0 & 0 & 10 \end{pmatrix} = 5(u_x\mathbf{a}_x - u_y\mathbf{a}_y)$$

By equating components, we get

$$\frac{du_x}{dt} = 5u_y \quad : \quad \frac{du_y}{dt} = -5u_x \quad : \quad \frac{\partial u_z}{\partial t} = 0 \quad \Rightarrow \quad u_z = C_o$$

We can eliminate u_x or u_y in above equations by taking second derivatives of one equation and making use of the other. Thus

$$\frac{\partial^2 u_x}{\partial t^2} = 5\frac{du_y}{dt} = -25u_x \quad \Rightarrow \quad \frac{\partial^2 u_x}{\partial t^2} + 25u_x = 0$$

which is a linear differential equation

$$u_x = C_1 \cos 5t + C_2 \sin 5t$$

$$5u_y = \frac{du_x}{dt} = -5C_1 \sin 5t + 5C_2 \cos 5t$$

or $$u_y = -C_1 \sin 5t + C_2 \cos 5t$$

We now determine constants C_o, C_1 and C_2 using the initial conditions. At $t = 0$, $\vec{u} = 3\mathbf{a}_y$, hence.

$$u_x = 0 \quad \Rightarrow \quad 0 = C_1.1 + C_2.0 \quad \Rightarrow \quad C_1 = 0$$
$$u_y = 3 \quad \Rightarrow \quad 3 = -C_1.0 + C_2.1 \quad \Rightarrow \quad C_2 = 3 \quad \& \quad u_z = 0 \quad \Rightarrow \quad C_o = 0$$

Substituting the values of C_o, C_1 and C_2 into equations, gives

$$\mathbf{u} = (u_x, u_y, u_z) = (3\sin 5t\,, 3\cos 5t\,, 0)$$
$$\mathbf{u}(t=4) = (3\sin 20\,, 3\cos 20\,, 0) = 2.739\,\mathbf{a}_x + 1.224\,\mathbf{a}_y \text{ m/s}$$
$$\mathbf{a} = \frac{d\vec{u}}{dt} = (15\cos 5t\,, -15\sin 5t\,, 0)$$

and $$\mathbf{a}(t=4) = 6.101\,\mathbf{a}_x - 13.703\,\mathbf{a}_y \text{ m/s}^2$$

(b) $$\vec{F} = \text{m}\mathbf{a} = 12.2\,\mathbf{a}_x - 27.4\,\mathbf{a}_y \; N$$

or, $\vec{F} = Q\mathbf{u} \times \mathbf{B} = (1)(2.739\,\mathbf{a}_x + 1.224\,\mathbf{a}_y)(10\,\mathbf{a}_z) = 12.2\,\mathbf{a}_x - 27.4\,\mathbf{a}_y \; N$

(c) Kinetic energy

$$K.E. = \frac{1}{2}m|\mathbf{u}|^2 = \frac{1}{2}(2)(2.739^2 + 1.224^2) = 9 \text{ J}$$

Example 8.6 [1]: A square loop of wire in the $z = 0$ plane carrying 2mA in the field of an infinite filament on the y axis, as shown in Figure (8.4). Find the total force on the loop.

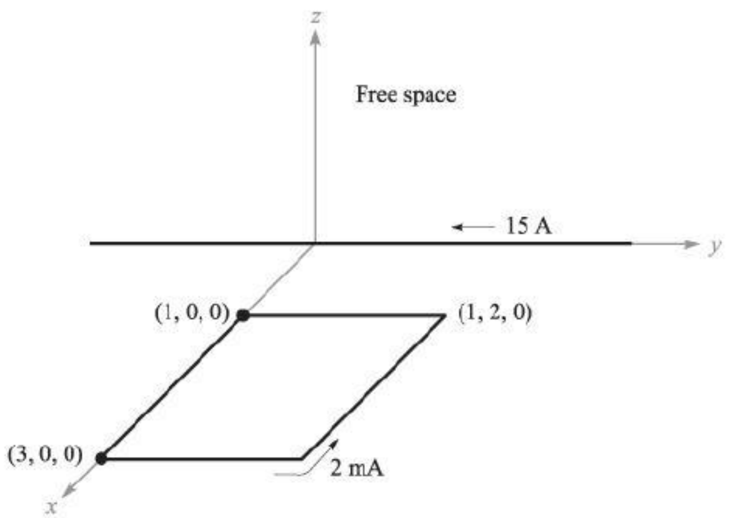


Figure (8.4)

Solution: The field produced in the plane of the loop by the straight filament is

$$\mathbf{H} = \frac{I}{4\pi\rho}\mathbf{a}_\phi = \frac{I}{4\pi x}\mathbf{a}_z = \frac{15}{4\pi x}\mathbf{a}_z \text{ A/m}$$

Therefore,

$$\mathbf{B} = \mu_o\mathbf{H} = 4\pi \times 10^{-7}\mathbf{H} = \frac{3 \times 10^{-6}}{x}\mathbf{a}_z \; T$$

We use the integral form,

$$\mathbf{F} = -I\oint \mathbf{B} \times d\mathbf{L}$$

The total force is the sum of the forces on the four sides.

$$\mathbf{F} = -2 \times 10^{-3} \int \left[\left(\frac{3 \times 10^{-6}}{x}\mathbf{a}_z\right) \times \left(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z\right)\right]$$

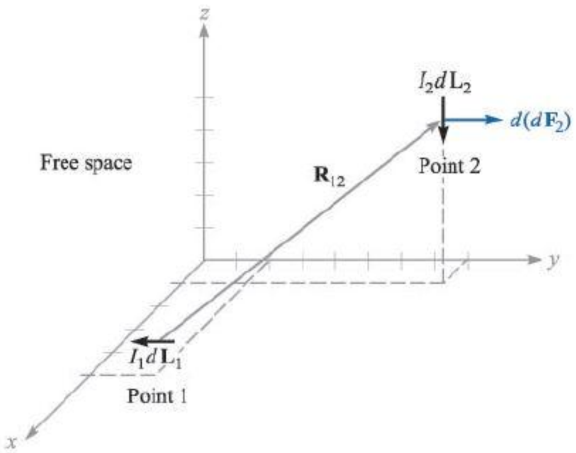
$$\mathbf{F} = -6 \times 10^{-9}\left[\int_{x=0}^{3} \frac{\mathbf{a}_z}{x} \times dx\mathbf{a}_x + \int_{y=0}^{2} \frac{\mathbf{a}_z}{3} \times dy\mathbf{a}_y + \int_{x=3}^{1} \frac{\mathbf{a}_z}{x} \times dx\mathbf{a}_x + \int_{y=0}^{2} \frac{\mathbf{a}_z}{1} \times dy\mathbf{a}_y\right]$$

$$\mathbf{F} = -2 \times 10^{-3} \times 3 \times 10^{-6}\left[\int_{x=0}^{3} \frac{dx}{x}\mathbf{a}_y + \int_{y=0}^{2} \frac{dy}{3}(-\mathbf{a}_x) + \int_{x=3}^{1} \frac{dx}{x}\mathbf{a}_y + \int_{y=0}^{2} dy(-\mathbf{a}_x)\right]$$

$$\mathbf{F} = -6 \times 10^{-9}\left[\ln x\Big|_{1}^{3}\mathbf{a}_y + \frac{1}{3}\Big[y\Big]_{0}^{2}(-\mathbf{a}_x) + \ln x\Big|_{3}^{1}\mathbf{a}_y + \Big[y\Big]_{2}^{0}(-\mathbf{a}_x)\right]$$

$$\mathbf{F} = -6 \times 10^{-9}\left[\ln(3)\,\mathbf{a}_y - \frac{2}{3}\mathbf{a}_x + \ln\left(\frac{1}{3}\right)\mathbf{a}_y + 2\mathbf{a}_x\right] = -8\mathbf{a}_x \text{ pN}$$

Example 8.7 [1]: Consider the two differential current elements shown in Figure (9.3). Determine the differential force on $d\mathbf{L}_2$.



Solution: We have $I_1 d\mathbf{L}_1 = -3\mathbf{a}_y$ A.m at $P_1(5, 2, 1)$, and $I_2 d\mathbf{L}_2 = -4\mathbf{a}_z$ A.m at $P_2(1, 8, 5)$. Thus, $\mathbf{R}_{12} = -4\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z$, $|\mathbf{R}_{12}| = \sqrt{(-4)^2 + (6)^2 + (4)^2}$ and we use this equation,

$$d(d\mathbf{F}_2) = \frac{\mu_o I_2\, d\mathbf{L}_2 \times \left(I_1\, d\mathbf{L}_1 \times \mathbf{a}_{R_{12}}\right)}{4\pi\, R_{12}^2}$$

$$d(d\mathbf{F}_2) = \frac{4\pi \times 10^{-7} \times (-4\mathbf{a}_z) \times \left[(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z)\right]}{4\pi(\,16 + 36 + 16)^{1.5}} = 8.56\mathbf{a}_y \text{ nN}$$

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\, d\mathbf{L}_1 \times \left(I_2\, d\mathbf{L}_2 \times \mathbf{a}_{R_{21}}\right)}{4\pi\, R_{21}^2}$$

$$d(d\mathbf{F}_1) = \frac{4\pi \times 10^{-7} \times (-3\mathbf{a}_y) \times \left[(-4\mathbf{a}_z) \times (4\mathbf{a}_x - 6\mathbf{a}_y - 4\mathbf{a}_z)\right]}{4\pi(\,16 + 36 + 16)^{1.5}} = 12.84\mathbf{a}_z \text{ nN}$$

8.3 MAGNETIC TORQUE AND MOMENT

The torque T (or mechanical moment of force) on the loop is the vector product of the force $\vec{F}$ and the moment arm **r**. That is,

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \quad \text{in (N.m)} \tag{8.16}$$

Let us apply this to a rectangular loop of length ℓ and width w placed in a uniform magnetic field **B** Fig. 8.5(a). We notice that $d\mathbf{L}$ is parallel to **B** along sides 1-2 and 3-4 of the loop and no force is exerted on those sides. Thus

$$\mathbf{F} = I\int_{2}^{3} d\mathbf{L} \times \mathbf{B} + I\int_{4}^{1} d\mathbf{L} \times \mathbf{B} = I\int_{0}^{\ell} dz\, \mathbf{a}_z \times \mathbf{B} + I\int_{\ell}^{0} dz\, \mathbf{a}_z \times \mathbf{B}$$

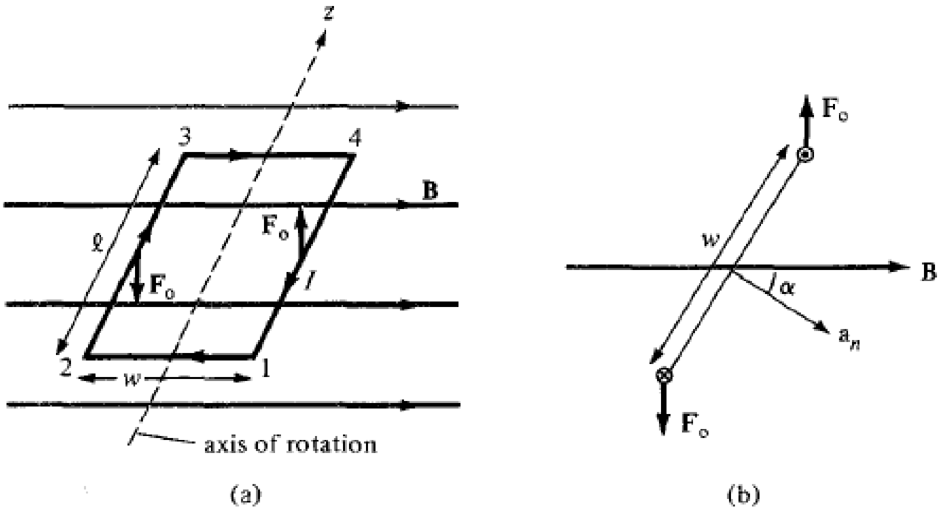


Fig. 8.5 Rectangular planar loop in a uniform magnetic field.

$$\mathbf{F} = \mathbf{F}_o - \mathbf{F}_o = 0 \tag{8.17}$$

where $|\mathbf{F}_o| = IB\ell$ because **B** is uniform.

- no force is exerted on the loop.
- $\mathbf{F}_o$ and $-\mathbf{F}_o$ act at different points on the loop, thereby creating a couple
- if the normal to the plane of the loop makes an angle α with **B**, as in Fig. 8.5(b).

The torque on the loop is

$$|\mathbf{T}| = |\mathbf{F}_o| w \sin\alpha$$

or

$$T = BI\ell w \sin\alpha \tag{8.18}$$

But $\ell w = S$, the area of the loop. Hence,

$$T = BIS \sin\alpha \tag{8.19}$$

We define the quantity

$$\mathbf{m} = IS\mathbf{a}_n \qquad (8.20)$$

This magnetic dipole moment (in A/m^2). $\mathbf{a}_n$ is a unit vector normal to the plane of the loop.

The magnetic dipole moment: is the product of current and area of the loop; its direction is normal to the loop. Thus,

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (8.21)$$

- This applicable in determining the torque on a planar loop.
- The magnetic field must be uniform.
- the torque is in the direction of the axis of rotation, it is directed such as to reduce α so that **m** and **B** are in the same direction.
- In an equilibrium position (when **m** and **B** are in the same direction), the loop is perpendicular to the magnetic field and the torque will be zero as well as the sum of the forces on the loop.

Example 8.6: A conductor located at $x = 0.4$ m, $y = 0$ and $0 < z < 2$ m carries a current of 5A in the $\mathbf{a}_z$ direction. Along the length of the conductor $\mathbf{B} = 2.5\,\mathbf{a}_x\,T$. Find the torque about the z-axis.

Solution:

$$\mathbf{F} = I(\mathbf{L} \times \mathbf{B}) = (5)(2\,\mathbf{a}_z \times 2.5\,\mathbf{a}_x) = 25\,\mathbf{a}_y \text{ N}$$

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} = (0.4\,\mathbf{a}_x) \times (25\,\mathbf{a}_y) = 10\,\mathbf{a}_z \text{ N.m}$$

8.4 A MAGNETIC DIPOLE

A bar magnet or a small filamentary current loop is usually referred to as a magnetic dipole. Let us determine the magnetic field **B** at an observation point $P(r, \theta, \phi)$ due to a circular loop carrying current I as in Fig. 8.6. The magnetic vector potential at P is

$$\mathbf{A} = \frac{\mu_o I}{4\pi} \oint \frac{d\mathbf{L}}{r} \tag{8.22}$$

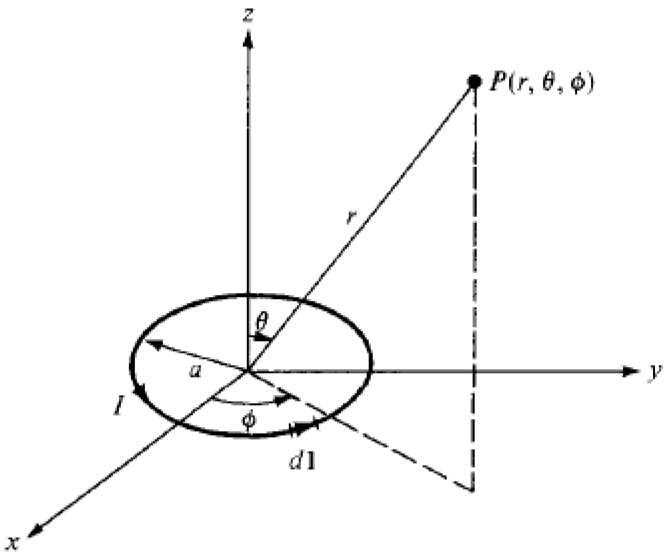


Fig. 8.6 Magnetic field at P due to a current loop.

At far field ($r >> a$, so that the loop appears small at the observation point), **A** has only ϕ-component and it is given by

$$\mathbf{A} = \frac{\mu_o I \pi a^2 \sin\theta \, \mathbf{a}_\phi}{4\pi \, r^2} \tag{8.23a}$$

or

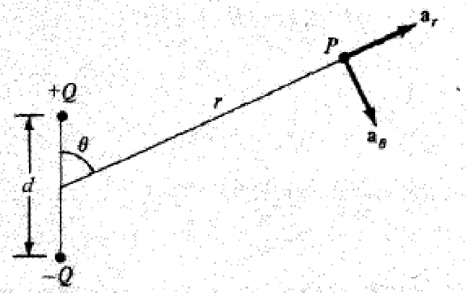
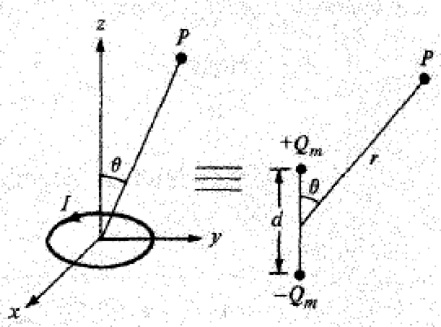
$$\mathbf{A} = \frac{\mu_o \mathbf{m} \times \mathbf{a}_r}{4\pi \, r^2} \tag{8.23b}$$

Where
$$\mathbf{m} = I\pi a^2 \mathbf{a}_z$$
$$\mathbf{a}_z \times \mathbf{a}_r = \sin\theta \, \mathbf{a}_\phi$$

And determine the magnetic flux density **B** from $\mathbf{B} = \nabla \times \mathbf{A}$ as

$$\mathbf{B} = \frac{\mu_o \mathbf{m}}{4\pi \, r^3}(2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta) \tag{8.24}$$

Table 8.1 Comparisons between Electric and Magnetic Monopoles and Dipoles

Electric	Magnetic
$V = \frac{Q}{4\pi\varepsilon_o r}$ $\mathbf{E} = \frac{Q\mathbf{a}_r}{4\pi\varepsilon_o r^2}$ 	Does not exist Q_m
Monopole (point charge)	Monopole (point charge)
$V = \frac{Q\cos\theta}{4\pi\varepsilon_o r^2}$ $\mathbf{E} = \frac{Qd}{4\pi\varepsilon_o r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$ 	$\mathbf{A} = \frac{\mu_o m\sin\theta\,\mathbf{a}_\phi}{4\pi r^2}$ $\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$ 
Dipole (two point charge)	Dipole (small current loop or bar magnet)

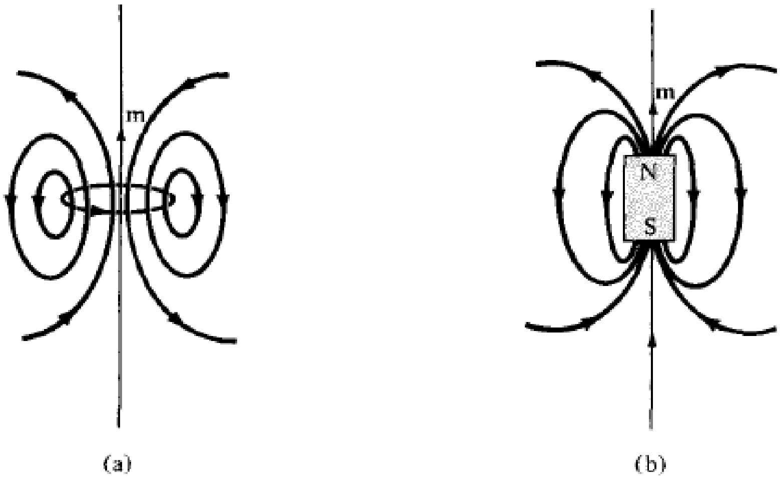


Fig. 8.7 The **B** lines due to magnetic dipoles: (a) a small current loop with $\mathbf{m} = IS$, (b) a bar magnet with $\mathbf{m} = Q_m\ell$.

Consider the bar magnet of Fig. 8.8. If Q_m is an isolated magnetic charge and ℓ is the length of the bar, the bar has a dipole moment $Q_m\ell$. (Notice that Q_m does exist; however, it does not exist without an associated Q_m. When the bar is in a uniform magnetic field **B**, it experiences a torque

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = Q_m \boldsymbol{\ell} \times \mathbf{B} \tag{8.25}$$

Where, ℓ points in the direction south-to-north. The torque tends to align the bar with the external magnetic field.
The force acting on the magnetic charge is given by

$$\mathbf{F} = Q_m \mathbf{B} \tag{8.26}$$

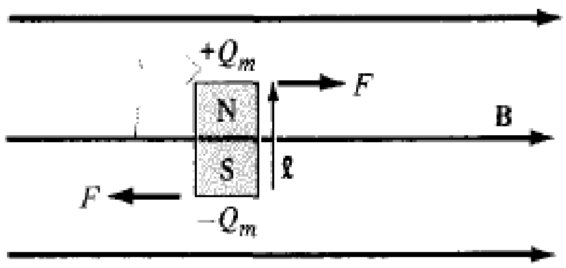


Fig. 8.8 A bar magnet in an external magnetic field.

Both a small current loop and a bar magnet produce magnetic dipoles, they are equivalent if they produce the same torque in a given **B** field; that is, when

$$T = Q_m \ell B = ISB \tag{8.27}$$

Hence, showing that they must have the same dipole moment.

$$Q_m \ell = IS \tag{8.28}$$

8.5 MAGNETIZATION IN MATERIALS

The material is composed of atoms. Each atom may be regarded as consisting of electrons orbiting about a central positive nucleus; the electrons also rotate (or spin) about their own axes. An internal magnetic field is produced by (electronic motions) electrons orbiting around the nucleus or electrons spinning as in Figure (8.9a,b), that are similar to the magnetic field produced by a current loop of Figure (8.10). The equivalent current loop has a magnetic moment of $\mathbf{m} = I_b S\, \mathbf{a}_n$, where S is the area of the loop and I_b is the bound current (bound to the atom). Without an external **B** field applied to the material, the sum of **m**'s is zero due to random orientation as in Figure (8.11a). When an external **B** field is applied, the magnetic moments of the electrons more or less align themselves with **B** so that the net magnetic moment is not zero, as illustrated in Figure (8.11b).

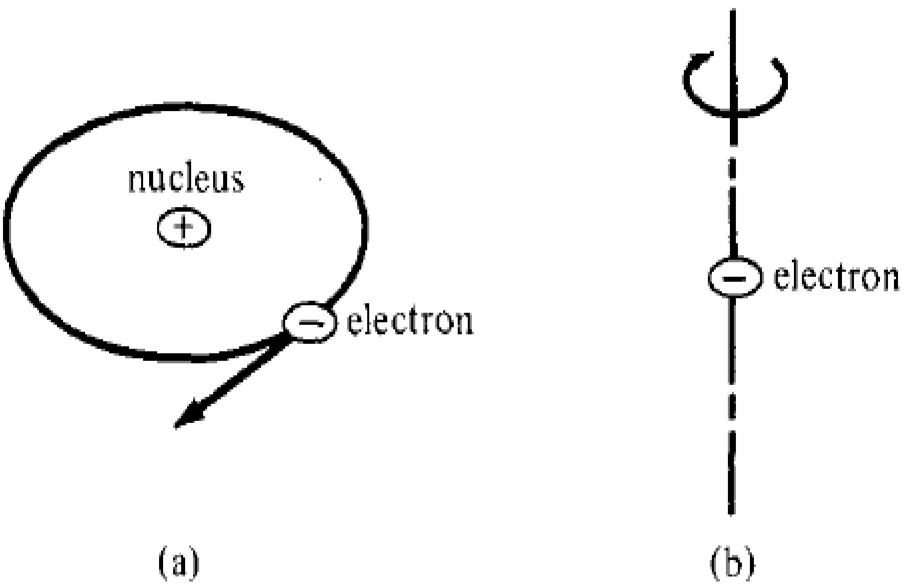


Fig. 8.9 (a) Electron orbiting around the nucleus; (b) electron spin.

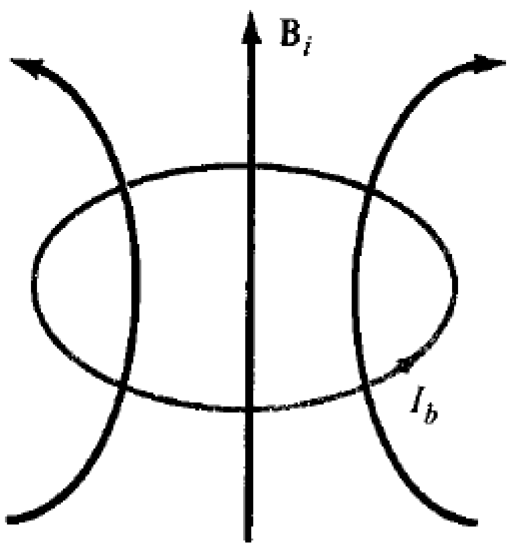


Fig 8.10 Circular current loop equivalent to electronic motion of Figure 8.9.

The magnetization **M** (in amperes/meter) is the magnetic dipole moment per unit volume.

If there are N atoms in a given volume Δv and the k^{th} atom has a magnetic moment $\mathbf{m}_k$,

$$\mathbf{m} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \mathbf{m}_k}{\Delta v} \tag{8.29}$$

For a differential volume dv', the magnetic moment is $d\mathbf{m} = \mathbf{M}dv'$. From eq. (8.23a), the vector magnetic potential due to $d\mathbf{m}$ is

$$d\mathbf{A} = \frac{\mu_o \mathbf{M} \times \mathbf{a}_R}{4\pi\, R^2} dv' = \frac{\mu_o \mathbf{M} \times \mathbf{R}}{4\pi\, R^3} dv'$$

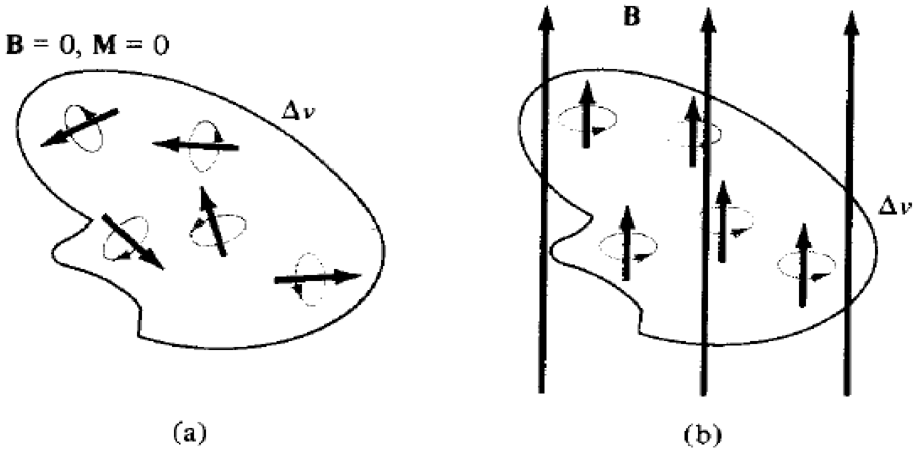


Fig. 8.11 Magnetic dipole moment in a volume (a) before B is applied, (b) after B is applied.

$$\frac{\mathbf{R}}{R^3} = \nabla' \frac{1}{R}$$

hence,

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int \mathbf{M} \times \nabla' \frac{1}{R} dv' \tag{8.30}$$

$$\mathbf{M} \times \nabla' \frac{1}{R} = \frac{1}{R} \nabla' \times \mathbf{M} - \nabla' \times \frac{\mathbf{M}}{R}$$

Substituting this into eq. (8.30) yields

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_o}{4\pi} \int_{v'} \nabla' \times \frac{\mathbf{M}}{R} dv'$$

Applying the vector identity

$$\int_{v'} \nabla' \times \mathbf{F}\, dv' = -\oint_{s'} \mathbf{F} \times d\boldsymbol{S}$$

to the second integral, we obtain

$$\mathbf{A} = \frac{\mu_o}{4\pi}\int_{v'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_o}{4\pi}\oint_{s'} \frac{\mathbf{M} \times \mathbf{a}_n}{R} dS'$$

$$\mathbf{A} = \frac{\mu_o}{4\pi}\int_{v'} \frac{\vec{J}_b}{R} dv' + \frac{\mu_o}{4\pi}\oint_{s'} \frac{\mathbf{K}_b}{R} dS' \qquad (8.31)$$

hence,

$$\vec{J}_b = \nabla \times \mathbf{M} \qquad (8.32)$$

And

$$\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n \qquad (8.33)$$

where $\vec{J}_b$ is the bound volume current density or magnetization volume current density (in $\mathrm{A/m^2}$), $\mathbf{K}_b$ is the bound surface current density (in $\mathrm{A/m}$). Equation (8.31) shows that the potential of a magnetic body is due to a volume current density $\vec{J}_b$ throughout the body and a surface current $\mathbf{K}_b$ on the surface of the body. The vector **M** is analogous to the polarization **P** in dielectrics and is sometimes called the magnetic polarization density of the medium. In another sense, **M** is analogous to **H** and they both have the same units. In this respect, as $\vec{J} = \nabla \times \mathbf{H}$, so is $\vec{J}_b = \nabla \times \mathbf{M}$.

In free space, $\mathbf{M} = 0$ and we have

$$\nabla \times \mathbf{H} = \vec{J}_f \quad \text{or} \quad \nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) = \vec{J}_f \qquad (8.34)$$

Where, $\vec{J}_f$ is the free current volume density.

In a material medium $\mathbf{M} \neq 0$, and as a result, **B** changes so that

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) = \vec{J}_f + \vec{J}_b = \vec{J}$$
$$= \nabla \times \mathbf{H} + \nabla \times \mathbf{M}$$

or

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) \qquad (8.35)$$

For linear materials, M (in $\mathrm{A/m}$) depends linearly on **H** such that

$$\mathbf{M} = \chi_m\,\mathbf{H} \qquad (8.36)$$

Where, χ_m is a dimensionless quantity (ratio of **M** to **H**) called magnetic susceptibility of the medium. Substituting eq. (8.35) into eq. (8.34) yields

$$\mathbf{B} = \mu_o(\mathbf{H} + \chi_m\, \mathbf{H})$$

$$\mathbf{B} = \mu_o(1 + \chi_m)\mathbf{H} = \mu_o\mu_r\, \mathbf{H} \tag{8.37}$$

or

$$\mathbf{B} = \mu\mathbf{H} \tag{8.38}$$

Where

$$\mu_r = (1 + \chi_m) = \frac{\mu}{\mu_o} \tag{8.39}$$

$$\mu = \mu_o\mu_r \tag{8.40}$$

- The quantity $\mu = \mu_o\mu_r$ is called the permeability of the material and is measured in henrys/meter.
- The dimensionless quantity μ_r is the ratio of the permeability of a given material to that of free space and is known as the relative permeability of the material.
- The relationships in equations (8.36) to (8.40) hold only for linear and isotropic materials. If the materials are anisotropic (e.g., crystals), equations (8.36) to (8.40) do not apply.

8.6 CLASSIFICATION OF MAGNETIC MATERIALS

We may use the magnetic susceptibility χ_m or the relative permeability μ_r to classify materials in terms of their magnetic property or behavior. A material is said to be nonmagnetic if $\chi_m = 0$ (or $\mu_r = 1$); it is magnetic otherwise. Free space, air, and materials with $\chi_m = 0$ (or $\mu_r = 1$) are regarded as nonmagnetic.

Magnetic materials may be grouped into three major classes:

- diamagnetic.
- paramagnetic.
- ferromagnetic.

This rough classification is depicted in Figure 8.12.

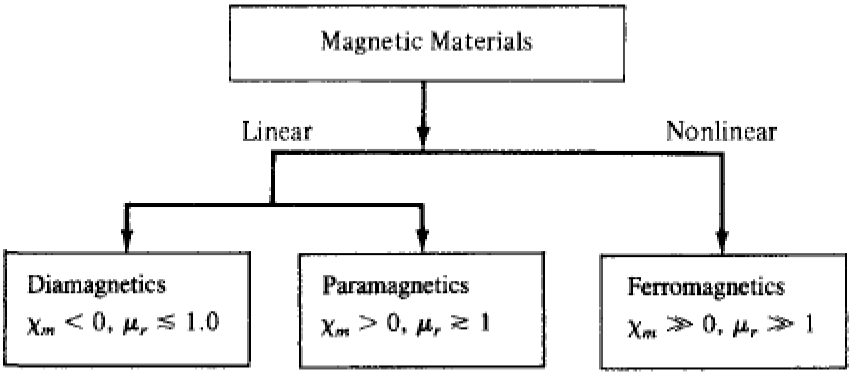


Fig. 8.12 Classification of magnetic materials.

Diamagnetism: occurs in materials where the magnetic fields due to electronic motions of orbiting and spinning completely cancel each other. For most diamagnetic materials (bismuth, lead, copper, silicon, diamond, sodium chloride).

Paramagnetism: occurs in materials where the magnetic fields produced by orbital and spinning electrons do not cancel completely. For most paramagnetic materials (air, platinum, tungsten, potassium).

Ferromagnetism: occurs in materials whose atoms have relatively large permanent magnetic moment. They are called ferromagnetic materials because the best known member is iron. Other members are cobalt, nickel, and their alloys.

Example 8.7: Region $0 \le z \le 2$ m is occupied by an infinite slab of permeable material ($\mu_r = 2.5$). If $\mathbf{B} = 10y\,\mathbf{a}_x - 5x\,\mathbf{a}_y$ mWb/m^2 within the slab, determine: (a) $\vec{J}$; (b) $\vec{J}_b$; (c) $\mathbf{M}$; (d) $\mathbf{K}_b$ on $z = 0$.

Solution:

(a) By definition,

$$\vec{J} = \nabla \times \mathbf{H} = \nabla \times \left(\frac{\mathbf{B}}{\mu_o \mu_r}\right) = \frac{1}{4\pi \times 10^{-7} \times (2.5)} \begin{bmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{bmatrix}$$

$$\vec{J} = \nabla \times \mathbf{H} = \frac{1}{10\pi \times 10^{-7}} \begin{bmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 10y & -5x & 0 \end{bmatrix} = \frac{1}{10\pi \times 10^{-7}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \mathbf{a}_z$$

$$\frac{10^6}{\pi}(-5 - 10)10^{-3}\,\mathbf{a}_z = -4.775\,\mathbf{a}_z \text{ kA}/m^2$$

(b) $$\vec{J}_b = \nabla \times \mathbf{M} = \nabla \times \chi_m \mathbf{H} = \chi_m (\nabla \times \mathbf{H}) = \chi_m \vec{J}$$

$$\vec{J}_b = \chi_m \vec{J} = (\mu_r - 1)\,\vec{J} = (2.5 - 1)\,(-4.775\,\mathbf{a}_z) = -7.163\,\mathbf{a}_z \text{ kA/m}^2$$

(c)

$$\mathbf{M} = \chi_m\,\mathbf{H} = \chi_m\,\frac{\mathbf{B}}{\mu_o\mu_r}$$

$$\mathbf{M} = 1.5\,\frac{(10y\,\mathbf{a}_x - 5x\,\mathbf{a}_y).10^{-3}}{4\pi \times 10^{-7} \times (2.5)} = (4.775y\,\mathbf{a}_x - 2.387x\,\mathbf{a}_y)\ \mathrm{kA/m}$$

(d) $$\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n$$

Since $z = 0$ is the lower side of the slab occupying $0 \leq z \leq 2$, $\mathbf{a}_n = -\mathbf{a}_z$. Hence,

$$\mathbf{K}_b = (4.775y\,\mathbf{a}_x - 2.387x\,\mathbf{a}_y) \times (-\mathbf{a}_z) = (2.387x\,\mathbf{a}_x + 4.775y\,\mathbf{a}_y)\ \mathrm{kA/m}$$

Example 8.8: In a certain region ($\mu = 4.6\,\mu_o$) and $\mathbf{B} = 10\,e^{-y}\,\mathbf{a}_z\ \mathrm{mWb/m^2}$.

Find: (a) χ_m, (b) $\mathbf{H}$, (c) $\mathbf{M}$.

Solution:

(a)

$$\mu_r = \frac{\mu}{\mu_o} = \frac{4.6\,\mu_o}{\mu_o} = 4.6$$

$$\chi_m = \mu_r - 1 = 4.6 - 1 = 3.6$$

(b)

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_o\mu_r} = \frac{10 \times 10^{-3}\,e^{-y}\,\mathbf{a}_z}{4\pi \times 10^{-7} \times (4.6)} = 1730\,e^{-y}\,\mathbf{a}_z\ \mathrm{A/m}$$

(c) $$\mathbf{M} = \chi_m\,\mathbf{H} = (3.6)\,(1730\,e^{-y}\,\mathbf{a}_z) = 6228\,e^{-y}\,\mathbf{a}_z\ \mathrm{A/m}$$

8.7 MAGNETIC BOUNDARY CONDITIONS

We define magnetic boundary conditions as the conditions that **H** (or **B**) field must satisfy at the boundary between two different media.

By using Gauss's law for magnetic fields.

$$\oint_S \mathbf{B} \,.\, d\mathbf{S} = 0 \tag{8.41}$$

and Ampere's circuit law

$$\oint_L \mathbf{E} \,.\, d\mathbf{L} = I \tag{8.42}$$

The boundary between two magnetic media 1 and 2, characterized, respectively, by μ_1 and μ_2 as in Fig. 8.13. Applying eq. (8.41) to the pillbox (Gaussian surface) of Fig. 8.13(a) and allowing $\Delta h \to 0$, we obtain

$$B_{1n}\Delta S - B_{2n}\Delta S = 0 \tag{8.43}$$

Thus,

$$\mathbf{B}_{1n} = \mathbf{B}_{2n} \tag{8.44a}$$

or

$$\mu_1 \mathbf{H}_{1n} = \mu_2 \mathbf{H}_{2n} \tag{8.44b}$$

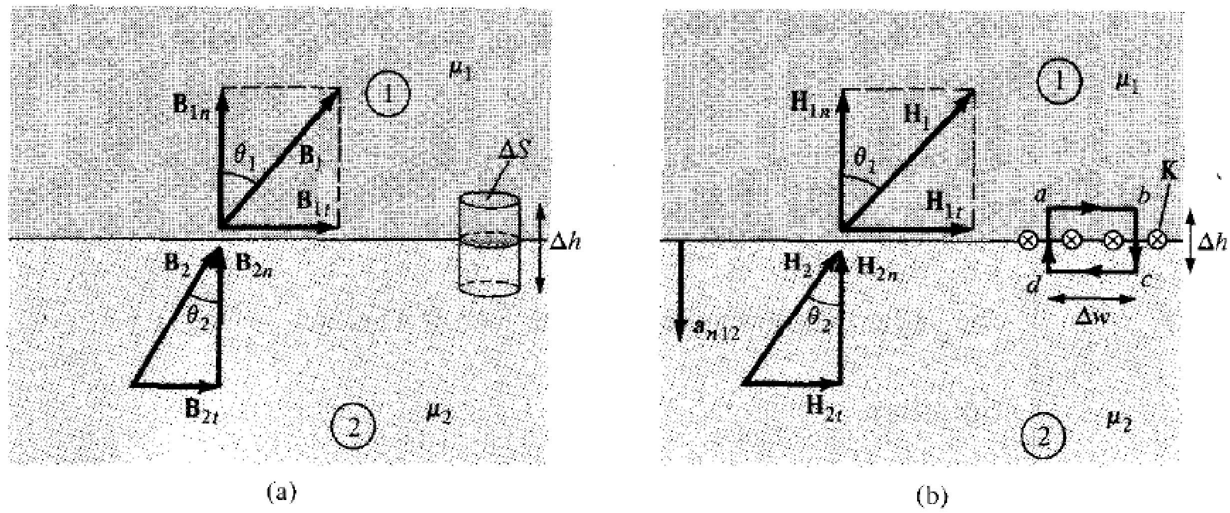


Fig. 8.13 Boundary conditions between two magnetic media: (a) for **B**, (b) for **H**.

Equation (8.44) shows that the normal component of **B** is continuous at the boundary and the normal component of **H** is discontinuous at the boundary.
Applying eq. (8.42) to the closed path (abcda) of Fig. 8.13(b) where surface current K on the boundary is assumed normal to the path. We obtain

$$K.\Delta\mathrm{W} = H_{1t}.\Delta\mathrm{W} + H_{1n}.\frac{\Delta\mathrm{h}}{2} + H_{2n}.\frac{\Delta\mathrm{h}}{2} - H_{2t}.\Delta\mathrm{W} - H_{2n}.\frac{\Delta\mathrm{h}}{2} - H_{1n}.\frac{\Delta\mathrm{h}}{2} \tag{8.45}$$

as $\Delta h \rightarrow 0$, eq. (8.45) leads to

$$H_{1t} - H_{2t} = K \tag{8.46}$$

This shows that the tangential component of H is also discontinuous. Equation (8.46) may be written in terms of B as

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K \tag{8.47}$$

In the general case, eq. (8.46) becomes

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \tag{8.48}$$

where $\mathbf{a}_{n12}$ is a unit vector normal to the interface and is directed from medium 1 to medium 2. If the boundary is free of current or the media are not conductors (for K is free current density), $K = 0$ and eq. (8.46) becomes

$$\mathbf{H}_{1t} = \mathbf{H}_{2t} \tag{8.49}$$

or

$$\frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\mathbf{B}_{2t}}{\mu_2} \tag{8.50}$$

Thus the tangential component of **H** is continuous while that of **B** is discontinuous at the boundary.

If the fields make an angle θ with the normal to the interface, eq. (8.44) results in

$$\mathbf{B}_1 \cos\theta_1 = \mathbf{B}_{1n} = \mathbf{B}_{2n} = \mathbf{B}_2 \cos\theta_2 \tag{8.51a}$$

while eq. (8.50) produces

$$\frac{\mathbf{B}_1}{\mu_1}\sin\theta_1 = H_{1t} = H_{2t} = \frac{\mathbf{B}_2}{\mu_2}\sin\theta_2 \tag{8.51b}$$

Dividing eq. (8.51b) by eq. (8.51a) gives

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} \tag{8.52}$$

Example 8.9: Given that $\mathbf{H}_1 = -2\,\mathbf{a}_x + 6\,\mathbf{a}_y + 4\,\mathbf{a}_z$ A/m in region $y - x - 2 \leq 0$ where $\mu_1 = 5\,\mu_o$, calculate

(a) $\mathbf{M}_1$ and $\mathbf{B}_1$ (b) $\mathbf{H}_2$ and $\mathbf{B}_2$ in region $y - x - 2 \geq 0$ where $\mu_2 = 2\,\mu_o$

Solution: Since $y - x - 2 = 0$ is a plane, $y - x \leq 2$ or $y \leq x + 2$ is region 1 in Fig. 8.14. A point in this region may be used to confirm this. For example, the origin (0, 0) is in this region since $0 - 0 - 2 < 0$. If we let the surface of the plane be described by $f(x, y) = y - x - 2$, a unit vector normal to the plane is given by

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$$

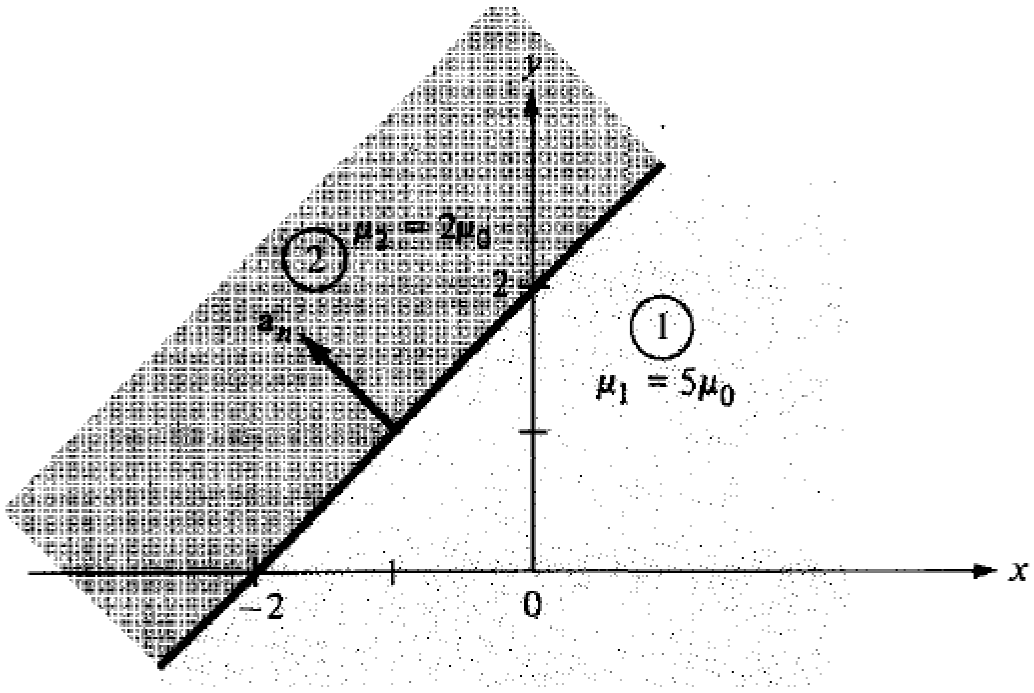


Figure 8.14 For Example 8.9.

(a) $\mathbf{M}_1 = \chi_{m1}\,\mathbf{H}_1 = (\mu_{r1} - 1)\mathbf{H}_1 = (5-1)(-2\,\mathbf{a}_x + 6\,\mathbf{a}_y + 4\,\mathbf{a}_z)$

$$\mathbf{M}_1 = -8\,\mathbf{a}_x + 24\,\mathbf{a}_y + 16\,\mathbf{a}_z \text{ A/m}$$

$$\mathbf{B}_1 = \mu_1\,\mathbf{H}_1 = \mu_o\mu_{r1}\,\mathbf{H}_1 = 4\pi \times 10^{-7}(5)(-2\,\mathbf{a}_x + 6\,\mathbf{a}_y + 4\,\mathbf{a}_z)$$

$$\mathbf{B}_1 = -12.57\,\mathbf{a}_x + 37.7\,\mathbf{a}_y + 25.13\,\mathbf{a}_z \ \ \mu\text{Wb/m}^2$$

(b) $\mathbf{H}_{1n} = (\mathbf{H}_1.\mathbf{a}_n)\mathbf{a}_n = \left[(-2\,\mathbf{a}_x + 6\,\mathbf{a}_y + 4\,\mathbf{a}_z).\left(\frac{-\mathbf{a}_x+\mathbf{a}_y}{\sqrt{2}}\right)\right]\left(\frac{-\mathbf{a}_x+\mathbf{a}_y}{\sqrt{2}}\right)$

$$\mathbf{H}_{1n} = -4\mathbf{a}_x + 4\mathbf{a}_y$$

But $\mathbf{H}_1 = \mathbf{H}_{1n} + \mathbf{H}_{1t}$

Hence, $\mathbf{H}_{1t} = \mathbf{H}_1 - \mathbf{H}_{1n} = (-2\,\mathbf{a}_x + 6\,\mathbf{a}_y + 4\,\mathbf{a}_z) - (-4\mathbf{a}_x + 4\mathbf{a}_y)$

$$\mathbf{H}_{1t} = 2\,\mathbf{a}_x + 2\,\mathbf{a}_y + 4\,\mathbf{a}_z$$

Using the boundary conditions, we have

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = 2\,\mathbf{a}_x + 2\,\mathbf{a}_y + 4\,\mathbf{a}_z$$

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} \ \Rightarrow \ \mu_2\mathbf{H}_{2n} = \mu_1\mathbf{H}_{1n}$$

or

$$\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2}\mathbf{H}_{1n} = \frac{5}{2}(-4\mathbf{a}_x + 4\mathbf{a}_y) = -10\mathbf{a}_x + 10\mathbf{a}_y$$

Thus,

$$\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t} = (-10\mathbf{a}_x + 10\mathbf{a}_y) + (2\,\mathbf{a}_x + 2\,\mathbf{a}_y + 4\,\mathbf{a}_z)$$

$$\mathbf{H}_2 = -8\,\mathbf{a}_x + 12\,\mathbf{a}_y + 4\,\mathbf{a}_z \text{ A/m}$$

and

$$\mathbf{B}_2 = \mu_2\mathbf{H}_2 = \mu_o\mu_{r2}\mathbf{H}_2 = (4\pi \times 10^{-7})(2)(-8\,\mathbf{a}_x + 12\,\mathbf{a}_y + 4\,\mathbf{a}_z)$$

$$\mathbf{B}_2 = -20.11\,\mathbf{a}_x + 30.16\,\mathbf{a}_y + 10.05\,\mathbf{a}_z \ \ \mu\text{Wb/m}^2$$

Example 8.10: The xy-plane serves as the interface between two different media. Medium 1 ($z < 0$) is filled with a material whose $\mu_r = 6$, and medium 2 ($z > 0$) is filled with a material whose $\mu_r = 4$. If the interface carries current $(\frac{1}{\mu_o})\,\mathbf{a}_y$ mA/m, and $\mathbf{B}_2 = 5\,\mathbf{a}_x + 8\,\mathbf{a}_z$ mWb/m^2 find $\mathbf{H}_1$ and $\mathbf{B}_1$.

Solution: In this example, $K \neq 0$, Consider the problem as illustrated in Figure 8.15. Let $\mathbf{B}_1 = (B_x, B_y, B_z)$ in mWb/m^2.

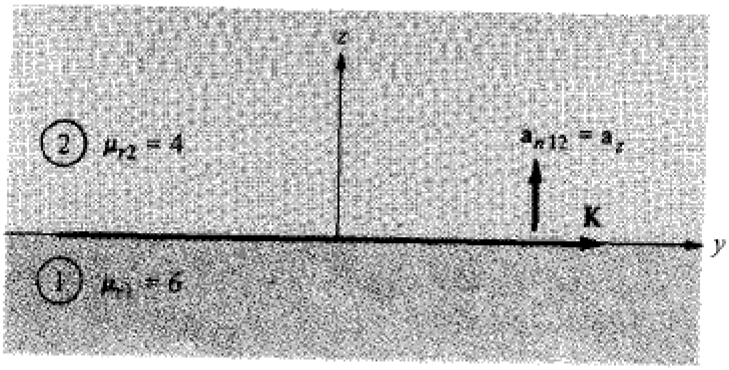


Figure 8.15 For Example 8.9.

$$\mathbf{B}_{1n} = \mathbf{B}_{2n} = 8\,\mathbf{a}_z \ \Rightarrow\ B_z = 8 \qquad (1)$$

But,

$$\mathbf{H}_2 = \frac{\mathbf{B}_2}{\mu_2} = \frac{1}{4\mu_o}(5\,\mathbf{a}_x + 8\,\mathbf{a}_z)\ \text{mA/m} \qquad (2)$$

and,

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{6\mu_o}\left(B_x\,\mathbf{a}_x + B_y\,\mathbf{a}_y + B_z\,\mathbf{a}_z\right)\ \text{mA/m} \qquad (3)$$

we can find the tangential components using

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$

or

$$\mathbf{H}_1 \times \mathbf{a}_{n12} = \mathbf{H}_2 \times \mathbf{a}_{n12} + \mathbf{K} \qquad (4)$$

Substituting equations (2) and (3) into equation (4) gives

$$\frac{1}{6\mu_o}\left(B_x\,\mathbf{a}_x + B_y\,\mathbf{a}_y + B_z\,\mathbf{a}_z\right) \times \mathbf{a}_z = \frac{1}{4\mu_o}(5\,\mathbf{a}_x + 8\,\mathbf{a}_z) \times \mathbf{a}_z + (\frac{1}{\mu_o})\,\mathbf{a}_y$$

$$\left(\frac{B_y}{6}\,\mathbf{a}_x - \frac{B_x}{6}\,\mathbf{a}_y + 0\,\mathbf{a}_z\right) = 0\,\mathbf{a}_x + \left(-\frac{5}{4} + 1\right)\mathbf{a}_y + 0\,\mathbf{a}_z$$

Equating components yields $\quad B_y = 0$

$$\frac{-B_x}{6} = \frac{-5}{4} + 1 \ \Rightarrow\ B_x = \frac{6}{4} = 1.5 \qquad (5)$$

From equations (1) and (5), $\mathbf{B}_1 = (1.5\,\mathbf{a}_x + 8\,\mathbf{a}_z)\ \text{mWb/m}^2$

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{\mu_o}(0.25\,\mathbf{a}_x + 1.33\,\mathbf{a}_z)\ \text{mA/m}$$

and

$$\mathbf{H}_1 = \frac{1}{\mu_o}(1.25\,\mathbf{a}_x + 2\,\mathbf{a}_z)\ \text{mA/m}$$

Example 8.11: Region 1, described by $3x + 4y \geq 10$, is free space whereas region 2, described by $3x + 4y \leq 10$, is a magnetic material for which $\mu = 10\mu_o$. Assuming that the boundary between the material and free space is current free find $\mathbf{B}_2$ if $\mathbf{B}_1 = 0.1\,\mathbf{a}_x + 0.4\,\mathbf{a}_y + 0.2\,\mathbf{a}_z\ \text{Wb/m}^2$

Solution:

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{3\mathbf{a}_x + 4\mathbf{a}_y}{5}$$

$$\mathbf{B}_{1n} = (\mathbf{B}_1.\mathbf{a}_n)\mathbf{a}_n = \left[(0.1\,\mathbf{a}_x + 0.4\,\mathbf{a}_y + 0.2\,\mathbf{a}_z).\left(\frac{3\mathbf{a}_x + 4\mathbf{a}_y}{5}\right)\right]\left(\frac{3\mathbf{a}_x + 4\mathbf{a}_y}{5}\right)$$

$$\mathbf{B}_{1n} = 0.228\,\mathbf{a}_x + 0.304\,\mathbf{a}_y = \mathbf{B}_{2n}$$

$$\mathbf{B}_{1t} = (\mathbf{B}_1 - \mathbf{B}_{1n}) = (0.1\,\mathbf{a}_x + 0.4\,\mathbf{a}_y + 0.2\,\mathbf{a}_z) - (0.228\,\mathbf{a}_x + 0.304\,\mathbf{a}_y)$$

$$\mathbf{B}_{1t} = -0.128\,\mathbf{a}_x + 0.096\,\mathbf{a}_y + 0.2\,\mathbf{a}_z$$

$$\mathbf{B}_{2t} = \frac{\mu_2}{\mu_1}\mathbf{B}_{1t} = \frac{10\mu_o}{\mu_o}(-0.128\,\mathbf{a}_x + 0.096\,\mathbf{a}_y + 0.2\,\mathbf{a}_z)$$

$$\mathbf{B}_{2t} = -1.28\,\mathbf{a}_x + 0.96\,\mathbf{a}_y + 2\,\mathbf{a}_z$$

$$\mathbf{B}_2 = \mathbf{B}_{2n} + \mathbf{B}_{2t} = (0.228\,\mathbf{a}_x + 0.304\,\mathbf{a}_y) + (-1.28\,\mathbf{a}_x + 0.96\,\mathbf{a}_y + 2\,\mathbf{a}_z)$$

$$\mathbf{B}_2 = -1.052\,\mathbf{a}_x + 1.264\,\mathbf{a}_y + 2\,\mathbf{a}_z\ Wb/m^2$$

Example 8.12: A unit normal vector from region 2 ($\mu = 2\mu_o$) to region 1 ($\mu = \mu_o$) is $\mathbf{a}_{n21} = 6\,\mathbf{a}_x + 2\,\mathbf{a}_y - 3\,\mathbf{a}_z$. If $\mathbf{H}_1 = 10\,\mathbf{a}_x + \mathbf{a}_y + 12\,\mathbf{a}_z$ A/m and $\mathbf{H}_2 = H_{2x}\,\mathbf{a}_x - 5\,\mathbf{a}_y + 4\,\mathbf{a}_z$ A/m, determine

(a) $\mathbf{H}_{2x}$ (b) The surface current density $\mathbf{K}$ on the interface (c) The angles $\mathbf{B}_1$ and $\mathbf{B}_2$ make with the normal to the interface.

Solution: (a) $\mathbf{B}_{1n} = \mathbf{B}_{2n} \Rightarrow \mu_1\mathbf{H}_{1n} = \mu_2\mathbf{H}_{2n}$ or $\mu_1\mathbf{H}_1.\mathbf{a}_{n21} = \mu_2\mathbf{H}_2.\mathbf{a}_{n21}$

$$\mu_o(10\mathbf{a}_x + \mathbf{a}_y + 12\mathbf{a}_z).\left(\frac{6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z}{7}\right) = 2\mu_o(H_{2x}\mathbf{a}_x - 5\mathbf{a}_y + 4\mathbf{a}_z).\left(\frac{6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z}{7}\right)$$

$$\mu_o\left(\frac{60 + 2 - 36}{7}\right) = 2\mu_o\left(\frac{6H_{2x} - 10 - 12}{7}\right)$$

$$35 = 6H_{2x} \quad\Rightarrow\quad H_{2x} = 5.833$$

(b)

$$\mathbf{K} = (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{a}_{n21} \times (\mathbf{H}_1 - \mathbf{H}_2)$$

$$(\mathbf{H}_1 - \mathbf{H}_2) = [(10\,\mathbf{a}_x + \mathbf{a}_y + 12\,\mathbf{a}_z) - (5.833\,\mathbf{a}_x - 5\,\mathbf{a}_y + 4\,\mathbf{a}_z)]$$

$$(\mathbf{H}_1 - \mathbf{H}_2) = (4.167\,\mathbf{a}_x + 6\,\mathbf{a}_y + 8\,\mathbf{a}_z)$$

$$\mathbf{K} = \mathbf{a}_{n21} \times (\mathbf{H}_1 - \mathbf{H}_2) = \frac{1}{7}\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 6 & 2 & -3 \\ 4.167 & 6 & 8 \end{vmatrix}$$

$$\mathbf{K} = 4.86\,\mathbf{a}_x - 8.64\,\mathbf{a}_y + 3.95\,\mathbf{a}_z \text{ A/m}$$

(c) Since $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{B}_1$ and $\mathbf{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos\theta_1 = \frac{\mathbf{H}_1 . \mathbf{a}_{n21}}{|\mathbf{H}_1|} = \frac{26}{7\sqrt{100 + 1 + 144}} = 0.2373 \qquad \Rightarrow \qquad \theta_1 = 76.27^{\text{o}}$$

$$\cos\theta_2 = \frac{\mathbf{H}_2 . \mathbf{a}_{n21}}{|\mathbf{H}_2|} = \frac{13}{7\sqrt{(5.833)^2 + 25 + 16}} = 0.2144 \qquad \Rightarrow \qquad \theta_2 = 77.62^{\text{o}}$$

8.8 INDUCTORS AND INDUCTANCES

Self-inductance of a circuit (L):

A circuit (or closed conducting path) carrying current I produces a magnetic field $\mathbf{B}$ which causes a flux $\psi = \int \mathbf{B} . d\mathbf{S}$ to pass through each turn of the circuit (Fig. 8.16). If the circuit has N identical turns, we define the flux linkage λ as

$$\lambda = N\psi \tag{8.53}$$

If the medium surrounding the circuit is linear, the flux linkage λ is proportional to the current I producing it; that is,

$$\lambda \propto I$$

or

$$\lambda = L\,I \tag{8.54}$$

- where L is a constant of proportionality called the inductance of the circuit.
- The unit of inductance is the henry (H) which is the same as webers/ampere.
- A circuit or part of a circuit that has inductance is called an inductor.

Inductance (L) of an inductor: is define as the ratio of the magnetic flux linkage λ to the current I through the inductor; that is,

$$L = \frac{\lambda}{I} = \frac{N\psi}{I} \tag{8.55}$$

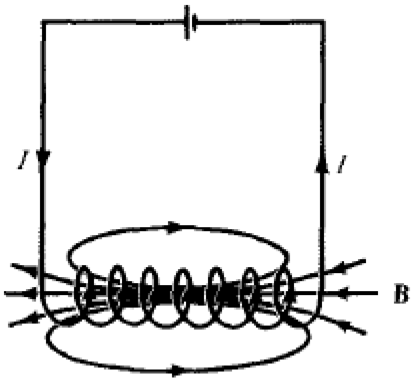


Figure 8.16 Magnetic field **B** produced by a circuit.

The magnetic energy W_m (in joules) stored in an inductor is expressed in circuit theory as:

$$W_m = \frac{1}{2} L\, I^2 \tag{8.56}$$

or

$$L = \frac{2\, W_m}{I^2} \tag{8.57}$$

Mutual inductance (M):

If we have two circuits carrying current I_1 and I_2 as shown in Fig. 8.17, a magnetic interaction exists between the circuits. Four component fluxes ψ_{11}, ψ_{12}, ψ_{21}, and ψ_{22} are produced. The flux ψ_{12}, is the flux passing through circuit 1 due to current I_2 in circuit 2. If $\mathbf{B}_2$ in the field due to I_2 and S_1 is the area of circuit 1, then

$$\psi_{12} = \int_{S_1} \mathbf{B}_2 \,.\, d\mathbf{S} \tag{8.58}$$

Mutual inductance M_{12}: is define as the ratio of the flux linkage $\lambda_{12} = N_1\psi_{12}$ on circuit 1 to current I_2, that is,

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\psi_{12}}{I_2} \tag{8.59a}$$

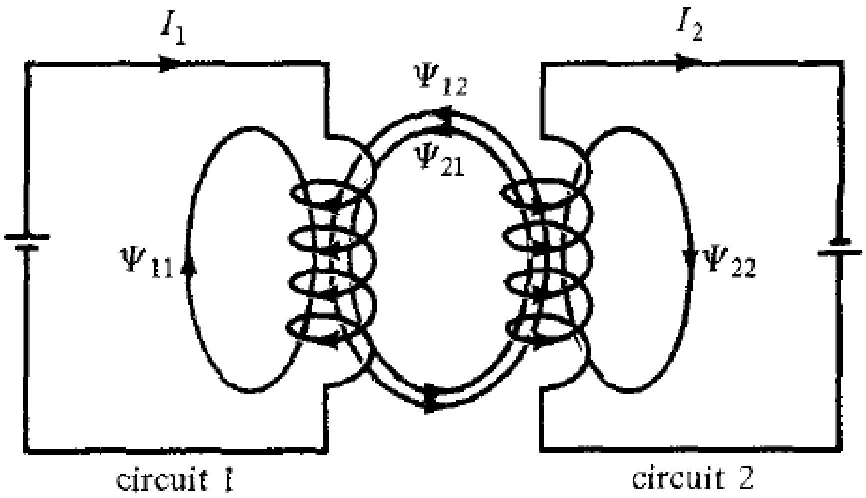


Figure 8.18 Magnetic interaction between two circuits.

Similarly, the mutual inductance M_{21} is defined as the flux linkages of circuit 2 per unit current I_1; that is,

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2\psi_{21}}{I_1} \quad (8.59b)$$

in the absence of ferromagnetic material

$$M_{12} = M_{21} \quad (8.59c)$$

We define the self-inductance of circuits 1 and 2, respectively, as

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1\psi_1}{I_1} \quad (8.60)$$

and

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2\psi_2}{I_2} \quad (8.61)$$

Where

$$\psi_1 = \psi_{11} + \psi_{12}$$

$$\psi_2 = \psi_{22} + \psi_{21}$$

The total energy in the magnetic field is the sum of the energies due to L_1, L_2 and M_{12} (or M_{21}); that is,

$$W_m = W_1 + W_2 + W_{12} = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 \pm M_{12}\, I_1I_2 \quad (8.62)$$

The positive sign is taken if currents I_1 and I_2 flow such that the magnetic fields of the two circuits strengthen each other. If the currents flow such that their magnetic fields oppose each other, the negative sign is taken.

In an inductor such as a coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the internal inductance L_{in}, while that produced by the flux external to it is called external inductance L_{ext}. The total inductance L is

$$L = L_{in} + L_{ext} \tag{8.63}$$

8.9 MAGNETIC ENERGY

From equation (8.56),

$$W_m = \frac{1}{2} L I^2 \tag{8.56}$$

The energy is stored in the magnetic field **B** of the inductor. We would like to express equation (8.56) in terms of **B** or **H**.

Consider a differential volume in a magnetic field as shown in Figure 8.19. Let the volume be covered with conducting sheets at the top and bottom surfaces with current ΔI.

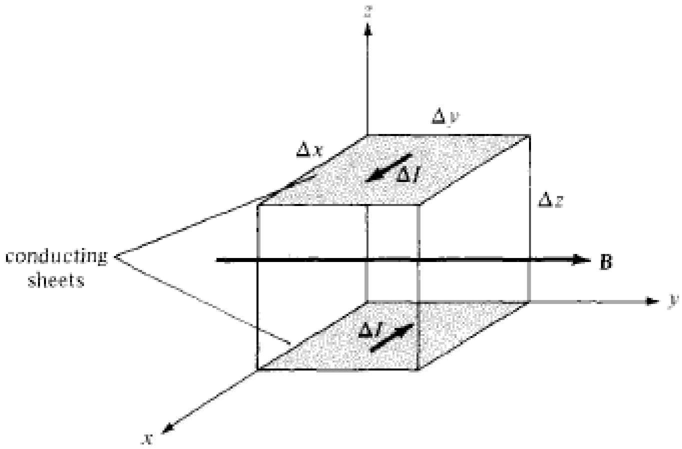


Figure 8.19 A differential volume in a magnetic field.

We assume that the whole region is filled with such differential volumes. From equation (8.55), each volume has an inductance

$$\Delta L = \frac{\Delta \psi}{\Delta I} = \frac{\mu H \,\Delta x \Delta z}{\Delta I} \tag{8.64}$$

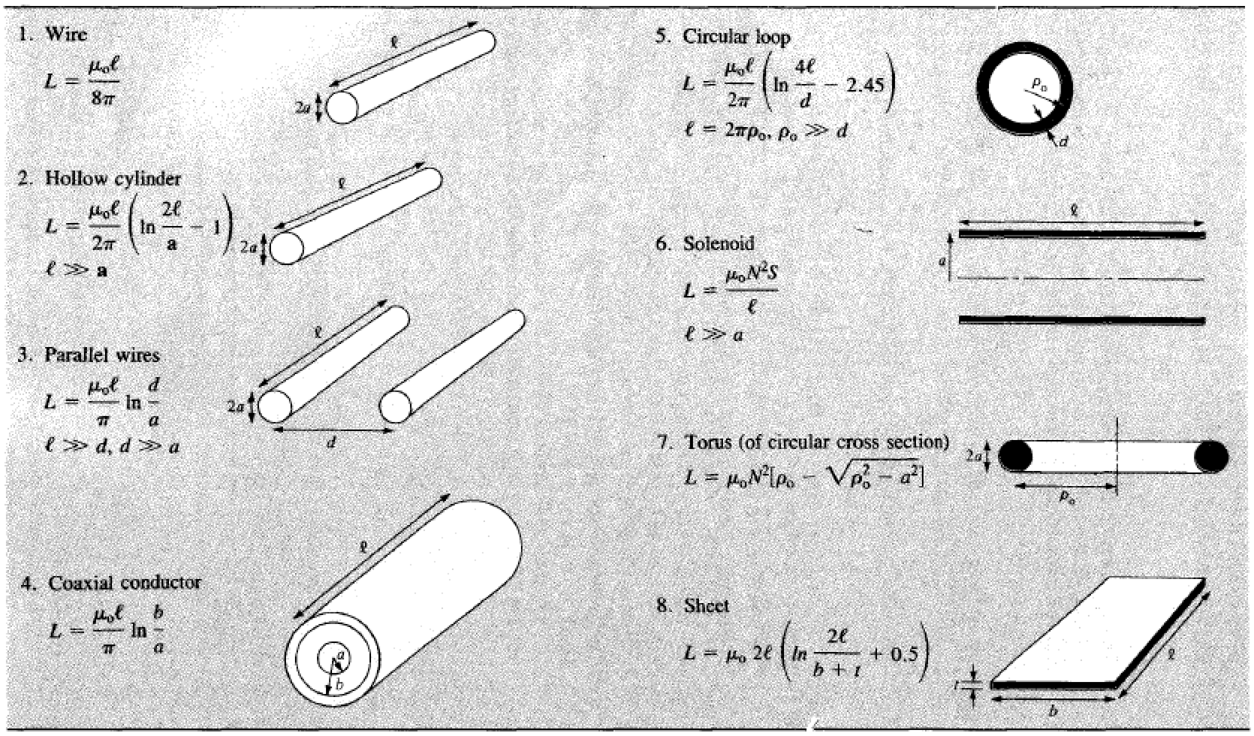
where $\Delta I = H\,\Delta y$. Substituting equation (8.64) into eq. (8.56), we have

$$\Delta W_m = \frac{1}{2}\,\Delta L\,\Delta I^2 = \frac{1}{2}\,\mu\,H^2\,\Delta x \Delta y \Delta z \tag{8.65a}$$

or

$$\Delta W_m = \frac{1}{2}\,\mu\,H^2\,\Delta v \tag{8.65b}$$

TABLE 8.2 A Collection of Formulas for Inductance of Common Elements



1. Wire: $L = \dfrac{\mu_o \ell}{8\pi}$
2. Hollow cylinder: $L = \dfrac{\mu_o \ell}{2\pi}\left(\ln\dfrac{2\ell}{a} - 1\right)$, $\ell \gg a$
3. Parallel wires: $L = \dfrac{\mu_o \ell}{\pi}\ln\dfrac{d}{a}$, $\ell \gg d,\ d \gg a$
4. Coaxial conductor: $L = \dfrac{\mu_o \ell}{\pi}\ln\dfrac{b}{a}$
5. Circular loop: $L = \dfrac{\mu_o \ell}{2\pi}\left(\ln\dfrac{4\ell}{d} - 2.45\right)$, $\ell = 2\pi\rho_o,\ \rho_o \gg d$
6. Solenoid: $L = \dfrac{\mu_o N^2 S}{\ell}$, $\ell \gg a$
7. Torus (of circular cross section): $L = \mu_o N^2[\rho_o - \sqrt{\rho_o^2 - a^2}]$
8. Sheet: $L = \mu_o 2\ell\left(\ln\dfrac{2\ell}{b+t} + 0.5\right)$

The magnetostatic energy density w_m (in $\mathrm{J/m^3}$) is defined as

$$w_m = \lim_{\Delta v \to 0}\frac{\Delta W_m}{\Delta v} = \frac{1}{2}\,\mu\,H^2$$

Hence,

$$w_m = \frac{1}{2}\,\mu\,H^2 = \frac{1}{2}\,B.H = \frac{B^2}{2\mu} \tag{8.66}$$

Thus the energy in a magnetostatic field in a linear medium is

$$W_m = \int w_m\,dv$$

or

$$W_m = \frac{1}{2}\int \mathbf{B}\,.\,\mathbf{H}\,dv = \frac{1}{2}\int \mu\,H^2\,dv \tag{8.67}$$

8.10 MAGNETIC CIRCUITS

Magnetic devices such as toroids, transformers, motors, generators, and relays may be considered as magnetic circuits.

The analysis of such circuits is made simple if an analogy between magnetic circuits and electric circuits is exploited.

The analogy between magnetic and electric circuits is summarized in table 8.3 and portrayed in Figure 8.20.

We define the magnetomotive force (mmf) $\mathcal{F}$ (in ampere-turns) as

$$\mathcal{F} = NI = \oint \mathbf{H}.d\mathbf{L} \tag{8.68}$$

We also define reluctance $\mathcal{R}$ (in ampere-turns/weber) as

$$\mathcal{R} = \frac{\ell}{\mu S} \tag{8.69}$$

TABLE 8.3 Analogy between Electric and Magnetic Circuits

Electric	Magnetic
Conductivity σ	Permeability μ
Field intensity E	Field intensity H
Current $I = \int \vec{J}.dS$	Magnetic flux $\psi = \int B.dS$
Current density $J = \frac{I}{S} = \sigma E$	Flux density $B = \frac{\psi}{S} = \mu H$
Electromotive force (emf) V	Magnetomotive force (mmf) $\mathcal{F}$
Resistance R	Reluctance $\mathcal{R}$
Conductance $G = \frac{1}{R}$	Permeance $\wp = \frac{1}{\mathcal{R}}$
Ohm's law $R = \frac{V}{I} = \frac{\ell}{\sigma S}$	Ohm's law $\mathcal{R} = \frac{\mathcal{F}}{\psi} = \frac{\ell}{\mu S}$
or $V = E\ell = IR$	or $\mathcal{F} = H\ell = \psi\mathcal{R} = NI$
Kirchoff's laws:	Kirchhoff's laws:
$\sum I = 0$	$\sum \psi = 0$
$\sum V - \sum IR = 0$	$\sum \mathcal{F} - \sum \psi\mathcal{R} = 0$

The reciprocal of reluctance is permeance $\wp$. The basic relationship for circuit elements is Ohm's law ($V = IR$):

$$\mathcal{F} = \psi\mathcal{R} \tag{8.70}$$

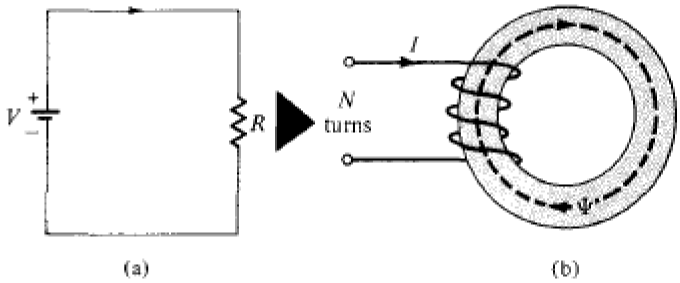


Figure 8.20 Analogy between (a) an electric circuit, and (b) a magnetic circuit.

The rules of adding voltages and for combining series and parallel resistances also hold for mmfs and reluctances. for n magnetic circuit elements in series

$$\psi_1 = \psi_2 = \psi_3 = \cdots = \psi_n \tag{8.71}$$

and

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \cdots + \mathcal{F}_n \tag{8.72}$$

For n magnetic circuit elements in parallel,

$$\psi = \psi_1 + \psi_2 + \psi_3 + \cdots + \psi_n \tag{8.73}$$

and

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \cdots = \mathcal{F}_n \tag{8.74}$$

The differences between electric and magnetic circuits.

- unlike an electric circuit where current I flows, magnetic flux does not flow.
- conductivity σ is independent of current density $\vec{J}$ in an electric circuit whereas permeability μ varies with flux density **B** in a magnetic circuit. this is because ferromagnetic (nonlinear) materials are normally used in most practical magnetic devices.

8.11 FORCE ON MAGNETIC MATERIALS

The magnetic force is useful in electromechanical systems such as electromagnets, relays, rotating machines, and magnetic levitation. Consider, for example, an electromagnet made of iron of constant relative permeability as shown in Fig. 8.21.

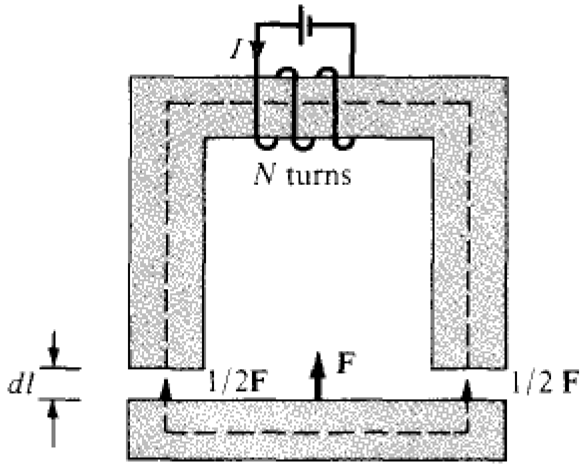


Figure 8.21 An electromagnet.

The work required to effect the displacement is equal to the change in stored energy in the air gap (assuming constant current), that is

$$-FdL = dW_m = 2\left[\frac{1}{2}\frac{B^2}{\mu_o}S\,dl\right] \qquad (8.75)$$

where

- S is the cross-sectional area of the gap,
- the factor 2 accounts for the two air gaps, and
- the negative sign indicates that the force acts to reduce the air gap (or that the force is attractive). Thus

$$F = -2\left[\frac{B^2 S}{2\mu_o}\right] \qquad (8.76)$$

The tractive force across a single gap can be obtained from eq. (8.75) as

$$F = -\frac{B^2 S}{2\mu_o} \qquad (8.77)$$

Equation (8.77) can be used to calculate the forces in many types of devices including relays, rotating machines, and magnetic levitation.

The tractive pressure (in N/m^2) in a magnetized surface is

$$P = \frac{F}{S} = \frac{B^2}{2\mu_o} = \frac{1}{2}\,BH \qquad (8.78)$$

which is the same as the energy density w_m in the air gap.