Heat Transfer: A Practical Approach
Second Edition
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Chapter 2
HEAT CONDUCTION EQUATION

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Objectives

- Understand multidimensionality and time dependence of heat transfer, and the conditions under which a heat transfer problem can be approximated as being one-dimensional.
- Obtain the differential equation of heat conduction in various coordinate systems, and simplify it for steady one-dimensional case.
- Identify the thermal conditions on surfaces, and express them mathematically as boundary and initial conditions.
- Solve one-dimensional heat conduction problems and obtain the temperature distributions within a medium and the heat flux.
- Analyze one-dimensional heat conduction in solids that involve heat generation.
- Evaluate heat conduction in solids with temperature-dependent thermal conductivity.

INTRODUCTION

- Although heat transfer and temperature are closely related, they are of a different nature.
- Temperature has only magnitude. It is a *scalar* quantity.
- Heat transfer has direction as well as magnitude. It is a *vector* quantity.
- We work with a coordinate system and indicate direction with plus or minus signs.

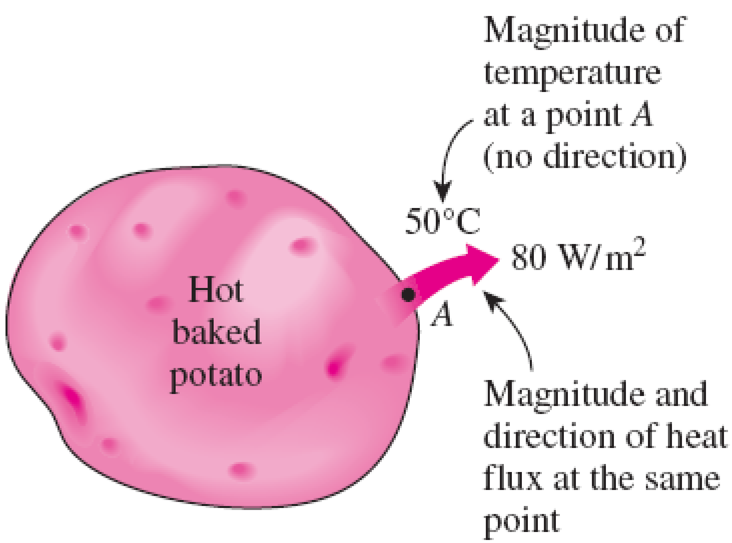


FIGURE 2–1
Heat transfer has direction as well as magnitude, and thus it is quantity.

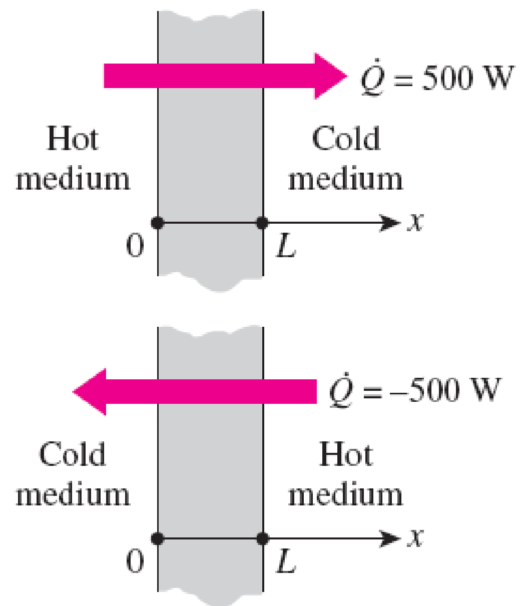
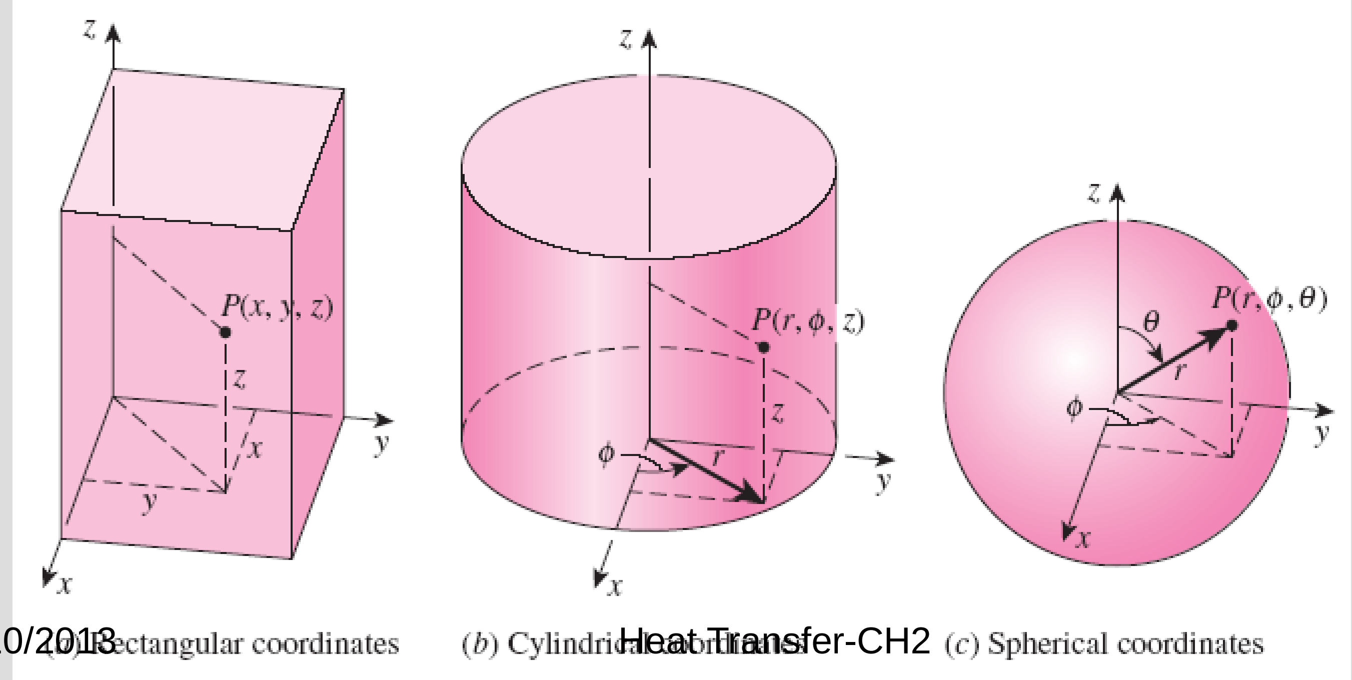


FIGURE 2–2
Indicating direction for heat transfer (positive in the positive direction; negative in the negative direction).

- The driving force for any form of heat transfer is the *temperature difference.*
- The larger the temperature difference, the larger the rate of heat transfer.
- Three prime coordinate systems:
  - ✓ rectangular $T(x, y, z, t)$
  - ✓ cylindrical $T(r, \phi, z, t)$
  - ✓ spherical $T(r, \phi, \theta, t)$.

FIGURE 2–3
The various distances and angles involved when describing the location of a point in different coordinate systems.



Steady versus Transient Heat Transfer

- ***Steady*** implies *no change* with time at any point within the medium
- ***Transient*** implies *variation with time* or *time dependence*
- In the special case of variation with time but not with position, the temperature of the medium changes *uniformly* with time. Such heat transfer systems are called **lumped systems**.

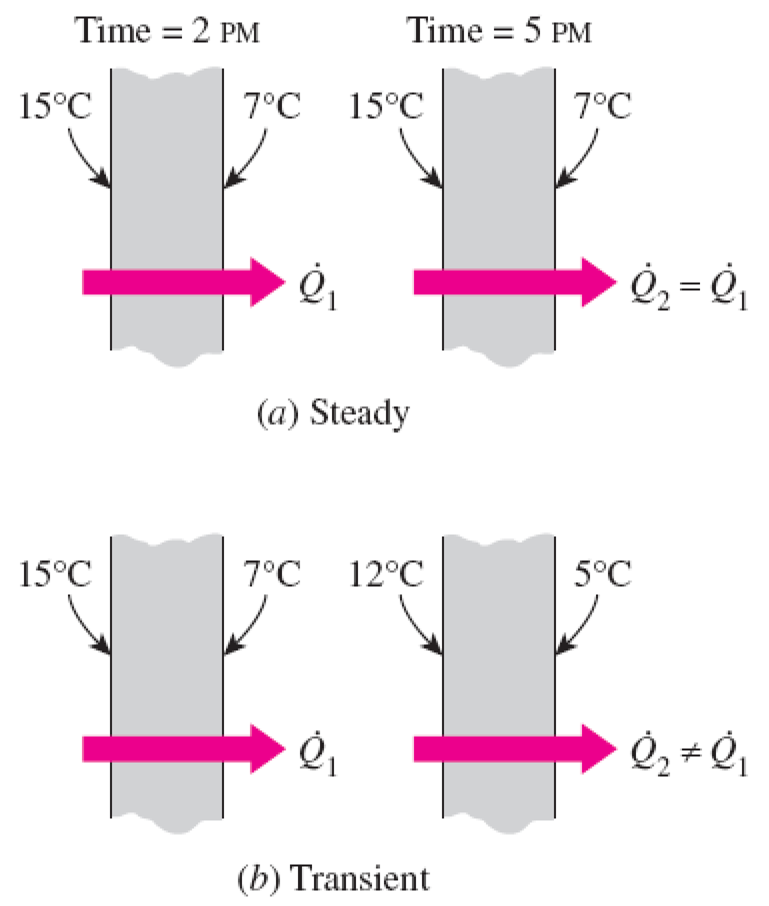


FIGURE 2–4

Transient and steady heat conduction in a plane wall.

Multidimensional Heat Transfer

- Heat transfer problems are also classified as being:
  - ✓ ***one-dimensional***
  - ✓ ***two dimensional***
  - ✓ ***three-dimensional***
- In the most general case, heat transfer through a medium is **three-dimensional**. However, some problems can be classified as two- or one-dimensional depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.
- **One-dimensional** if the temperature in the medium varies in one direction only and thus heat is transferred in one direction, and the variation of temperature and thus heat transfer in other directions are negligible or zero.
- **Two-dimensional** if the temperature in a medium, in some cases, varies mainly in two primary directions, and the variation of temperature in the third direction (and thus heat transfer in that direction) is negligible.

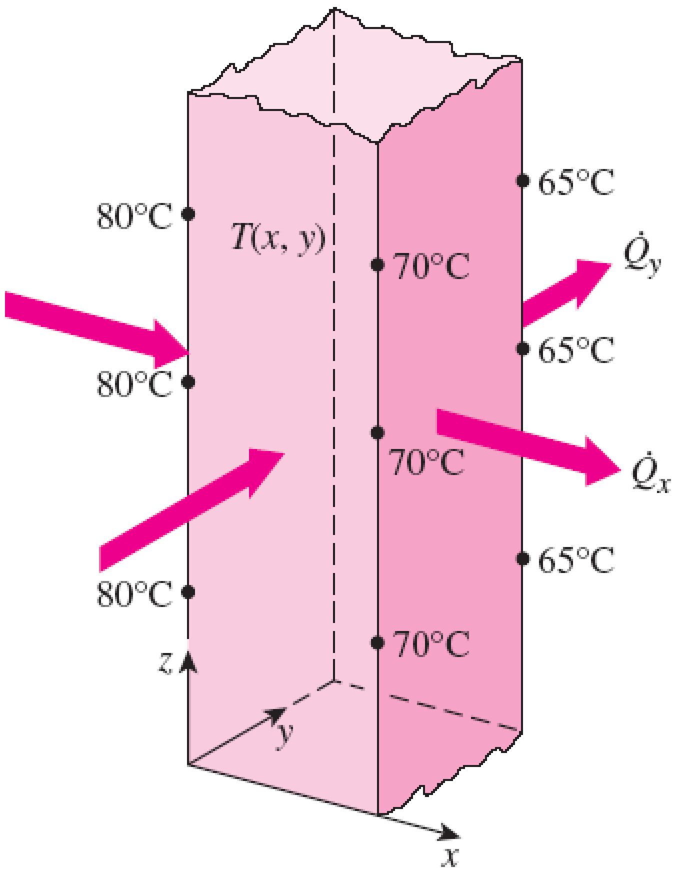


FIGURE 2–5
Two-dimensional heat transfer in a long rectangular bar.

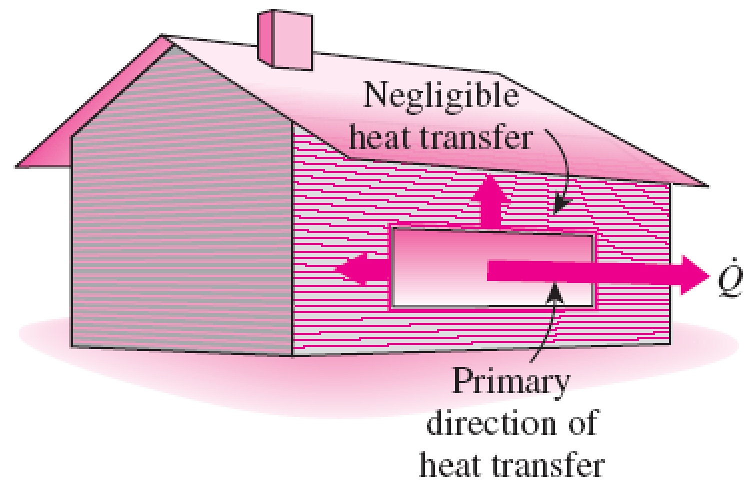


FIGURE 2–6
Heat transfer through the window of a house can be taken to be one-dimensional.

- The rate of heat conduction through a medium in a specified direction (say, in the *x*-direction) is expressed by **Fourier's law of heat conduction** for one-dimensional heat conduction as:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive *x* -direction.

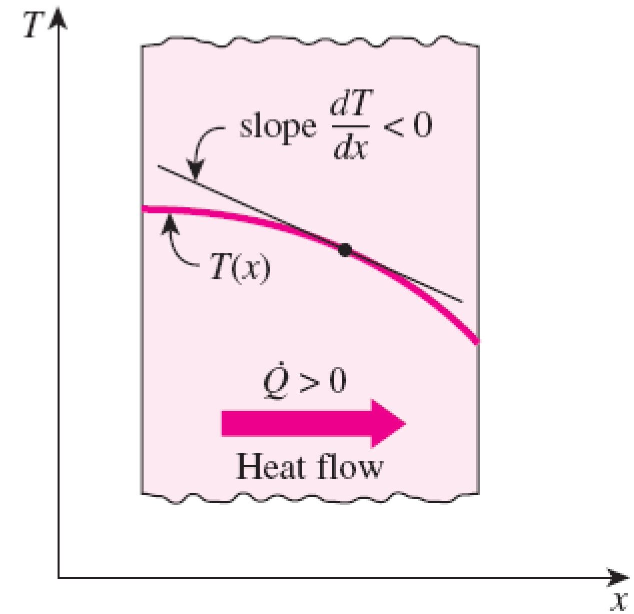


FIGURE 2–7

The temperature gradient dT/dx is simply the slope of the temperature curve on a T-x diagram.

- The heat flux vector at a point *P* on the surface of the figure must be perpendicular to the surface, and it must point in the direction of decreasing temperature
- If *n* is the normal of the isothermal surface at point *P*, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{\partial T}{\partial n} \qquad \text{(W)}$$

$$\vec{\dot{Q}}_n = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

$$\dot{Q}_x = -kA_x\frac{\partial T}{\partial x}, \qquad \dot{Q}_y = -kA_y\frac{\partial T}{\partial y},$$

$$\dot{Q}_z = -kA_z\frac{\partial T}{\partial z}$$

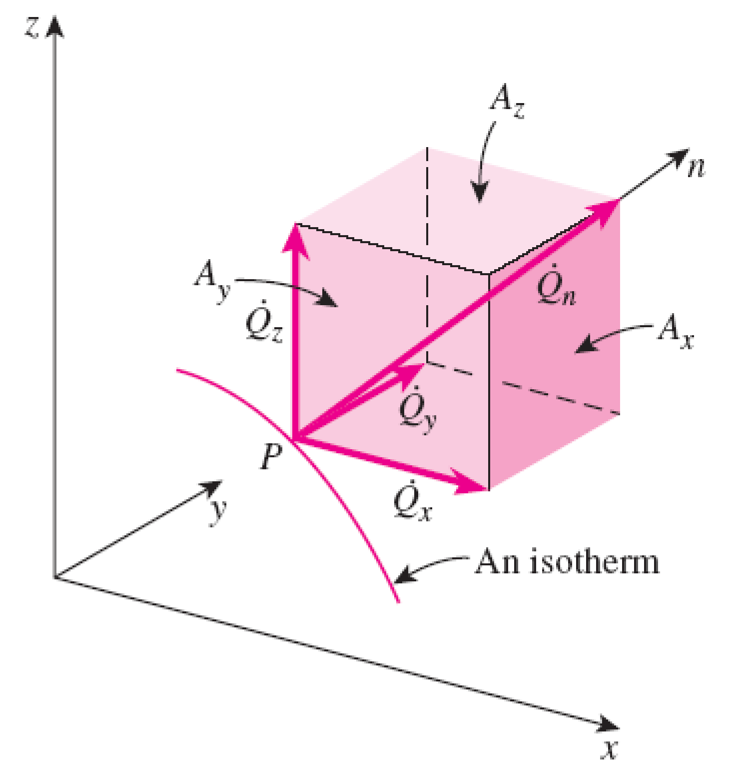


FIGURE 2–8
The heat transfer vector is always normal to an isothermal surface and can be resolved into its components like any other vector.

Heat Generation

- Examples:
  - ✓ electrical energy being converted to heat at a rate of I^2R,
  - ✓ fuel elements of nuclear reactors,
  - ✓ exothermic chemical reactions.
- Heat generation is a *volumetric phenomenon.*
- The rate of heat generation units : W/m^3 or $Btu/h\cdot ft^3$.
- The rate of heat generation in a medium may vary with time as well as position within the medium.

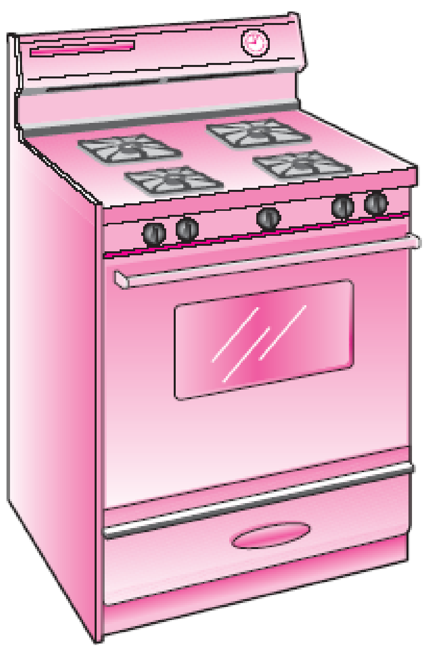


FIGURE 2–9
Heat is generated in the heating coils of an electric range as a result of the conversion of electrical energy to heat.

$$\dot{E}_{gen} = \int_V \dot{e}_{gen}\, dV \qquad (W)$$

$$\dot{E}_{gen} = \dot{e}_{gen} V,$$

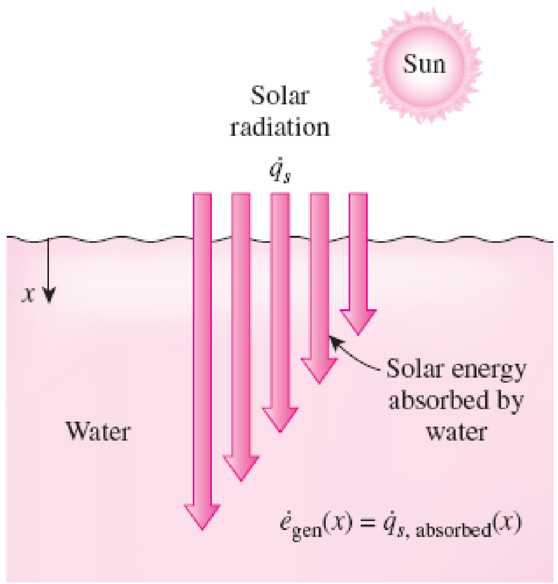


FIGURE 2–10
The absorption of solar radiation by water can be treated as heat generation.

ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

Consider heat conduction through a large plane wall such as the wall of a house, the glass of a single pane window, the metal plate at the bottom of a pressing iron, a cast-iron steam pipe, a cylindrical nuclear fuel element, an electrical resistance wire, the wall of a spherical container, or a spherical metal ball that is being quenched or tempered.

Heat conduction in these and many other geometries can be approximated as being *one-dimensional* since heat conduction through these geometries is dominant in one direction and negligible in other directions.

Next we develop the onedimensional heat conduction equation in rectangular, cylindrical, and spherical coordinates.

Heat Conduction Equation in a Large Plane Wall

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \textbf{(2-6)}$$

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta x (T_{t+\Delta t} - T_t)$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} A \Delta x$$

Substituting into Eq. 2–6, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{\text{gen}} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $A\Delta$x gives

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{e}_{\text{gen}} = \rho c \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$$\lim_{\Delta x \to 0} \frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{\partial \dot{Q}}{\partial x} = \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)$$

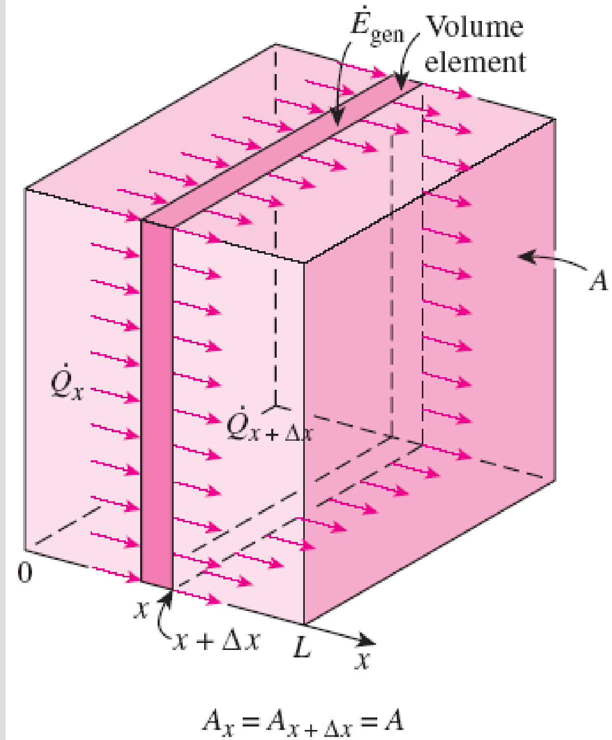


FIGURE 2–12

One-dimensional heat conduction through a volume element in a large plane wall.

Variable conductivity: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:*
($\partial/\partial t = 0$)
$$\frac{d^2T}{dx^2} + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

(2) *Transient, no heat generation:*
($\dot{e}_{\text{gen}} = 0$)
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:*
($\partial/\partial t = 0$ and $\dot{e}_{\text{gen}} = 0$)
$$\frac{d^2T}{dx^2} = 0$$

General, one-dimensional:

No generation — Steady-state

$$\frac{\partial^2 T}{\partial x^2} + \cancelto{0}{\frac{\dot{e}_{\text{gen}}}{k}} = \cancelto{0}{\frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

Steady, one-dimensional:

$$\frac{d^2T}{dx^2} = 0$$

The simplification of the one-dimensional heat conduction equation in a plane wall for the case of constant conductivity for steady conduction with no heat generation.

Heat Conduction Equation in a Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta r (T_{t+\Delta t} - T_t)$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} A \Delta r$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{\text{gen}} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{e}_{\text{gen}} = \rho c \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta r \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$$\lim_{\Delta r \to 0} \frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{\partial \dot{Q}}{\partial r} = \frac{\partial}{\partial r}\left(-kA\frac{\partial T}{\partial r}\right)$$

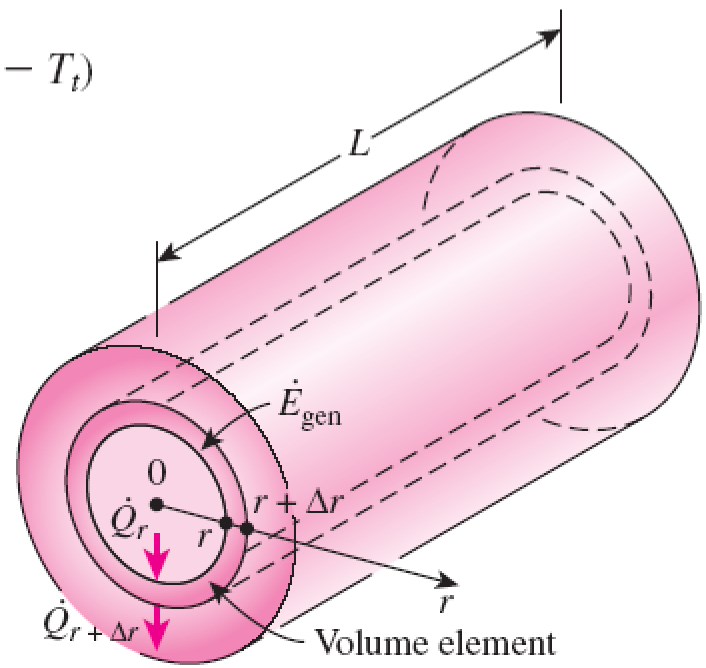


FIGURE 2–14

One-dimensional heat conduction through a volume element in a long cylinder.

Variable conductivity:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:* $(\partial/\partial t = 0)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{e}_{gen} = 0)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{e}_{gen} = 0)$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

(*a*) The form that is ready to integrate

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

(*b*) The equivalent alternative form

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = 0$$

Two equivalent forms of the differential equation for the one-dimensional steady heat conduction in a cylinder with no heat generation.

Heat Conduction Equation in a Sphere

Variable conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

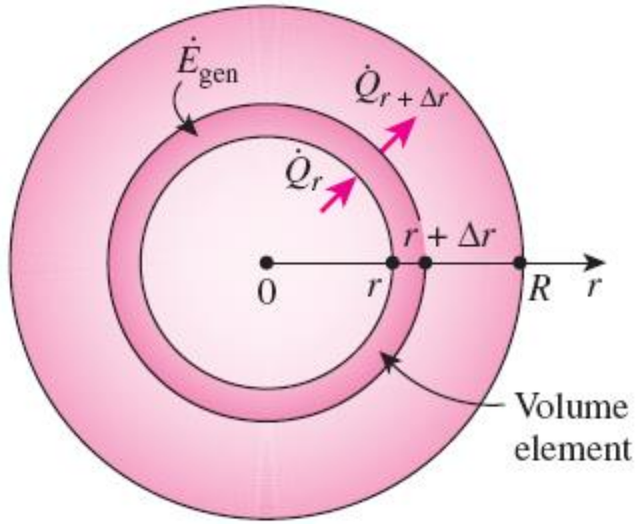


FIGURE 2–16

One-dimensional heat conduction through a volume element in a sphere.

(1) *Steady-state:* $(\partial/\partial t = 0)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{e}_{\text{gen}} = 0)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{e}_{\text{gen}} = 0)$

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \quad \text{or} \quad r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$$

Combined One-Dimensional Heat Conduction Equation

An examination of the one-dimensional transient heat conduction equations for the plane wall, cylinder, and sphere reveals that all three equations can be expressed in a compact form as

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k \frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$n = 0$ for a plane wall

$n = 1$ for a cylinder

$n = 2$ for a sphere

In the case of a plane wall, it is customary to replace the variable r by x.

This equation can be simplified for steady-state or no heat generation cases as described before.

GENERAL HEAT CONDUCTION EQUATION

In the last section we considered one-dimensional heat conduction and assumed heat conduction in other directions to be negligible.

Most heat transfer problems encountered in practice can be approximated as being one-dimensional, and we mostly deal with such problems in this text.

However, this is not always the case, and sometimes we need to consider heat transfer in other directions as well.

In such cases heat conduction is said to be *multidimensional*, and in this section we develop the governing differential equation in such systems in rectangular, cylindrical, and spherical coordinate systems.

Rectangular Coordinates

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction at}\\ x,\ y,\ \text{and } z\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x,\\ y+\Delta y, \text{ and } z+\Delta z\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

or

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \text{(2–36)}$$

Noting that the volume of the element is $V_{\text{element}} = \Delta x \Delta y \Delta z$, the change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c \Delta x \Delta y \Delta z (T_{t+\Delta t} - T_t)$$
$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} \Delta x \Delta y \Delta z$$

Substituting into Eq. 2–36, we get

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{e}_{\text{gen}} \Delta x \Delta y \Delta z = \rho c \Delta x \Delta y \Delta z \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $\Delta x \Delta y \Delta z$ gives

$$-\frac{1}{\Delta y \Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} - \frac{1}{\Delta x \Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} - \frac{1}{\Delta x \Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} + \dot{e}_{\text{gen}} =$$
$$\rho c \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(2–37)}$$

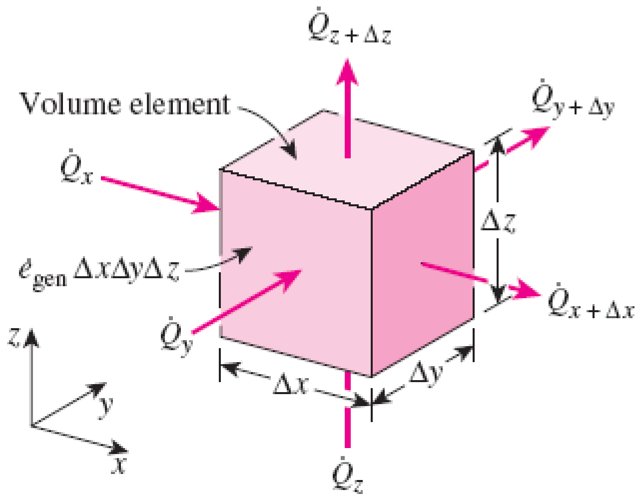


FIGURE 2–20

Three-dimensional heat conduction through a rectangular volume element.

Noting that the heat transfer areas of the element for heat conduction in the x, y, and z directions are $A_x = \Delta y \Delta z$, $A_y = \Delta x \Delta z$, and $A_z = \Delta x \Delta y$, respectively, and taking the limit as Δx, Δy, Δz and $\Delta t \rightarrow 0$ yields

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t} \qquad \textbf{(2–38)}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x \to 0} \frac{1}{\Delta y \Delta z} \frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{1}{\Delta y \Delta z}\frac{\partial Q_x}{\partial x} = \frac{1}{\Delta y \Delta z}\frac{\partial}{\partial x}\left(-k\Delta y \Delta z \frac{\partial T}{\partial x}\right) = -\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

$$\lim_{\Delta y \to 0} \frac{1}{\Delta x \Delta z} \frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} = \frac{1}{\Delta x \Delta z}\frac{\partial Q_y}{\partial y} = \frac{1}{\Delta x \Delta z}\frac{\partial}{\partial y}\left(-k\Delta x \Delta z \frac{\partial T}{\partial y}\right) = -\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

$$\lim_{\Delta z \to 0} \frac{1}{\Delta x \Delta y} \frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} = \frac{1}{\Delta x \Delta y}\frac{\partial Q_z}{\partial z} = \frac{1}{\Delta x \Delta y}\frac{\partial}{\partial z}\left(-k\Delta x \Delta y \frac{\partial T}{\partial z}\right) = -\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

Eq. 2–38 is the general heat conduction equation in rectangular coordinates. In the case of constant thermal conductivity, it reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \textbf{(2–39)}$$

where the property $\alpha = k/\rho c$ is again the *thermal diffusivity* of the material. Eq. 2–39 is known as the **Fourier-Biot equation**, and it reduces to these forms under specified conditions:

(1) *Steady-state:*
(called the **Poisson equation**)

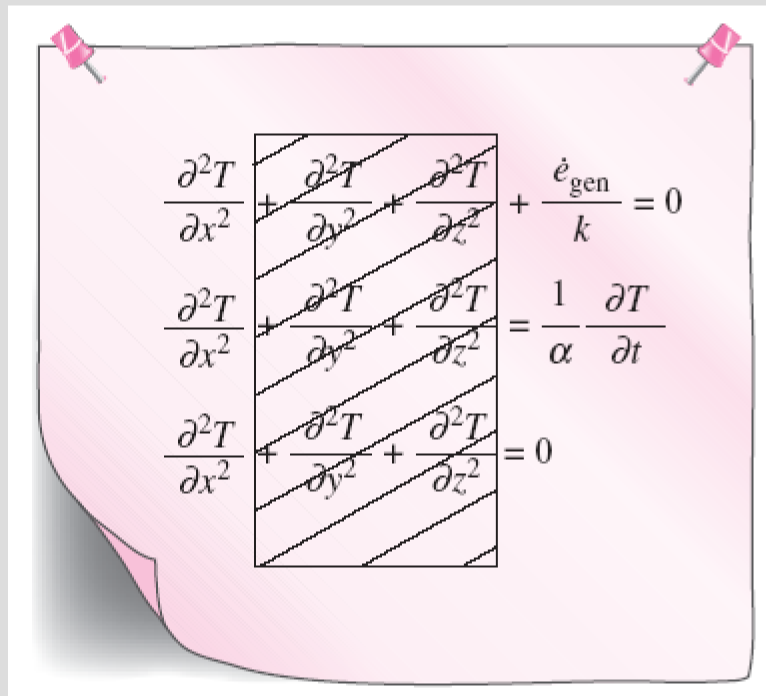
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{gen}}{k} = 0$$

(2) *Transient, no heat generation:*
(called the **diffusion equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:*
(called the **Laplace equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$



The three-dimensional heat conduction equations reduce to the one-dimensional ones when the temperature varies in one dimension only.

Cylindrical Coordinates

Relations between the coordinates of a point in rectangular and cylindrical coordinate systems:

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad \text{and} \qquad z = z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{\text{gen}} = \rho c\frac{\partial T}{\partial t}$$

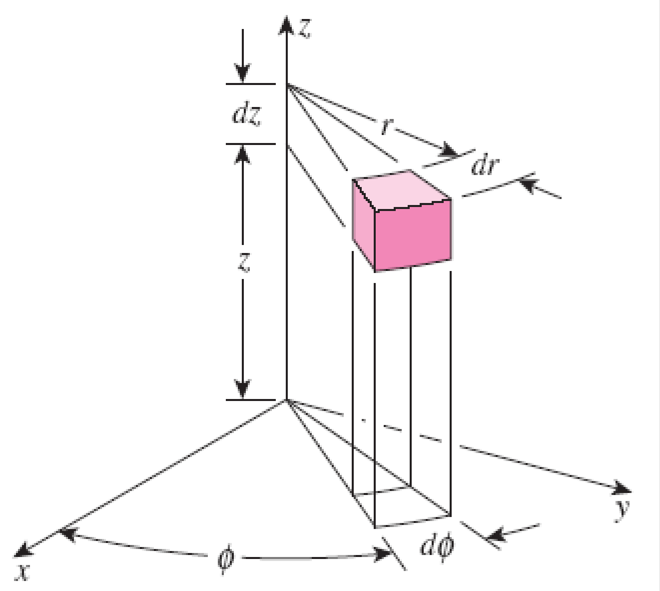


FIGURE 2–22

A differential volume element in cylindrical coordinates.

Spherical Coordinates

Relations between the coordinates of a point in rectangular and spherical coordinate systems:

$$x = r\cos\phi\sin\theta, \qquad y = r\sin\phi\sin\theta, \qquad \text{and} \qquad z = \cos\theta$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{e}_{\text{gen}} = \rho c\frac{\partial T}{\partial t}$$

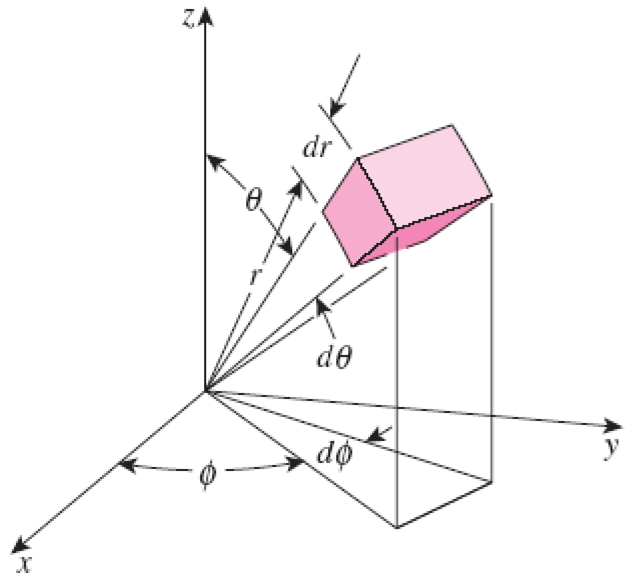


FIGURE 2–23

A differential volume element in spherical coordinates.

BOUNDARY AND INITIAL CONDITIONS

The description of a heat transfer problem in a medium is not complete without a full description of the thermal conditions at the bounding surfaces of the medium.

Boundary conditions: The *mathematical expressions* of the thermal conditions at the boundaries.

The temperature at any point on the wall at a specified time depends on the condition of the geometry at the beginning of the heat conduction process.

Such a condition, which is usually specified at time ***t* = 0**, is called the **initial condition**, which is a mathematical expression for the temperature distribution of the medium initially.

$$T(x, y, z, 0) = f(x, y, z)$$

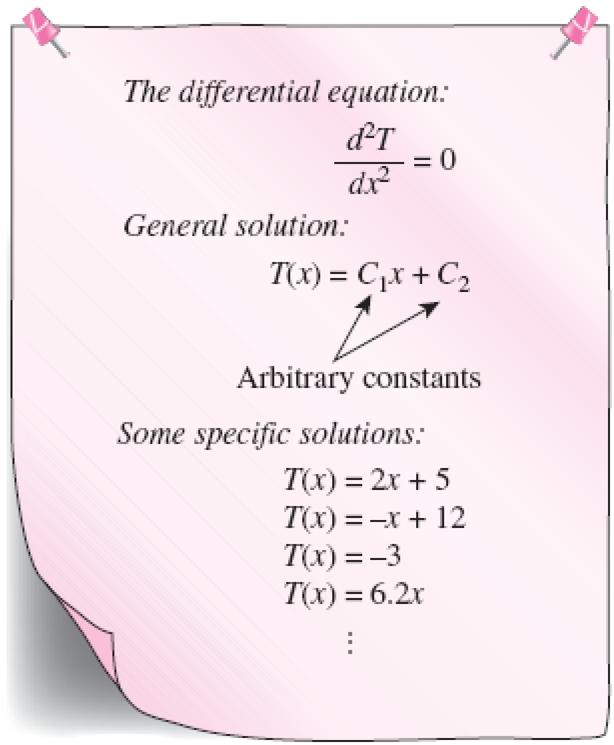


FIGURE 2–25

The general solution of a typical differential equation involves arbitrary constants, and thus an infinite number of solutions.

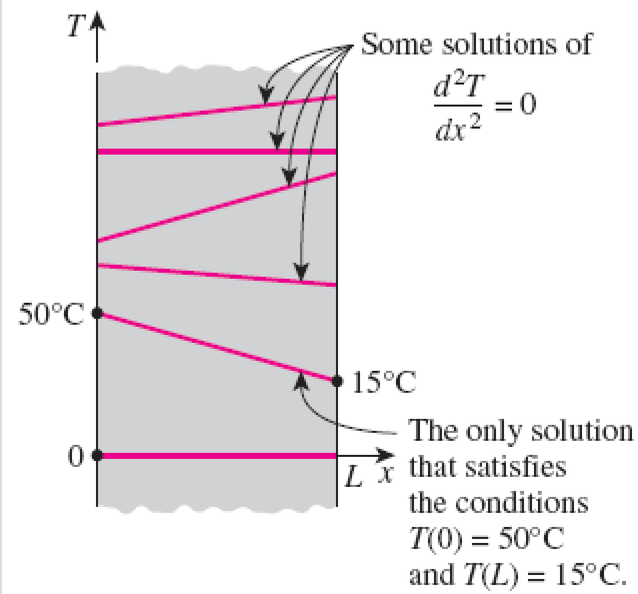


FIGURE 2–26

To describe a heat transfer problem completely, two boundary conditions must be given for each direction along which heat transfer is significant.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

1 Specified Temperature Boundary Condition

The *temperature* of an exposed surface can usually be measured directly and easily.

Therefore, one of the easiest ways to specify the thermal conditions on a surface is to specify the temperature.

For one-dimensional heat transfer through a plane wall of thickness *L*, for example, the specified temperature boundary conditions can be expressed as

$$T(0, t) = T_1$$
$$T(L, t) = T_2$$

where T_1 and T_2 are the specified temperatures at surfaces at $x = 0$ and $x = L$, respectively.

The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.

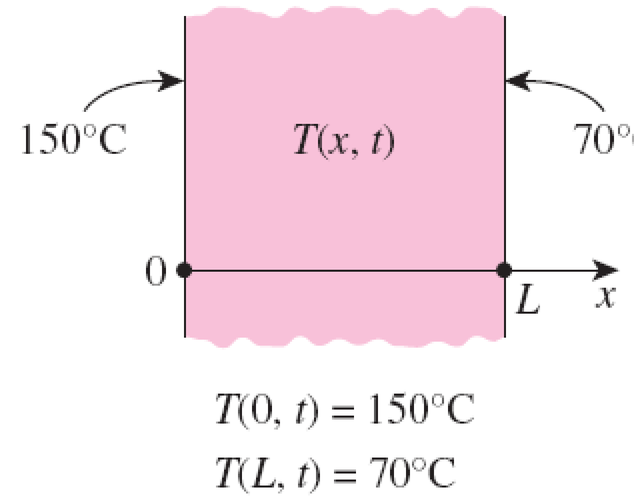


FIGURE 2–27

Specified temperature boundary conditions on both surfaces of a plane wall.

2 Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by

$$\dot{q} = -k\frac{\partial T}{\partial x} = \left(\begin{array}{c}\text{Heat flux in the} \\ \text{positive } x - \text{direction}\end{array}\right) \qquad (\text{W/m}^2)$$

For a plate of thickness *L* subjected to heat flux of 50 W/m² into the medium from both sides, for example, the specified heat flux boundary conditions can be expressed as

$$-k\frac{\partial T(0, t)}{\partial x} = 50 \qquad \text{and} \qquad -k\frac{\partial T(L, t)}{\partial x} = -50$$

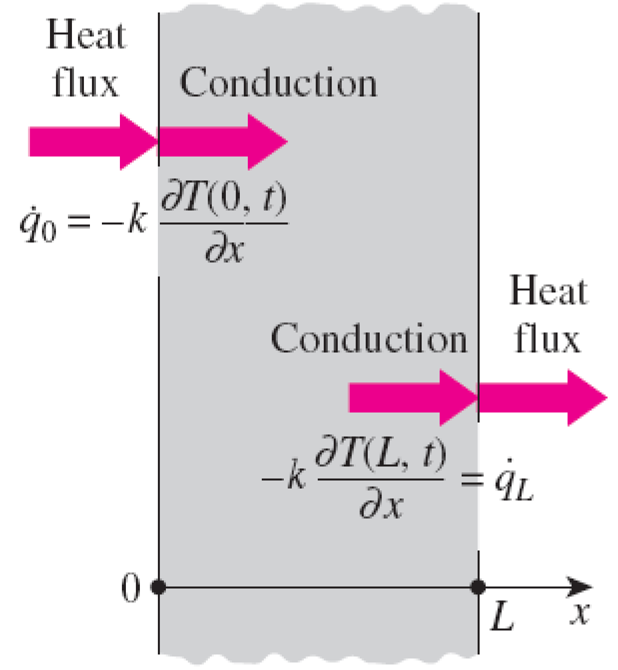


FIGURE 2–28

Specified heat flux boundary conditions on both surfaces of a plane wall.

Special Case: Insulated Boundary

A well-insulated surface can be modeled as a surface with a specified heat flux of zero. Then the boundary condition on a perfectly insulated surface (at x = 0, for example) can be expressed as

$$k\frac{\partial T(0,t)}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial T(0,t)}{\partial x} = 0$$

On an insulated surface, the first derivative of temperature with respect to the space variable (the temperature gradient) in the direction normal to the insulated surface is zero.

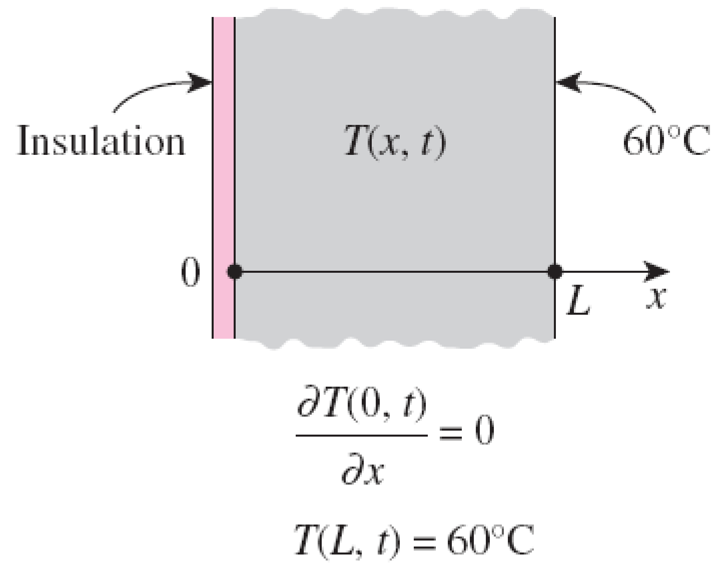


FIGURE 2–29

A plane wall with insulation and specified temperature boundary conditions.

Another Special Case: Thermal Symmetry

Some heat transfer problems possess *thermal symmetry* as a result of the symmetry in imposed thermal conditions.

For example, the two surfaces of a large hot plate of thickness *L* suspended vertically in air is subjected to the same thermal conditions, and thus the temperature distribution in one half of the plate is the same as that in the other half.

That is, the heat transfer problem in this plate possesses thermal symmetry about the center plane at $x = L/2$.

Therefore, the center plane can be viewed as an insulated surface, and the thermal condition at this plane of symmetry can be expressed as

$$\frac{\partial T(L/2, t)}{\partial x} = 0$$

which resembles the *insulation* or *zero heat flux* boundary condition.

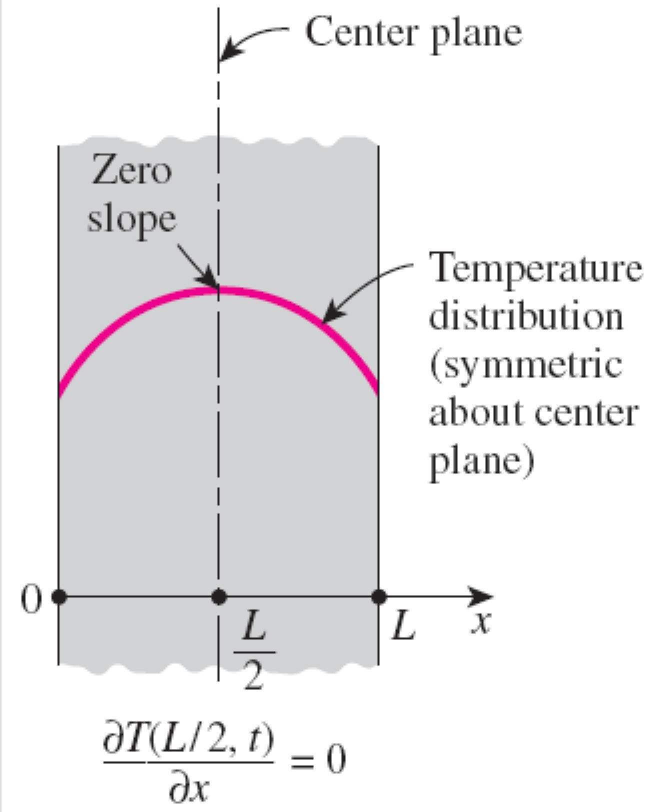


FIGURE 2–30

Thermal symmetry boundary condition at the center plane of a plane wall.

3 Convection Boundary Condition

For one-dimensional heat transfer in the *x*-direction in a plate of thickness *L*, the convection boundary conditions on both surfaces:

$$\begin{pmatrix}\text{Heat conduction}\\ \text{at the surface in a}\\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Heat convection}\\ \text{at the surface in}\\ \text{the same direction}\end{pmatrix}$$

$$-k\frac{\partial T(0,t)}{\partial x} = h_1[T_{\infty 1} - T(0,t)]$$

$$-k\frac{\partial T(L,t)}{\partial x} = h_2[T(L,t) - T_{\infty 2}]$$

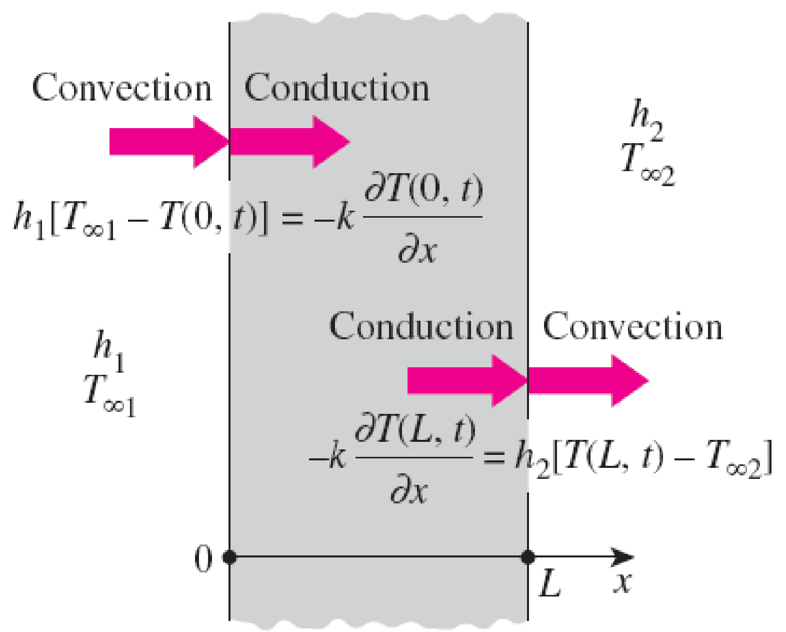


FIGURE 2–32

Convection boundary conditions on the two surfaces of a plane wall.

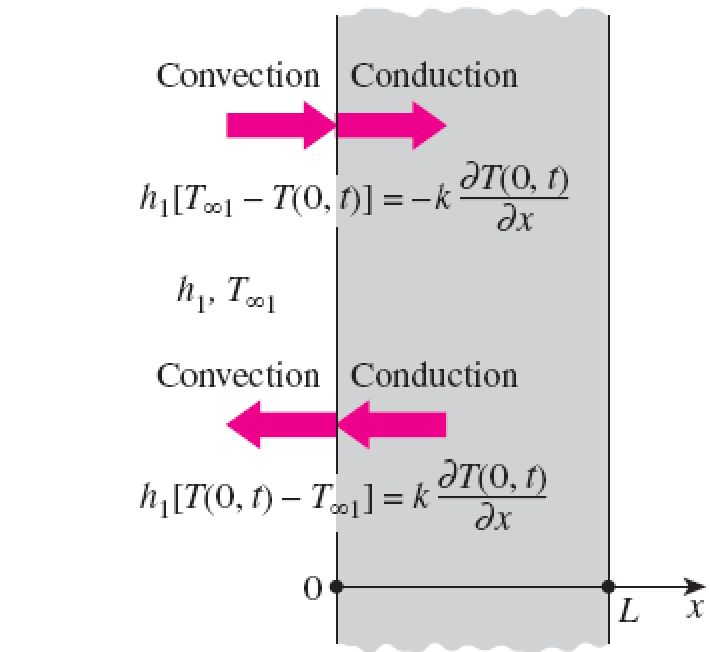


FIGURE 2–33

The assumed direction of heat transfer at a boundary has no effect on the boundary condition expression.

4 Radiation Boundary Condition

Radiation boundary condition on a surface:

$$\begin{pmatrix}\text{Heat conduction}\\ \text{at the surface in a}\\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Radiation exchange}\\ \text{at the surface in}\\ \text{the same direction}\end{pmatrix}$$

For one-dimensional heat transfer in the *x*-direction in a plate of thickness *L*, the radiation boundary conditions on both surfaces can be expressed as

$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\sigma[T^4_{\text{surr},1} - \mathrm{T}(0,t)^4]$$

$$-k\frac{\partial T(L,t)}{\partial x} = \varepsilon_2\sigma[T(L,t)^4 - T^4_{\text{surr},2}]$$

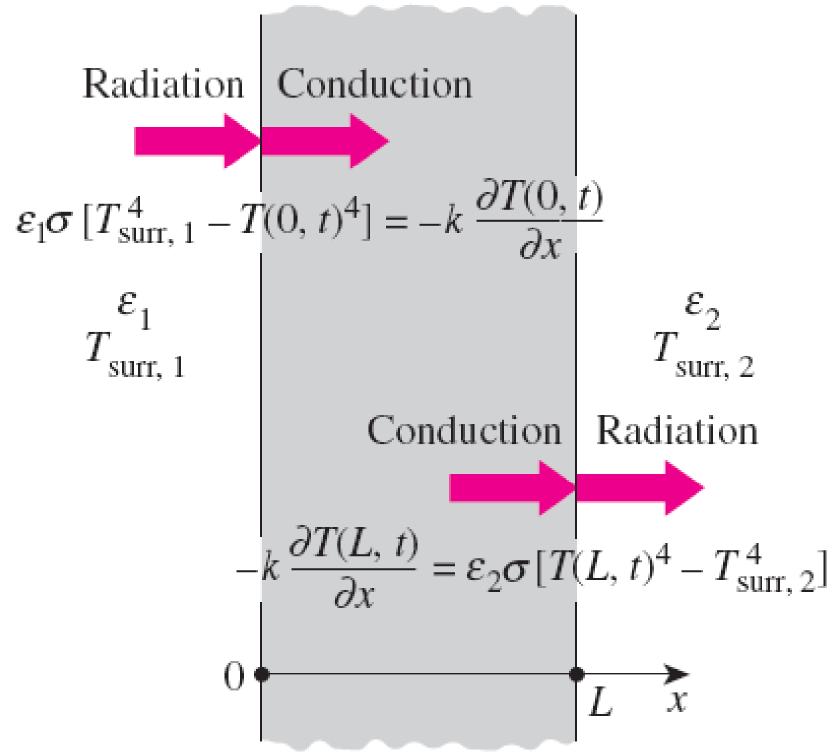


FIGURE 2–35

Radiation boundary conditions on both surfaces of a plane wall.

5 Interface Boundary Conditions

The boundary conditions at an interface are based on the requirements that

(1) two bodies in contact must have the *same temperature* at the area of contact and

(2) an interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same.*

The boundary conditions at the interface of two bodies *A* and *B* in perfect contact at $x = x_0$ can be expressed as

$$T_A(x_0, t) = T_B(x_0, t)$$

$$-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x}$$

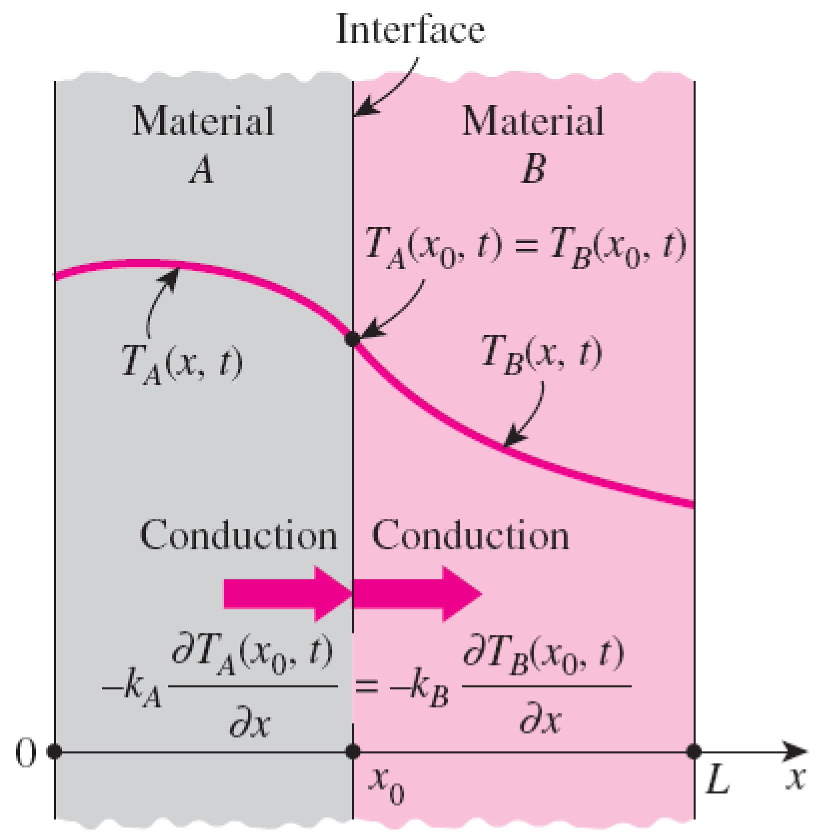


FIGURE 2–36

Boundary conditions at the interface of two bodies in perfect contact.

6 Generalized Boundary Conditions

In general, however, a surface may involve convection, radiation, *and* specified heat flux simultaneously.

The boundary condition in such cases is again obtained from a surface energy balance, expressed as

$$\begin{pmatrix} \text{Heat transfer} \\ \text{to the surface} \\ \text{in all modes} \end{pmatrix} = \begin{pmatrix} \text{Heat transfer} \\ \text{from the surface} \\ \text{in all modes} \end{pmatrix}$$

SOLUTION OF STEADY ONE-DIMENSIONAL HEAT CONDUCTION PROBLEMS

In this section we will solve a wide range of heat conduction problems in rectangular, cylindrical, and spherical geometries.

We will limit our attention to problems that result in *ordinary differential equations* such as the *steady one-dimensional* heat conduction problems. We will also assume *constant thermal conductivity.*

The solution procedure for solving heat conduction problems can be summarized as

(1) *formulate* the problem by obtaining the applicable differential equation in its simplest form and specifying the boundary conditions,

(2) Obtain the *general solution* of the differential equation, and

(3) apply the *boundary conditions* and determine the arbitrary constants in the general solution.

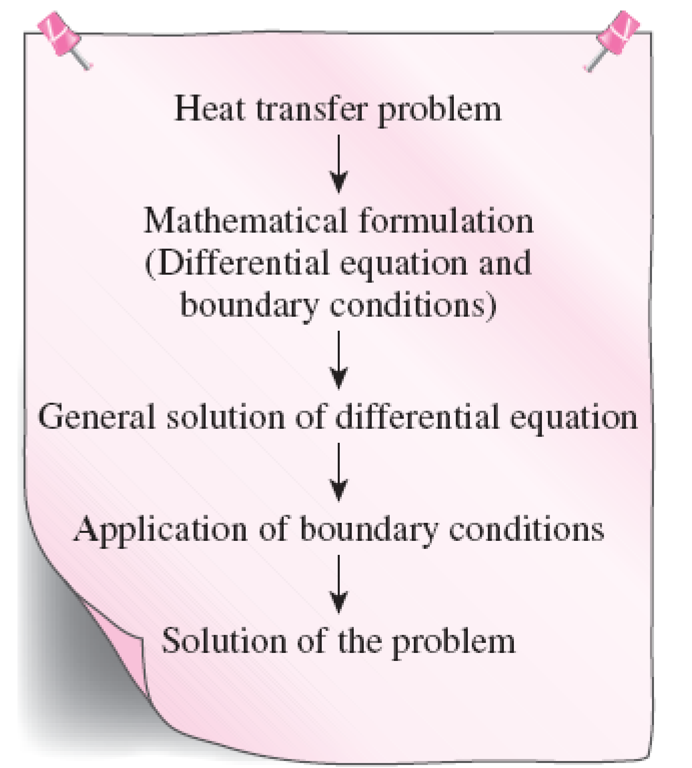


FIGURE 2–39

Basic steps involved in the solution of heat transfer problems.

EXAMPLE 2–10 Heat Conduction in a Plane Wall

Consider a large plane wall of thickness $L = 0.2$ m, thermal conductivity $k = 1.2$ W/m·K, and surface area $A = 15$ m^2. The two sides of the wall are maintained at constant temperatures of $T_1 = 120°C$ and $T_2 = 50°C$, respectively, as shown in Fig. 2–40. Determine (*a*) the variation of temperature within the wall and the value of temperature at $x = 0.1$ m and (*b*) the rate of heat conduction through the wall under steady conditions.

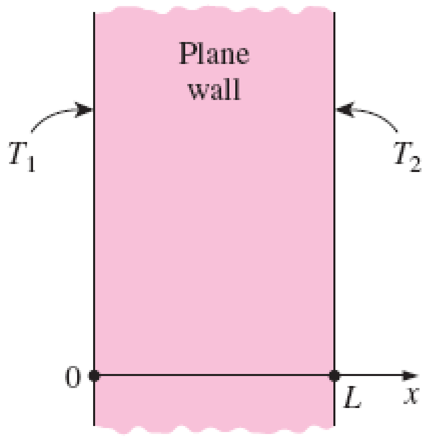


FIGURE 2–40
Schematic for Example 2–10.

SOLUTION A plane wall with specified surface temperatures is given. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat conduction is steady. **2** Heat conduction is one-dimensional since the wall is large relative to its thickness and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 1.2$ W/m·K.

Analysis (*a*) Taking the direction normal to the surface of the wall to be the *x*-direction, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 120°C$$
$$T(L) = T_2 = 50°C$$

The differential equation is linear and second order, and a quick inspection of it reveals that it has a single term involving derivatives and no terms involving the unknown function T as a factor. Thus, it can be solved by direct integration. Noting that an integration reduces the order of a derivative by one, the general solution of the differential equation above can be obtained by two simple successive integrations, each of which introduces an integration constant.

Integrating the differential equation once with respect to x yields

$$\frac{dT}{dx} = C_1$$

where C_1 is an arbitrary constant. Notice that the order of the derivative went down by one as a result of integration. As a check, if we take the derivative of this equation, we will obtain the original differential equation. This equation is not the solution yet since it involves a derivative.

Integrating one more time, we obtain

$$T(x) = C_1 x + C_2$$

which is the general solution of the differential equation (Fig. 2–41). The general solution in this case resembles the general formula of a straight line whose slope is C_1 and whose value at $x = 0$ is C_2. This is not surprising since the second derivative represents the change in the slope of a function, and a zero second derivative indicates that the slope of the function remains constant. Therefore, *any straight line* is a solution of this differential equation.

The general solution contains two unknown constants C_1 and C_2, and thus we need two equations to determine them uniquely and obtain the specific solution. These equations are obtained by forcing the general solution to satisfy the specified boundary conditions. The application of each condition yields one equation, and thus we need to specify two conditions to determine the constants C_1 and C_2.

When applying a boundary condition to an equation, *all occurrences of the dependent and independent variables and any derivatives are replaced by the specified values.* Thus the only unknowns in the resulting equations are the arbitrary constants.

The first boundary condition can be interpreted as *in the general solution, replace all the x's by zero and T(x) by T_1.* That is (Fig. 2–42),

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

Differential equation:

$$\frac{d^2T}{dx^2} = 0$$

Integrate:

$$\frac{dT}{dx} = C_1$$

Integrate again:

$$T(x) = C_1 x + C_2$$

General solution

Arbitrary constants

FIGURE 2–41

Obtaining the general solution of a simple second order differential equation by integration.

The second boundary condition can be interpreted as *in the general solution, replace all the x's by L and T(x) by T_2*. That is,

$$T(L) = C_1L + C_2 \quad \rightarrow \quad T_2 = C_1L + T_1 \quad \rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

Substituting the C_1 and C_2 expressions into the general solution, we obtain

$$T(x) = \frac{T_2 - T_1}{L}x + T_1 \qquad \textbf{(2–56)}$$

which is the desired solution since it satisfies not only the differential equation but also the two specified boundary conditions. That is, differentiating Eq. 2–56 with respect to x twice will give d^2T/dx^2, which is the given differential equation, and substituting $x = 0$ and $x = L$ into Eq. 2–56 gives $T(0) = T_1$ and $T(L) = T_2$, respectively, which are the specified conditions at the boundaries.

Substituting the given information, the value of the temperature at $x = 0.1$ m is determined to be

$$T(0.1 \text{ m}) = \frac{(50 - 120)°\text{C}}{0.2 \text{ m}}(0.1 \text{ m}) + 120°\text{C} = \mathbf{85°C}$$

(*b*) The rate of heat conduction anywhere in the wall is determined from Fourier's law to be

$$\dot{Q}_{\text{wall}} = -kA\frac{dT}{dx} = -kAC_1 = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} \qquad \textbf{(2–57)}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (1.2 \text{ W/m·K})(15 \text{ m}^2)\frac{(120 - 50)°\text{C}}{0.2 \text{ m}} = \mathbf{6300 \text{ W}}$$

Discussion Note that under steady conditions, the rate of heat conduction through a plane wall is constant.

Boundary condition:

$$T(0) = T_1$$

General solution:

$$T(x) = C_1x + C_2$$

Applying the boundary condition:

$$\underbrace{T(\underset{\substack{\uparrow\\0}}{x})}_{T_1} = C_1\underset{\substack{\uparrow\\0}}{x} + C_2$$

Substituting:

$$T_1 = C_1 \times 0 + C_2 \rightarrow C_2 = T_1$$

It cannot involve x or $T(x)$ after the boundary condition is applied.

FIGURE 2–42

When applying a boundary condition to the general solution at a specified point, all occurrences of the dependent and independent variables should be replaced by their specified values at that point.

EXAMPLE 2–14 Heat Loss through a Steam Pipe

Consider a steam pipe of length $L = 20$ m, inner radius $r_1 = 6$ cm, outer radius $r_2 = 8$ cm, and thermal conductivity $k = 20$ W/m·K, as shown in Fig. 2–49. The inner and outer surfaces of the pipe are maintained at average temperatures of $T_1 = 150°C$ and $T_2 = 60°C$, respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.

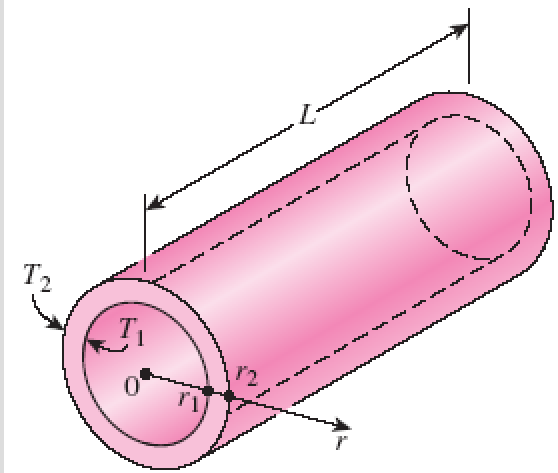


FIGURE 2–49
Schematic for Example 2–14.

SOLUTION A steam pipe is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction, and thus $T = T(r)$. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 20$ W/m·K.

Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 150°C$$
$$T(r_2) = T_2 = 60°C$$

Integrating the differential equation once with respect to r gives

$$r\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Again integrating with respect to r gives (Fig. 2–50)

$$T(r) = C_1 \ln r + C_2 \qquad (a)$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in Eq. (a) with the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad C_1 \ln r_1 + C_2 = T_1$$
$$T(r_2) = T_2 \quad \rightarrow \quad C_1 \ln r_2 + C_2 = T_2$$

which are two equations in two unknowns, C_1 and C_2. Solving them simultaneously gives

$$C_1 = \frac{T_2 - T_1}{\ln(r_2/r_1)} \qquad \text{and} \qquad C_2 = T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)} \ln r_1$$

Substituting them into Eq. (a) and rearranging, the variation of temperature within the pipe is determined to be

$$T(r) = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_2 - T_1) + T_1 \qquad \textbf{(2–58)}$$

The rate of heat loss from the steam is simply the total rate of heat conduction through the pipe, and is determined from Fourier's law to be

$$\dot{Q}_{\text{cylinder}} = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{C_1}{r} = -2\pi kLC_1 = 2\pi kL\frac{T_1 - T_2}{\ln(r_2/r_1)} \qquad \textbf{(2–59)}$$

The numerical value of the rate of heat conduction through the pipe is determined by substituting the given values

$$\dot{Q} = 2\pi(20\ \text{W/m·K})(20\ \text{m})\frac{(150 - 60)°\text{C}}{\ln(0.08/0.06)} = \mathbf{786\ kW}$$

Discussion Note that the total rate of heat transfer through a pipe is constant, but the heat flux $\dot{q} = \dot{Q}/(2\pi rL)$ is not since it decreases in the direction of heat transfer with increasing radius.

Differential equation:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Integrate:

$$r\frac{dT}{dr} = C_1$$

Divide by r ($r \neq 0$):

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Integrate again:

$$T(r) = C_1 \ln r + C_2$$

which is the general solution.

FIGURE 2–50

Basic steps involved in the solution of the steady one-dimensional heat conduction equation in cylindrical coordinates.

EXAMPLE 2–15 **Heat Conduction through a Spherical Shell**

Consider a spherical container of inner radius $r_1 = 8$ cm, outer radius $r_2 = 10$ cm, and thermal conductivity $k = 45$ W/m·K, as shown in Fig. 2–51. The inner and outer surfaces of the container are maintained at constant temperatures of $T_1 = 200°C$ and $T_2 = 80°C$, respectively, as a result of some chemical reactions occurring inside. Obtain a general relation for the temperature distribution inside the shell under steady conditions, and determine the rate of heat loss from the container.

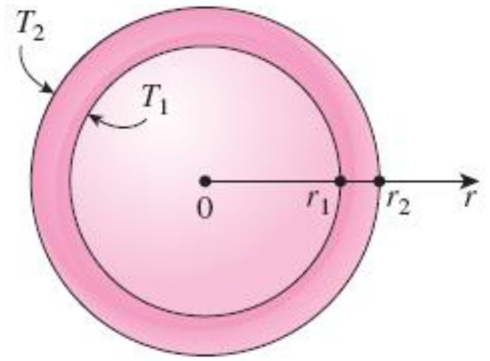


FIGURE 2–51
Schematic for Example 2–15.

SOLUTION A spherical container is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the midpoint, and thus $T = T(r)$. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 45$ W/m·K.

Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 200°C$$
$$T(r_2) = T_2 = 80°C$$

Integrating the differential equation once with respect to r yields

$$r^2\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r^2 to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r^2}$$

Again integrating with respect to r gives

$$T(r) = -\frac{C_1}{r} + C_2 \tag{a}$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in the relation above by the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad -\frac{C_1}{r_1} + C_2 = T_1$$

$$T(r_2) = T_2 \quad \rightarrow \quad -\frac{C_1}{r_2} + C_2 = T_2$$

which are two equations in two unknowns, C_1 and C_2. Solving them simultaneously gives

$$C_1 = -\frac{r_1 r_2}{r_2 - r_1}(T_1 - T_2) \qquad \text{and} \qquad C_2 = \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1}$$

Substituting into Eq. (a), the variation of temperature within the spherical shell is determined to be

$$T(r) = \frac{r_1 r_2}{r(r_2 - r_1)}(T_1 - T_2) + \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1} \tag{2–60}$$

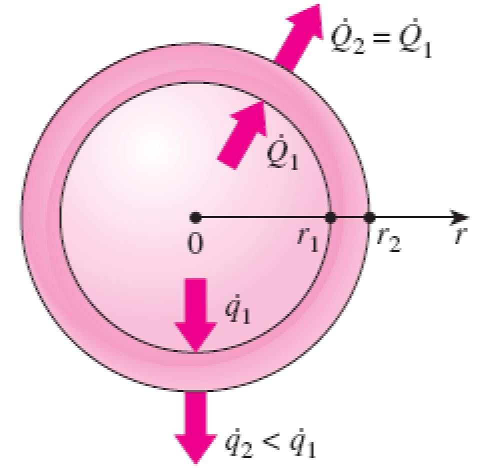
The rate of heat loss from the container is simply the total rate of heat conduction through the container wall and is determined from Fourier's law

$$\dot{Q}_{\text{sphere}} = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{C_1}{r^2} = -4\pi k C_1 = 4\pi k r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1} \tag{2–61}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = 4\pi(45\ \text{W/m·K})(0.08\ \text{m})(0.10\ \text{m})\frac{(200 - 80)°\text{C}}{(0.10 - 0.08)\ \text{m}} = \mathbf{27.1\ kW}$$

Discussion Note that the total rate of heat transfer through a spherical shell is constant, but the heat flux $\dot{q} = \dot{Q}/4\pi r^2$ is not since it decreases in the direction of heat transfer with increasing radius as shown in Fig. 2–52.



$$\dot{q}_1 = \frac{\dot{Q}_1}{A_1} = \frac{27.1\ \text{kW}}{4\pi(0.08\ \text{m})^2} = 337\ \text{kW/m}^2$$

$$\dot{q}_2 = \frac{\dot{Q}_2}{A_2} = \frac{27.1\ \text{kW}}{4\pi(0.10\ \text{m})^2} = 216\ \text{kW/m}^2$$

FIGURE 2–52

During steady one-dimensional heat conduction in a spherical (or cylindrical) container, the total rate of heat transfer remains constant, but the heat flux decreases with increasing radius.

HEAT GENERATION IN A SOLID

Many practical heat transfer applications involve the conversion of some form of energy into *thermal* energy in the medium.

Such mediums are said to involve internal *heat generation*, which manifests itself as a rise in temperature throughout the medium.

Some examples of heat generation are

- *resistance heating* in wires,

- exothermic *chemical reactions* in a solid, and

- *nuclear reactions* in nuclear fuel rods

where electrical, chemical, and nuclear energies are converted to heat, respectively.

Heat generation in an electrical wire of outer radius r_o and length L can be expressed as

$$\dot{e}_{\text{gen}} = \frac{\dot{E}_{\text{gen, electric}}}{V_{\text{wire}}} = \frac{I^2 R_e}{\pi r_o^2 L} \qquad (\text{W/m}^3)$$

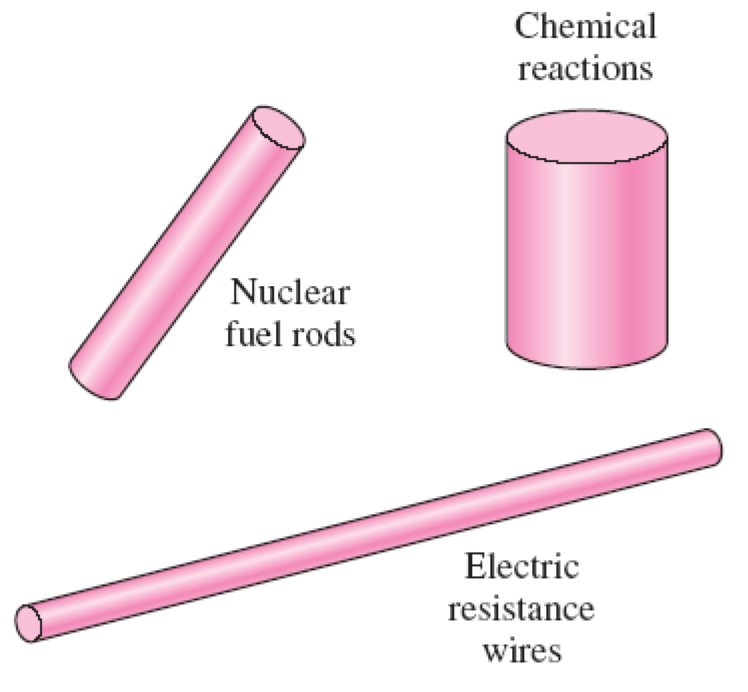


FIGURE 2–53
Heat generation in solids is commonly encountered in practice.

The quantities of major interest in a medium with heat generation are the surface temperature T_s and the maximum temperature T_{max} that occurs in the medium in *steady* operation.

$$\begin{pmatrix}\text{Rate of}\\ \textit{heat transfer}\\ \text{from the solid}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \textit{energy generation}\\ \text{within the solid}\end{pmatrix}$$

$$\dot{Q} = \dot{e}_{\text{gen}}V \qquad \text{(W)}$$

$$\dot{Q} = hA_s(T_s - T_\infty) \qquad \text{(W)}$$

$$T_s = T_\infty + \frac{\dot{e}_{\text{gen}}V}{hA_s}$$

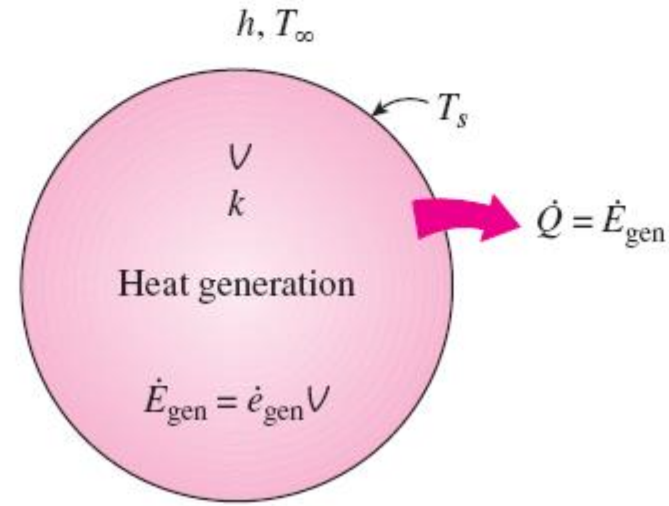


FIGURE 2–54

At steady conditions, the entire heat generated in a solid must leave the solid through its outer surface.

For a large *plane wall* of thickness $2L$ ($A_s = 2A_{\text{wall}}$ and $V = 2LA_{\text{wall}}$) with both sides of the wall maintained at the same temperature T_s, a long solid *cylinder* of radius r_o ($A_s = 2\pi r_o L$ and $V = \pi r_o^2 L$), and a solid *sphere* of radius r_o ($A_s = 4\pi r_o^2$ and $V = \frac{4}{3}\pi r_o^3$), Eq. 2–66 reduces to

$$T_{s,\text{ plane wall}} = T_\infty + \frac{\dot{e}_{\text{gen}}L}{h}$$

$$T_{s,\text{ cylinder}} = T_\infty + \frac{\dot{e}_{\text{gen}}r_o}{2h}$$

$$T_{s,\text{ sphere}} = T_\infty + \frac{\dot{e}_{\text{gen}}r_o}{3h}$$

$$-kA_r\frac{dT}{dr}=\dot{e}_{gen}V_r \qquad A_r=2\pi rL \qquad V_r=\pi r^2L \qquad -k(2\pi rL)\frac{dT}{dr}=\dot{e}_{gen}(\pi r^2 L) \;\rightarrow\; dT=-\frac{\dot{e}_{gen}}{2k}rdr$$

Integrating from $r = 0$ where $T(0) = T_0$ to $r = r_o$ where $T(r_o) = T_s$ yields

$$\Delta T_{\text{max, cylinder}} = T_0 - T_s = \frac{\dot{e}_{gen}r_o^2}{4k} \qquad \Delta T_{\text{max, plane wall}} = \frac{\dot{e}_{gen}L^2}{2k} \qquad \Delta T_{\text{max, sphere}} = \frac{\dot{e}_{gen}r_o^2}{6k}$$

$$T_{center} = T_0 = T_s + \Delta T_{max}$$

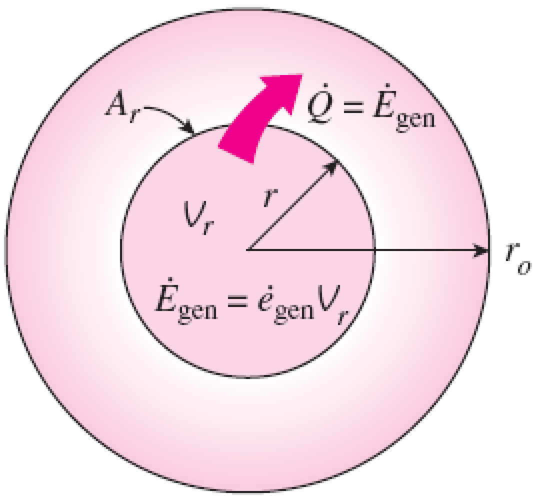


FIGURE 2–55

Heat conducted through a cylindrical shell of radius r is equal to the heat generated within a shell.

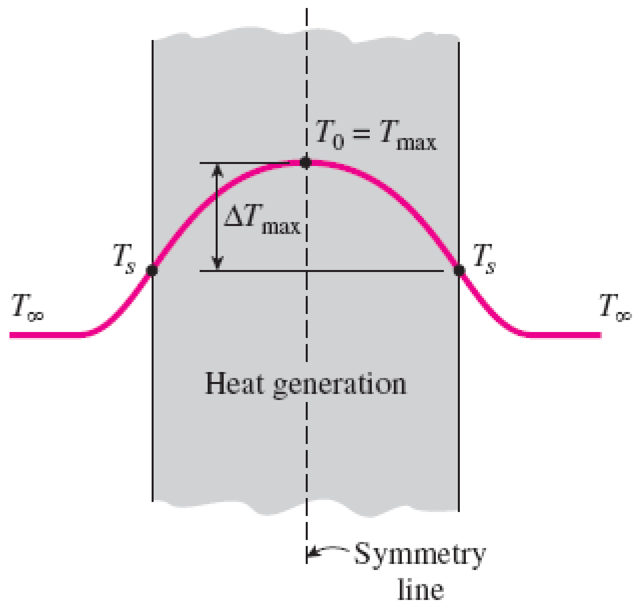


FIGURE 2–56

The maximum temperature in a symmetrical solid with uniform heat generation occurs at its center.

VARIABLE THERMAL CONDUCTIVITY, *k*(*T*)

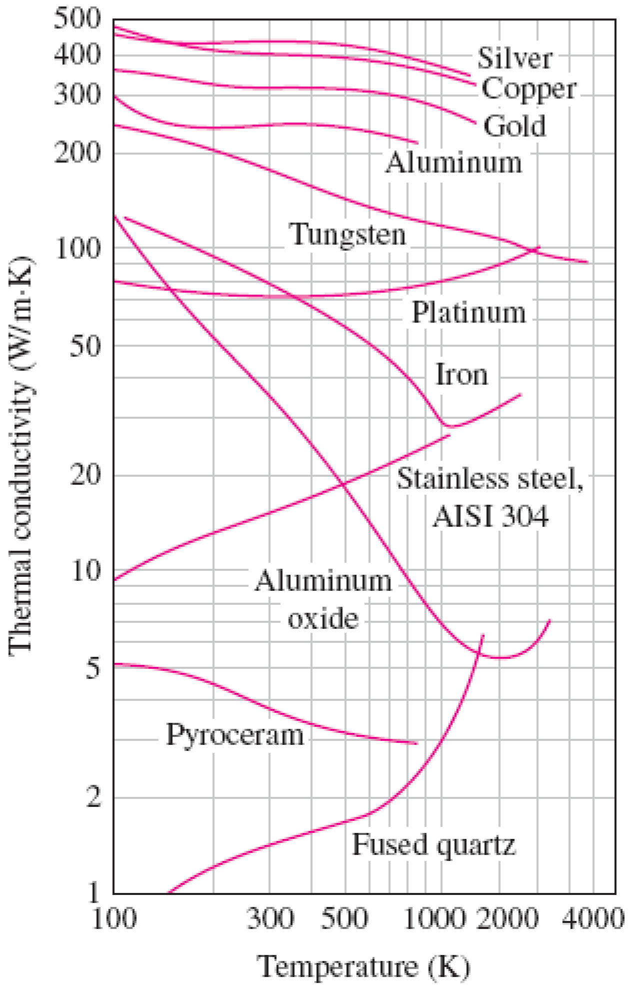


FIGURE 2–62

Variation of the thermal conductivity of some solids with temperature.

When the variation of thermal conductivity with temperature in a specified temperature interval is large, it may be necessary to account for this variation to minimize the error.

When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{\text{avg}} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

$$\dot{Q}_{\text{plane wall}} = k_{\text{avg}} A \frac{T_1 - T_2}{L} = \frac{A}{L} \int_{T_2}^{T_1} k(T)dT$$

$$\dot{Q}_{\text{cylinder}} = 2\pi k_{\text{avg}} L \frac{T_1 - T_2}{\ln(r_2/r_1)} = \frac{2\pi L}{\ln(r_2/r_1)} \int_{T_2}^{T_1} k(T)dT$$

$$\dot{Q}_{\text{sphere}} = 4\pi k_{\text{avg}} r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1} = \frac{4\pi r_1 r_2}{r_2 - r_1} \int_{T_2}^{T_1} k(T)dT$$

The variation in thermal conductivity of a material with temperature in the temperature range of interest can often be approximated as a linear function and expressed as

$$k(T) = k_0(1 + \beta T)$$

β temperature coefficient of thermal conductivity.

The *average* value of thermal conductivity in the temperature range T_1 to T_2 in this case can be determined from

$$k_{avg} = \frac{\int_{T_1}^{T_2} k_0(1 + \beta T)dT}{T_2 - T_1} = k_0\left(1 + \beta \frac{T_2 + T_1}{2}\right) = k(T_{avg})$$

The *average thermal conductivity* in this case is equal to the thermal conductivity value at the *average temperature.*

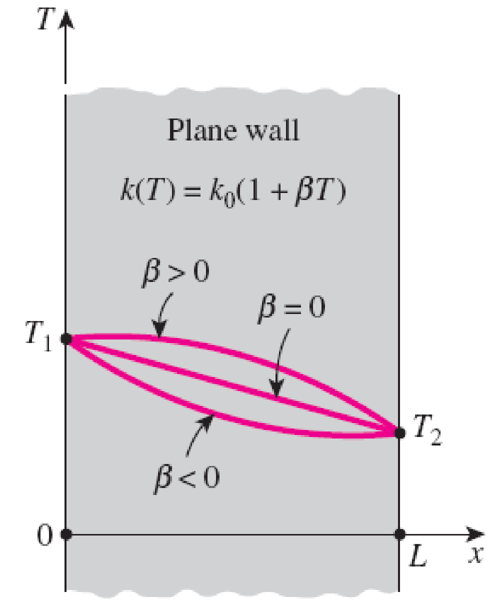


FIGURE 2–63
The variation of temperature in a plane wall during steady one-dimensional heat conduction for the cases of constant and variable thermal conductivity.

Summary

- Introduction
  - ✓ Steady versus Transient Heat Transfer
  - ✓ Multidimensional Heat Transfer
  - ✓ Heat Generation
- One-Dimensional Heat Conduction Equation
  - ✓ Heat Conduction Equation in a Large Plane Wall
  - ✓ Heat Conduction Equation in a Long Cylinder
  - ✓ Heat Conduction Equation in a Sphere
  - ✓ Combined One-Dimensional Heat Conduction Equation
- General Heat Conduction Equation
  - ✓ Rectangular Coordinates
  - ✓ Cylindrical Coordinates
  - ✓ Spherical Coordinates
- Boundary and Initial Conditions
- Solution of Steady One-Dimensional Heat Conduction Problems
- Heat Generation in a Solid
- Variable Thermal Conductivity $k(T)$