EE421/521 Image Processing

Lecture 6
SAMPLING & RESAMPLING

1

Introduction

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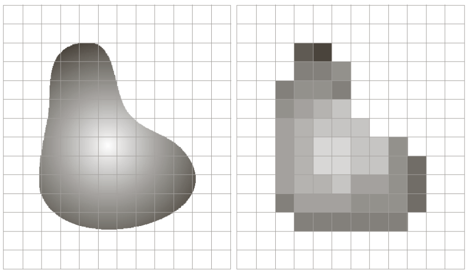
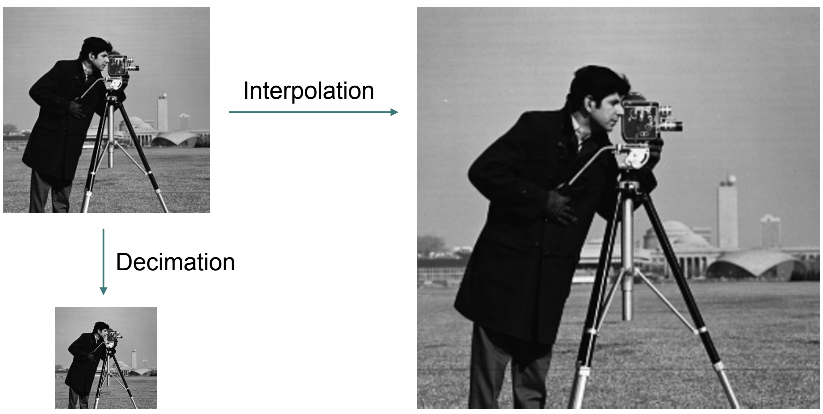
Sampling

- Analog to digital conversion (with a given sampling rate)

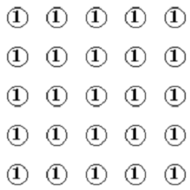


Resampling

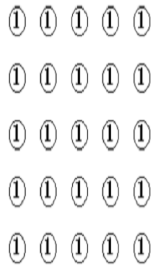
- Sampling rate conversion (digital to digital)



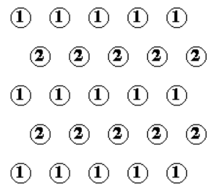
Sampling Structure



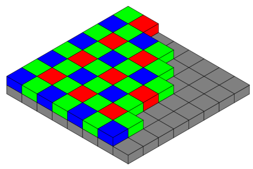
Square



Rectangular



Quincunx



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Recall Fourier Spectrum

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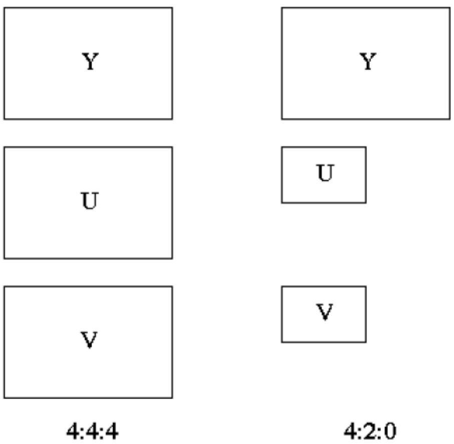
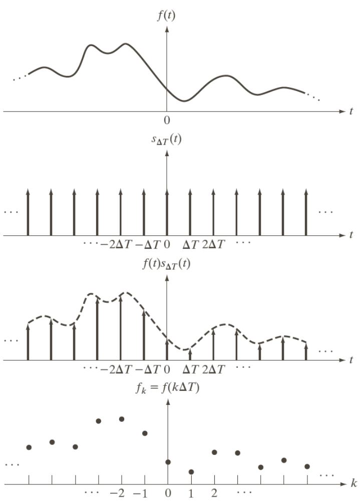
Smaller Bandwidth Enables Coarser Sampling

MPEG Sampling Format

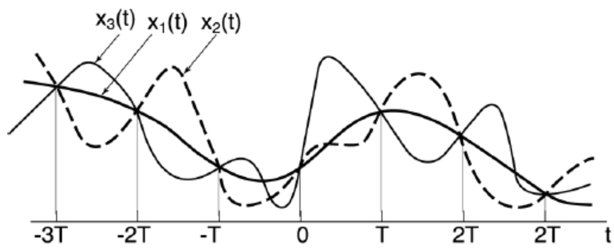


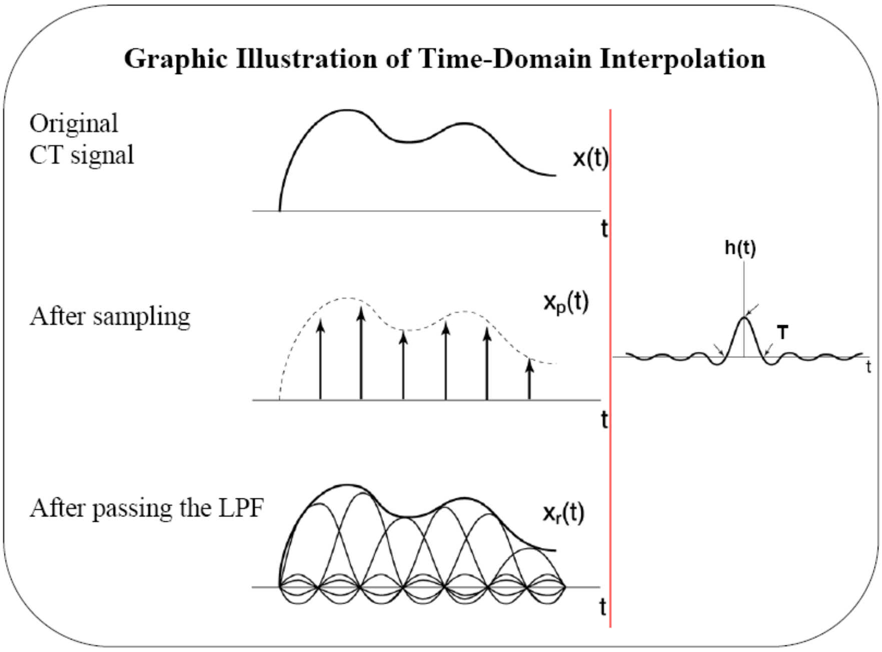
Fourier Analysis of Sampling

Sampling

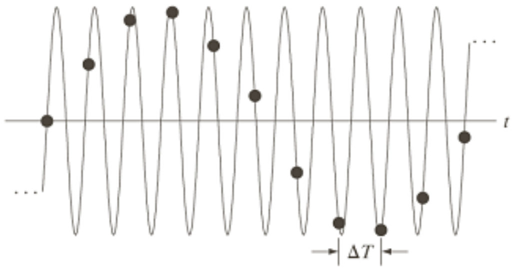


Implied Reconstruction is a Low Frequency Signal

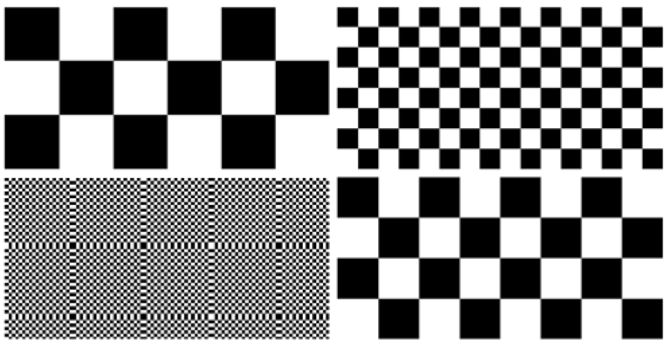




Undersampling Results in Aliasing



Aliasing: High Frequency Parts Appear at Low Frequencies

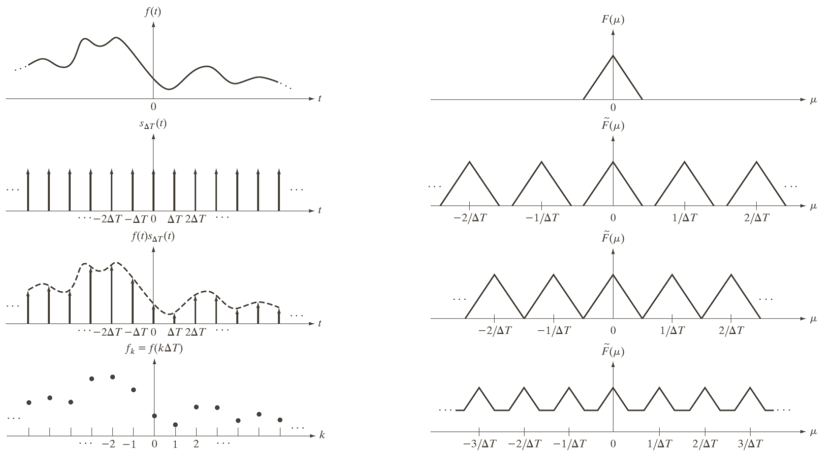


Square sizes: 16, 6, 0.9, 0.5, left-to-right & up-to-down, respectively.

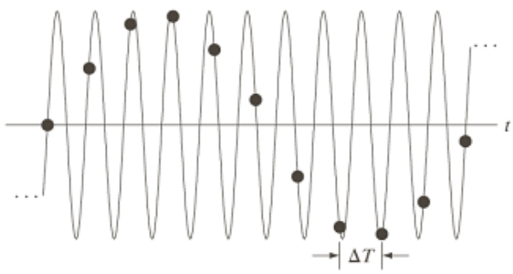
Aliasing Caused by Downsizing

Anti-Aliased Image

Forier Analysis of Aliasing

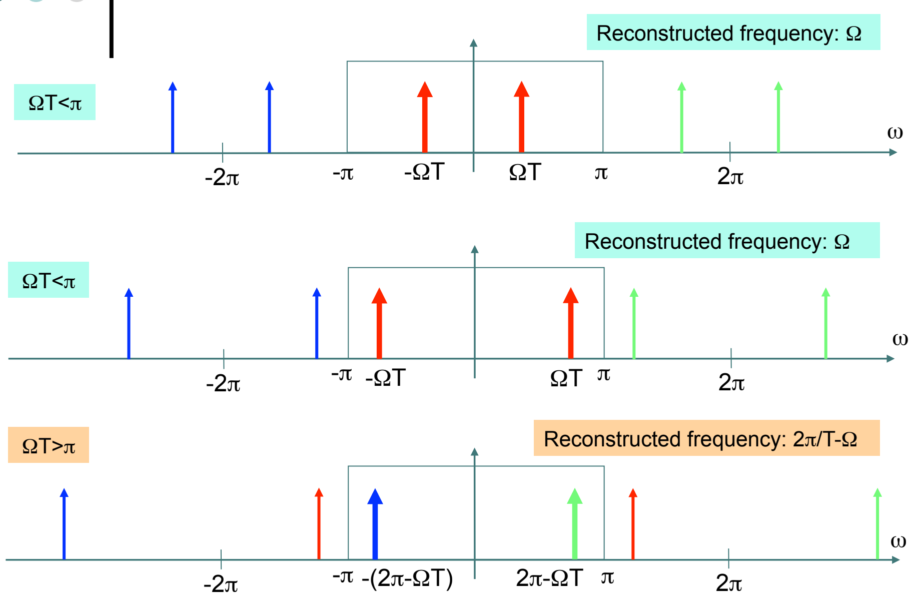


Aliasing for a Sinusoidal



Aliasing

Ω: Frequency of the sinusoidal T: Sampling interval



Resampling: Interpolation & Decimation

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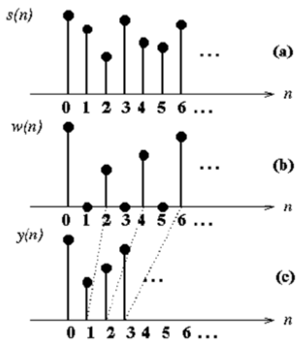
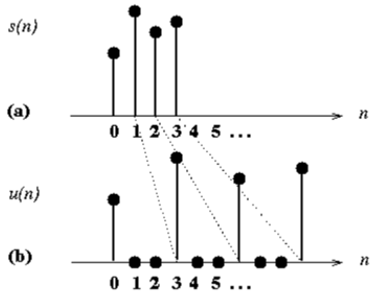
Downsampling & Upsampling

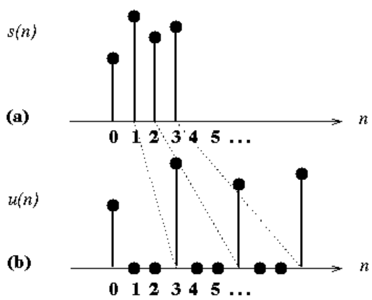


2:1 Downsampling



1:3 Upsampling

Interpolation: Upsampling

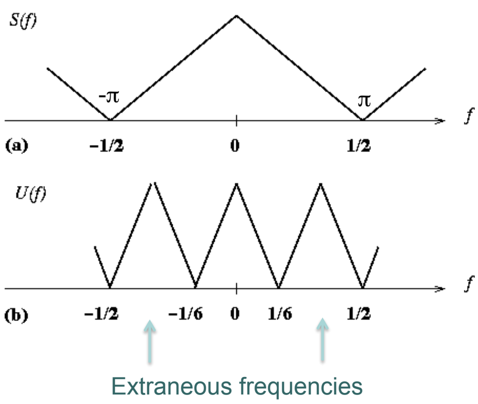


Time domain

$$u(n) = \begin{cases} s(\frac{n}{L}) & \text{for } n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$

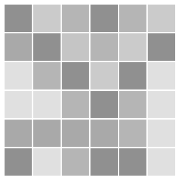
Frequency domain

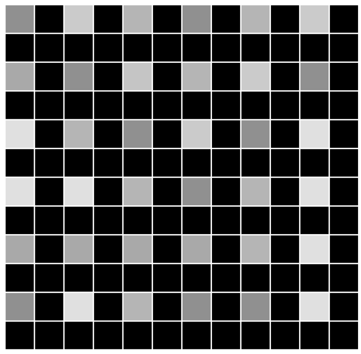
$$U(f) = \sum_{n=-\infty}^{\infty} s(n) \exp\{-j2\pi f L n\} = S(fL)$$



1:2 Upsampling Example

- Insert zeroes





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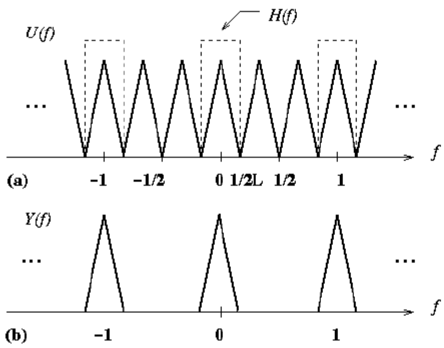
Interpolation: LP Filtering

- Ideal interpolation filter is an ideal lowpass filter.



$$h(n) = \frac{\sin(\frac{\pi n}{L})}{\frac{\pi n}{L}}$$

Interpolation by L=3.

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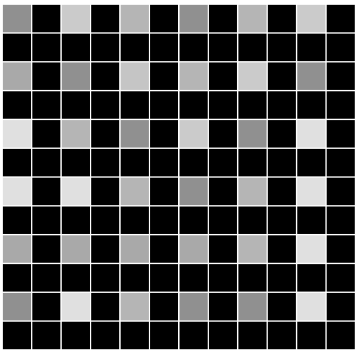
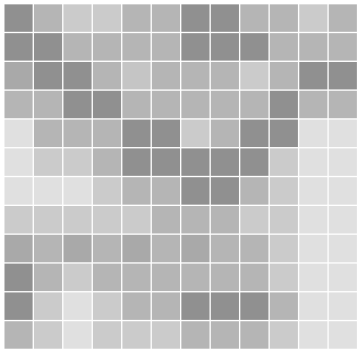
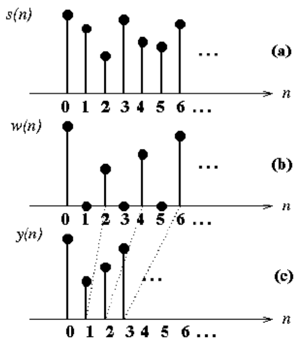
1:2 Interpolation Example

- Low-pass filter fills in for zeroes





1-D Decimation: Downsampling

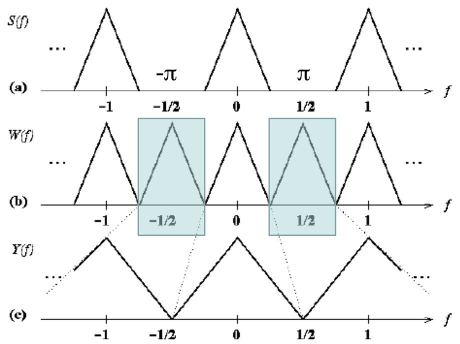


$$w(n) = s(n) \sum_{k=-\infty}^{\infty} \delta(n - kM)$$

$$y(n) = w(Mn)$$

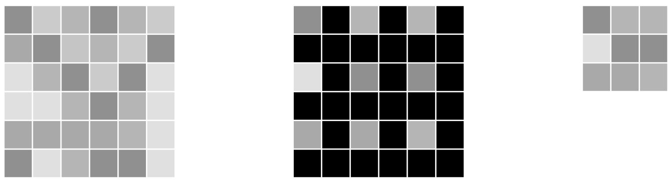
$$W(f) = \frac{1}{M} \sum_{k=-\infty}^{\infty} S(f - \frac{k}{M})$$

$$Y(f) = \sum_{n=-\infty}^{\infty} w(Mn) \exp\{-j2\pi fn\} = W(\frac{f}{M})$$



2:1 Downsampling Example

1. Set every other sample to zero
2. Remove those zero samples



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Low-pass Filtering for Resampling

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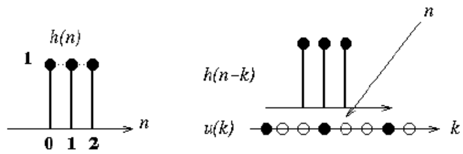
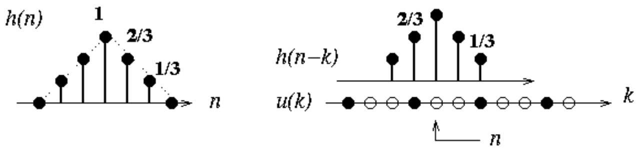
Practical Lowpass Filters

- Zero-order hold (sample repeat)



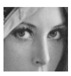
The impulse response for L=3.

- Linear interpolation



The impulse response for L=3.

Simple lowpass filtering examples



Original

Zero-order hold (box)
Pixel replication

Bilinear (triangular)

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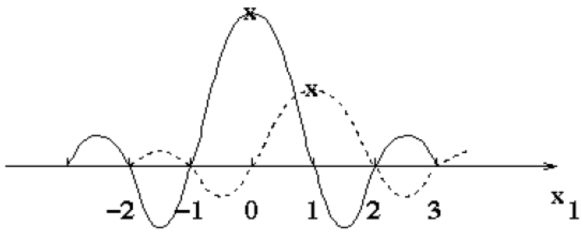
Ideal LP Filtering in Time



- The impulse response of the ideal interpolation filter is a sinc function.
- Because of its periodic zero-crossings it will not alter existing signal samples, while replacing zero valued samples with non-zero ones in the upsampled signal.

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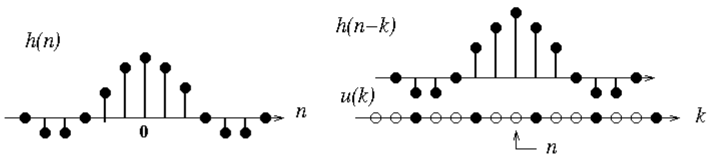
Approximation to Ideal Lowpass Filter

- Cubic Spline Interpolation
  - Approximate the impulse response of the ideal lowpass filter (sinc function) by three cubic polynomials.
  - The frequency response is better than that of the truncated sinc function.



The impulse response for L=3.

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Bilinear vs. Cubic spline (Bicubic) Close-ups

Bilinear

Bicubic

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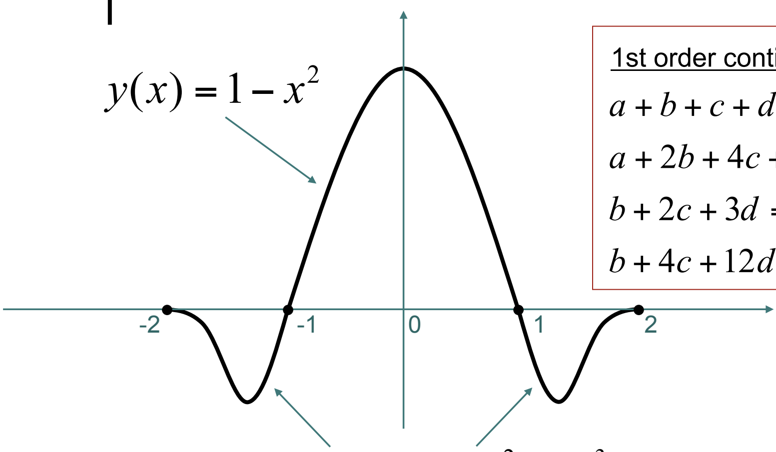
Cubic Spline Filter

$$y(x) = 1 - x^2$$



$$y(x) = a + bx + cx^2 + dx^3$$

<u>1st order continuity:</u>

$$a + b + c + d = 0$$
$$a + 2b + 4c + 8d = 0$$
$$b + 2c + 3d = -2$$
$$b + 4c + 12d = 0$$

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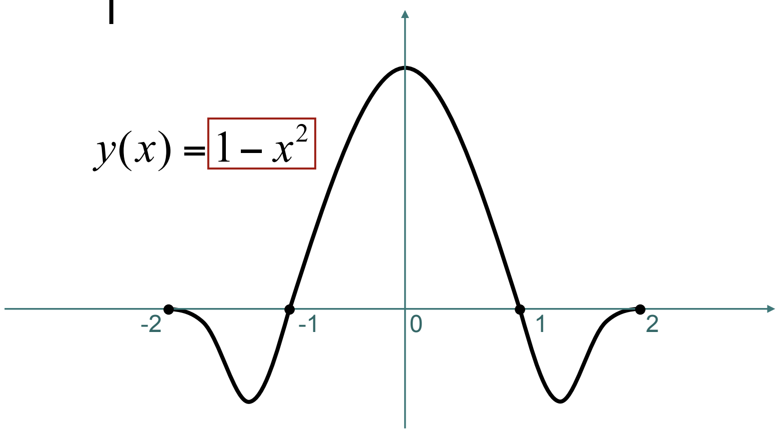
Cubic Spline (C-Spline) Filter

$$y(x) = \boxed{1 - x^2}$$



$$y(x) = 8 - 16x + 10x^2 - 2x^3 = \boxed{-2(x-1)(x-2)^2}$$

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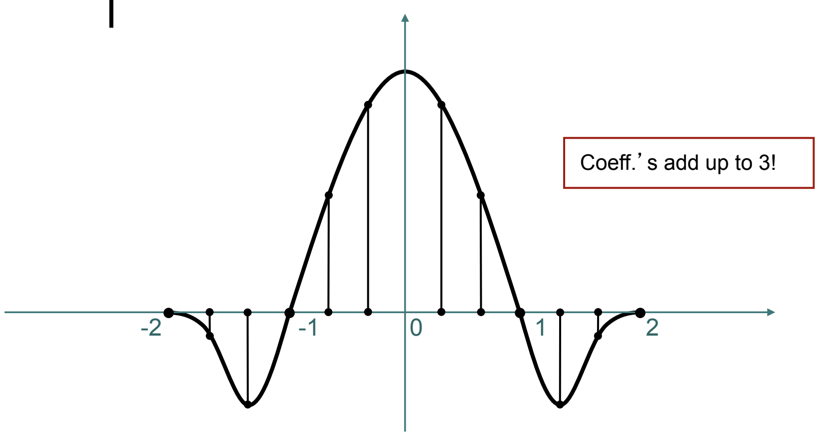
3:1 C-Spline Filter Coefficients



$$h[n] = \left\{ -\frac{4}{27}, -\frac{8}{27}, 0, \frac{5}{9}, \frac{8}{9}, 1, \frac{8}{9}, \frac{5}{9}, 0, -\frac{8}{27}, -\frac{4}{27} \right\}$$

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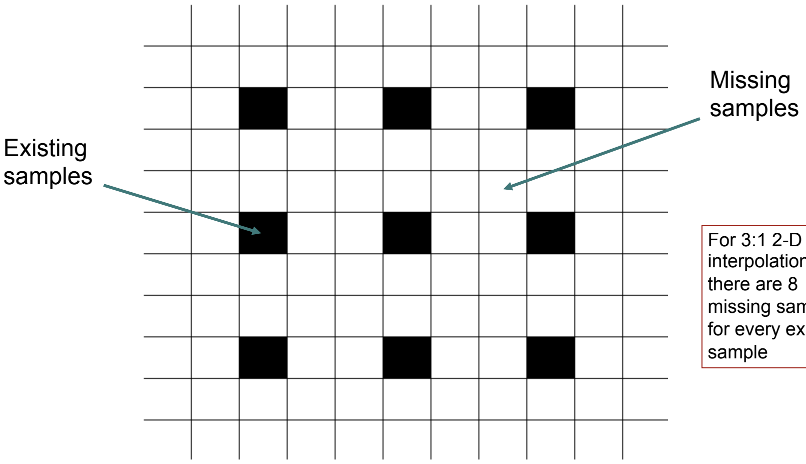
2-D Interpolation



For 3:1 2-D interpolation, there are 8 missing samples for every existing sample

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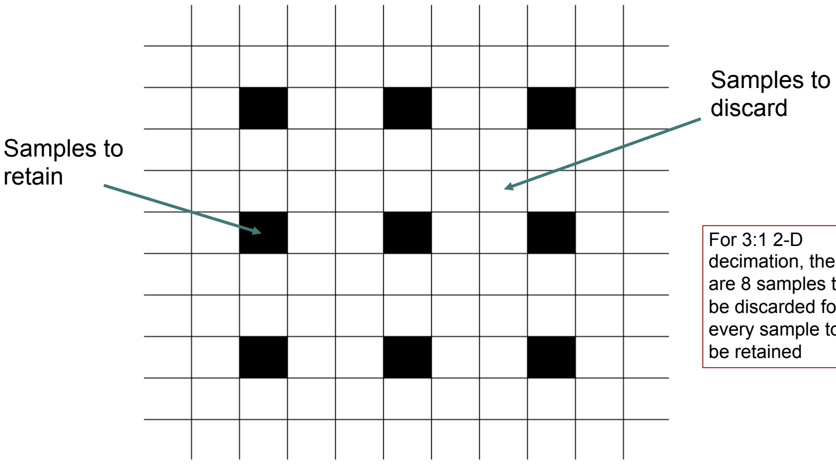
2-D Decimation



For 3:1 2-D decimation, there are 8 samples to be discarded for every sample to be retained

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3:1 2-D C-Spline Filter Coeff.’s

$$h[m] = \left\{-\frac{4}{27}, -\frac{8}{27}, 0, \frac{5}{9}, \frac{8}{9}, 1, \frac{8}{9}, \frac{5}{9}, 0, -\frac{8}{27}, -\frac{4}{27}\right\}$$

$$h[m,n] = h[m]h[n],$$
$$m,n = -5,\cdots,0,\cdots,5$$

2-D Filter is Separable!

	-0,15	-0,30	0,00	0,56	0,89	1,00	0,89	0,56	0,00	-0,30	-0,15	
-0,15	0,02	0,04	0,00	-0,08	-0,13	-0,15	-0,13	-0,08	0,00	0,04	0,02	-5
-0,30	0,04	0,09	0,00	-0,16	-0,26	-0,30	-0,26	-0,16	0,00	0,09	0,04	-4
0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	-3
0,56	-0,08	-0,16	0,00	0,31	0,49	0,56	0,49	0,31	0,00	-0,16	-0,08	-2
0,89	-0,13	-0,26	0,00	0,49	0,79	0,89	0,79	0,49	0,00	-0,26	-0,13	-1
1,00	-0,15	-0,30	0,00	0,56	0,89	1,00	0,89	0,56	0,00	-0,30	-0,15	0
0,89	-0,13	-0,26	0,00	0,49	0,79	0,89	0,79	0,49	0,00	-0,26	-0,13	1
0,56	-0,08	-0,16	0,00	0,31	0,49	0,56	0,49	0,31	0,00	-0,16	-0,08	2
0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	3
-0,30	0,04	0,09	0,00	-0,16	-0,26	-0,30	-0,26	-0,16	0,00	0,09	0,04	4
-0,15	0,02	0,04	0,00	-0,08	-0,13	-0,15	-0,13	-0,08	0,00	0,04	0,02	5
	-5	-4	-3	-2	-1	0	1	2	3	4	5	

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Implementation of a Separable 2-D Filter

$$y[m,n] = \sum_i \sum_j h[m-i]h[n-j]x[i,j]$$
$$= \sum_j h[n-j] \underbrace{\sum_i h[m-i]x[i,j]}_{} \longleftarrow \text{rows}$$
$$= \sum_j h[n-j]\bar{x}[m,j] \longleftarrow \text{columns}$$

Two 1-D convolution computations: $2N \log N$
Rather than one 2-D convolution: $N^2 \log N^2$

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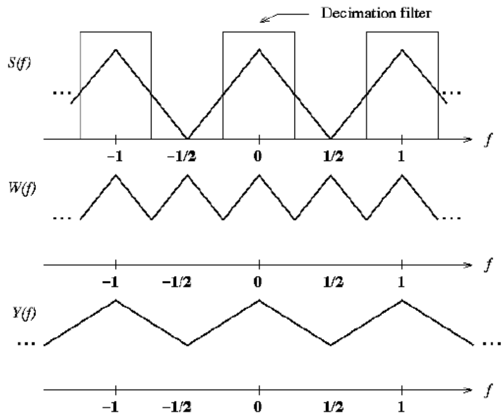
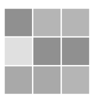
1-D Decimation: LP Filtering

- To avoid aliasing, the signal is low pass filtered before decimation.

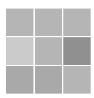


Antialias filtering for M=2

2:1 Decimation Example

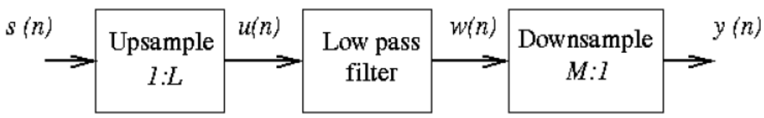


without prefiltering
before downsampling



with prefiltering
before downsampling

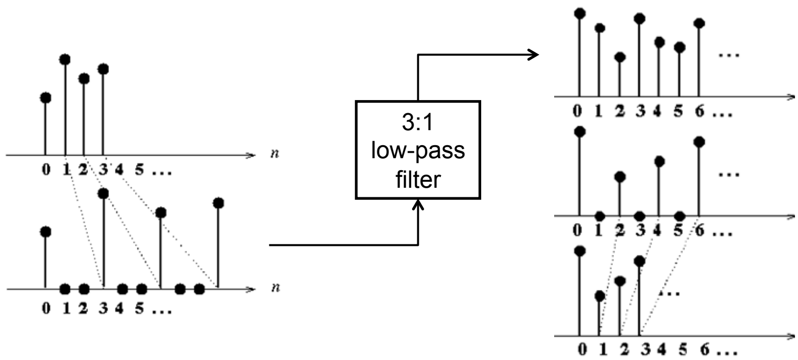
Sampling Rate Change by a Fraction



Rate change by a factor of L/M.

- A single lowpass filter with cutoff frequency $f_c = \min\{ 1/2M, 1/2L \}$ is sufficient.
- When L > M, the requirement to preserve the values of existing samples must be incorporated into the filter design.

Sampling conversion by 3/2



3:1 upsampling

2:1 downsampling

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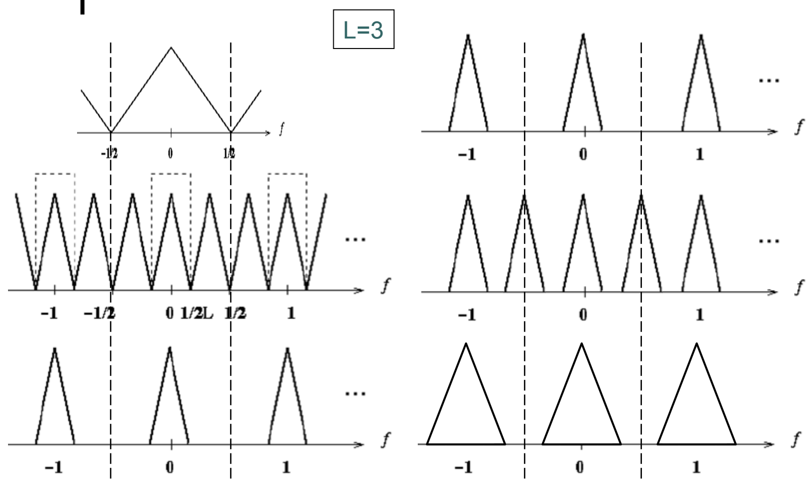
Sampling conversion by **3/2**



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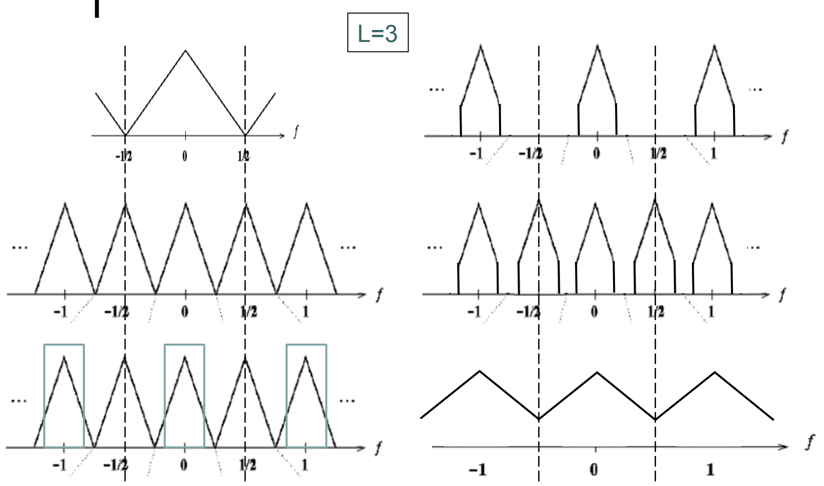
Sampling conversion by **2/3**



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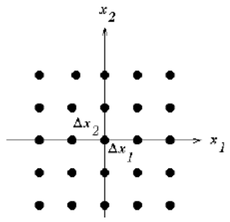
Sampling Structure (Lattice)

Rectangular Sampling



- 2D rectangular still image: $s_c(x_1, x_2)$
- Sample at locations: $x_1 = n_1 \Delta x_1$

  $x_2 = n_2 \Delta x_2$
- Sampled signal in dicrete coordinates:

$$s(n_1, n_2) = s_c(n_1 \Delta x_1, n_2 \Delta x_2)$$

- Sampled signal in continuous coordinates:

$$s_p(x_1, x_2) = s_c(x_1, x_2) \sum_{n_1} \sum_{n_2} \delta(x_1 - n_1 \Delta x_1, x_2 - n_2 \Delta x_2)$$
$$= \sum_{n_1} \sum_{n_2} s(n_1, n_2) \delta(x_1 - n_1 \Delta x_1, x_2 - n_2 \Delta x_2)$$

Rectangular Sampling

- Rectangular grid is separable. Let the horizontal and vertical sampling intervals be Δx_1 and Δx_2, respectively.

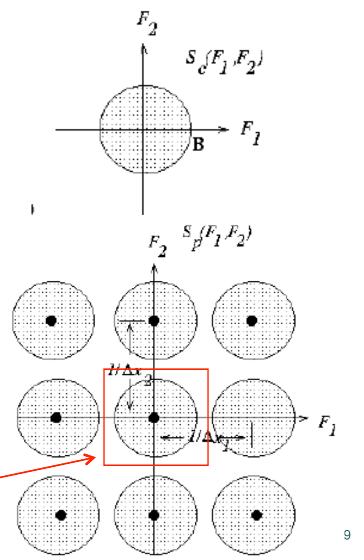
$$\boxed{S_p(F_1,F_2)=\frac{1}{\Delta x_1 \Delta x_2}\sum_{k_1}\sum_{k_2} S_c\left(F_1-\frac{k_1}{\Delta x_1},F_2-\frac{k_2}{\Delta x_2}\right)}$$

The spectrum of the sampled image is periodic with the horizontal and vertical periods $1/\Delta x_1$ and $1/\Delta x_2$.

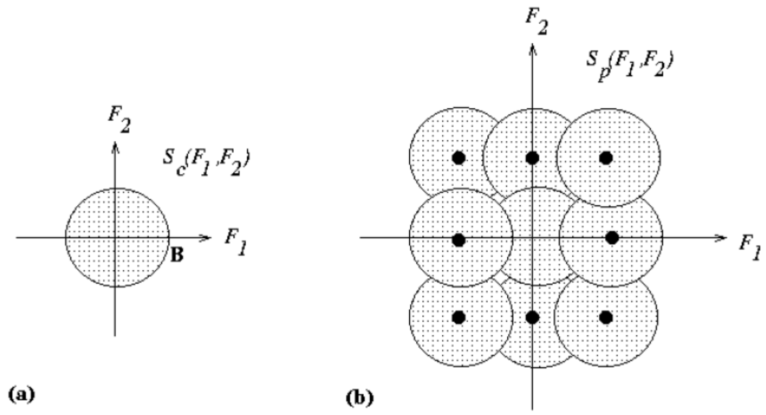
Rectangular Sampling

- As a result of sampling, the Fourier spectrum replicates in the 2D frequency plane.
- There is no aliasing if the spectrum of the original image is within the rectangle given as

$$\left[-\frac{1}{2\Delta x_1},\frac{1}{2\Delta x_1}\right]\times\left[-\frac{1}{2\Delta x_2},\frac{1}{2\Delta x_2}\right]$$



Aliasing in 2-D



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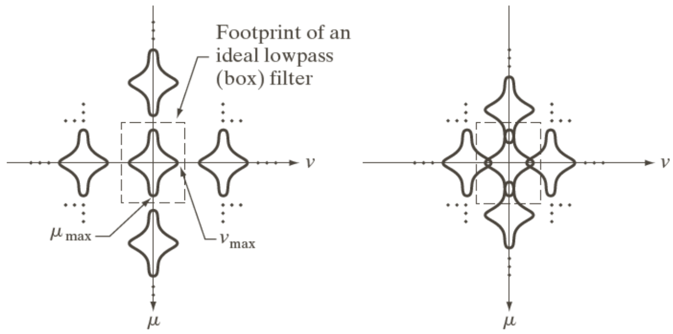
Low-Pass Filtering in 2-D



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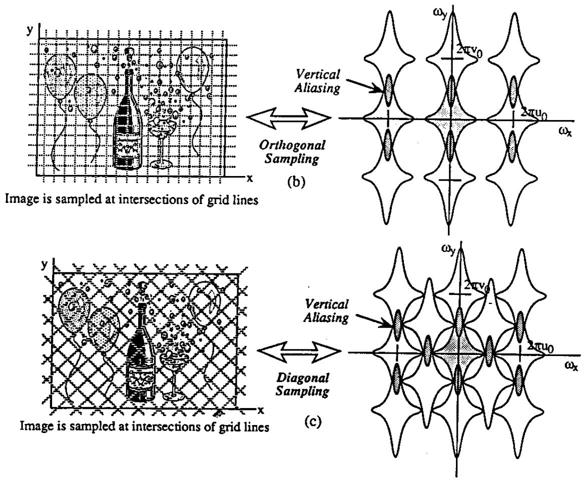
Rectangular vs. Non-Rectangular Sampling



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Non-Rectangular Sampling

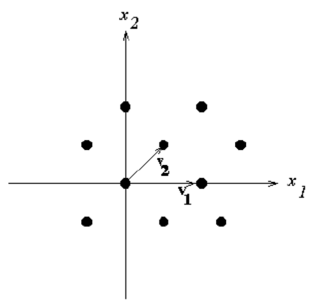
- Define 2 basis vectors

$$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$$

- Such that

Sampling locations $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = V \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}}_{\text{Sampling matrix}}$$



Arbitrary periodic sampling geometry.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{v}_1 m_1 + \vec{v}_2 m_2$$

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Sampling Lattice

- A lattice Λ^M in $\mathbf{R}^M$ is the set of all linear combinations of M linearly independent vectors in $\mathbf{R}^M$.

$$\Lambda^M = \{ n_1 \mathbf{v}_1 + n_2 \mathbf{v}_2 + \cdots + n_M \mathbf{v}_M = \mathbf{V}\mathbf{n} \mid n_1, n_2, \cdots, n_M \in \mathbf{Z} \}$$

- $\mathbf{V} = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \ldots \mid \mathbf{v}_M]$ is called the sampling matrix and $|\det(\mathbf{V})|$ is the reciprocal of the sampling density.
- The sampling matrix $\mathbf{V}$ is not unique for a given sampling grid, but $\det(\mathbf{V})$ is.
- The sampled signal is given by

$$\begin{aligned} s(\mathbf{n}) &= s_c(\mathbf{V}\mathbf{n}), \ \mathbf{n} \in \mathbf{Z}^M \\ &= s_c(\mathbf{x}), \quad \mathbf{x} \in \Lambda^M \end{aligned}$$

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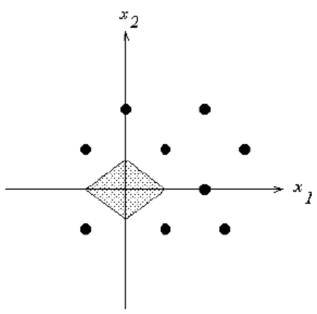
Reciprocal Lattice

- The set of all vectors $\mathbf{r}$ such that $\mathbf{r}^T \mathbf{x}$ is an integer for all $\mathbf{x} \in \Lambda$ is called the reciprocal lattice Λ^*.
- A basis for Λ^* is the set of vectors $u_1, u_2, \ldots, u_M$ such that $u_i^T v_j = \delta_{ij}$, $i,j = 1,2,\ldots, M$ or $\mathbf{U}^T \ \mathbf{V} = \mathbf{I}$
- Unit Cell (Voronoi cell)
  The set of points that are closer to the origin than to any other sample point.



Spectrum of the Sampled Signal

- Spectrum of the sampled signal is equal to an infinite sum of copies of the analog spectrum shifted according to the reciprocal lattice Λ^*

$$S_p(\mathbf{F}) = \frac{1}{|\det(\mathbf{V})|}\sum_{\mathbf{k}} S_c(\mathbf{F}-\mathbf{U}\mathbf{k})$$

where the periodicity matrix $\mathbf{U} = [\,u_1 \mid u_2 \mid \ldots \mid u_M\,]$ satisfies $\mathbf{U}^T\mathbf{V} = \mathbf{I}$ and $u_1, u_2, \ldots, u_M$ are called periodicity vectors.

- These expressions are also valid for rectangular sampling with **V** and **U** being diagonal matrices.

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Hence, in Fourier Domain…

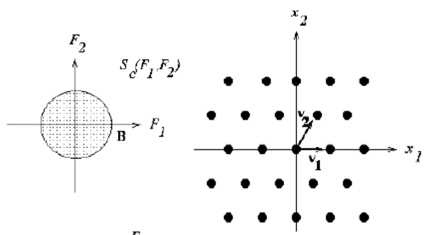
$$S_p(\vec{F}) = \frac{1}{|\det V|}\sum_{k} S_c(\vec{F}-U\vec{k}))$$

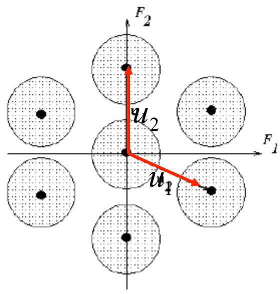
$$U^T V = I$$

$$[\vec{u}_1 \quad \vec{u}_2]^T[\vec{v}_1 \quad \vec{v}_2] = 1$$

$$\Rightarrow \vec{u}_1^T\vec{v}_2 = 0, \vec{u}_2^T\vec{v}_1 = 0$$

$$\Rightarrow \vec{u}_1 \perp \vec{v}_2, \vec{u}_2 \perp \vec{v}_1$$





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Nyquist Sampling

- Sampling operation does not result in loss of information if the analog signal spectrum $S_c(\mathbf{F})=0$ outside the unit cell of Λ^{M*}.
- Given spectral support $S_c(\mathbf{F})$ of a bandlimited analog signal, select a lattice Λ^{M} or equivalently a sampling matrix $\mathbf{V}$.
- Given a sampling lattice Λ^{M}, design an anti-alias filter that confines the spectrum $S_c(\mathbf{F})$ within the unit cell of Λ^{M*}.

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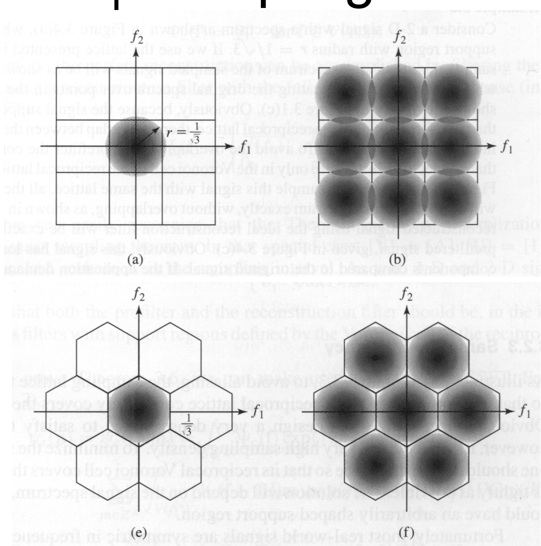
Sampling Efficiency



$$V=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow U=V^{-T}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\text{sampling density}=\frac{1}{|\det(V)|}=1$$

$$V=\begin{bmatrix}\sqrt{3}/2 & 0\\ -1/2 & 1\end{bmatrix}$$

$$U^T=\begin{bmatrix}2/\sqrt{3} & 0\\ 1/\sqrt{3} & 1\end{bmatrix}$$

$$\text{sampling density}=2/\sqrt{3}>1$$

Sampling is more efficient if the Voronoi cell of the reciprocal matrix encloses the signal spectrum as tightly as possible

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Sampling Structure Conversion

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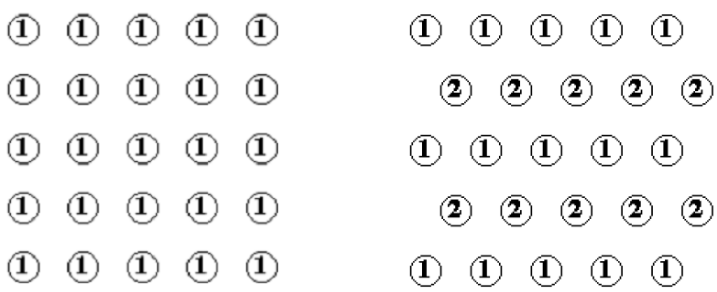
Sampling Conversion Generalized to Lattices



Progressive   Quincunx

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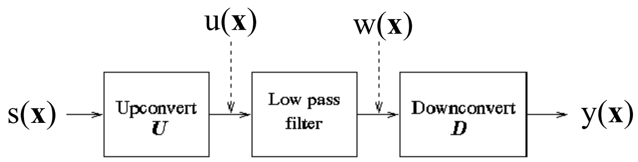
Sampling Conversion Extended to Lattices



Define

$$u_p(\mathbf{x}) = U\, s_p(\mathbf{x}) = \begin{cases} s_p(\mathbf{x}) & \mathbf{x} \in \Lambda_1 \\ 0 & \mathbf{x} \notin \Lambda_1,\ \mathbf{x} \in \Lambda_1 + \Lambda_2 \end{cases}$$

insert 0's

and

$$y_p(\mathbf{x}) = D\, w_p(\mathbf{x}) = w_p(\mathbf{x}) \quad \mathbf{x} \in \Lambda_2$$

remove samples

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Lattice Algebra

- Sum of two lattices

$$\Lambda_1 + \Lambda_2 = \{\, \mathbf{x}_1 + \mathbf{x}_2 \mid \mathbf{x}_1 \in \Lambda_1 \text{ and } \mathbf{x}_2 \in \Lambda_2 \,\}$$

  – lattice for zero filling

- Intersection of two lattices

$$\Lambda_1 \cap \Lambda_2 = \{\, \mathbf{x} \mid \mathbf{x} \in \Lambda_1 \text{ and } \mathbf{x} \in \Lambda_2 \,\}$$

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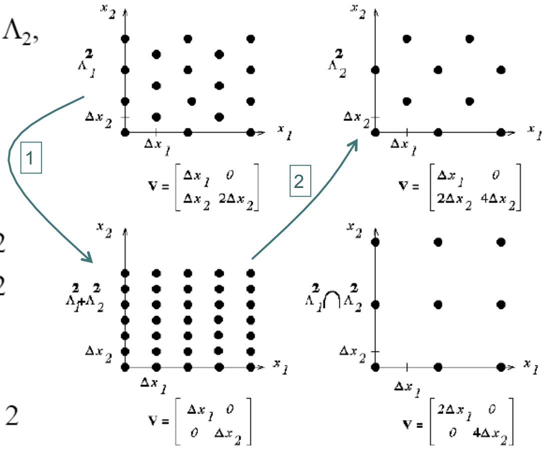
Example: Conversion from Λ_1 to Λ_2

- The lattices Λ_1, Λ_2, $\Lambda_1 + \Lambda_2$ and $\Lambda_1 \cap \Lambda_2$.

$d(\Lambda_1) = 2$ X_1 X_2
$d(\Lambda_2) = 4$ X_1 X_2

$Q = (\Lambda_1 + \Lambda_2 : \Lambda_1)$
$= (\Lambda_2 : \Lambda_1 \cap \Lambda_2) = 2$



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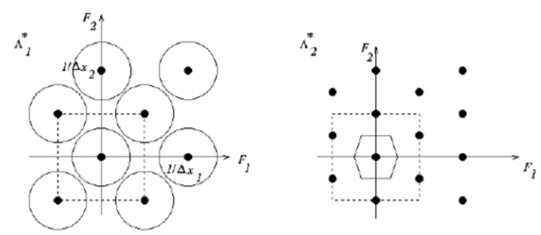
Fourier Spectrum

- One period of the filter frequency response is given by the unit cell of $(\Lambda_1 + \Lambda_2)^*$.
- In order to avoid aliasing, the passband of the lowpass filter is restricted to the Voronoi cell of $(\Lambda_1 + \Lambda_2)^*$



$$U_1 = \begin{bmatrix} 1/\Delta x_1 & -1/2\Delta x_1 \\ 0 & 1/2\Delta x_2 \end{bmatrix} \qquad U_2 = \begin{bmatrix} 1/\Delta x_1 & -1/2\Delta x_1 \\ 0 & 1/4\Delta x_2 \end{bmatrix}$$

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Project 1.6

Resampling

Due 14.11.2013

Project 1.6

1. Select an arbitrary image.
2. Downsample it by a factor of 8.
3. Decimate it by a factor of 8.
4. Decimate it by a factor of 4 and then interpolate by a factor of 4.
5. Interpolate it by a factor of 4 and then decimate by a factor of 4.
6. Compare the images obtained in Steps 2 and 3, and comment on their differences.
7. Calculate the RMSE values between the original image and the images obtained in Steps 4 and 5. Comment on the results.

Next Lecture

- GEOMETRIC TRANSFORMS