University of Diyala
College of Engineering
Dep. of Civil Engineering

Applied Mathematics – I

2^{nd} Stage

Dr. Huda M. Mubarak

Analytic Geometry

Rectangular Coordinates

The points in a plane may be placed in one-to-one correspondence with pairs of real numbers. A common method is to use perpendicular lines that are horizontal and vertical and intersect at a point called the *origin*. These two lines constitute the coordinate axes; the horizontal line is the x-axis and the vertical line is the y-axis. The positive direction of the x-axis is to the right, whereas the positive direction of the y-axis is up.

Thus, point P is associated with the pair of real numbers (x_1, y_1) and is denoted $P(x_1, y_1)$. The coordinate axes divide the plane into quadrants I, II, III, and IV.

Distance between Two Points; Slope

The distance d between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In the special case when P_1 and P_2 are both on one of the coordinate axes, for instance, the x-axis,

$$d = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$$

or on the y-axis,

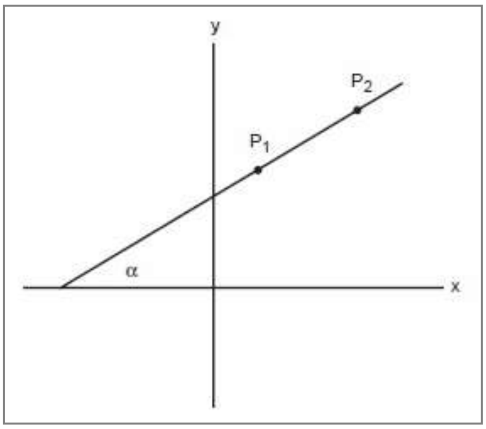
$$d = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$$

The midpoint of the line segment P_1P_2 is

$$\left(\frac{x_1 + x_1}{2}, \frac{y_1 + y_2}{2}\right)$$

The slope of the line segment P_1P_2, provided it is not vertical, is denoted by m and is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



The slope is related to the angle of inclination α by

$$m = \tan\alpha$$

Two lines (or line segments) with slopes m_1 and m_2 are perpendicular if $\boldsymbol{m_1 = -1/m_2}$ and are parallel if $\boldsymbol{m_1 = m_2}$.

Equations of Straight Lines

A vertical line has an equation of the form

$$x = c$$

where (c, 0) is its intersection with the x-axis. A line of slope m through point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1)$$

Thus, a *horizontal line* (slope = 0) through point (x_1, y_1) is given by

$$y = y_1$$

A nonvertical line through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by either

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Or

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

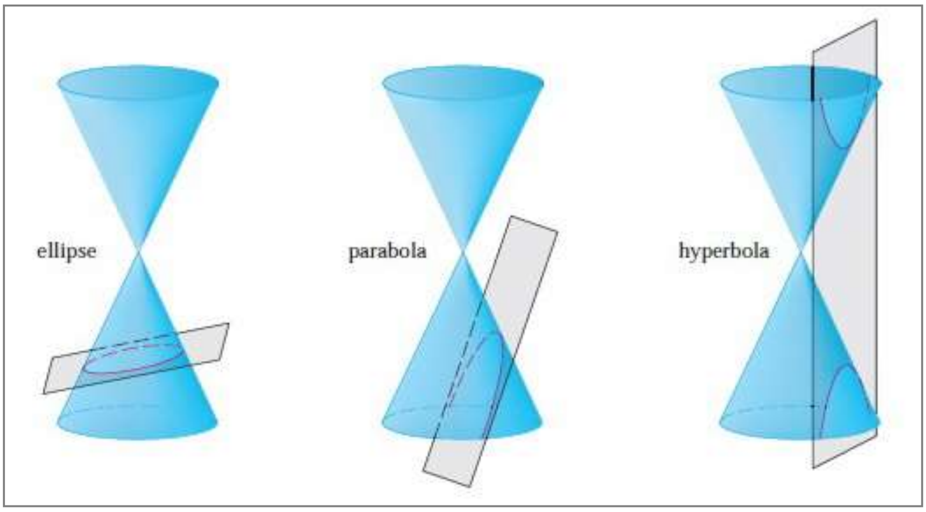
Circle

The general equation of a circle of radius r and center at P(x_1, y_1) is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

Conic Sections

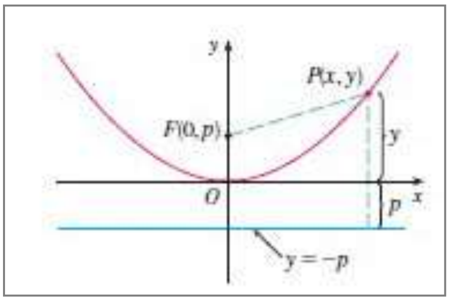
The conic sections are called conics because they result from intersecting a cone with a plane as shown in figure



Parabola

A parabola is the set of all points (x, y) in the plane that are equidistant from a given line called the directrix and a given point called the focus. The parabola is symmetric about a line that contains the focus and is perpendicular to the directrix. The line of symmetry intersects the parabola at its vertex. The eccentricity e = 1.

We obtain a particularly simple equation for a parabola if we place its vertex at the origin *O* and its directrix parallel to the -axis as in Figure below. If the focus is the point (0, p), then the directrix has the equation $y = -p$. If $P(x,y)$ is any point on the parabola, then the distance from P to the focus is



$$|PF| = \sqrt{x^2 + (y-p)^2}$$

and the distance from P to the directrix is $|y+p|$

The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y-p)^2} = |y+p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y-p)^2 = |y+p|^2 = (y+p)^2$$

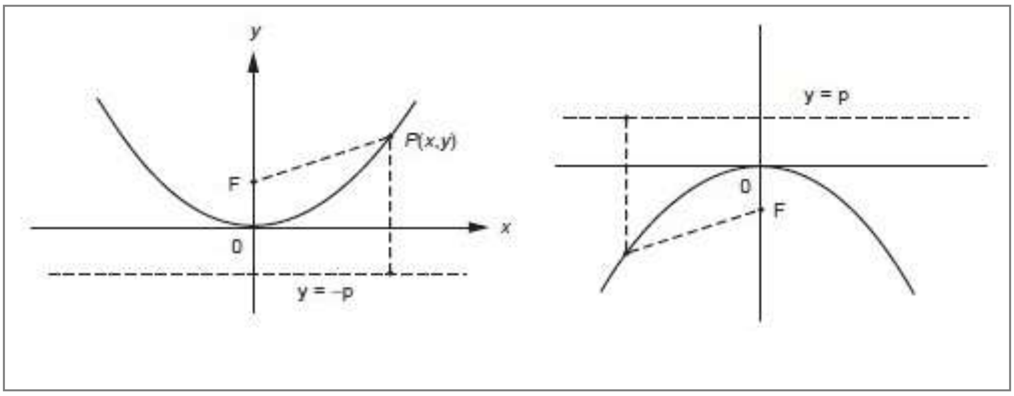
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

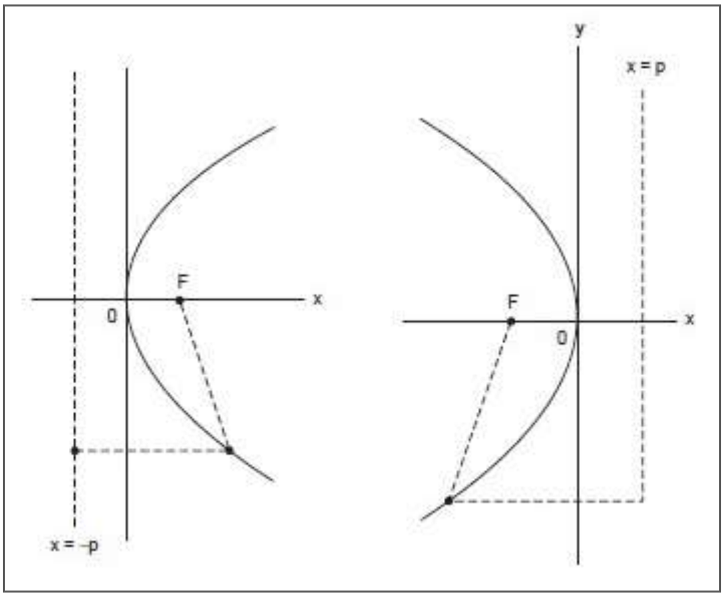
An equation of the parabola with focus $(0,p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

The distance between the focus and the vertex, or vertex and directrix, is denoted by p (> 0) and leads to one of the following equations of a parabola with vertex at the origin.



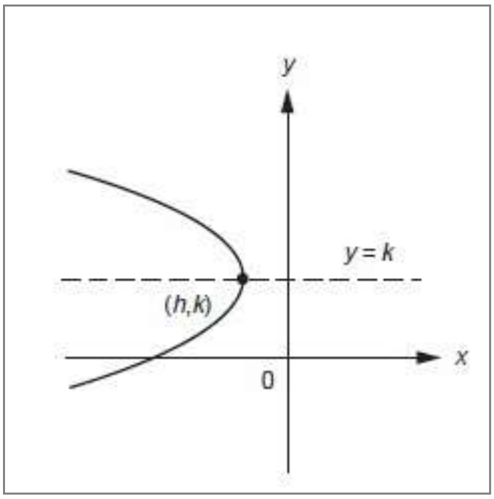
$x^2 = 4py\ , p > 0$ $\quad\quad$ $x^2 = 4py\ , p < 0$



$y^2 = 4px\ , p > 0$ $\quad\quad$ $y^2 = 4px\ , p < 0$

For each of the four orientations shown in Figures, the corresponding parabola with vertex (h, k) is obtained by replacing x by $x - h$ and y by $y - k$. Thus, the parabola in Figure below has the equation

$$(y-k)^2 = -4p(x-h)$$



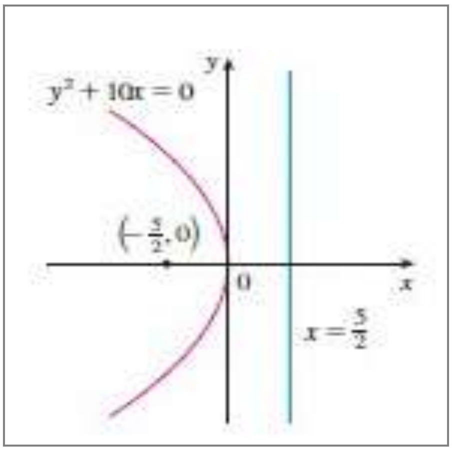
Example

Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

Solution

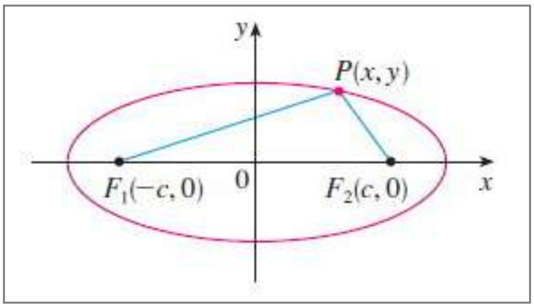
If we write the equation as $y^2 = -10x$ and compare it with Equation, we see that $4p = -10$, so $p = -5/2$. Thus the focus is $(p,0) = (-5/2, 0)$ and the directrix is x = 5/2 .

The sketch is shown in Figure below



Ellipse

An ellipse is the set of all points in the plane such that the sum of their distances from two fixed points, called *foci,* is a given constant $2a$. The distance between the foci is denoted $2c$; the length of the major axis is $2a$, whereas the length of the minor axis is $2b$. The eccentricity of an ellipse, e, is < 1.

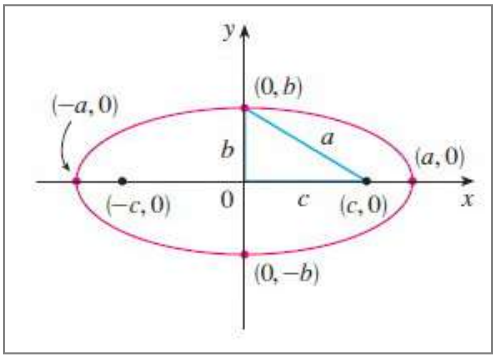


The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \mp a$. The corresponding points (a,0) and (-a,0) are called the vertices of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \mp b$. The line segment joining (0,b) and (0,-b) is the **minor axis**.

We summarize this discussion as follows

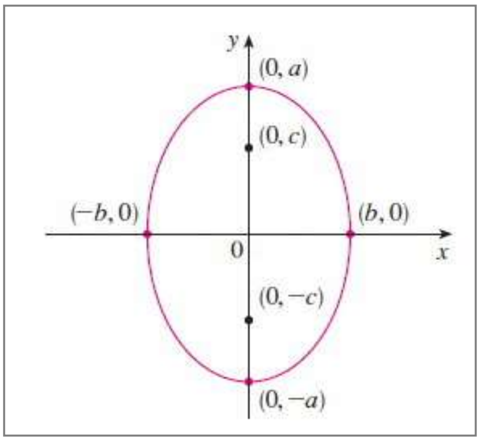
1- The ellipse has **foci** $(\mp \boldsymbol{c}, \mathbf{0})$, $\boldsymbol{where\ c^2 = a^2 - b^2, and\ vertices}$ $(\mp \boldsymbol{a}, \mathbf{0})$

$$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} = \mathbf{1} \qquad \boldsymbol{a} \geq \boldsymbol{b} > 0$$



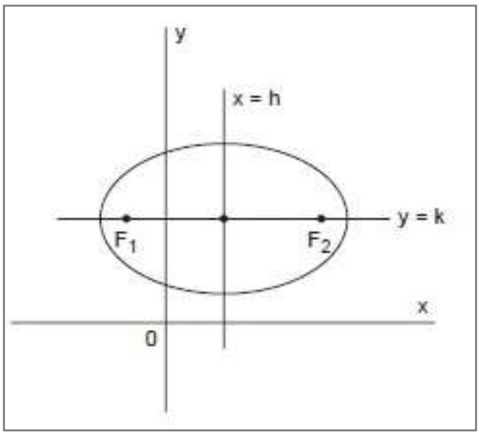
2- The ellipse has **foci** $(\mathbf{0}, \mp \boldsymbol{c})$, $\boldsymbol{where\ c^2 = a^2 - b^2, and\ vertices}$ $(\mathbf{0}, \mp \boldsymbol{a})$

$$\frac{\boldsymbol{x^2}}{\boldsymbol{b^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{a^2}} = \mathbf{1} \qquad \boldsymbol{a} \geq \boldsymbol{b} > 0$$



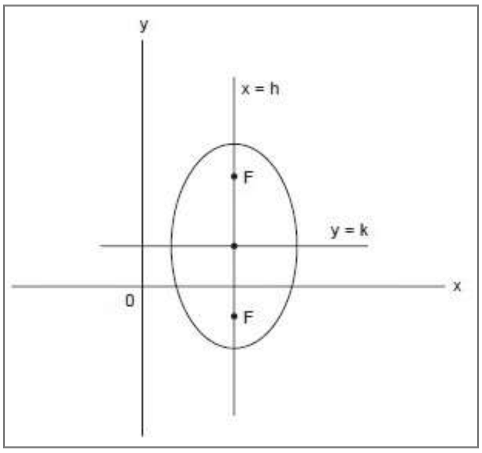
An ellipse with center at point (h, k) and major axis *parallel to the x-axis* is given by the equation

$$\frac{(\boldsymbol{x} - \boldsymbol{h})^2}{\boldsymbol{a^2}} + \frac{(\boldsymbol{y} - \boldsymbol{k})^2}{\boldsymbol{b^2}} = \mathbf{1}$$



An ellipse with center at (h, k) and major axis parallel to the y-axis is given by the equation

$$\frac{(\boldsymbol{y}-\boldsymbol{k})^2}{\boldsymbol{a}^2}+\frac{(\boldsymbol{x}-\boldsymbol{h})^2}{\boldsymbol{b}^2}=\mathbf{1}$$

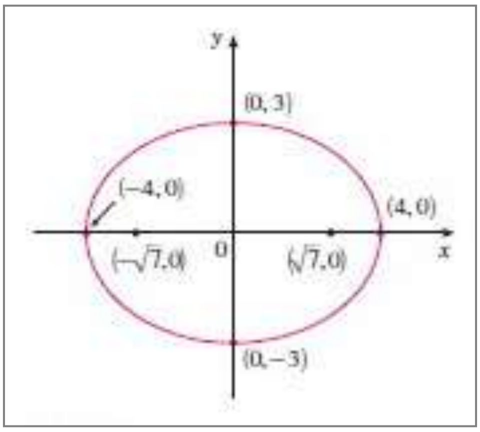


Example Sketch the graph of $9x^2+16y^2=144$ and locate the foci.

Solution Divide both sides of the equation by 144:

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16, b^2 = 9, a = 4,$ *and* $b = 3$. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$, the foci are $(\mp\sqrt{7},0)$. The graph is sketched as



Example Find an equation of the ellipse with foci $(0,\mp2)$ and vertices $(0,\mp3)$.

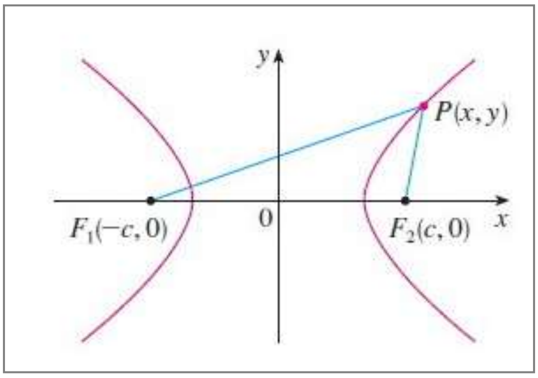
Solution we have $c = 2$ and $a = 3$. Then we obtain $b^2=\mathrm{a}^2-\mathrm{c}^2=9-4=5$, so an equation of the ellipse is

$$\frac{x^2}{5}+\frac{y^2}{9}=1$$

Another way of writing the equation is $9x^2+5y^2=45$.

Hyperbola

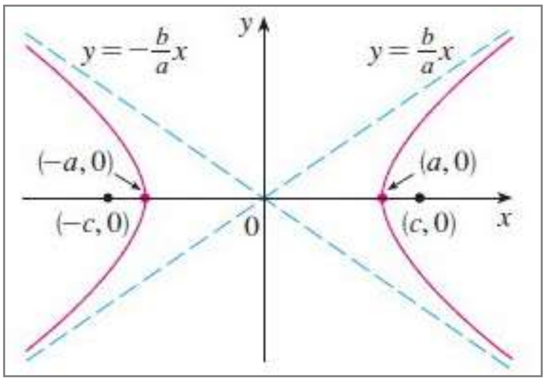
A hyperbola is the set of all points in the plane such that the difference of its distances from two fixed points (foci) is a given positive constant denoted $2a$. The distance between the two foci is $2c$ and that between the two vertices is $2a$. The eccentricity of a hyperbola is (e > 1).



When we draw a hyperbola it is useful to first draw its asymptotes, which are the dashed lines in figures below

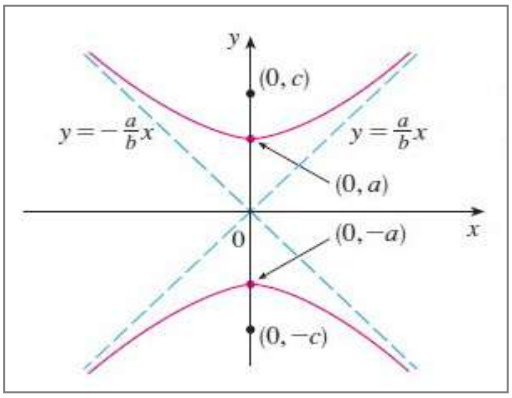
1- The Hyperbola has **foci** $(\mp \boldsymbol{c}, \mathbf{0}), \boldsymbol{where}\ \boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2, \boldsymbol{vertices}\ (\mp \boldsymbol{a}, \mathbf{0}), \boldsymbol{and\ asymptotes}\ \boldsymbol{y} = \mp \left(\frac{\boldsymbol{b}}{\boldsymbol{a}}\right) \boldsymbol{x}$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



2- The Hyperbola has **foci** $(\mathbf{0}, \mp \boldsymbol{c}), \boldsymbol{where}\ \boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2, \boldsymbol{vertices}\ (\mathbf{0}, \mp \boldsymbol{a}), \boldsymbol{and\ asymptotes}\ \boldsymbol{y} = \mp \left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right) \boldsymbol{x}$

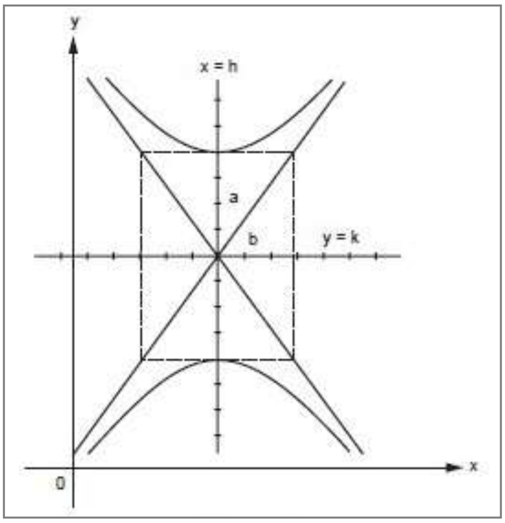
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

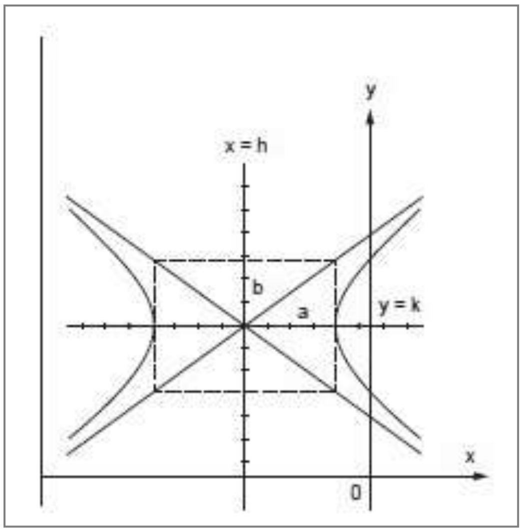


When the focal axis is parallel to the *y*-axis, the equation of the hyperbola with center (*h*, *k*)

$$\frac{(\boldsymbol{y}-\boldsymbol{k})^2}{\boldsymbol{a}^2}-\frac{(\boldsymbol{x}-\boldsymbol{h})^2}{\boldsymbol{b}^2}=\mathbf{1}$$

If the focal axis is parallel to the *x*-axis and center (*h*, *k*), then

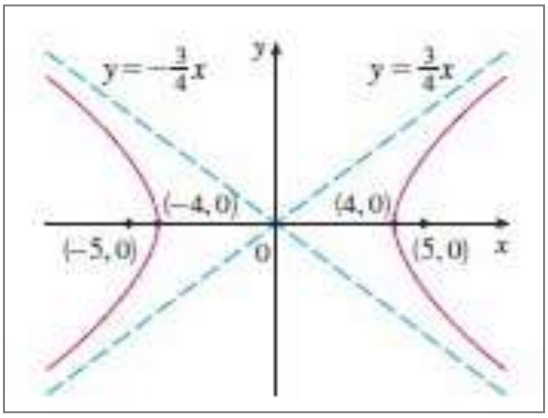




Example Find the foci and asymptotes of the hyperbola $9x^2-16y^2=144$ and sketch its graph.
Solution
If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

which is of the form with *a* = *4* and *b* = *3*. Since $c^2 = 16 + 9 = 25$, so *c* = 5, the foci are $(\mp 5,0)$. The asymptotes are the lines $y=\frac{3}{4}\,x$ and $y=-\frac{3}{4}\,x$. The graph is shown in Figure



Example Find the foci and equation of the hyperbola with vertices $(0,\mp 1)$ and asymptote *y* = *2x*

Solution we see that *a* = *1* and *b* = *2*. Thus *b* = *a/2* = *1/2* and $c^2 = a^2 + b^2 = 5/4$. The foci are $(0,\mp\sqrt{5}/2)$ and the equation of the hyperbola is $y^2-4x^2=1$

Translated conics

1- Parabolas with vertex (h, k) and axis parallel to x-axis

$$(y-k)^2 = 4p(x-h) \qquad \text{[Opens right]}$$

$$(y-k)^2 = -4p(x-h) \qquad \text{[Opens left]}$$

2- Parabolas with vertex (h, k) and axis parallel to y-axis

$$(x-h)^2 = 4p(y-k) \qquad \text{[Opens up]}$$

$$(x-h)^2 = -4p(y-k) \qquad \text{[Opens down]}$$

3- Ellipse with center (h, k) and major axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad [b < a]$$

4- Ellipse with center (h, k) and major axis parallel to y-axis

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1 \qquad [b < a]$$

5- Hyperbola with center (h, k) and focal axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

6- Hyperbola with center (h, k) and focal axis parallel to y-axis

<u>Example</u> Find an equation for the parabola that has its vertex at $(1, 2)$ and its focus at $(4, 2)$.

<u>Solution</u> Since the focus and vertex are on a horizontal line, and since the focus is to the right of the vertex, the parabola opens to the right and its equation has the form

$$(y-k)^2 = 4p(x-h)$$

Since the vertex and focus are 3 units apart, we have $p = 3$, and since the vertex is at $(h, k) = (1, 2)$, we obtain

$$(y-2)^2 = 12(x-1)$$

<u>Example</u> Describe the graph of the equation

$$y^2 - 8x - 6y - 23 = 0$$

<u>Solution</u> The equation involves quadratic terms in y but none in x, so we first take all of the y-terms to one side:

$$y^2 - 6y = 8x + 23$$

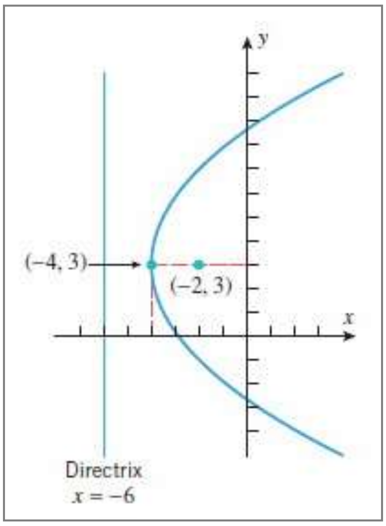
Next, we complete the square on the y-terms by adding 9 to both sides:

$$(y-3)^2 = 8x + 32$$

Finally, we factor out the coefficient of the x-term to obtain

$$(y-3)^2 = 8(x+4)$$

This equation is with $h = -4$, $k = 3$, and $p = 2$, so the graph is a parabola with vertex $(-4, 3)$ opening to the right. Since $p = 2$, the focus is 2 units to the right of the vertex, which places it at the point $(-2, 3)$; and the directrix is 2 units to the left of the vertex, which means that its equation is $x = -6$.



Example Describe the graph of the equation

$$16x^2 + 9y^2 - 64x - 54y + 1 = 0$$

Solution This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(16x^2 - 64x) + (9y^2 - 54y) = -1$$

Next, factor out the coefficients of x^2 and y^2 and complete the squares:

$$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$$

or

$$16(x - 2)^2 + 9(y - 3)^2 = 144$$

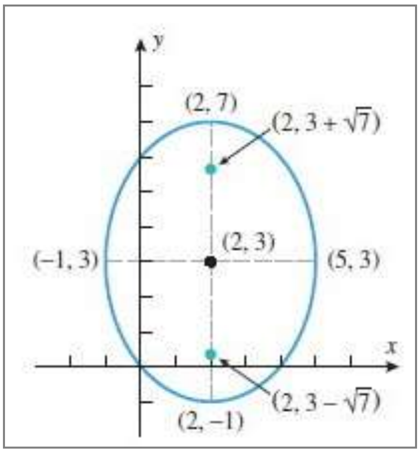
Finally, divide through by 144 to introduce a 1 on the right side:

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$

This is an equation with $h = 2$, $k = 3$, $a^2 = 16$, and $b^2 = 9$. Thus, the graph of the equation is an ellipse with center $(2, 3)$ and major axis parallel to the y-axis. Since $a = 4$, the major axis extends 4 units above and 4 units below the center, so its endpoints are $(2, 7)$ and $(2,-1)$. Since $b = 3$, the minor axis extends 3 units to the left and 3 units to the right of the center, so its endpoints are $(-1, 3)$ and $(5, 3)$. Since

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

the foci lie $\sqrt{7}$ units above and below the center, placing them at the points $(2, 3 + \sqrt{7})$ and $(2, 3 - \sqrt{7})$



Example Describe the graph of the equation

$$x^2 - y^2 - 4x + 8y - 21 = 0$$

Solution This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(x^2 - 4x) - (y^2 - 8y) = 21$$

by completing the squares that this equation can be written as $\dfrac{(x-2)^2}{9} - \dfrac{(y-4)^2}{9} = 1$

This is an equation with $h = 2$, $k = 4$, $a^2 = 9$, and $b^2 = 9$. Thus, the equation represents a hyperbola with center $(2, 4)$ and focal axis parallel to the x-axis. Since $a = 3$, the vertices are located 3 units to the left and 3 units to the right of the center, or at the points $(-1, 4)$ and $(5, 4)$. $c = 3\sqrt{2}$, so the foci are located $3\sqrt{2}$ units to the left and right of the center, or at the points $(2 - 3\sqrt{2}, 4)$ and $(2 + 3\sqrt{2}, 4)$

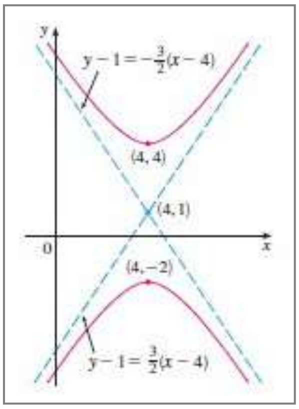
The equations of the asymptotes may be found

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 0$$

This can be written as $y - 4 = \pm(x - 2)$, which yields the asymptotes

$$y = x + 2 \quad \text{and} \quad y = -x + 6$$

With the aid of a box extending $a = 3$ units left and right of the center and $b = 3$ units above and below the center,



HYPERBOLIC FUNCTIONS

Definitions of Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. The function e^x can be expressed in the following way as the sum of an even function and an odd function:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{Even}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{Odd}}$$

These functions are sufficiently important that there are names and notation associated with them: the odd function is called the hyperbolic sine of x and the even function is called the hyperbolic cosine of x. They are denoted by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad and \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Where *sinh* is pronounced "cinch" or "shine" and *cosh* rhymes with "gosh." From these two building blocks we can create four more functions to produce the following set of six hyperbolic functions.

Hyperbolic sine $\boldsymbol{sinh\, x = \frac{e^x - e^{-x}}{2}}$

Hyperbolic cosine $\boldsymbol{cosh\, x = \frac{e^x + e^{-x}}{2}}$

Hyperbolic tangent $\boldsymbol{tanh\, x = \frac{sinh}{cosh} = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$

Hyperbolic cotangent $\boldsymbol{coth\, x = \frac{cosh}{sinh} = \frac{e^x + e^{-x}}{e^x - e^{-x}}}$

Hyperbolic secant $\boldsymbol{sech\, x = \frac{1}{cosh\, x} = \frac{2}{e^x + e^{-x}}}$

Hyperbolic cosecant $\boldsymbol{csch\, x = \frac{1}{sinh\, x} = \frac{2}{e^x - e^{-x}}}$

Example

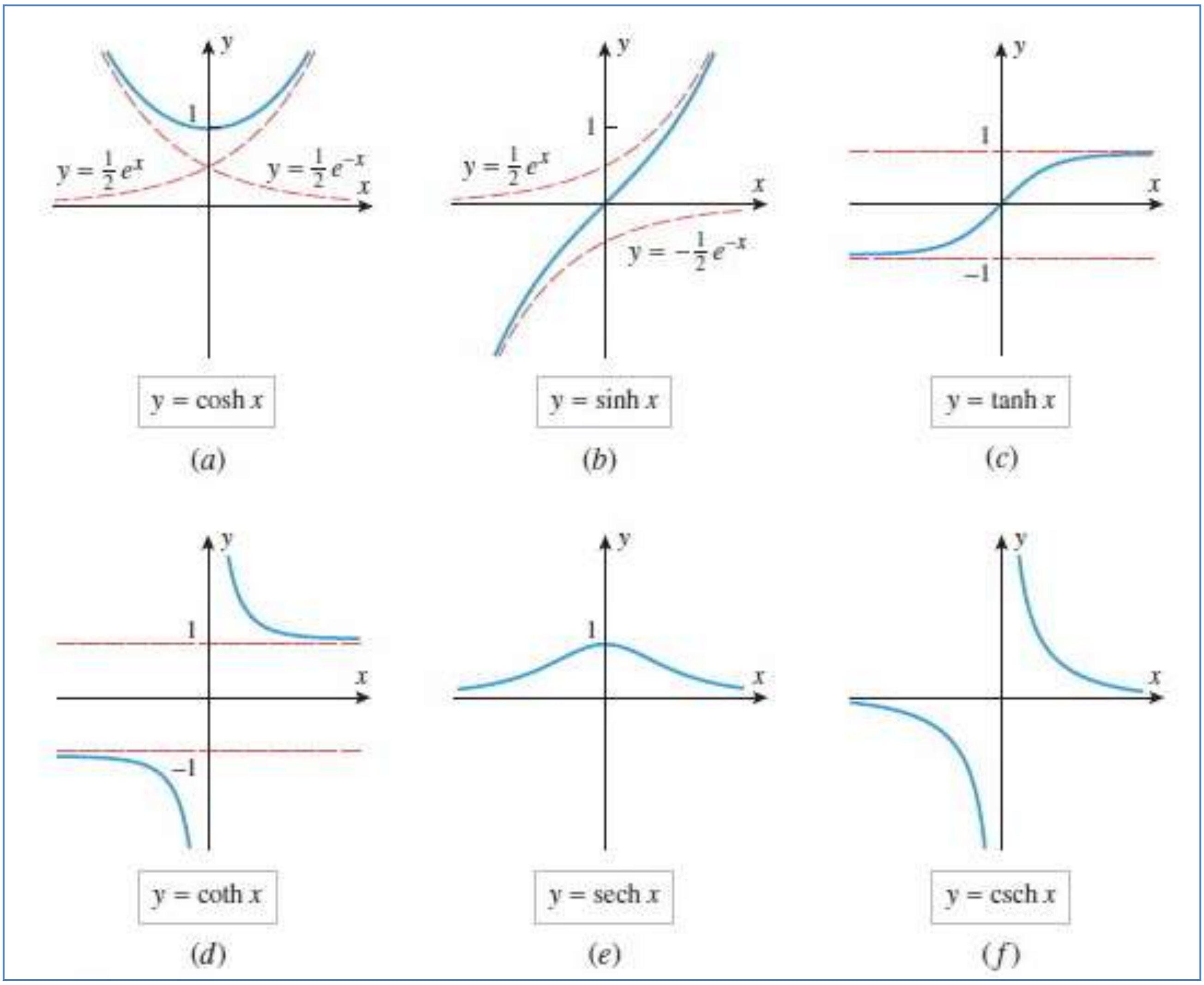
$$\boldsymbol{sinh\, 0 = \frac{e^0 - e^{-0}}{2} = \frac{1 - 1}{2} = 0}$$

$$\boldsymbol{cosh\, 0 = \frac{e^0 + e^{-0}}{2} = \frac{1 - 1}{2} = 1}$$

$$\boldsymbol{sinh\, 2 = \frac{e^2 - e^{-2}}{2} = 3.6269}$$

Note: The term "tanh", "sech", and "csch" are pronounced "tanch", "seech", and "coseech" respectively

Graphs of the Hyperbolic Functions



Observe that sinh x has a domain $(-\infty, +\infty)$ and range of $(-\infty, +\infty)$ where cosh x has a domain of $(-\infty, +\infty)$ and a range of $[1, +\infty)$.

Hyperbolic Identities

The hyperbolic functions satisfy various identities that are similar to identities for trigonometric functions. The most fundamental of these is

$\cosh^2 x - sinh^2 x = 1$ which can proved by writing

$$\cosh^2 x - sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$$
$$= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) = e^x . e^{-x} = 1$$

The following theorem summarizes some of the more useful hyperbolic identities:

$$\cosh x + \sinh x = e^x \qquad \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh x - \sinh x = e^{-x} \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\coth^2 x - 1 = \operatorname{csch}^2 x \qquad \sinh 2x = 2\sinh x \cosh x$$

$$\cosh(-x) = \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh(-x) = -\sinh x \qquad \cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$$

<u>Derivative and Integral Formulas</u>

Derivative formulas for sinh x and cosh x can be obtained by expressing these functions in terms of e^x and e^{-x}:

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \frac{d}{dx}[\sinh x] - \sinh x \frac{d}{dx}[\cosh x]}{\cosh^2 x} = \frac{cosh^2x - sinh^2x}{cosh^2x}$$

$$= \frac{1}{cosh^2x} = sech^2x$$

The following theorem provides a complete list of the generalized derivative formulas and corresponding integration formulas for the hyperbolic functions.

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Example

1- $\frac{d}{dx}[\cosh(x^3)] = \sinh(x^3)\,.\frac{d}{dx}[x^3] = 3x^2\sinh(x^3)$

$2-\dfrac{d}{dx}[\ln(\tanh x)] = \dfrac{1}{\tanh x}\cdot\dfrac{d}{dx}[\tanh x] = \dfrac{sech^2 x}{\tanh x}$

$3-\frac{d}{dx}\left[\cosh\sqrt{x}\right] = \sinh\sqrt{x}\,.\frac{d}{dx}\left[\sqrt{x}\right] = \frac{\sinh\sqrt{x}}{2\sqrt{x}}$

$4-\dfrac{d}{dt}\left[\tanh\sqrt{1+t^2}\right] = \operatorname{sech}^2\sqrt{1+t^2}.\dfrac{d}{dt}\sqrt{1+t^2} = sech^2\sqrt{1+t^2}.\dfrac{t}{\sqrt{1+t^2}} = \dfrac{t}{\sqrt{1+t^2}}.sech^2\sqrt{1+t^2}$

Example

$1-\int sinh^5 x\, cosh\, x\, dx = \dfrac{1}{6}sinh^6 x + C$

$2-\int tanh\, x\, dx = \int \dfrac{\sinh x}{\cosh x}dx = \ln|\cosh x| + C = \ln\cosh x + C$

$$3-\int_0^1 sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2}dx$$

$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 = \frac{\sinh 2}{2} - \frac{1}{2} \approx 0.4067$$

$$4-\int_0^{ln2} 4e^x \sinh x\, dx = \int_0^{ln2} 4e^x\,\frac{e^x - e^{-x}}{2}\, dx$$

$$= \int_0^{ln2}(2e^{2x} - 2)dx = [e^{2x} - 2x]\begin{matrix} ln2 \\ 0\end{matrix} = (e^{2ln2} - 2ln2) - (1-0) = 4 - 2ln2 - 1$$

$$= 3 - 1.386 \approx 1.6137$$

Inverses of Hyperbolic Functions

The graphs of the six inverse hyperbolic functions in Figure below were obtained by reflecting the graphs of the hyperbolic functions (with the appropriate restrictions) about the line $y = x$.

Useful Identities

1- $\mathbf{sech}^{-1}\boldsymbol{x} = \mathbf{cosh}^{-1}\frac{\mathbf{1}}{\boldsymbol{x}}$ **2-** $\mathbf{csch}^{-1}\boldsymbol{x} = \mathbf{sinh}^{-1}\frac{\mathbf{1}}{\boldsymbol{x}}$ **3-** $\mathbf{coth}^{-1}\boldsymbol{x} = \mathbf{tanh}^{-1}\frac{\mathbf{1}}{\boldsymbol{x}}$

Example Prove $\mathbf{coth}^{-1}\boldsymbol{x} = \mathbf{tanh}^{-1}\frac{\mathbf{1}}{\boldsymbol{x}}$

$y = \coth^{-1} x$ $x = \coth y$

$\dfrac{1}{x} = \dfrac{1}{\coth y} = \tanh y$

$y = \tanh^{-1}\dfrac{1}{x}$

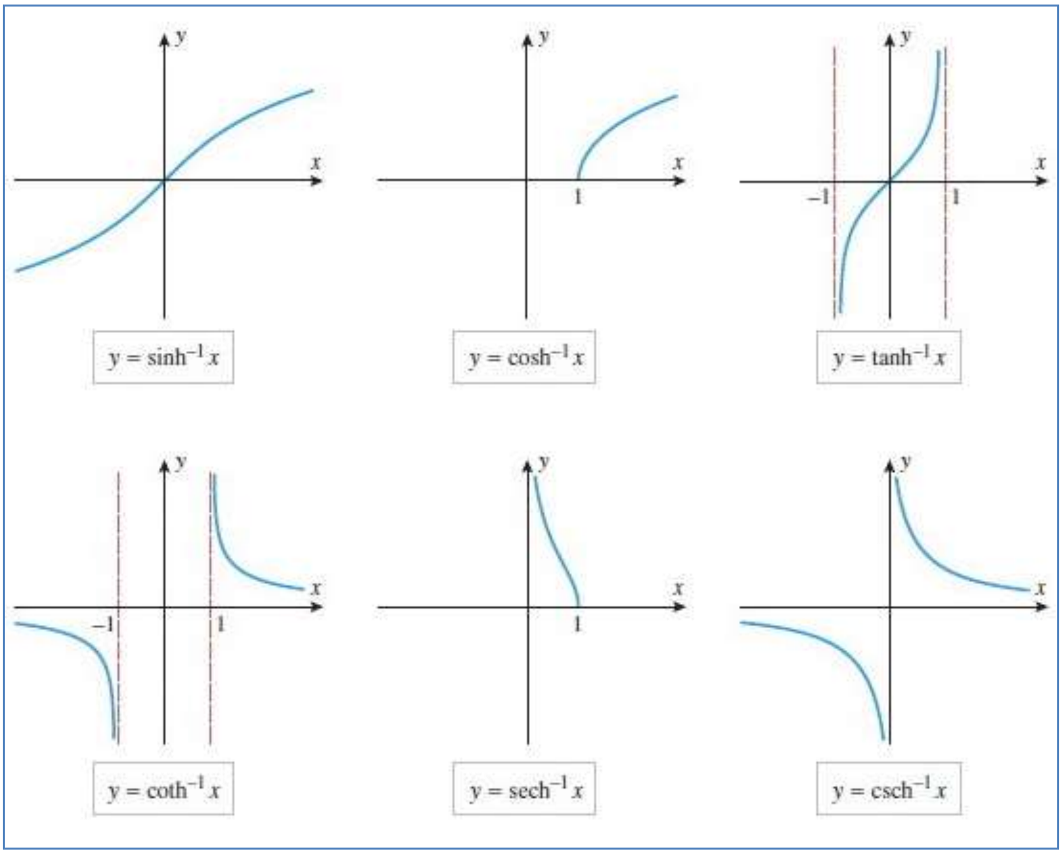


Table below summarizes the basic properties of the inverse Hyperbolic Functions.

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sinh^{-1} x$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < +\infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < +\infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, +\infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \geq 0$ $\cosh(\cosh^{-1} x) = x$ if $x \geq 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, +\infty)$	$\tanh^{-1}(\tanh x) = x$ if $-\infty < x < +\infty$ $\tanh(\tanh^{-1} x) = x$ if $-1 < x < 1$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$ $\coth(\coth^{-1} x) = x$ if $x < -1$ or $x > 1$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = x$ if $x \geq 0$ $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$ if $0 < x \leq 1$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = x$ if $x < 0$ or $x > 0$ $\operatorname{csch}(\operatorname{csch}^{-1} x) = x$ if $x < 0$ or $x > 0$

Logarithmic Forms of Inverse Hyperbolic Functions

Because the hyperbolic functions are expressible in terms of *ex* , it should not be surprising that the inverse hyperbolic functions are expressible in terms of natural logarithms; the next theorem shows that this is so.

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Example

Prove $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

$y = \sinh^{-1} x$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$e^y - 2x - e^{-y} = 0 \qquad \textit{multiplying by } e^y \ \rightarrow \ e^{2y} - 2xe^y - 1 = 0$

And applying the quadratic formula yields

$$e^y = \frac{2x \mp \sqrt{4x^2+4}}{2} = x \mp \sqrt{x^2+1} \quad \textit{since } e^y > 0, \textit{Thus } e^y = x + \sqrt{x^2+1}$$

Taking natural logarithms yields

$$y = \ln\left(x + \sqrt{x^2+1}\right) \textit{ or } \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

Example

$$\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2+1}\right) = \ln(1+\sqrt{2}) \approx 0.8814$$

$$\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln 3 \approx 0.5493$$

H.W: Find $\sinh^{-1} 2 = 1.443$

$\tanh^{-1} 0.25 = 0.255$

Derivatives and Integrals Involving Inverse Hyperbolic Functions

The following two theorems list the generalized derivative formulas and corresponding integration formulas for the inverse hyperbolic functions.

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1 \qquad \frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1 \qquad \frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \;\; or \;\; \ln(u + \sqrt{u^2+a^2}) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \;\; or \;\; \ln(u + \sqrt{u^2-a^2}) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & |u| < a \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & |u| > a \end{cases} \;\; or \;\; \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \;\; or \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \;\; or \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

Example Show that $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+u^2}}\frac{d}{dx}$

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right)] = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) = \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})(\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}$$

H.W: Prove above theorem as Example

Example:

1- Find $\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1-(\sin x)^2} \cdot \frac{d}{dx}\sin x = \frac{1}{1-sin^2 x}\cos x = \frac{\cos x}{cos^2 x} = \sec x$

2- Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}, x > \frac{3}{2}$

Let $u = 2x, Thus\ du = 2dx\ and$

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{2dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-3^2}} = \frac{1}{2}\cosh^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

3- $\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{dx}{\sqrt{x^2-4x+3+1-1}} = \int \frac{dx}{\sqrt{x^2-4x+4-1}} = \int \frac{dx}{\sqrt{(x-2)^2-1}}$

Let $u = x\text{-}2 \quad du = dx$

$= \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C = \cosh^{-1}(x-2) + C$

4- $\int_1^2 \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2}\int_1^2 \frac{dx}{\frac{1}{2}x\sqrt{1+(\frac{x}{2})^2}} = -\frac{1}{2}\operatorname{csch}^{-1}\frac{x}{2}\Big]_1^2 = -\operatorname{csch}^{-1} 1 + \operatorname{csch}^{-1}\frac{1}{2}$

Hanging Cables

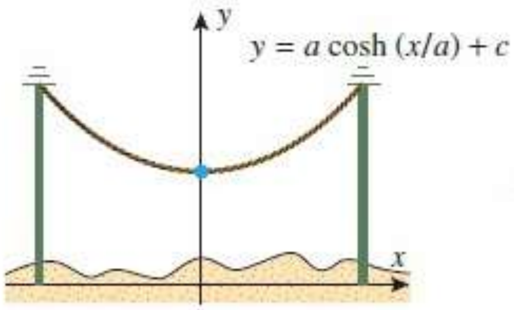
Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a ***catenary*** (from the Latin *catena*, meaning "chain"). If, as in Figure, a coordinate system is introduced so that the low point of the cable lies on the *y*-axis, then it can be shown using principles of physics that the cable has an equation of the form

$$y = a\cosh\left(\frac{x}{a}\right) + c$$

Where the parameters *a* and *c* are determined by the distance between the poles and the composition of the cable.

Glen Allison/Stone/Getty Images

The design of the Gateway Arch near St. Louis is based on an inverted hyper-



Example:

A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

Solution From above, the wire forms a catenary curve with equation

$$y = a\cosh\left(\frac{x}{a}\right) + c$$

Where the origin is on the ground midway between the poles. Using Formula for the length of the catenary, we have

$$100 = \int_{-45}^{45}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\int_0^{45}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \;\; by\ summetry\ about\ y-axis$$

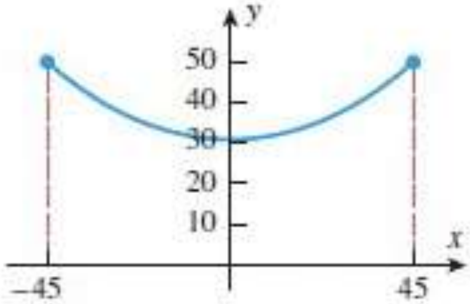
$$= 2\int_0^{45}\sqrt{1+sinh^2\left(\frac{x}{a}\right)}\,dx = 2\int_0^{45}\cosh\left(\frac{x}{a}\right)dx = 2a\sinh\left(\frac{x}{a}\right)\Big]_0^{45} = 2a\sinh\left(\frac{45}{a}\right)$$

Using a calculating utility's numeric solver to solve

$$100 = 2a\sinh\left(\frac{45}{a}\right)\ for\ a\ gives\ a \approx 56.01\,.then$$

$$50 = y(45) = 56.01\cosh\left(\frac{45}{56.01}\right) + c \approx 75.08 + c \quad so\ c \approx -25.08$$

Thus, the middle of the wire is $y(0) \approx 56.01 - 25.08 = 30.93\ ft\ above\ the\ ground.$



PARTIAL DERIVATIVES

Definition

If is a function of two variables x *and* y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of *f* with respect to *x* at *(a,b)*** and denote it by $f_x(a, b)$.

Similarly, the **partial derivative of *f* with respect to *y* at *(a,b)***, denoted by $f_y(a, b)$, is obtained by keeping x fixed $(x = a)$ and finding the ordinary derivative at b of the function $G(y) = f(a, y)$.

If f is a function of two variables, its **partial derivatives** are the functions f_x *and* f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1: If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ *and* $f_y(2,1)$

$$f_x = 3x^2 + 2xy^3$$

$$f_x(2,1) = 3 * 2^2 + 2 * 2 * 1^3 = 16$$

$$f_y = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3 * 2^2 * 1^2 - 4 * 1 = 8$$

Example 2: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$

$$\frac{\partial z}{\partial x} = x^4 \cos(xy^3) * y^3 + \sin(xy^3) * 4x^3 = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

$$\frac{\partial z}{\partial y} = x^4 \cos(xy^3) * 3xy^2 + \sin(xy^3) * 0 = 3x^5 y^2 \cos(xy^3)$$

Example 3: If $f(x, y) = \sin\left(\frac{x}{1+y}\right), calculate\ \frac{\partial f}{\partial x}\ and\ \frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right).\left(\frac{1}{1+y}\right)$$

$$\frac{\partial f}{\partial y} = -\cos\left(\frac{x}{1+y}\right)\frac{x}{(1+y)^2}$$

Example 4: Find $fx, fy\ for\ f(x, y) = \frac{2y}{y+\cos x}$

$$fx = \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2} = \frac{2y\ sin\ x}{(y+\cos x)^2}$$

$$fy = \frac{(y+\cos x)(2) - 2y(1)}{(y+\cos x)^2}$$

Example 5: Find $\frac{\partial z}{\partial x}\ and\ \frac{\partial z}{\partial y}\ if\ z\ is\ defind\ implicity\ as\ afunction\ of\ x\ and\ y\ by\ the\ equation$

$$x^3 + y^3 + z^3 + 6xyz = 1$$

To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\frac{\partial z}{\partial x}$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Example 6: If $z = x^2 + xf(xy)$ show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x^2 + z$$

$$\frac{\partial z}{\partial x} = 2x + xyf'(xy) + f(xy)$$

$$\frac{\partial z}{\partial y} = x^2 f'(xy)$$

$$x\big(2x + xyf'(xy) + f(xy)\big) - y\big(x^2 f'(xy)\big)$$

$$= 2x^2 + x^2 y f'^{(xy)} + xf(xy) - yx^2 f'^{(xy)}$$

$$= x^2 + x^2 + xf(xy) = x^2 + z$$

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y , and z , then its partial derivative with respect to x is defined as

$$\boldsymbol{f_x(x, y, z) = \lim_{h\to 0}\frac{f(x+h, y, z) - f(x, y, z)}{h}}$$

Example:

1- Find f_x, f_y and f_z if $f(x, y, z) = e^{xy}\ln z$

$$f_x = ye^{xy}\ln z \qquad f_y = xe^{xy}\ln z \qquad and \qquad f_z = \frac{e^{xy}}{z}$$

2- If $f(x, y, z) = x^3y^2z^4 + 2xy + z, \quad then$

$$f_x(x, y, z) = 3x^2y^2z^4 + 2y$$

$$f_y(x, y, z) = 2x^3yz^4 + 2x$$

$$f_z(x, y, z) = 4x^3y^2z^3 + 1$$

3- If $f(\rho, \theta, \emptyset) = \rho^2\cos\emptyset\sin\theta, then$

$$f_\rho(\rho, \theta, \emptyset) = 2\rho\cos\emptyset\sin\theta$$

$$f_\theta(\rho, \theta, \emptyset) = \rho^2\cos\emptyset\cos\theta$$

$$f_\emptyset(\rho, \theta, \emptyset) = -\rho^2\sin\emptyset\sin\theta$$

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $= f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Warning: Observe that the two notations for the mixed second partials have opposite conventions for the order of differentiation. In the "∂" notation the derivatives are taken right to left, and in the "subscript" notation they are taken left to right. The conventions are logical if you insert parentheses:

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \textbf{Righ to left. Differentiate inside the parentheses first}$$

$$(f_x)_y = f_{xy} \qquad \textbf{Left to right. Differentiate inside the parentheses first}$$

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example: Find the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$f_x(x,y) = 3x^2 + 2xy^3$$

$$f_y(x,y) = 3x^2y^2 + 4y$$

$$f_{xx} = 6x + 2y^3 \qquad\qquad f_{xy} = 6xy^2$$

$$f_{yx} = 6xy^2 \qquad\qquad f_{yy=}6yx^2 - 4$$

Example: Find the second –order partial derivatives of $f(x,y) = x^2y^3 + x^4y$

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \qquad and \qquad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\boldsymbol{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}} = 2y^3 + 12x^2y$$

$$\boldsymbol{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}} = 6x^2y$$

$$\boldsymbol{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y}} = 6xy^2 + 4x^3$$

$$\boldsymbol{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}} = 6xy^2 + 4x^3$$

Example:

1- Let $f(x,y) = y^2e^x + y \quad find\ f_{xyy}$

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = 2e^x$$

2- Calculate $f_{xxyz}\ if\ f(x,y,z) = \sin(3x + yz)$

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

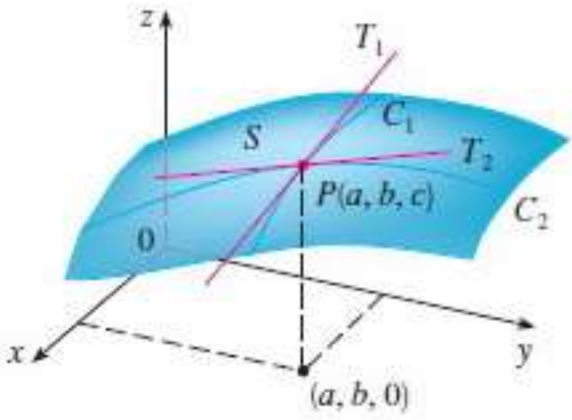
$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface *S* (the graph of *f*). If $f(a, b) = c$, then the point *P (a,b,c)* lies on *S*. By fixing $= b$, we are restricting our attention to the curve $C1$ in which the vertical plane $y = b$ intersects *S*. (In other words, C_1 is the trace of *S* in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects *S* in a curve C_2. Both of the curves C_1 *and* C_2 pass through the point *P*. (See Figure)

Notice that the curve C_1 is the graph of the function$g(x) = f(x, b)$, so the slope of its tangent T_1 at *P* is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at *P* is $G'(b) = f_y(a, b)$. Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at *P(a,b,c)* to the traces C_1 *and* C_2 of *S* in the planes $y = b$ and $x = a$.



As we have seen in the case of the heat index function, partial derivatives can also be interpreted as rates of change. If $z = f(x, y)$, then $\frac{\partial z}{\partial x}$ represents the rate of change of *z* with respect to *x* when *y* is fixed. Similarly, $\frac{\partial z}{\partial y}$ represents the rate of change of *z* with respect to *y* when *x* is fixed.

Example: If $f(x, y) = 4 - x^2 - 2y^2, find\ f_x(1,1)\ and\ f_y(1,1)$ and interpret these numbers slopes.

$$f_x = -2x$$

$$f_y = -4y$$

$$f_x\,(1,1) = -2$$

$$f_y(1,1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point (1,1,1) is $f_x(1,1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $= 3 - 2y^2, x = 1$, and the slope of the tangent line at (1,1,1) is $f_y(1,1) = -4$.

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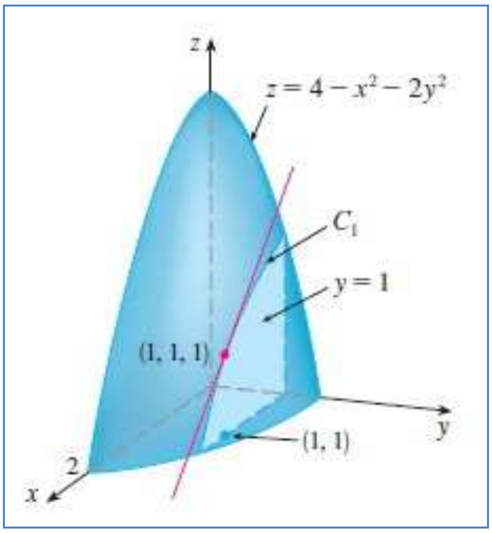
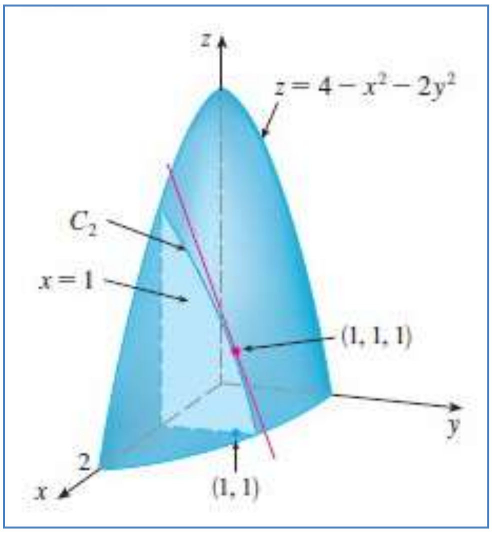




Example:

Recall that the wind chill temperature index is given by the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16}$$

Compute the partial derivative of W with respect to v at the point $(T, v) = (25, 10)$ and interpret this partial derivative as a rate of change.

$$\frac{\partial W}{\partial v}(T, v) = 0 + 0 + (0.4275T - 35.75)(0.16)v^{0.16-1} = (0.4275T - 35.75)(0.16)v^{-0.84}$$

Substituting $T = 25$ *and* $v = 10$ gives

$$\frac{\partial W}{\partial v}(25{,}10) = (-4.01)10^{-0.84} \approx -0.58\frac{°\mathrm{F}}{mi/h}$$

Example:

Let $f(x, y) = x^2y + 5y^3$

a- Find the slope of the surface $z = f(x, y)$ in the x-direction at the point (1,-2).

b- Find the slope of the surface $z = f(x, y)$ in the y-direction at the point (1,-2).

a- Differentiating f with respect to x with y held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1,-2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

b- Differentiating f with respect to y with x held fixed yields

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y-direction is $f_y(1,-2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y.

Example: Suppose that $D = \sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths x and y that are allowed to vary. Find a formula for the rate of change of D with respect to x if x varies with y held constant, and use this formula to find the rate of change of D with respect to x at the point where $x = 3$ and $y = 4$.

Solution. Differentiating both sides of the equation $D^2 = x^2 + y^2$ with respect to x yields

$$2D\frac{\partial D}{\partial x} = 2x$$

$$D\frac{\partial D}{\partial x} = x$$

Since $D = 5$ when $x = 3$ and $y = 4$, it follows that

$$5\frac{\partial D}{\partial x} = 3 \quad at\ x = 3, y = 4$$

$$\frac{\partial D}{\partial x} = \frac{3}{5}$$

Thus, D is increasing at a rate of $\frac{3}{5}$ unit per unit increase in x at the point (3, 4).

Total Differential

Definition:

If we move from (x_o,y_o) to a point (x_o+dx,y_o+dy) nearby the resulting change

$$\boldsymbol{df = f_x(x_o, y_o)\,dx + f_y(x_o, y_o)\,dy}$$

In the linearization of f is called the total differential of f

Often we take $dx = \Delta x = x - x_o\ and\ dy = \Delta y = y - y_o$

Example: Estimating Change in Volume

Suppose that a cylindrical can is designed to have a radius of 1 in and a height of 5 in. but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution:

$$V = \pi r^2 h \qquad \Delta V = dV = V_r(r_o, h_o)d_r + V_h(r_o, h_o)dh$$

$$Vr = 2\pi rh \quad and \quad Vh = \pi r^2$$

$$dV = 2\pi r_o h_o dr + \pi r_o^2 dh$$

$$= 2\pi(1)(5)(0.03) + \pi 1^2(-0.1)$$

$$= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63\ in^3$$

The relative change is estimated by

$$\frac{df}{f(x_o, y_o)} = \frac{dV}{V(r_o, h_o)} = \frac{0.2\pi}{\pi r_o^2 h_o} = \frac{0.2\pi}{\pi(1)^2(5)} = 0.04$$

Giving 4 % as an estimate of the percentage change.

Example: Sensitivity to Change

Your company manufactures right circular cylindrical molasses storage tanks that are 25 ft high with a radius of 5 ft. how sensitive are the tanks volumes to small variations in height and radius?

$$V = \pi r^2 h \qquad dV = V_r(5,25)d_r + V_h(5,25)dh$$

$$= 2\pi rh(5,25)d_r + (\pi r^2)(5,25)d_h$$

$$= 250\pi d_r + 25\pi dh$$

Thus a 1 unit change in r will change V by about 250π units. A 1 unit change in h will change V by about 25π units. The tanks volume is 10 times more sensitive to a small change in r than it is to a small change of equal size in h.

The Chain Rule

(Chain Rules for Derivatives) *If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and*

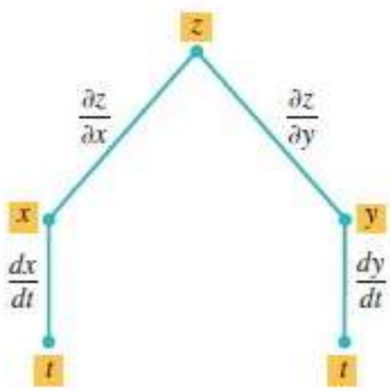
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example:

1- **Suppose that $z = x^2 y, \quad x = t^2, \quad y = t^3$ Find $\frac{\partial z}{\partial t}$**

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy)(2t) + (x^2)(3t^2) = (2t^5)(2t) + (t^4)(3t^2) = 7t^6$$

Alternatively, we can express z directly as a function of t,

$$z = x^2 y = (t^2)^2(t^3) = t^7$$

2- Suppose that $w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta$

Use the chain rule to find $\frac{dw}{d\theta}$ when $\theta = \frac{\pi}{4}$

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial \theta}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2x)(-\sin\theta)$$

$$+ \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2y)(\cos\theta) + \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2z)(sec^2\theta)$$

When $x = \cos\frac{\pi}{4} = 1/\sqrt{2} \qquad y = \sin\frac{\pi}{4} = 1/\sqrt{2} \qquad z = \tan\frac{\pi}{4} = 1$

$$\frac{dw}{d\theta} = \sqrt{2}$$

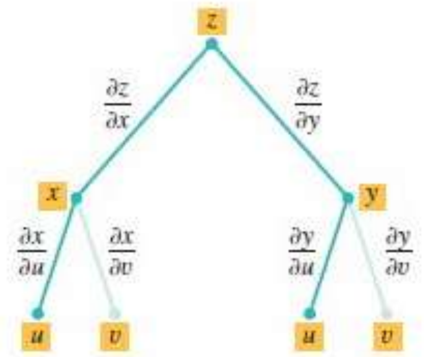
<u>Chain Rules For Partial Derivatives</u>

(Chain Rules for Partial Derivatives) *If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v), and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first-order partial derivatives at the point (u, v) given by*

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

If each function $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ has first-order partial derivatives at the point (u, v), and if the function $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(u, v), y(u, v), z(u, v))$, then $w = f(x(u, v), y(u, v), z(u, v))$ has first-order partial derivatives at the point (u, v) given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$



$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

Example: Given that $z = e^{xy}$ $\quad x = 2u + v,$ $\quad y = \frac{u}{v}$

Find $\frac{\partial z}{\partial u}$ $\quad and \quad$ $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=(ye^{xy})(2)+(xe^{xy})\left(\frac{1}{v}\right)=\left[2y+\frac{x}{v}\right]e^{xy}$$

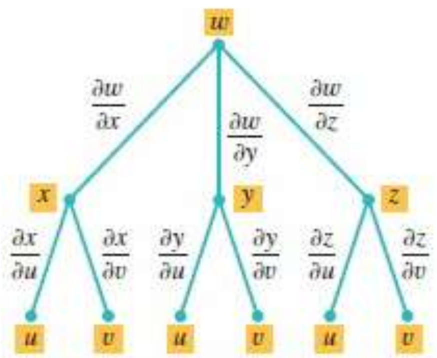
$$=\left[\frac{2u}{v}+\frac{2u+v}{v}\right]e^{(2u+v)(\frac{u}{v})}=[\frac{4u}{v}+1]e^{(2u+v)(\frac{u}{v})}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=(ye^{xy})(1)+(xe^{xy})\left(-\frac{u}{v^2}\right)=\left[y-x(\frac{u}{v^2})\right]e^{xy}$$

$$=\left[\frac{u}{v}-(2u+v)(\frac{u}{v^2})\right]e^{(2u+v)(\frac{u}{v})}=-2u^2/v^2\ e^{(2u+v)(\frac{u}{v})}$$

H.W: Suppose that $w = e^{xyz}$, $\quad x = 3u + v$, $\quad y = 3u - v,$ $\quad z = u^2 v$

Find $\frac{\partial w}{\partial u}$ $\quad and \quad$ $\frac{\partial w}{\partial v}$



$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Example: Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin \emptyset \cos \theta \qquad y = \rho \sin \emptyset \sin \theta \qquad z = \rho \cos \emptyset$$

Find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$

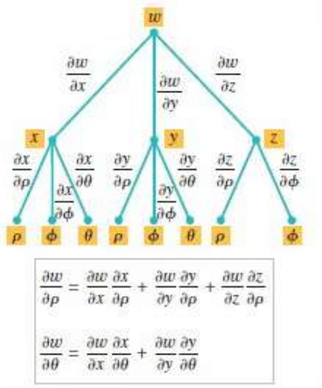
$$\frac{\partial w}{\partial \rho} = 2x sin\ \emptyset \cos \theta + 2y \sin \emptyset \sin \theta - 2z \cos \emptyset$$

$$= 2\rho\ sin^2\emptyset\ cos^2\theta + 2\rho\ sin^2\ \emptyset\ sin^2\theta - 2\rho\ cos^2\emptyset$$

$$= 2\rho\ sin^2\emptyset\ (cos^2\theta + sin^2\theta) - 2\rho\ cos^2\emptyset$$

$$= 2\rho(sin^2\emptyset - cos^2\emptyset) = -2\rho\ cos2\emptyset$$

$$\frac{\partial w}{\partial \theta} = (2x)(-\rho \sin \emptyset \sin \theta + (2y) \sin \emptyset \cos \theta\ = -2\rho^2 sin^2\emptyset\ sin\theta\ cos\theta + 2\rho^2 sin^2\emptyset\ sin\theta\ cos\theta = 0$$



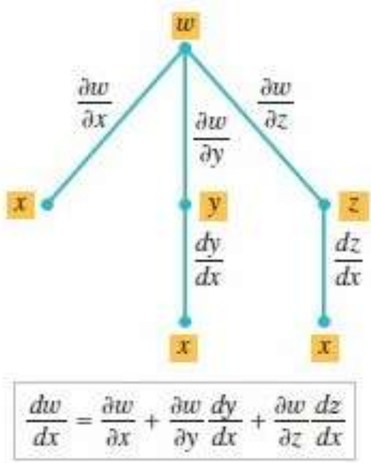
Example: Suppose that $w = xy + yz, \qquad y = \sin x, \qquad z = e^x$

Find dw/dx

$$\frac{dw}{dx} = y + (x + z) \cos x + y\ e^x = \sin x + (x + e^x) \cos x + e^x \sin x$$

This result can also be first expressing w explicitly in terms of x as

$$w = x \sin x + e^x \sin x$$



Implicit Differentiation

1- If the equation

$f(x, y) = c$ $defines\ y\ implicitly\ as\ a\ differentialble\ function\ of\ x, and\ if\ \frac{\partial f}{\partial y} \neq 0, then$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

2- If the equation

$f(x, y, z) = c$ defines z implicitly as a differentialble function of $x, and\ y, and\ if\ \frac{\partial f}{\partial z} \neq 0, then$

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad and \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

Example:

1- Given that $x^3 + y^2 x - 3 = 0$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Alternatively, differentiating implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad or \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

2- Consider the sphere $x^2 + y^2 + z^2 = 1$ Find $\partial z/\partial x\ \ and\ \ \partial z/\partial y$ at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$

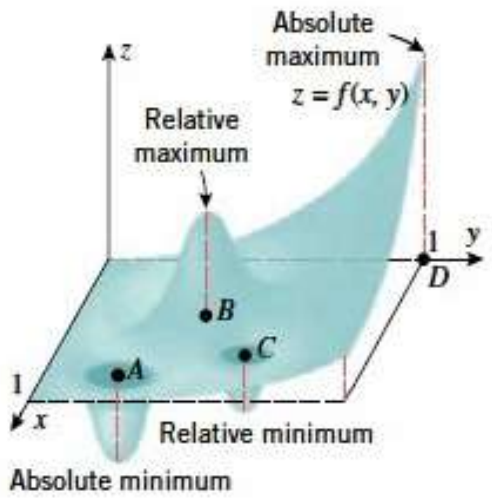
$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z} = -1$$

$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z} = -\frac{1}{2}$$

Maxima and Minima of Functions of Two Variables

Definition: A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

Definition: A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.



Extreme-value theorem:

If $f(x,y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and absolute minimum

Definition: *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*

$$f_x(x_0, y_0) = 0 \quad and \quad f_y(x_0, y_0) = 0$$

Definition: A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

Definition: *(The Second Partials Test)* *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(a) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(b) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(c) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(d) *If $D = 0$, then no conclusion can be drawn.*

Example: Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

$$f_x = 6x - 2y \ \ and \ f_y = -2x + 2y - 8$$

The critical points of *f* satisfy the equations

$6x - 2y = 0$

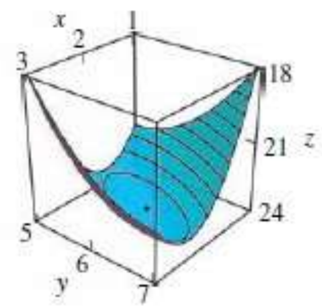
$-2x + 2y - 8 = 0$

$x = 2, y = 6$, *so* $(2,6)$ *is the only critical point.*

$f_{xx}(x, y) = 6, \qquad f_{yy}(x, y) = 2, \qquad f_{xy}(x, y) = -2$

$D = f_{xx}(2,6) f_{yy}(2,6) - \ {f_{xy}}^2(2,6) = (6)(2) - (-2)^2 = 8 > 0$

$f_{xx}(2,6) = 6 > 0 \quad so\ f\ has\ a\ relative\ minimum\ at\ (2,6)$



$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

Example : Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

$$f_x(x, y) = 4y - 4x^3 \quad and\ f_y(x, y) = 4x - 4y^3$$

The critical points of *f* satisfy the equations

$4y - 4x^3 = 0 \quad or\ \ y = x^3$

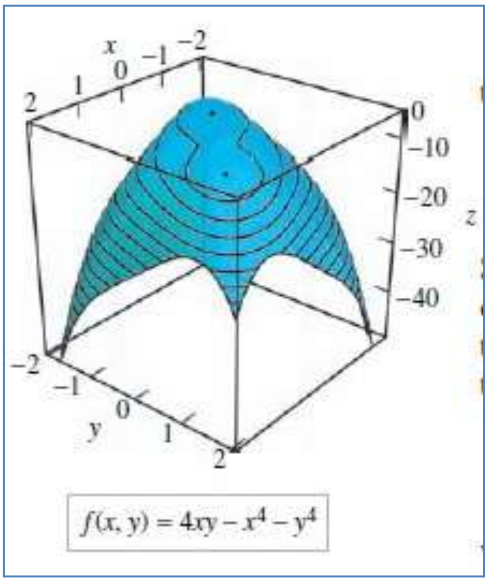
$4x - 4y^3 = 0 \ \ or\ x = y^3$

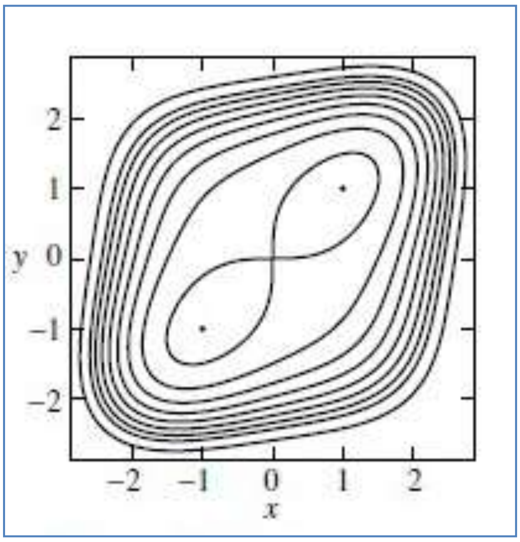
$x = (x^3)^3 \quad \rightarrow \quad x^9 - x = 0\ or\ x(x^8 - 1) = 0 \quad x = 0, x = 1, x = -1$

$y = 0, y = 1, y = -1$

$f_{xx}(x, y) = -12x^2, \qquad f_{yy}(x, y) = -12y^2, \qquad f_{xy}(x, y) = 4$

Critical points (x_o,y_o)	$f_{xx}(x_o,y_o)$	$f_{yy}(x_o,y_o)$	$f_{xy}(x_o,y_o)$	$D = f_{xx}f_{yy} - f^2{}_{xy}$	Notes
(0,0)	0	0	4	-16	Saddle point
(1,1)	-12	-12	4	128	Relative Maximum
(-1,-1)	-12	-12	4	128	Relative Maximum





<u>Finding Absolute Extrema On Closed And Bounded Sets</u>

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R

Step 1. Find the critical points of f that lie in the interior of R.

Step 2. Find all boundary points at which the absolute extrema can occur.

Step 3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

<u>Example</u> Find the absolute maximum and minimum values of

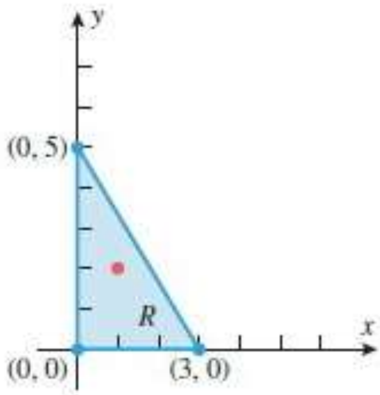
$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

The region R is shown in Figure ,We have

$$\frac{\partial f}{\partial x} = 3y - 6 \quad and \quad \frac{\partial f}{\partial y} = 3x - 3$$

So all critical points occur where



$3y - 6 = 0 \; and \; 3x - 3 = 0$

Solving these equations yields

$x = 1 \; and \; y = 2,$ So (1,2) is the critical point

* The line segment between *(*0, 0*) and (*3, 0*)*: On this line segment we have *y* = 0, so $f(x, y)$ simplifies to a function of the single variable *x*,

$$u(x) = f(x, 0) = -6x + 7, \quad 0 \le x \le 3$$

This function has no critical points because $u'(x) = -6$ is nonzero for all *x*. Thus the extreme values of *u(x)* occur at the endpoints *x* = 0 and *x* = 3, which correspond to the points (0, 0) and (3, 0) of *R*.
* The line segment between (0, 0) and (0, 5): On this line segment we have x = 0, so $f(x, y)$ simplifies to a function of the single variable *y*,

$$v(y) = f(0, y) = -3y + 7, \quad 0 \le y \le 5$$

This function has no critical points because $v'(y) = -3$ is nonzero for all *y*. Thus, the extreme values of *v(y)* occur at the endpoints *y = 0 and y = 5*, which correspond to the points (0, 0) and (0, 5) of R.
* The line segment between (3, 0) and (0, 5): In the xy-plane, an equation for this line segment is

$$y = -\frac{5}{3}x + 5, \quad 0 \le x \le 3$$

so $f(x, y)$ simplifies to a function of the single variable x,

$$w(x) = f(x, -\frac{5}{3}x + 5) = 3x\left(-\frac{5}{3}x + 5\right) - 6x - 3\left(-\frac{5}{3}x + 5\right) + 7$$
$$= -5x^2 + 14x - 8, \quad 0 \le x \le 3$$

Since

$w'(x) = -10x + 14$, the $equation \; w'(x) = 0 \; yields \; x = \frac{7}{5}$ as the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{7}{5}$ or at the endpoints x = 0 and x = 3. The endpoints correspond to the points (0, 5) and (3, 0) of R, and from the critical point corresponds to $\left(\frac{7}{5}, \frac{8}{3}\right)$.

(x, y)	(0,0)	(3,0)	(0,5)	$(\frac{7}{5}, \frac{8}{3})$	(1,2)
$f(x, y)$	7	-11	-8	$\frac{9}{5}$	1
	Absolute Maximum	Absolute Minimum			

Example: Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution: Let

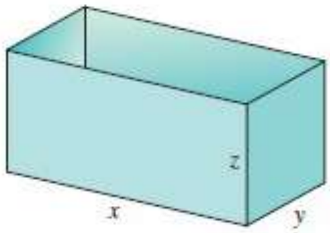
x = length of the box (in feet)

y = width of the box (in feet)

z = height of the box (in feet)

S = surface area of the box (in square feet)

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area



Two sides each have area *xz*.
Two sides each have area *yz*.
The base has area *xy*.

$$S = xy + 2xz + 2yz$$

The volume requirement $xyz = 32$

We obtain $z = 32/xy$, so S can be rewritten as

$$S = xy + \frac{64}{y} + \frac{64}{x}$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \qquad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

$$y - \frac{64}{x^2} = 0, \qquad x - \frac{64}{y^2} = 0$$

$$y = \frac{64}{x^2}$$

$$x - \frac{64}{(64/x^2)^2} = 0$$

$$x\left(1 - \frac{x^2}{64}\right) = 0$$

$x = 0 \;\; and \; x = 4 \qquad y = 4$

If $x = y = 4$, $z = 2 \;\; so \; S = 48$

So the box using the last material has a height of 2 ft and a square base whose edges are 4 ft long.

Lagrange Multipliers

Joseph Louis Lagrange (1736–1813) French–Italian mathematician and astronomer. Lagrange, the son of a public official, was born in Turin, Italy. (Baptismal records list his name as Giuseppe Lodovico Lagrangia.) Although his father wanted him to be a lawyer, Lagrange was attracted to mathematics and astronomy after reading a memoir by the astronomer Halley. At age 16 he began to study mathematics on his own and by age 19 was appointed to a professorship at the Royal Artillery School in Turin. The following year Lagrange sent Euler solutions to some famous problems using new methods that eventually blossomed into a branch of mathematics called calculus of variations.

Definition ***Three-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

Definition ***Two-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

Let us assume that a constrained relative maximum or minimum occurs at the point (x_0, y_0), and for simplicity let us further assume that the equation $g(x, y) = 0$ can be smoothly parametrized as

$$x = x(s), \qquad y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Thus, the quantity

$$z = f(x(s), y(s))$$

has a relative maximum or minimum at $s = 0$, and this implies that $dz/ds = 0$ at that point.
From the chain rule, this equation can be expressed as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \left(\frac{\partial f}{\partial y}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right).\left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) = 0$$

It then follows that there is some scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

This scalar is called a ***Lagrange multiplier***. Thus, the ***method of Lagrange multipliers*** for finding constrained relative extrema is to look for points on the constraint curve $g(x, y) = 0$ at which Equation above is satisfied for some scalar λ.

The notation $\boldsymbol{\nabla f}$ is read "grad *f*" as well as "gradient of *f*" and "del *f*". The symbol ∇ by itself is read "del". Another notation for the gradient is grad f, read the way it is written.

Theorem (*Constrained-Extremum Principle for Two Variables and One Constraint*) *Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel; that is, there is some number λ such that*

$$\boldsymbol{\nabla} f(x_o, y_o) = \lambda\, \boldsymbol{\nabla} g(x_o, y_o)$$

<u>Example:</u> At what point or points on the circle $x^2 + y^2 = 1\ does\ f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution: The circle $x^2 + y^2 = 1$ is a closed and bounded set and $f(x, y) = xy$ is a continuous function, so it follows from the Extreme-Value Theorem, that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.
We want to maximize $f(x, y) = xy$ subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0$$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\boldsymbol{i} + x\boldsymbol{j} \quad \text{and} \quad \nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad or \quad y\boldsymbol{i} + x\boldsymbol{j} = \lambda(2x\boldsymbol{i} + 2y\boldsymbol{j})$$

Which is equivalent to the pair of equations $\quad y = 2x\lambda \quad$ and $\quad x = 2y\lambda$
It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

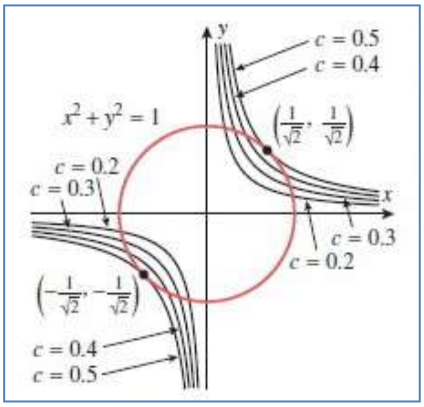
$$\lambda = \frac{y}{2x} \quad and \quad \lambda = \frac{x}{2y}$$

From which we obtain

$$\frac{y}{2x} = \frac{x}{2y} \quad or \quad y^2 = x^2$$

Substituting this yields

$$2x^2 - 1 = 0$$



From which we obtain $x = \mp 1/\sqrt{2}$. Each of these values, when substituted in Equation, produces y values of $y = \mp 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), and\ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	1/2	-1/2	-1/2	1/2
Point	Absolute Maximum	Absolute Minimum	Absolute Minimum	Absolute Maximum

Example: Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

Solution: Let
x = length of the rectangle,
y = width of the rectangle,
A = area of the rectangle
We want to maximize $A = xy$ on the line segment

$$2x + 2y = p, \qquad 0 \leq x, y$$

that corresponds to the perimeter constraint. This segment is a closed and bounded set, and since $f(x, y) = xy$ is a continuous function, it follows from the Extreme-Value Theorem. That f has an absolute maximum on this segment. This absolute maximum must also be a constrained relative maximum since f is 0 at the endpoints of the segment and positive elsewhere on the segment.
If $g(x, y) = 2x + 2y$, then we have

$$\nabla f = y\boldsymbol{i} + x\boldsymbol{j} \quad \text{and} \quad \nabla g = 2\boldsymbol{i} + 2\boldsymbol{j}$$

Noting that $\nabla g \neq \mathbf{0}$, it follows that

$$y\boldsymbol{i} + x\boldsymbol{j} = \lambda(2\boldsymbol{i} + 2\boldsymbol{j})$$

at a constrained relative maximum. This is equivalent to the two equations

$$y = 2\lambda \quad and \quad x = 2\lambda$$

Eliminating λ from these equations we obtain $x = y$, which shows that the rectangle is actually a square. Using this condition and constraint, we obtain $x = p/4,\ y = p/4$.

Theorem (***Constrained-Extremum Principle for Three Variables and One Constraint***) *Let f and g be functions of three variables with continuous first partial derivatives on some open set containing the constraint surface $g(x, y, z) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this surface. If f has a constrained relative extremum, then this extremum occurs at a point (x_o, y_o, z_o) on the constraint surface at which the gradient vectors $\nabla f(x_o, y_o, z_o)$ and $\nabla g(x_o, y_o, z_o)$ are parallel; that is, there is some number λ such that*

$$\nabla f\,(x_o, y_o, z_o) = \lambda\, \nabla g(x_o, y_o, z_o)$$

Example: Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point (1, 2, 2).

Solution: To avoid radicals, we will find points on the sphere that minimize and maximize the *square* of the distance to (1, 2, 2). Thus, we want to find the relative extrema of

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2$$

Subject to the constraint

$$x^2 + y^2 + z^2 = 36$$

If we let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$. Thus, $\nabla g = \mathbf{0}$ if and only if $x = y = z = 0$. It follows that $\nabla g \neq \mathbf{0}$ at any point of the sphere, and hence the constrained relative extrema must occur at points where

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

That is,

$$2(x - 1)\boldsymbol{i} + 2(y - 2)\boldsymbol{j} + 2(z - 2)\boldsymbol{k} = \lambda(2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k})$$

Which leads to the equations

$$2(x - 1) = 2x\lambda, \quad 2(y - 2) = 2y\lambda, \quad 2(z - 2) = 2z\lambda$$

We may assume that x, y, and z are nonzero since $x = 0$ does not satisfy the first equation, $y = 0$ does not satisfy the second, and $z = 0$ does not satisfy the third. Thus, we can rewrite

$$\frac{x-1}{x} = \lambda, \quad \frac{y-2}{y} = \lambda, \quad \frac{z-2}{z} = \lambda$$

The first two equations imply that

$$\frac{x-1}{x} = \frac{y-2}{y}$$

from which it follows that

$$y = 2x, \quad z = 2x$$

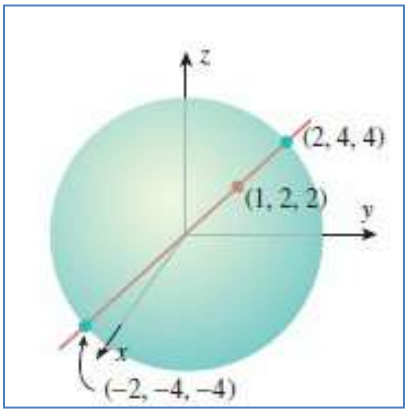
Substituting, we obtain

$$9x^2 = 36 \quad or \quad x = \mp 2$$

Substituting these values in equation yields two points:

$$(2, 4, 4)\ and\ (-2, -4, -4)$$

Since $f(2, 4, 4) = 9\ and\ f(-2, -4, -4) = 81$, it follows that $(2, 4, 4)$ is the point on the sphere closest to (1, 2, 2), and $(-2,-4,-4)$ is the point that is farthest.



<u>Example:</u> Use Lagrange multipliers to determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution: the problem is to minimize the surface area

$$S = xy + 2xz + 2yz$$

subject to the volume constraint

$$xyz = 32$$

If we let $f(x, y, z) = xy + 2xz + 2yz$ and $g(x, y, z) = xyz,$ then

$$\nabla f = (y + 2z)\boldsymbol{i} + (x + 2z)\boldsymbol{j} + (2x + 2y)\boldsymbol{k} \quad and \quad \nabla g = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$$

It follows that $\nabla g \neq \mathbf{0}$ at any point on the surface $xyz = 32, since\ x, y, and\ z$ are all nonzero on this surface. Thus, at a constrained relative extremum we must have $\nabla f = \lambda \nabla g$, that is,

$$(y + 2z)\boldsymbol{i} + (x + 2z)\boldsymbol{j} + (2x + 2y)\boldsymbol{k} = \lambda(yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k})$$

This condition yields the three equations

$$y + 2z = \lambda yz, \qquad x + 2z = \lambda xz, \qquad 2x + 2y = \lambda xy$$

Because x, y, and z are nonzero, these equations can be rewritten as

$$\frac{1}{z} + \frac{2}{y} = \lambda, \qquad \frac{1}{z} + \frac{2}{x} = \lambda, \qquad \frac{2}{y} + \frac{2}{x} = \lambda$$

From the first two equations,

$$y = x$$

and from the first and third equations,

$$z = \frac{1}{2}x$$

Substituting (y) and (z) in the volume constraint yields

$$12x^3 = 32$$

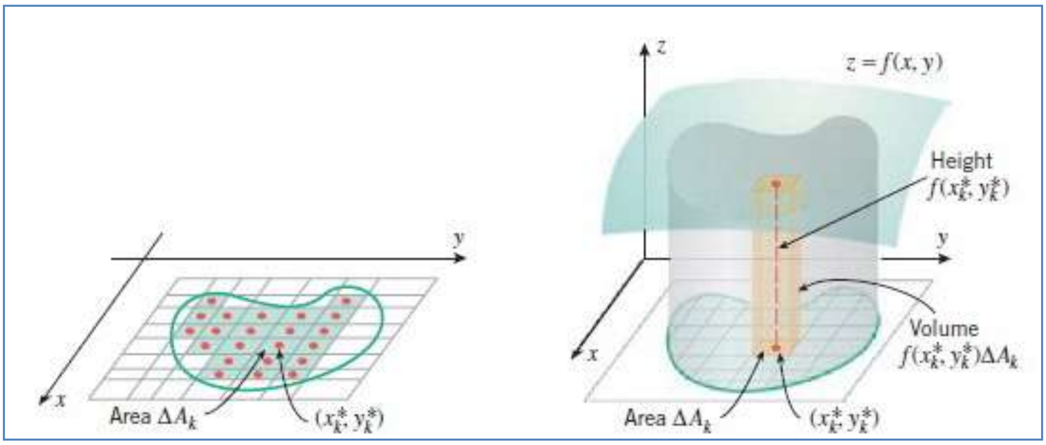
$$x = 4, \qquad y = 4, \qquad z = 2$$

MULTIPLE INTEGRALS

<u>Double Integrals</u>

DEFINITION (*Volume Under a Surface*) If f is a function of two variables that is continuous and nonnegative on a region R in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R is defined by

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \iint_R f(x,y)dA$$



The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, ***partial integration***. The symbols

$$\int_a^b f(x,y)dx \qquad and \qquad \int_c^d f(x,y)dy$$

denote ***partial definite integrals***; the first integral, called the ***partial definite integral with respect to x***, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the ***partial definite integral with respect to y***, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x. This two-stage integration process is called iterated (or repeated) integration. We introduce the following notation:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

<u>Example</u>

a:

$$\int_1^3 \int_2^4 (40-2xy)\,dy\,dx = \int_1^3 \left[\int_2^4 (40-2xy)dy\right] dx = \int_1^3 (40y-2xy)]_{y=2}^{4}\,dx$$

$$= \int_1^3 [(160-16x)-(80-4x)]dx \ = \int_1^3 (80-12x)dx = (80x-6x^2)]_1^3 = 112$$

b:

$$\int_2^4 \int_1^3 (40-2xy)\,dx\,dy = \int_2^4 \left[\int_1^3 (40-2xy)dx\right] dy = \int_2^4 (40x-x^2y)]_{x=1}^{3}\,dy$$

$$= \int_2^4 [(120-9y)-(40-y)]dy = \int_2^4 (80-8y)dy = (80y-4y^2)]_2^4 = 112$$

Theorem (*Fubini's Theorem*) Let R be the rectangle defined by the inequalities

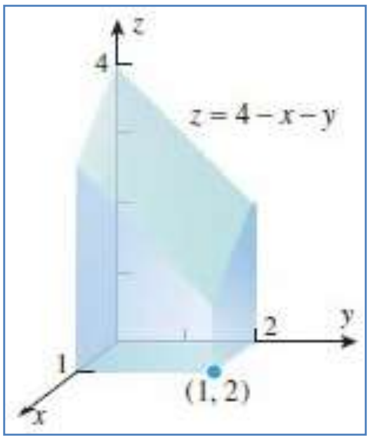
$$a \le x \le b,\ c \le y \le d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx$$

<u>Example</u> Use a double integral to find the volume of the solid that is bounded above by the plane

$z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$



Solution:

$\int_0^2 \int_0^1 (4-x-y)dx\,dy \quad or \qquad \int_0^1 \int_0^2 (4-x-y)dy\,dx$

V=$\int_0^2 \int_0^1 (4-x-y)dx\,dy = \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy =$
$\int_0^2 \left(\frac{7}{2} - y\right) dy = [\frac{7}{2}y - \frac{y^2}{2}]_0^2 = 5$

<u>H.w</u> check this result by evaluating the second integrals

<u>Example</u> Evaluate the double integral

$$\iint_R y^2 x\,dA$$

Over the rectangle $R = \{(x,y): -3 \le x \le 2, 0 \le y \le 1)\}$

$$\iint_R y^2x\,dA = \int_0^1 \int_{-3}^2 y^2x\,dx\,dy = \ \int_0^1 [\frac{1}{2}y^2x^2]_{x=-3}^{2}dy = \int_0^1 (-\frac{5}{2}y^2)\,dy = -\frac{5}{6}y^3]_0^1 \ = -\frac{5}{6}$$

Properties of Double Integrals

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA \quad (c \text{ a constant})$$

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint_R [f(x,y)-g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$$

It is evident intuitively that if *f(x, y)* is nonnegative on a region *R*, then subdividing *R* into two regions R_1 and R_2 has the effect of subdividing the solid between *R* and $z = f(x, y)$ into two solids, the sum of whose volumes is the volume of the entire solid.

This suggests the following result, which holds even if *f* has negative values:

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

Double Integrals over Nonrectangular Regions

Iterated Integrals with Nonconstant Limits Of Integration

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

Example: Evaluate

(a) $\int_0^1 \int_{-x}^{x^2} y^2x\,dy\,dx$ (b) $\int_0^{\pi/3} \int_0^{\cos y} x\sin y\,dx\,dy$

Solution: **(a)** $\int_0^1 \int_{-x}^{x^2} y^2x\,dy\,dx = \int_0^1 [\int_{-x}^{x^2} y^2x\,dy]\,dx = \int_0^1 \frac{y^3x}{3}]_{y=-x}^{x^2}\,dx$

$$= \int_0^1 [\frac{x^7}{3} + \frac{x^4}{3}]\,dx = \left(\frac{x^8}{24} + \frac{x^5}{15}\right)]_0^1 = \frac{13}{120}$$

$$\textbf{(b)} \int_0^{\pi/3}\int_0^{\cos y} x\sin y\,dx\,dy = \int_0^{\pi/3}\Big[\int_0^{\cos y} x\,\sin y\,dx\Big]dy = \int_0^{\pi/3} \frac{x^2}{2}\sin y\Big]_{x=0}^{\cos y} dy =$$

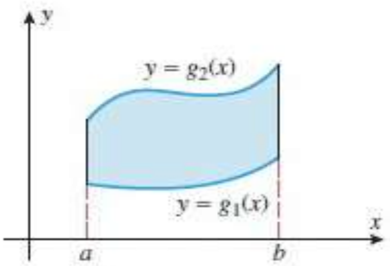
$$\int_0^{\pi/3}[\frac{1}{2}cos^2 y\sin y]\,dy = -\frac{1}{6}cos^3 y]_0^{\pi/3} = \frac{7}{48}$$

<u>Double Integrals over Nonrectangular Regions</u>

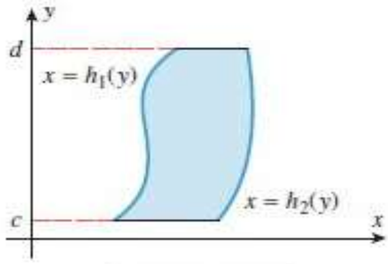
<u>Definition</u>

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$,
where $g_1(x) \le g_2(x)$ for $a \le x \le b$

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$



A type I region
(a)



A type II region
(b)

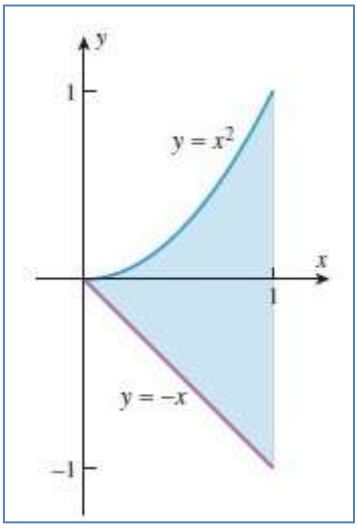
(a) *If R is a type I region on which $f(x, y)$ is continuous, then*

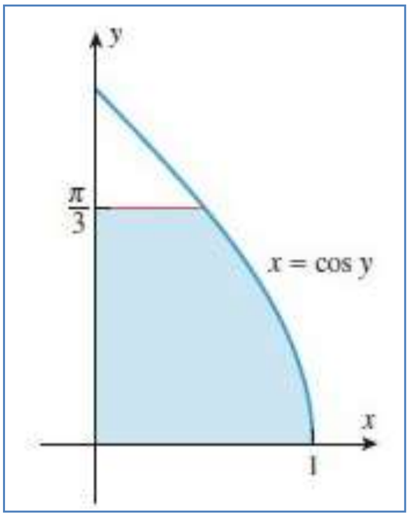
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(b) *If R is a type II region on which $f(x, y)$ is continuous, then*

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

The integral in Example (a) is the double integral of the function $f(x, y) = y^2x$ over the type I region R bounded on the left and right by the vertical lines $x = 0$ and $x = 1$ and bounded below and above by the curves $y = -x$ and $y = x^2$.
The integral in Example (b) is the double integral of the function $f(x, y) = x \sin y$ over the type II region R bounded below and above by the horizontal lines $y = 0$ and $y = \pi/3$ and bounded on the left and right by the curves $x = 0$ and $x = \cos y$





Determining Limits of Integration: Type I Region

Step 1. Since x is held fixed for the first integration, we draw a vertical line through the region R at an arbitrary fixed value x. This line crosses the boundary of R twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration.

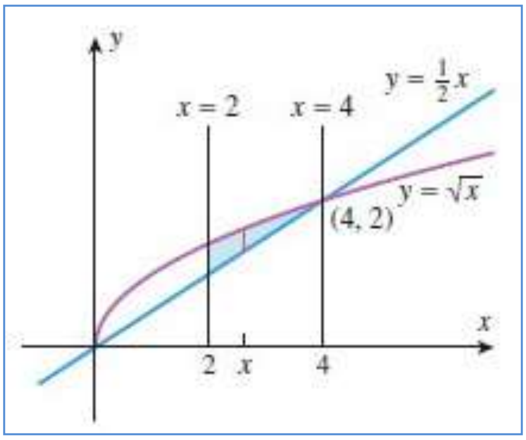
Step 2. Imagine moving the line drawn in Step 1 first to the left and then to the right. The leftmost position where the line intersects the region R is $x = a$, and the rightmost position where the line intersects the region R is $x = b$. This yields the limits for the x-integration.

Example: Evaluate

$$\iint_R xy\, dA$$

over the region R enclosed between $y = \frac{1}{2}x,\ y = \sqrt{x}$ and

x = 2 and x = 4



$$\iint_R xy\, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx$$

$$= \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx = \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4$$

$$= \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}$$

Determining Limits of Integration: Type II Region

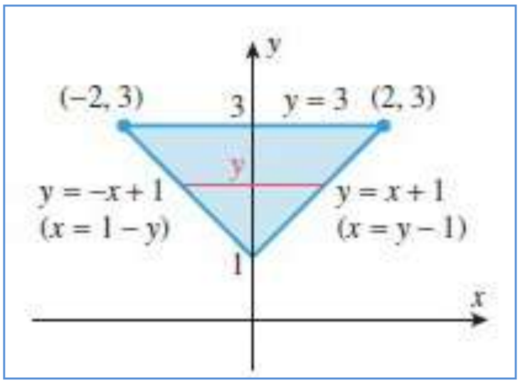
Step 1. Since y is held fixed for the first integration, we draw a horizontal line through the region R at a fixed value y. This line crosses the boundary of R twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration.

Step 2. Imagine moving the line drawn in Step 1 first down and then up. The lowest position where the line intersects the region R is $y = c$, and the highest position where the line intersects the region R is $y = d$. This yields the y-limits of integration.

Example: Evaluate

$$\iint_R (2x - y^2)dA$$

Over the triangle region R enclosed between the lines $y = -x + 1, y = x + 1, and\ y = 3$



We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in Figure. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration.

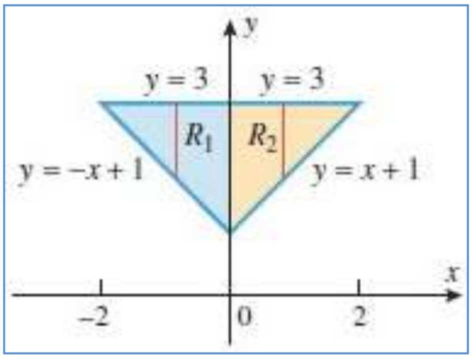
Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\iint_R (2x - y^2)dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]dy$$

$$= \int_1^3 (2y^2 - 2y^3)dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}$$

H.W Resolve above example as type I region



Example: Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane

$$z = 4 - 4x - 2y$$

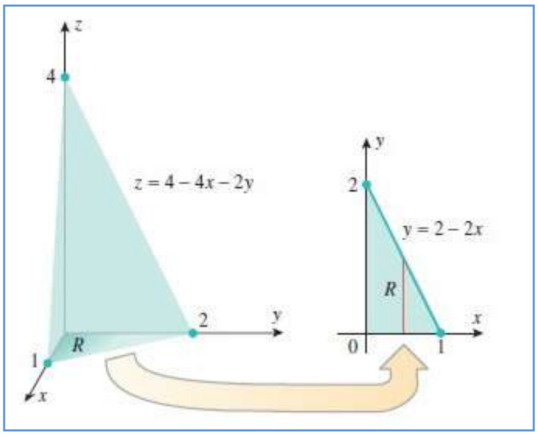
and below by the triangular region R shown in Figure. Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y)\, dA$$

The region R is bounded by the x-axis, the y-axis, and the line $y = 2 - 2x$ [set $z = 0$], so that treating R as a type I region yields

$$V = \iint_R (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y) dy\, dx =$$

$$\int_0^1 [4y - 4yx - y^2]_{y=0}^{2-2x}\, dx = \int_0^1 (4 - 8x + 4x^2) dx = \frac{4}{3}$$



Example: Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ *and* $z = 0$.

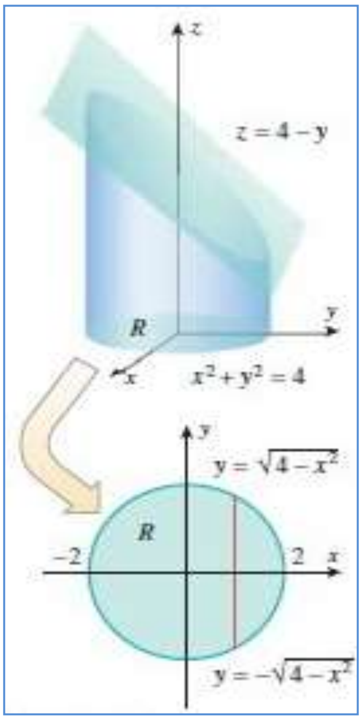
Solution. The solid shown in Figure is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y) dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi$$



Reversing the Order of Integration

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration. The next example illustrates how this is done.

Example: Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$$

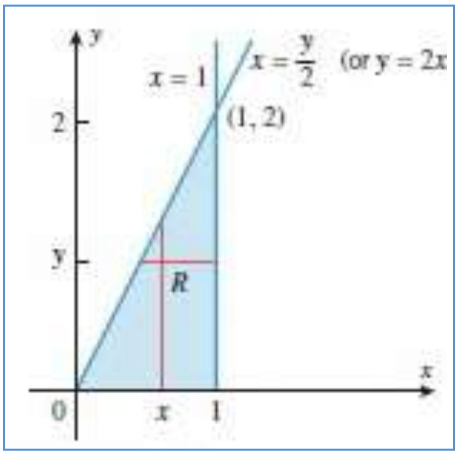
cannot be evaluated by performing the *x*-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

Solution: For the inside integration, y is fixed and x varies from the line $x = y/2$ to the line $x = 1$. For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R.
To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dx\, dy = \iint_R e^{x^2}\, dA$$

$$= \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 \left[e^{x^2} y\right]_{y=0}^{2x} dx$$

$$= \int_0^1 2xe^{x^2} dx = e^{x^2}\Big]_0^1 = e - 1$$



Area Calculated as a Double Integral

We stated that the volume V of a right cylinder with cross-sectional area A and height h is

$$V = A \cdot h$$

Now suppose that we are interested in finding the area A of a region R in the *xy*-plane. If we translate the region R upward 1 unit, then the resulting solid will be a right cylinder that has cross-sectional area A, base R, and the plane $z = 1$ as its top. Thus, it follows that

$$\iint_R 1\, dA = (area\ of\ R).1$$

Which we can rewrite as

$$area\ of\ R = \iint_R 1\, dA = \iint_R dA$$

Example: Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution: The region R may be treated equally well as type I or type II.

(a) Treating R as type I yields

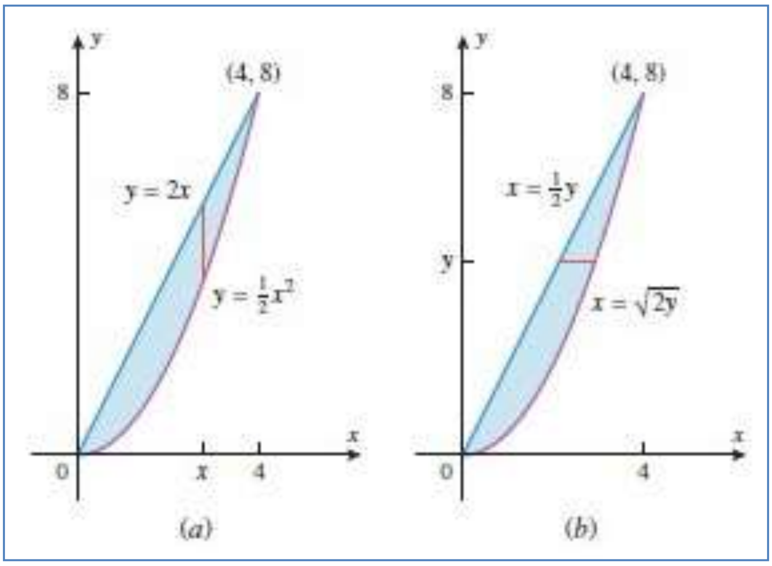
$$area\ of\ R = \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 [y]_{y=\frac{x^2}{2}}^{2x}\ dx$$

$$= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

(b) Treating R as type II yields

$$area\ of\ R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 [x]_{x=\frac{y}{2}}^{\sqrt{2y}}\ dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = 16/3$$



Hint

$$y = \frac{1}{2}x^2, \qquad y = 2x$$

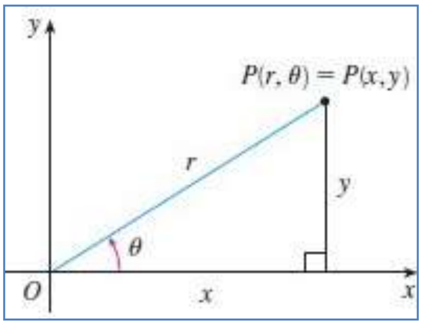
$$2x = \frac{1}{2}x^2 \quad \rightarrow \quad x = 0 \quad y = 0, \qquad x = 4 \quad y = 8$$

Double Integrals in Polar Coordinates

Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.

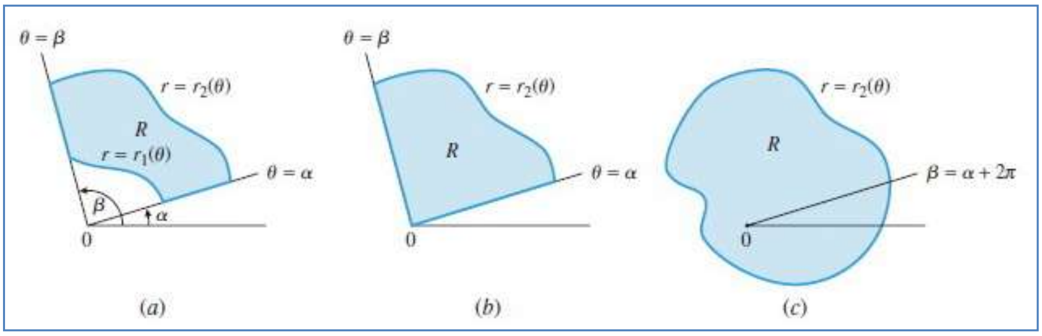
Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates.

Recall from Figure shown that the polar coordinates of a point are related to the rectangular coordinates by the equations



$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y\_ = r\sin\theta$

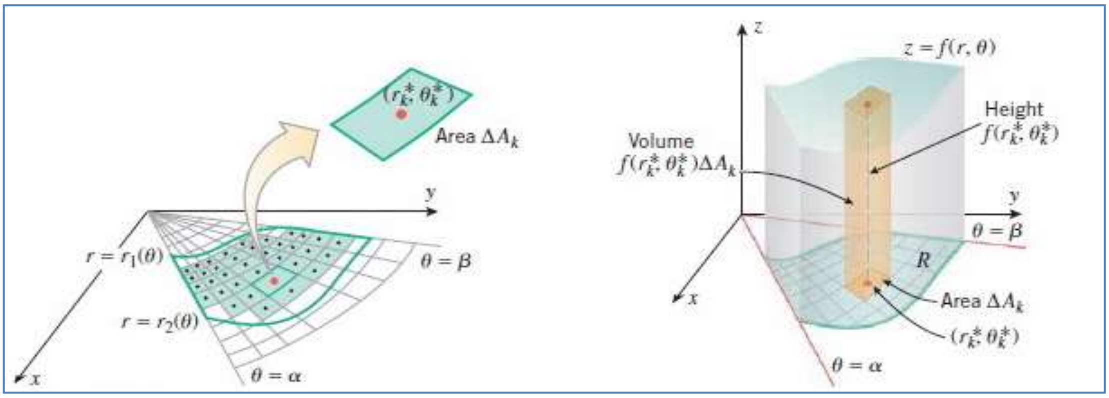
A region R in a polar coordinate system that is enclosed between two rays, θ = α and θ = β, and two polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$.



Definition A ***simple polar region*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \le \beta$ (ii) $\beta - \alpha \le 2\pi$ (iii) $0 \le r_1(\theta) \le r_2(\theta)$

The volume problem in polar coordinates Given a function $f(r, \theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$



Which is called the ***polar double integral*** of *f(r, θ)* over *R*. If *f(r, θ)* is continuous and nonnegative on *R*, then the volume can be expressed as

$$V = \iint_R f(r,\theta)dA = \lim_{n\to\infty}\sum_{k=1}^{n} f(r_k^*,\theta_k^*)\Delta A_k$$

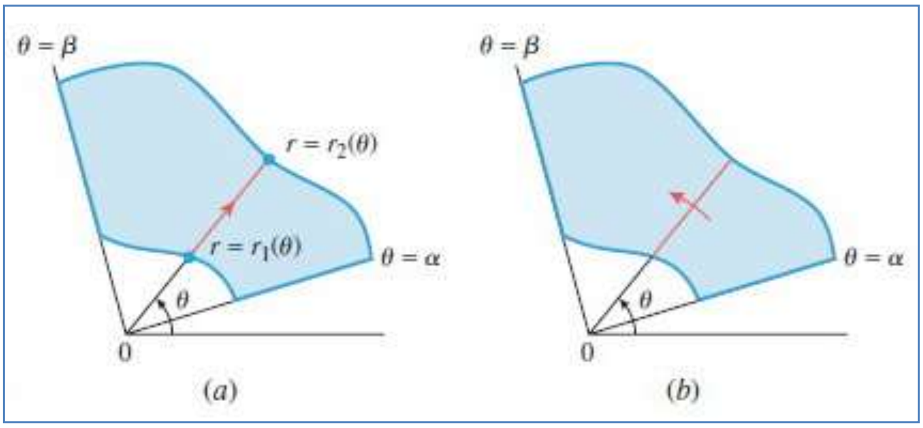
The volume V can be expressed as the iterated integral

$$V = \iint_R f(r,\theta)dA = \int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)r\,dr\,d\theta$$

Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ. This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration.

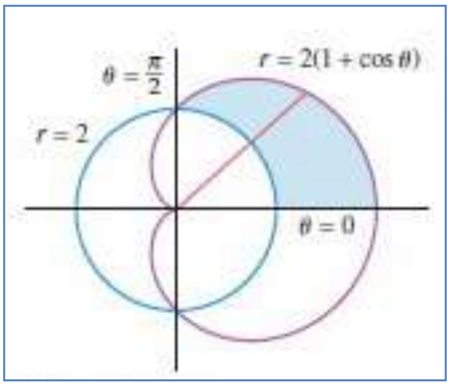
Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$ and the greatest angle is $\theta = \beta$. This determines the θ-limits of integration.



Example: Evaluate

$$\iint_R \sin\theta\, dA$$

Where R is the region in the first quadrant that is outside the circle r = 2 and inside the cardioid $r = 2(1 + \cos\theta)$.



Solution:

$$\iint_R \sin\theta \, dA = \int_0^{\frac{\pi}{2}} \int_2^{2(1+\cos\theta)} (\sin\theta) r \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{1}{2} r^2 sin\,\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - sin\theta] \, d\theta = 2[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta]_0^{\frac{\pi}{2}}$$

$$= 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$$

Example:

The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

Solution:

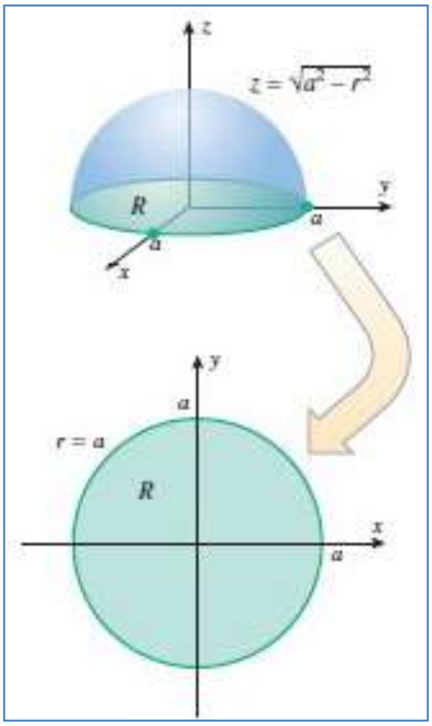
In cylindrical coordinates the upper hemisphere is given by the equation

$$z = a^2 - r^2$$

So the volume enclosed by the entire sphere is

$$V = 2\iint_R \sqrt{a^2 - r^2} \, dA$$

Where R is the circular region shown in figure. Thus,



$$V = 2\iint_R \sqrt{a^2 - r^2} \, dA$$

$$= \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \, (2r) dr \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{2}{3}(a^2 - r^2)^{\frac{3}{2}}\right]_{r=0}^a d\theta = \int_0^{2\pi} \frac{2}{3} a^3 d\theta$$

$$= \left[\frac{2}{3} a^3 \theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3$$

Finding Areas Using Polar Double Integrals

$$area\ of\ R = \iint_R 1 \, dA = \iint_R dA$$

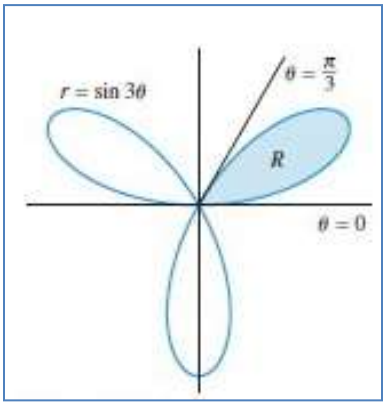
Example: Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.

Solution: The rose is sketched in Figure. We will calculate the area of the petal R in the first quadrant and multiply by three.

$$A = 3\iint_R dA = 3\int_0^{\frac{\pi}{3}}\int_0^{\sin 3\theta} r\,dr\,d\theta =$$

$$\frac{3}{2}\int_0^{\frac{\pi}{3}} sin^2\,3\theta\,d\theta =$$

$$\frac{3}{4}\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta = \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{1}{4}\pi$$



Example: Use polar coordinates to evaluate

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}(x^2+y^2)^{3/2}dy\,dx$$

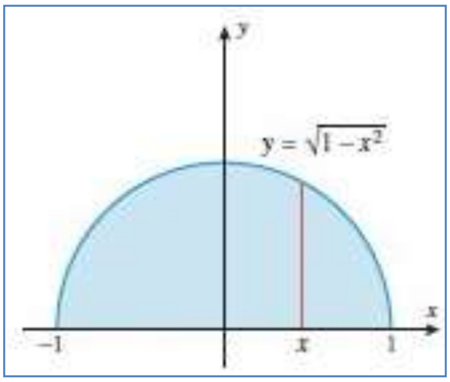
Solution: In this problem we are starting with an iterated integral in rectangular coordinates rather than a double integral, so before we can make the conversion to polar coordinates we will have to identify the region of integration. To do this, we observe that for fixed x the y-integration runs from $y = 0$ to $y = \sqrt{1-x^2}$, which tells us that the lower boundary of the region is the x-axis and the upper boundary is a semicircle of radius 1 centered at the origin. From the x-integration we see that x varies from −1 to 1, so we conclude that the region of integration is as shown in Figure. In polar coordinates, this is the region swept out as r varies between 0 and 1 and θ varies between 0 and π. Thus,

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}(x^2+y^2)^{\frac{3}{2}}dy\,dx = \iint_R (x^2+y^2)^{\frac{3}{2}}\,dA$$

$$= \int_0^{\pi}\int_0^{1}(r^3)r\,dr\,d\theta = \int_0^{\pi}\frac{1}{5}d\theta = \frac{\pi}{5}$$

Hint:

$$x = r\,cos\,\theta,\quad y = r\,sin\,\theta,\quad x^2+y^2 = r^2$$



Example: Evaluate

$$\iint_R (3x + 4y^2)\, dA$$

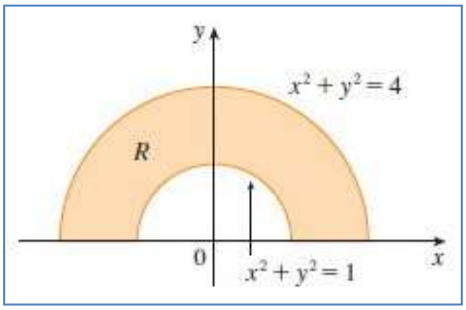
, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution The region R can be described as

$$R = \{(x, y) | y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$

It is the half-ring shown in Figure, and in polar coordinates it is given by $1 \leq r \leq 2, 0 \leq \theta \leq \pi$. Therefore,

$$\iint_R (3x + 4y^2)\, dA = \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2 \sin^2\theta) r\, dr\, d\theta$$



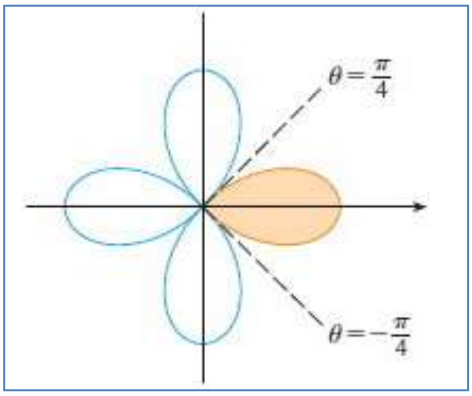
$$= \int_0^{\pi} [r^3 \cos\theta + r^4 \sin^2\theta]_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} \left[7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta]_0^{\pi} = \frac{15\pi}{2}$$

Example: Use a double integral to find the area enclosed by one loop of the four leaved rose $= \cos 2\theta$.

Solution: From the sketch of the curve in Figure, we see that a loop is given by the region



$$R = \{(r, \theta) \,|\, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq \cos 2\theta\}$$

$$\text{Area of } R = \iint_R dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\, dr d\theta$$

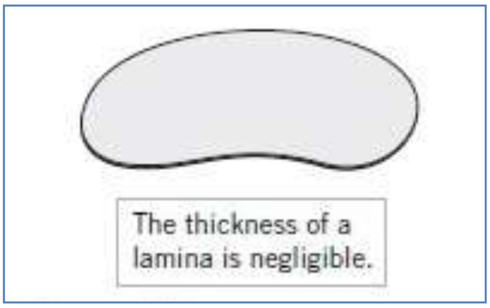
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2} r^2\right]_0^{\cos 2\theta} d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta$$

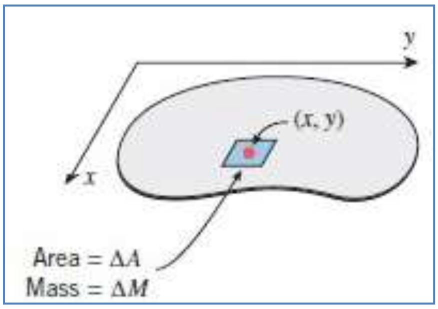
$$= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\, d\theta = \frac{1}{4} [\theta + \frac{1}{4}\sin 4\theta]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

Centers of Gravity Using Multiple Integrals

Moments and Centers of Mass

An idealized flat object that is thin enough to be viewed as a two-dimensional plane region is called a ***lamina***. A lamina is called ***homogeneous*** if its composition is uniform throughout and ***inhomogeneous*** otherwise. The ***density*** of a *homogeneous* lamina was defined to be its mass per unit area. Thus, the density δ of a homogeneous lamina of mass M and area A is given by $\boldsymbol{\delta = M/A}$.
The density at a point (x, y) can be specified by a function δ(x, y), called the density function,





Mass of a lamina If a lamina with a continuous density function $\delta(x, y)$ occupies a region R in the xy-plane, then its total mass M is given by

$$M = \lim_{n\to+\infty} \sum_{k=1}^{n} \delta(x_k^*, y_k^*)\Delta A_k = \iint_R \delta(x,y)dA$$

Example: A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has density function

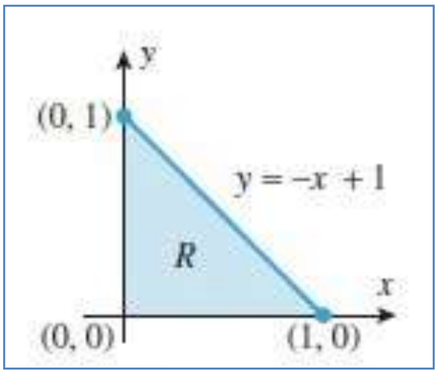
$\delta(x,y) = xy$. Find its total mass.

Solution: the mass M of the lamina is

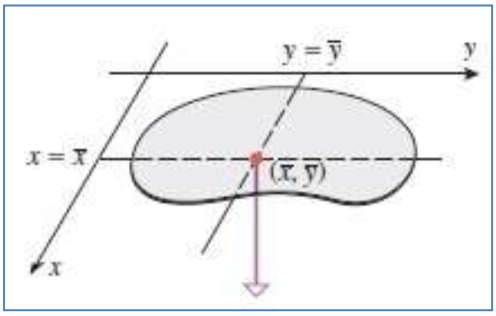
$$M = \iint_R \delta(x,y)dA = \iint_R xy\, dA$$

$$= \int_0^1 \int_0^{-x+1} xy\, dy\, dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{-x+1} dx$$

$$= \int_0^1 \left[\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right] dx = \frac{1}{24}\ (unit\ of\ mass)$$



<u>The Center of Gravity</u> of a lamina occupying a region *R* in the horizontal *xy*-plane is the point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is "equivalent" to that of a single force acting at $(\bar{x}, \bar{y})$. If $(\bar{x}, \bar{y})$ is in *R*, then the lamina will balance horizontally on a point of support placed at $(\bar{x}, \bar{y})$.



Suppose that a lamina with a continuous density function δ(x, y) occupies a region R in a horizontal xy-plane. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity.

Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{\iint_R x\,\delta(x,y)dA}{\iint_R \delta(x,y)dA}, \qquad \bar{y} = \frac{\iint_R y\,\delta(x,y)dA}{\iint_R \delta(x,y)dA}$$

Alternative Formulas for Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{M_y}{M} = \frac{1}{mass\ of\ R}\iint_R x\,\delta(x,y)dA, \qquad \bar{y} = \frac{M_x}{M} = \frac{1}{mass\ of\ R}\iint_R y\,\delta(x,y)dA$$

My is called the ***first moment of the lamina about the y-axis***

Mx is called the ***first moment of the lamina about the x-axis***.

<u>Example:</u> Find the center of gravity of the triangular lamina with vertices (0, 0), (0, 1), and (1, 0) and density function *δ(x, y) = xy*.

<u>Solution:</u> The lamina is shown in previous Example, we found the mass of the lamina to be

$$M = \iint_R \delta(x,y)dA = \iint_R xy\,dA = \frac{1}{24}$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\,\delta(x,y)dA = \iint_R x^2y\,dA = \int_0^1\int_0^{-x+1} x^2y\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left[\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2\right] dx = \frac{1}{60}$$

The moment of the lamina about the x-axis is

$$M_x = \iint_R y\,\delta(x,y)dA = \iint_R x\,y^2\,dA = \int_0^1 \int_0^{-x+1} x\,y^2\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{3}xy^3\right]_{y=0}^{-x+1} dx = \int_0^1 \left[-\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x\right] dx = \frac{1}{60}$$

$$\bar{x} = \frac{M_y}{M} = \frac{1/60}{1/24} = \frac{2}{5}, \qquad \bar{y} = \frac{M_x}{M} = \frac{1/60}{1/24} = \frac{2}{5}$$

So the center of gravity is $(\frac{2}{5}, \frac{2}{5})$

the center of gravity of a *homogeneous* lamina is called the ***centroid of the lamina*** or sometimes the ***centroid of the region R***. Because the density function δ is constant for a homogeneous lamina, the factor δ may be moved through the integral and canceled. The centroid $(\bar{x}, \bar{y})$ is a geometric property of the region R and is given by the following formulas:

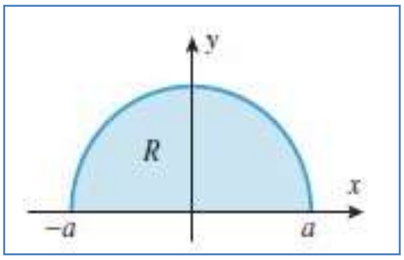
$$\bar{\boldsymbol{x}} = \frac{\iint_R \boldsymbol{x}\,\boldsymbol{dA}}{\iint_R \boldsymbol{dA}} = \frac{\mathbf{1}}{\boldsymbol{area\ of\ R}} \iint_R \boldsymbol{x}\,\boldsymbol{dA}$$

$$\bar{\boldsymbol{y}} = \frac{\iint_R \boldsymbol{y}\,\boldsymbol{dA}}{\iint_R \boldsymbol{dA}} = \frac{\mathbf{1}}{\boldsymbol{area\ of\ R}} \iint_R \boldsymbol{y}\,\boldsymbol{dA}$$

Example: Find the centroid of the semicircular region in Figure.

Solution: By symmetry, $\bar{x} = 0$ since the y-axis is obviously a line of balance.

$$\bar{y} = \frac{1}{area\ of\ R} \iint_R y\,dA = \frac{1}{\frac{1}{2}\pi a^2} \iint_R y\,dA$$



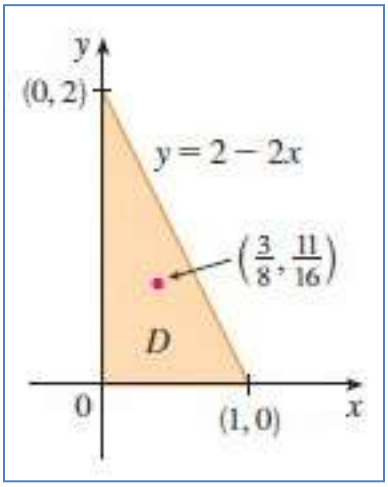
$$= \frac{1}{\frac{1}{2}\pi a^2} \int_0^{\pi} \int_0^{a} (r\sin\theta) r\,dr\,d\theta = \frac{1}{\frac{1}{2}\pi a^2} \int_0^{\pi} \left[\frac{1}{3}r^3 \sin\theta\right]_{r=0}^{a} d\theta$$

$$= \frac{1}{\frac{1}{2}\pi a^2} \left(\frac{1}{3}a^3\right) \int_0^{\pi} \sin\theta\,d\theta = \frac{1}{\frac{1}{2}\pi a^2} \left(\frac{2}{3}a^3\right) = \frac{4a}{3\pi}$$

Example: Find the mass and center of mass of a triangular lamina with vertices (0,0), (1,0), and (0,2) if the density function is $\delta(x,y) = 1 + 3x + y$.

Solution: The triangle is shown in Figure. (Note that the equation of the upper boundary is

$y = 2 - 2x$) The mass of the lamina is



$$\boldsymbol{M} = \iint_R \delta(x,y)dA = \iint_R (1 + 3x + y)\, dA$$

$$= \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx = \int_0^1 \left[y + 3xy + \frac{y^2}{2}\right]_{y=0}^{y=2-2x} dx$$

$$= 4\int_0^1 [1 - x^2]dx = 4\left[x - \frac{x^3}{3}\right] = \frac{8}{3}\ (unit\ of\ mass)$$

$$\bar{\boldsymbol{x}} = \frac{1}{M}\iint_R x\, \delta(x,y)dA = \frac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2}\right]_{y=0}^{2-2x} dx = \frac{3}{2}\int_0^1 (x - x^3)dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$$

$$\bar{\boldsymbol{y}} = \frac{1}{M}\iint_R y\, \delta(x,y)dA = \frac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{2-2x} dx = \frac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)dx$$

$$= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$

The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$.

Example: The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

Solution: Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$. Then the distance from a point (x,y) to the center of the circle (the origin) is $\sqrt{x^2+y^2}$. Therefore the density function is

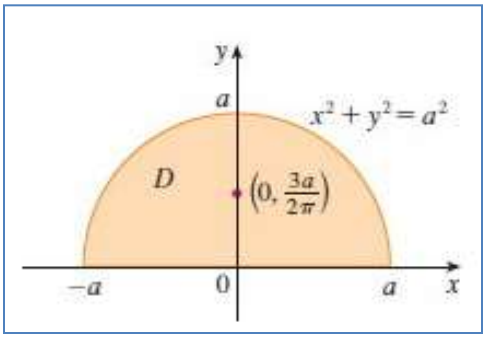
$$\delta(x,y) = K\sqrt{x^2+y^2}$$

Where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2+y^2} = r$ and the region R is given by $0 \le r \le a, 0 \le \theta \le \pi$. Thus the mass of the lamina is

$$\boldsymbol{M} = \iint\limits_R \delta(x,y)dA = \iint\limits_R K\sqrt{x^2+y^2}dA$$

$$= \int_0^{\pi}\int_0^a (Kr)r\,dr\,d\theta = K\int_0^{\pi} d\theta \int_0^a r^2 dr = K\pi \frac{r^3}{3}\Big]_0^a = \frac{K\pi a^3}{3}$$

Both the lamina and the density function are symmetric with respect to the -axis, so the center of mass must lie on the *y*-axis, that is $\boldsymbol{\bar{x} = 0}$. The *y*-coordinate is given by



$$\boldsymbol{\bar{y}} = \frac{1}{M}\iint\limits_R y\,\delta(x,y)dA = \frac{3}{K\pi a^3}\int_0^{\pi}\int_0^a r\sin\theta(Kr)r\,dr\,d\theta$$

$$= \frac{3}{\pi a^3}\int_0^{\pi}\sin\theta\,d\theta\int_0^a r^3 dr = \frac{3}{\pi a^3}[-\cos\theta]_0^{\pi}\left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{3}{\pi a^3}\frac{2a^4}{4} = \frac{3a}{2\pi}$$

Therefore the center of mass is located at the point $\boldsymbol{(0, \frac{3a}{2\pi})}$.