2
DISCRETE-TIME SIGNALS AND SYSTEMS

2.0 INTRODUCTION

The term *signal* is generally applied to something that conveys information. Signals generally convey information about the state or behavior of a physical system, and often, signals are synthesized for the purpose of communicating information between humans or between humans and machines. Although signals can be represented in many ways, in all cases the information is contained in some pattern of variations. Signals are represented mathematically as functions of one or more independent variables. For example, a speech signal is represented mathematically as a function of time, and a photographic image is represented as a brightness function of two spatial variables. A common convention—and one that usually will be followed in this book—is to refer to the independent variable of the mathematical representation of a signal as time, although in specific examples the independent variable may in fact not represent time.

The independent variable in the mathematical representation of a signal may be either continuous or discrete. *Continuous-time signals* are defined along a continuum of times and thus are represented by a continuous independent variable. Continuous-time signals are often referred to as *analog signals*. *Discrete-time signals* are defined at discrete times, and thus, the independent variable has discrete values; i.e., discrete-time signals are represented as sequences of numbers. Signals such as speech or images may have either a continuous- or a discrete-variable representation, and if certain conditions hold, these representations are entirely equivalent. Besides the independent variables being either continuous or discrete, the signal amplitude may be either continuous or discrete. *Digital signals* are those for which both time and amplitude are discrete.

Signal-processing systems may be classified along the same lines as signals. That is, continuous-time systems are systems for which both the input and the output are continuous-time signals, and discrete-time systems are those for which both the input and the output are discrete-time signals. Similarly, a digital system is a system for which both the input and the output are digital signals. Digital signal processing, then, deals with the transformation of signals that are discrete in both amplitude and time. The principal focus in this book is on discrete-time (rather than digital) signals and systems. However, the theory of discrete-time signals and systems is also exceedingly useful for digital signals and systems, particularly if the signal amplitudes are finely quantized. The effects of signal amplitude quantization are considered in Sections 4.8, 6.7–6.9, and 9.7.

Discrete-time signals may arise by sampling a continuous-time signal, or they may be generated directly by some discrete-time process. Whatever the origin of the discrete-time signals, discrete-time signal-processing systems have many attractive features. They can be realized with great flexibility with a variety of technologies, such as charge transport devices, surface acoustic wave devices, general-purpose digital computers, or high-speed microprocessors. Complete signal-processing systems can be implemented using VLSI techniques. Discrete-time systems can be used to simulate analog systems or, more importantly, to realize signal transformations that cannot be implemented with continuous-time hardware. Thus, discrete-time representations of signals are often desirable when sophisticated and flexible signal processing is required.

In this chapter, we consider the fundamental concepts of discrete-time signals and signal-processing systems for one-dimensional signals. We emphasize the class of linear time-invariant discrete-time systems. Many of the properties and results that we derive in this and subsequent chapters will be similar to properties and results for linear time-invariant continuous-time systems, as presented in a variety of texts. (See, for example, Oppenheim and Willsky, 1997.) In fact, it is possible to approach the discussion of discrete-time systems by treating sequences as analog signals that are impulse trains. This approach, if implemented carefully, can lead to correct results and has formed the basis for much of the classical discussion of sampled data systems. (See, for example, Phillips and Nagle, 1995.) However, not all sequences arise from sampling a continuous-time signal, and many discrete-time systems are not simply approximations to corresponding analog systems. Furthermore, there are important and fundamental differences between discrete- and continuous-time systems. Therefore, rather than attempt to force results from continuous-time system theory into a discrete-time framework, we will derive parallel results starting within a framework and with notation that is suitable to discrete-time systems. Discrete-time signals will be related to continuous-time signals only when it is necessary and useful to do so.

2.1 DISCRETE-TIME SIGNALS: SEQUENCES

Discrete-time signals are represented mathematically as sequences of numbers. A sequence of numbers x, in which the nth number in the sequence is denoted $x[n]$,[1] is formally written as

$$x = \{x[n]\}, \qquad -\infty < n < \infty, \tag{2.1}$$

where n is an integer. In a practical setting, such sequences can often arise from periodic

[1]A sequence is simply a function whose domain is the set of integers. Note that we use [ ] to enclose the independent variable of such functions, and we use ( ) to enclose the independent variable of continuous-variable functions.

sampling of an analog signal. In this case, the numeric value of the nth number in the sequence is equal to the value of the analog signal, $x_a(t)$, at time nT; i.e.,

$$x[n] = x_a(nT), \qquad -\infty < n < \infty. \tag{2.2}$$

The quantity T is called the *sampling period*, and its reciprocal is the *sampling frequency*. Although sequences do not always arise from sampling analog waveforms, it is convenient to refer to $x[n]$ as the "nth sample" of the sequence. Also, although, strictly speaking, $x[n]$ denotes the nth number in the sequence, the notation of Eq. (2.1) is often unnecessarily cumbersome, and it is convenient and unambiguous to refer to "the sequence $x[n]$" when we mean the entire sequence, just as we referred to the "analog signal $x_a(t)$." Discrete-time signals (i.e., sequences) are often depicted graphically as shown in Figure 2.1. Although the abscissa is drawn as a continuous line, it is important to recognize that $x[n]$ is defined only for integer values of n. It is not correct to think of $x[n]$ as being zero for n is not an integer; $x[n]$ is simply undefined for noninteger values of n.

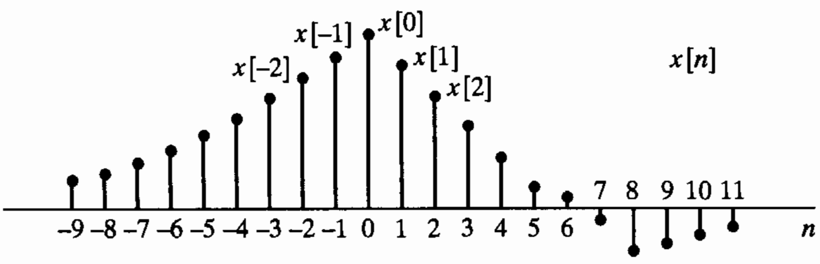


Figure 2.1 Graphical representation of a discrete-time signal.

As an example, Figure 2.2(a) shows a segment of a speech signal corresponding to acoustic pressure variation as a function of time, and Figure 2.2(b) presents a sequence

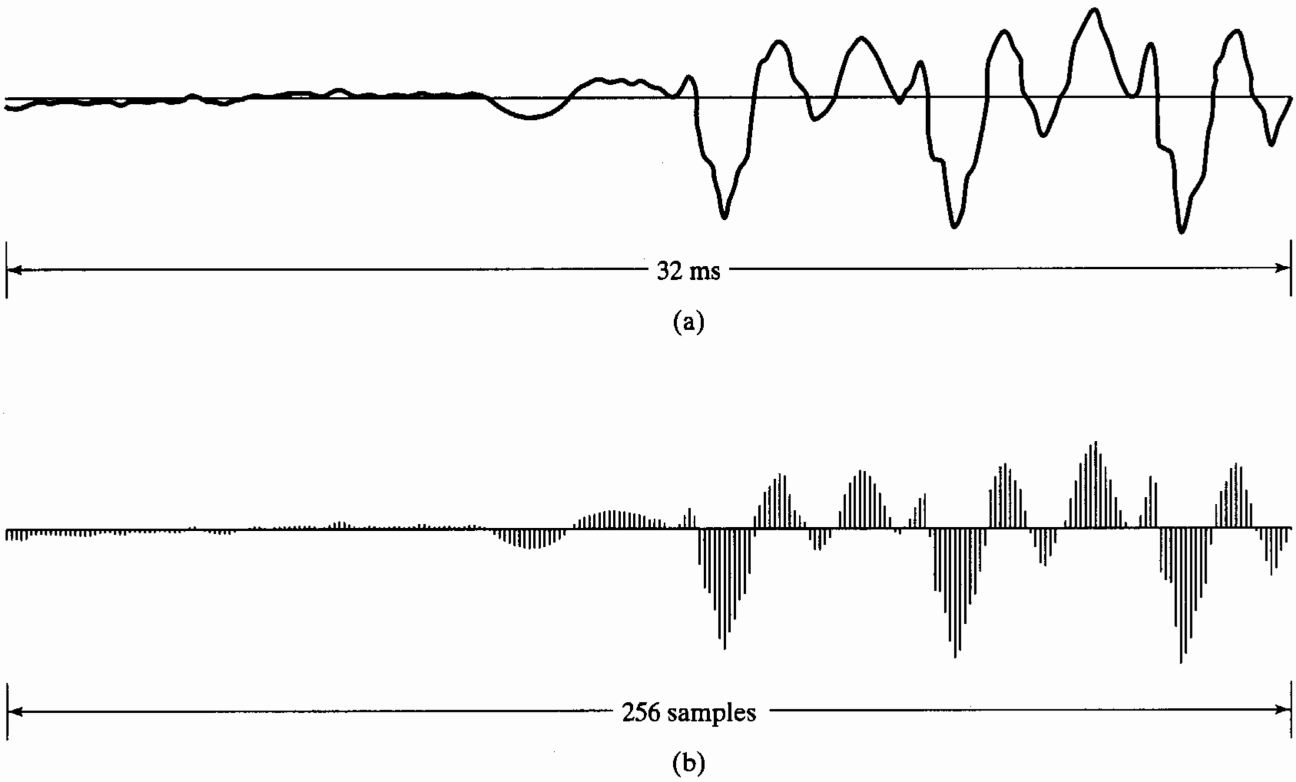


Figure 2.2 (a) Segment of a continuous-time speech signal. (b) Sequence of samples obtained from part (a) with $T = 125\ \mu s$.

of samples of the speech signal. Although the original speech signal is defined at all values of time t, the sequence contains information about the signal only at discrete instants. From the sampling theorem, discussed in Chapter 4, the original signal can be reconstructed as accurately as desired from a corresponding sequence of samples if the samples are taken frequently enough.

2.1.1 Basic Sequences and Sequence Operations

In the analysis of discrete-time signal-processing systems, sequences are manipulated in several basic ways. The product and sum of two sequences $x[n]$ and $y[n]$ are defined as the sample-by-sample product and sum, respectively. Multiplication of a sequence $x[n]$ by a number α is defined as multiplication of each sample value by α. A sequence $y[n]$ is said to be a delayed or shifted version of a sequence $x[n]$ if

$$y[n] = x[n - n_0], \tag{2.3}$$

where n_0 is an integer.

In discussing the theory of discrete-time signals and systems, several basic sequences are of particular importance. These sequences are shown in Figure 2.3 and are discussed next.

The *unit sample sequence* (Figure 2.3a) is defined as the sequence

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases} \tag{2.4}$$

As we will see, the unit sample sequence plays the same role for discrete-time signals and systems that the unit impulse function (Dirac delta function) does for continuous-time signals and systems. For convenience, the unit sample sequence is often referred to as a *discrete-time impulse* or simply as an *impulse*. It is important to note that a discrete-time impulse does not suffer from the mathematical complications of the continuous-time impulse; its definition is simple and precise.

As we will see in the discussion of linear systems, one of the important aspects of the impulse sequence is that an arbitrary sequence can be represented as a sum of scaled, delayed impulses. For example, the sequence $p[n]$ in Figure 2.4 can be expressed as

$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7]. \tag{2.5}$$

More generally, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{2.6}$$

We will make specific use of Eq. (2.6) in discussing the representation of discrete-time linear systems.

The *unit step sequence* (Figure 2.3b) is given by

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{2.7}$$

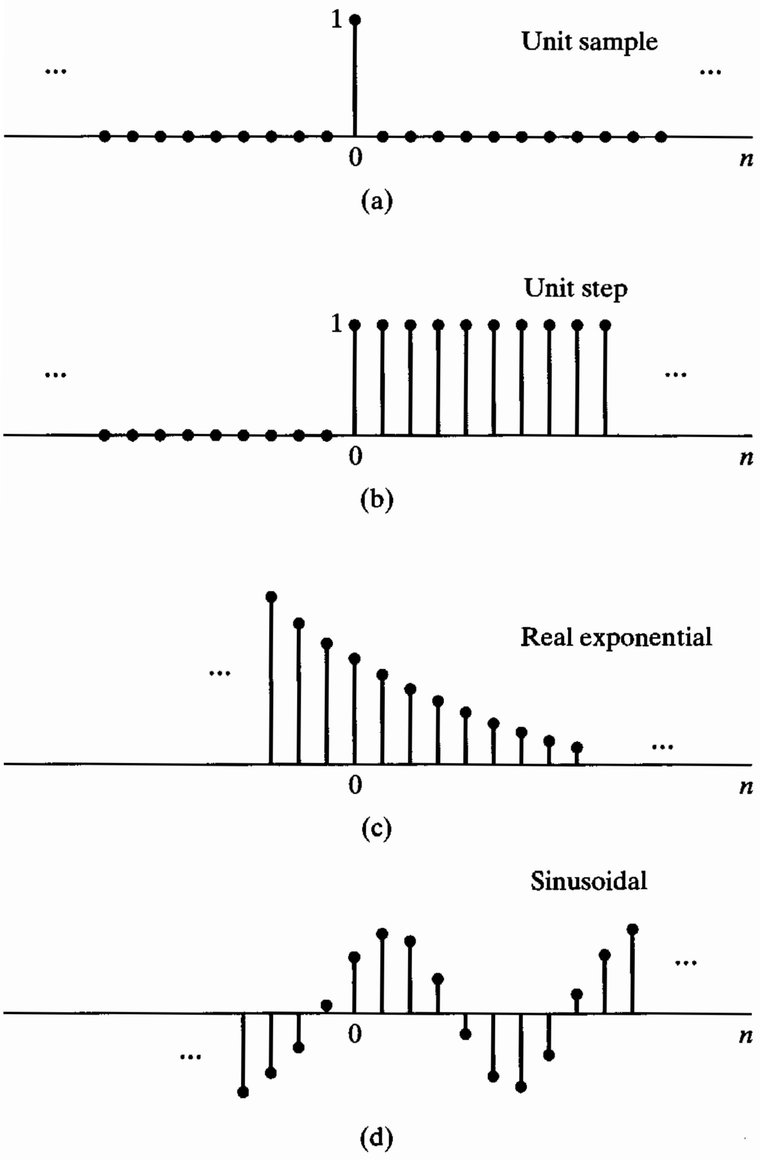


Figure 2.3 Some basic sequences. The sequences shown play important roles in the analysis and representation of discrete-time signals and systems.

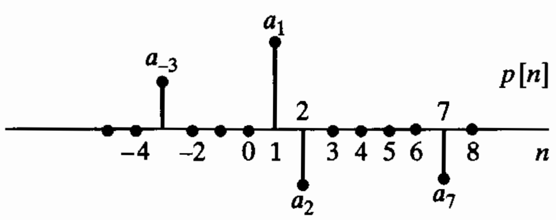


Figure 2.4 Example of a sequence to be represented as a sum of scaled, delayed impulses.

The unit step is related to the impulse by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]; \tag{2.8}$$

that is, the value of the unit step sequence at (time) index n is equal to the accumulated sum of the value at index n and all previous values of the impulse sequence. An alternative representation of the unit step in terms of the impulse is obtained by interpreting the unit step in Figure 2.3(b) in terms of a sum of delayed impulses as in Eq. (2.6). In this case, the nonzero values are all unity, so

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots \tag{2.9a}$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{2.9b}$$

Conversely, the impulse sequence can be expressed as the first backward difference of the unit step sequence, i.e.,

$$\delta[n] = u[n] - u[n-1]. \tag{2.10}$$

Exponential sequences are extremely important in representing and analyzing linear time-invariant discrete-time systems. The general form of an exponential sequence is

$$x[n] = A\alpha^n. \tag{2.11}$$

If A and α are real numbers, then the sequence is real. If $0 < \alpha < 1$ and A is positive, then the sequence values are positive and decrease with increasing n, as in Figure 2.3(c). For $-1 < \alpha < 0$, the sequence values alternate in sign, but again decrease in magnitude with increasing n. If $|\alpha| > 1$, then the sequence grows in magnitude as n increases.

Example 2.1 Combining Basic Sequences

We often combine basic sequences to form simple representations of other sequences. If we want an exponential sequence that is zero for $n < 0$, we can write this as the somewhat cumbersome expression

$$x[n] = \begin{cases} A\alpha^n, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{2.12}$$

A much simpler expression is $x[n] = A\alpha^n u[n]$.

Sinusoidal sequences are also very important. A sinusoidal sequence has the general form

$$x[n] = A\cos(\omega_0 n + \phi), \qquad \text{for all } n, \tag{2.13}$$

with A and ϕ real constants, and is illustrated in Figure 2.3(d).

The exponential sequence $A\alpha^n$ with complex α has real and imaginary parts that are exponentially weighted sinusoids. Specifically, if $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$, the sequence $A\alpha^n$ can be expressed in any of the following ways:

$$\begin{aligned} x[n] = A\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} \\ &= |A|\,|\alpha|^n e^{j(\omega_0 n + \phi)} \\ &= |A|\,|\alpha|^n \cos(\omega_0 n + \phi) + j|A|\,|\alpha|^n \sin(\omega_0 n + \phi). \end{aligned} \tag{2.14}$$

The sequence oscillates with an exponentially growing envelope if $|\alpha| > 1$ or with an exponentially decaying envelope if $|\alpha| < 1$. (As a simple example, consider the case $\omega_0 = \pi$.)

When $|\alpha| = 1$, the sequence is referred to as a *complex exponential sequence* and has the form

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi); \tag{2.15}$$

that is, the real and imaginary parts of $e^{j\omega_0 n}$ vary sinusoidally with n. By analogy with the continuous-time case, the quantity ω_0 is called the *frequency* of the complex sinusoid or complex exponential, and ϕ is called the *phase*. However, note that n is a dimensionless integer. Thus, the dimension of ω_0 must be radians. If we wish to maintain a closer analogy with the continuous-time case, we can specify the units of ω_0 to be radians per sample and the units of n to be samples.

The fact that n is always an integer in Eq. (2.15) leads to some important differences between the properties of discrete-time and continuous-time complex exponential sequences and sinusoidal sequences. An important difference between continuous-time and discrete-time complex sinusoids is seen when we consider a frequency $(\omega_0 + 2\pi)$. In this case,

$$\begin{aligned} x[n] &= Ae^{j(\omega_0+2\pi)n} \\ &= Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}. \end{aligned} \tag{2.16}$$

More generally, we can easily see that complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, where r is an integer, are indistinguishable from one another. An identical statement holds for sinusoidal sequences. Specifically, it is easily verified that

$$\begin{aligned} x[n] &= A\cos[(\omega_0 + 2\pi r)n + \phi] \\ &= A\cos(\omega_0 n + \phi). \end{aligned} \tag{2.17}$$

The implications of this property for sequences obtained by sampling sinusoids and other signals will be discussed in Chapter 4. For now, we simply conclude that, when discussing complex exponential signals of the form $x[n] = Ae^{j\omega_0 n}$ or real sinusoidal signals of the form $x[n] = A\cos(\omega_0 n + \phi)$, we need only consider frequencies in an interval of length 2π, such as $-\pi < \omega_0 \le \pi$ or $0 \le \omega_0 < 2\pi$.

Another important difference between continuous-time and discrete-time complex exponentials and sinusoids concerns their periodicity. In the continuous-time case, a sinusoidal signal and a complex exponential signal are both periodic, with the period equal to 2π divided by the frequency. In the discrete-time case, a periodic sequence is a sequence for which

$$x[n] = x[n + N], \qquad \text{for all } n, \tag{2.18}$$

where the period N is necessarily an integer. If this condition for periodicity is tested for the discrete-time sinusoid, then

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi), \tag{2.19}$$

which requires that

$$\omega_0 N = 2\pi k, \tag{2.20}$$

where k is an integer. A similar statement holds for the complex exponential sequence

$Ce^{j\omega_0 n}$; that is, periodicity with period N requires that

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \tag{2.21}$$

which is true only for $\omega_0 N = 2\pi k$, as in Eq. (2.20). Consequently, complex exponential and sinusoidal sequences are not necessarily periodic in n with period $(2\pi/\omega_0)$ and, depending on the value of ω_0, may not be periodic at all.

Example 2.2 Periodic and Aperiodic Discrete-Time Sinusoids

Consider the signal $x_1[n] = \cos(\pi n/4)$. This signal has a period of $N = 8$. To show this, note that $x[n+8] = \cos(\pi(n+8)/4) = \cos(\pi n/4 + 2\pi) = \cos(\pi n/4) = x[n]$, satisfying the definition of a discrete-time periodic signal. Contrary to our intuition from continuous-time sinusoids, increasing the frequency of a discrete-time sinusoid does not necessarily decrease the period of the signal. Consider the discrete-time sinusoid $x_2[n] = \cos(3\pi n/8)$, which has a higher frequency than $x_1[n]$. However, $x_2[n]$ is not periodic with period 8, since $x_2[n+8] = \cos(3\pi(n+8)/8) = \cos(3\pi n/8 + 3\pi) = -x_2[n]$. Using an argument analogous to the one for $x_1[n]$, we can show that $x_2[n]$ has a period of $N = 16$. Thus, increasing the frequency from $\omega_0 = 2\pi/8$ to $\omega_0 = 3\pi/8$ also increases the period of the signal. This occurs because discrete-time signals are defined only for integer indices n.

The integer restriction on n causes some sinusoidal signals not to be periodic at all. For example, there is no integer N such that the signal $x_3[n] = \cos(n)$ satisfies the condition $x_3[n+N] = x_3[n]$ for all n. These and other properties of discrete-time sinusoids that run counter to their continuous-time counterparts are caused by the limitation of the time index n to integers for discrete-time signals and systems.

When we combine the condition of Eq. (2.20) with our previous observation that ω_0 and $(\omega_0 + 2\pi r)$ are indistinguishable frequencies, it becomes clear that there are N distinguishable frequencies for which the corresponding sequences are periodic with period N. One set of frequencies is $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. These properties of complex exponential and sinusoidal sequences are basic to both the theory and the design of computational algorithms for discrete-time Fourier analysis, and they will be discussed in more detail in Chapters 8 and 9.

Related to the preceding discussion is the fact that the interpretation of high and low frequencies is somewhat different for continuous-time and discrete-time sinusoidal and complex exponential signals. For a continuous-time sinusoidal signal $x(t) = A\cos(\Omega_0 t + \phi)$, as Ω_0 increases, $x(t)$ oscillates more and more rapidly. For the discrete-time sinusoidal signal $x[n] = A\cos(\omega_0 n + \phi)$, as ω_0 increases from $\omega_0 = 0$ toward $\omega_0 = \pi$, $x[n]$ oscillates more and more rapidly. However, as ω_0 increases from $\omega_0 = \pi$ to $\omega_0 = 2\pi$, the oscillations become slower. This is illustrated in Figure 2.5. In fact, because of the periodicity in ω_0 of sinusoidal and complex exponential sequences, $\omega_0 = 2\pi$ is indistinguishable from $\omega_0 = 0$, and, more generally, frequencies around $\omega_0 = 2\pi$ are indistinguishable from frequencies around $\omega_0 = 0$. As a consequence, for sinusoidal and complex exponential signals, values of ω_0 in the vicinity of $\omega_0 = 2\pi k$ for any integer value of k are typically referred to as low frequencies (relatively slow oscillations), while values of ω_0 in the vicinity of $\omega_0 = (\pi + 2\pi k)$ for any integer value of k are typically referred to as high frequencies (relatively rapid oscillations).

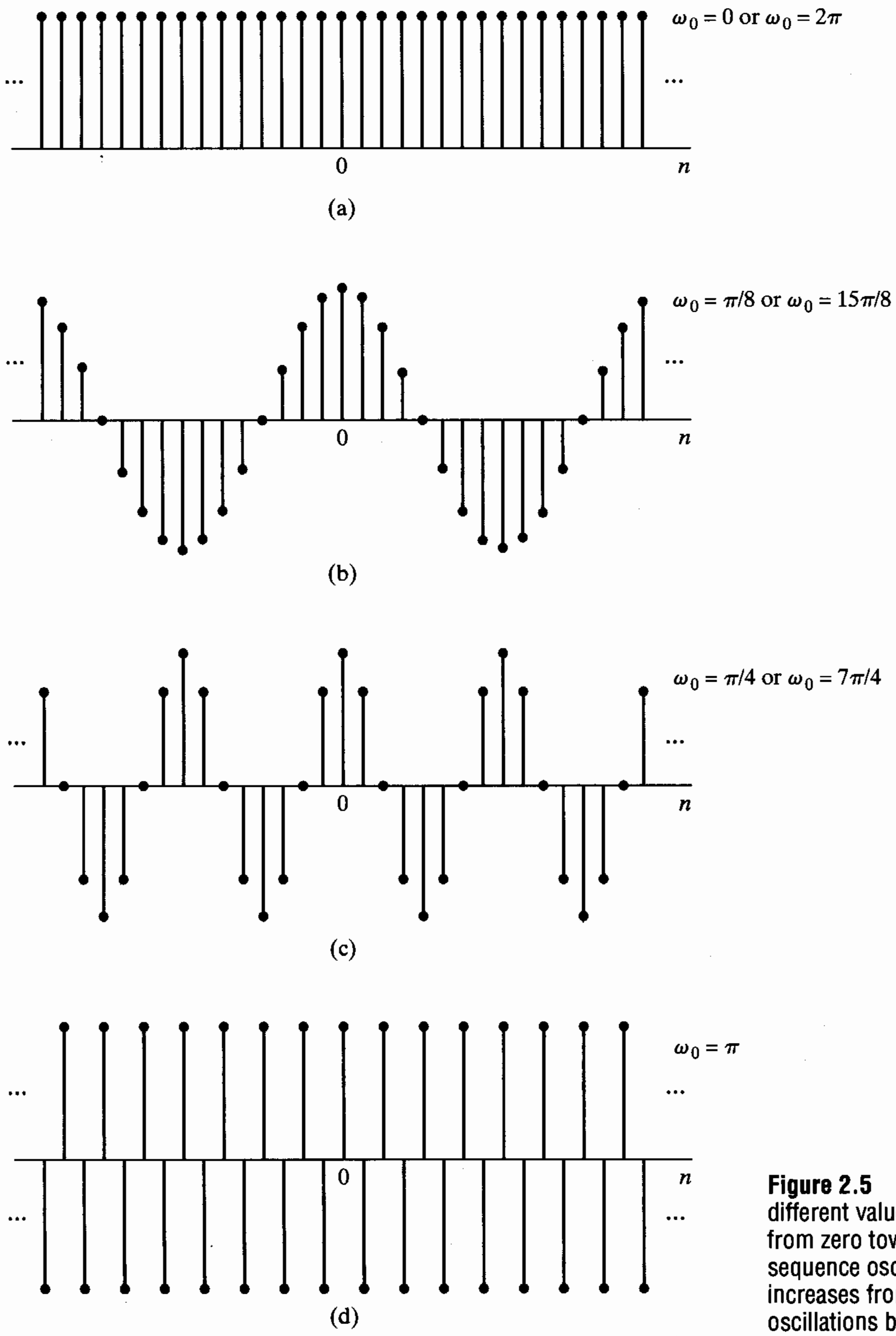


Figure 2.5 $\cos \omega_0 n$ for several different values of ω_0. As ω_0 increases from zero toward π (parts a–d), the sequence oscillates more rapidly. As ω_0 increases from π to 2π (parts d–a), the oscillations become slower.

2.2 DISCRETE-TIME SYSTEMS

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$. This can be denoted as

$$y[n] = T\{x[n]\} \tag{2.22}$$

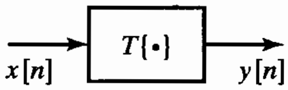


Figure 2.6 Representation of a discrete-time system, i.e., a transformation that maps an input sequence $x[n]$ into a unique output sequence $y[n]$.

and is indicated pictorially in Figure 2.6. Equation (2.22) represents a rule or formula for computing the output sequence values from the input sequence values. It should be emphasized that the value of the output sequence at each value of the index n may depend on $x[n]$ for all values of n. The following examples illustrate some simple and useful systems.

Example 2.3 The Ideal Delay System

The ideal delay system is defined by the equation

$$y[n] = x[n - n_d], \qquad -\infty < n < \infty, \tag{2.23}$$

where n_d is a fixed positive integer called the delay of the system. In words, the ideal delay system simply shifts the input sequence to the right by n_d samples to form the output. If, in Eq. (2.23), n_d is a fixed negative integer, then the system would shift the input to the left by $|n_d|$ samples, corresponding to a time advance.

In Example 2.3, only one sample of the input sequence is involved in determining a certain output sample. In the following example, this is not the case.

Example 2.4 Moving Average

The general moving-average system is defined by the equation

$$\begin{aligned} y[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \\ &= \frac{1}{M_1 + M_2 + 1} \{x[n + M_1] + x[n + M_1 - 1] + \cdots + x[n] \\ &\quad + x[n-1] + \cdots + x[n - M_2]\}. \end{aligned} \tag{2.24}$$

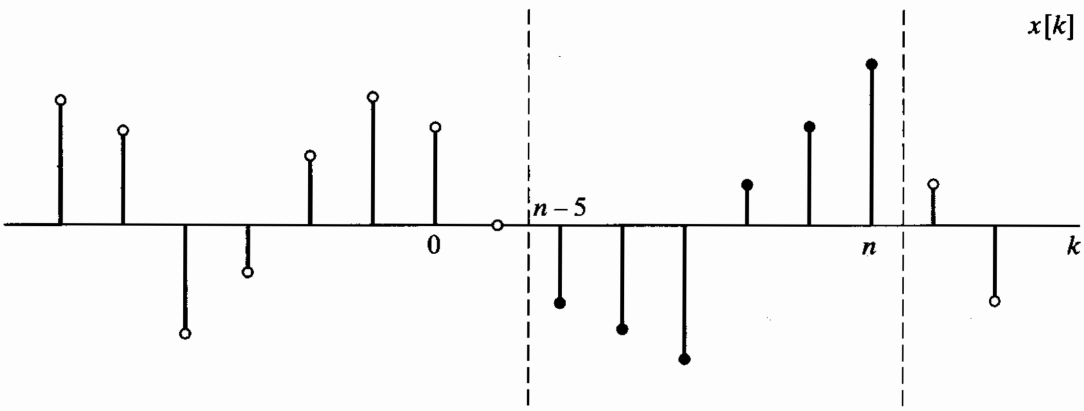


Figure 2.7 Sequence values involved in computing a causal moving average.

This system computes the nth sample of the output sequence as the average of $(M_1 + M_2 + 1)$ samples of the input sequence around the nth sample. Figure 2.7 shows an input sequence plotted as a function of a dummy index k and the samples involved in the computation of the output sample $y[n]$ for $n = 7$, $M_1 = 0$, and $M_2 = 5$. The output sample $y[7]$ is equal to one-sixth of the sum of all the samples between the vertical dotted lines. To compute $y[8]$, both dotted lines would move one sample to the right.

Classes of systems are defined by placing constraints on the properties of the transformation $T\{\cdot\}$. Doing so often leads to very general mathematical representations, as we will see. Of particular importance are the system constraints and properties, discussed in Sections 2.2.1–2.2.5.

2.2.1 Memoryless Systems

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n.

Example 2.5 A Memoryless System

An example of a memoryless system is a system for which $x[n]$ and $y[n]$ are related by

$$y[n] = (x[n])^2, \qquad \text{for each value of } n. \tag{2.25}$$

The system in Example 2.3 is not memoryless unless $n_d = 0$; in particular, this system is referred to as having "memory" whether n_d is positive (a time delay) or negative (a time advance). The system in Example 2.4 is not memoryless unless $M_1 = M_2 = 0$.

2.2.2 Linear Systems

The class of *linear systems* is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if

$$T\{x_1[n] + x_2[n]\} = T\{x_2[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \tag{2.26a}$$

and

$$T\{ax[n]\} = aT\{x[n]\} = ay[n], \tag{2.26b}$$

where a is an arbitrary constant. The first property is called the *additivity property,* and the second is called the *homogeneity* or *scaling property*. These two properties can be combined into the principle of superposition, stated as

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\} \tag{2.27}$$

for arbitrary constants a and b. This equation can be generalized to the superposition of many inputs. Specifically, if

$$x[n] = \sum_k a_k x_k[n], \tag{2.28a}$$

then the output of a linear system will be

$$y[n] = \sum_k a_k y_k[n], \tag{2.28b}$$

where $y_k[n]$ is the system response to the input $x_k[n]$.

By using the definition of the principle of superposition, we can easily show that the systems of Examples 2.3 and 2.4 are linear systems. (See Problem 2.23.) An example of a nonlinear system is the system in Example 2.5.

Example 2.6 The Accumulator System

The system defined by the input–output equation

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{2.29}$$

is called the *accumulator* system, since the output at time n is just the sum of the present and all previous input samples. The accumulator system is a linear system. In order to prove this, we must show that it satisfies the superposition principle for all inputs, not just any specific set of inputs. We begin by defining two arbitrary inputs $x_1[n]$ and $x_2[n]$ and their corresponding outputs

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k], \tag{2.30}$$

$$y_2[n] = \sum_{k=-\infty}^{n} x_2[k]. \tag{2.31}$$

When the input is $x_3[n] = ax_1[n] + bx_2[n]$, the superposition principle requires the output $y_3[n] = ay_1[n] + by_2[n]$ for all possible choices of a and b. We can show this by starting from Eq. (2.29):

$$y_3[n] = \sum_{k=-\infty}^{n} x_3[k], \tag{2.32}$$

$$= \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]), \tag{2.33}$$

$$= a\sum_{k=-\infty}^{n} x_1[k] + b\sum_{k=-\infty}^{n} x_2[k], \tag{2.34}$$

$$= ay_1[n] + by_2[n]. \tag{2.35}$$

Thus, the accumulator system of Eq. (2.29) satisfies the superposition principle for all inputs and is therefore linear.

In general, it may be simpler to prove that a system is not linear (if it is not) than to prove that it is linear (if it is). We simply must find an input or set of inputs for which the system does not satisfy the conditions of linearity.

Example 2.7 A Nonlinear System

Consider the system defined by

$$w[n] = \log_{10}(|x[n]|). \tag{2.36}$$

This system is not linear. In order to prove this, we only need to find one counterexample—that is, one set of inputs and outputs which demonstrates that the system violates the superposition principle, Eq. (2.27). The inputs $x_1[n] = 1$ and $x_2[n] = 10$ are a counterexample. The output for the first signal is $w_1[n] = 0$, while for the second, $w_2[n] = 1$. The scaling property of linear systems requires that, since $x_2[n] = 10x_1[n]$, if the system is linear, it must be true that $w_2[n] = 10w_1[n]$. Since this is not so for Eq. (2.36) for this set of inputs and outputs, the system is *not* linear.

2.2.3 Time-Invariant Systems

A time-invariant system (often referred to equivalently as a shift-invariant system) is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence. Specifically, suppose that a system transforms the input sequence with values $x[n]$ into the output sequence with values $y[n]$. Then the system is said to be time invariant if, for all n_0, the input sequence with values $x_1[n] = x[n - n_0]$ produces the output sequence with values $y_1[n] = y[n - n_0]$.

As in the case of linearity, proving that a system is time invariant requires a general proof making no specific assumptions about the input signals. All of the systems in Examples 2.3–2.7 are time invariant. The style of proof for time invariance is illustrated in Examples 2.8 and 2.9.

Example 2.8 The Accumulator as a Time-Invariant System

Consider the accumulator from Example 2.6. We define $x_1[n] = x[n - n_0]$. To show time invariance, we solve for both $y[n-n_0]$ and $y_1[n]$ and compare them to see whether they are equal. First,

$$y[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]. \tag{2.37}$$

Next, we find

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] \tag{2.38}$$

$$= \sum_{k=-\infty}^{n} x[k - n_0]. \tag{2.39}$$

Substituting the change of variables $k_1 = k - n_0$ into the summation gives

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n - n_0]. \tag{2.40}$$

Thus, the accumulator is a time-invariant system.

The following example illustrates a system that is not time invariant.

Example 2.9 The Compressor System

The system defined by the relation

$$y[n] = x[Mn], \qquad -\infty < n < \infty, \tag{2.41}$$

with M a positive integer, is called a *compressor*. Specifically, it discards $(M - 1)$ samples out of M; i.e., it creates the output sequence by selecting every Mth sample. This system is not time invariant. We can show that it is not by considering the response $y_1[n]$ to the input $x_1[n] = x[n - n_0]$. In order for the system to be time invariant, the output of the system when the input is $x_1[n]$ must be equal to $y[n - n_0]$. The output $y_1[n]$ that results from the input $x_1[n]$ can be directly computed from Eq. (2.41) to be

$$y_1[n] = x_1[Mn] = x[Mn - n_0]. \tag{2.42}$$

Delaying the output $y[n]$ by n_0 samples yields

$$y[n - n_0] = x[M(n - n_0)]. \tag{2.43}$$

Comparing these two outputs, we see that $y[n - n_0]$ is not equal to $y_1[n]$ for all M and n_0, and therefore, the system is not time invariant.

It is also possible to prove that a system is not time invariant by finding a single counterexample that violates the time-invariance property. For instance, a counterexample for the compressor is the case when $M = 2$, $x[n] = \delta[n]$, and $x_1[n] = \delta[n - 1]$. For this choice of inputs and M, $y[n] = \delta[n]$, but $y_1[n] = 0$; thus, it is clear that $y_1[n] \neq y[n - 1]$ for this system.

2.2.4 Causality

A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leq n_0$. This implies that if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$. That is, the system is *nonanticipative*. The system of Example 2.3 is causal for $n_d \geq 0$ and is noncausal for $n_d < 0$. The system of Example 2.4 is causal if $-M_1 \geq 0$ and $M_2 \geq 0$; otherwise it is noncausal. The system of Example 2.5 is causal, as is the accumulator of Example 2.6 and the nonlinear system in Example 2.7. However, the system of Example 2.9 is noncausal if $M > 1$, since $y[1] = x[M]$. Another noncausal system is given in the following example.

Example 2.10 The Forward and Backward Difference Systems

Consider the *forward difference system* defined by the relationship

$$y[n] = x[n + 1] - x[n]. \tag{2.44}$$

This system is not causal, since the current value of the output depends on a future value of the input. The violation of causality can be demonstrated by considering the two inputs $x_1[n] = \delta[n - 1]$ and $x_2[n] = 0$ and their corresponding outputs $y_1[n] = \delta[n] - \delta[n - 1]$ and $y_2[n] = 0$. Note that $x_1[n] = x_2[n]$ for $n \leq 0$, so the definition of causality requires that $y_1[n] = y_2[n]$ for $n \leq 0$, which is clearly not the case for $n = 0$. Thus, by this counterexample, we have shown that the system is not causal.

The *backward difference system,* defined as

$$y[n] = x[n] - x[n - 1], \tag{2.45}$$

has an output that depends only on the present and past values of the input. Because there is no way for the output at a specific time $y[n_0]$ to incorporate values of the input for $n > n_0$, the system is causal.

2.2.5 Stability

A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \leq B_x < \infty, \qquad \text{for all } n. \tag{2.46}$$

Stability requires that, for every bounded input, there exist a fixed positive finite value B_y such that

$$|y[n]| \le B_y < \infty, \qquad \text{for all } n. \tag{2.47}$$

It is important to emphasize that the properties we have defined in this section are properties of *systems,* not of the inputs to a system. That is, we may be able to find inputs for which the properties hold, but the existence of the property for some inputs does not mean that the system has the property. For the system to have the property, it must hold for *all* inputs. For example, an unstable system may have some bounded inputs for which the output is bounded, but for the system to have the property of stability, it must be true that for *all* bounded inputs, the output is bounded. If we can find just one input for which the system property does not hold, then we have shown that the system does *not* have that property. The following example illustrates the testing of stability for several of the systems that we have defined.

Example 2.11 Testing for Stability or Instability

The system of Example 2.5 is stable. To see this, assume that the input $x[n]$ is bounded such that $|x[n]| \le B_x$ for all n. Then $|y[n]| = |x[n]|^2 \le B_x^2$. Thus, we can choose $B_y = B_x^2$ and prove that $y[n]$ is bounded.

Likewise, we can see that the system defined in Example 2.7 is unstable, since $y[n] = \log_{10}(|x[n]|) = -\infty$ for any values of the time index n at which $x[n] = 0$, even though the output will be bounded for any input samples that are not equal to zero.

The accumulator, as defined in Example 2.6 by Eq. (2.29), is also not stable. For example, consider the case when $x[n] = u[n]$, which is clearly bounded by $B_x = 1$. For this input, the output of the accumulator is

$$y[n] = \sum_{k=-\infty}^{n} u[k] \tag{2.48}$$

$$= \begin{cases} 0, & n < 0, \\ (n+1), & n \ge 0. \end{cases} \tag{2.49}$$

There is no finite choice for B_y such that $(n+1) \le B_y < \infty$ for all n; thus, the system is unstable.

Using similar arguments, it can be shown that the systems in Examples 2.3, 2.4, 2.9 and 2.10 are all stable.

2.3 LINEAR TIME-INVARIANT SYSTEMS

A particularly important class of systems consists of those that are linear and time invariant. These two properties in combination lead to especially convenient representations for such systems. Most important, this class of systems has significant signal-processing applications. The class of linear systems is defined by the principle of superposition in Eq. (2.27). If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses as in Eq. (2.6), it follows that a linear system can be completely characterized by its impulse response. Specifically, let $h_k[n]$ be the response of the system to $\delta[n-k]$, an impulse occurring at $n = k$. Then, from Eq. (2.6),

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}. \tag{2.50}$$

From the principle of superposition in Eq. (2.27), we can write

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]. \tag{2.51}$$

According to Eq. (2.51), the system response to any input can be expressed in terms of the responses of the system to the sequences $\delta[n-k]$. If only linearity is imposed, $h_k[n]$ will depend on both n and k, in which case the computational usefulness of Eq. (2.51) is limited. We obtain a more useful result if we impose the additional constraint of time invariance.

The property of time invariance implies that if $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n-k]$ is $h[n-k]$. With this additional constraint, Eq. (2.51) becomes

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{2.52}$$

As a consequence of Eq. (2.52), a linear time-invariant system (which we will sometimes abbreviate as LTI) is completely characterized by its impulse response $h[n]$ in the sense that, given $h[n]$, it is possible to use Eq. (2.52) to compute the output $y[n]$ due to *any* input $x[n]$.

Equation (2.52) is commonly called the *convolution sum*. If $y[n]$ is a sequence whose values are related to the values of two sequences $h[n]$ and $x[n]$ as in Eq. (2.52), we say that $y[n]$ is the convolution of $x[n]$ with $h[n]$ and represent this by the notation

$$y[n] = x[n] * h[n]. \tag{2.53}$$

The operation of discrete-time convolution takes two sequences $x[n]$ and $h[n]$ and produces a third sequence $y[n]$. Equation (2.52) expresses each sample of the output sequence in terms all of the samples of the input and impulse response sequences.

The derivation of Eq. (2.52) suggests the interpretation that the input sample at $n = k$, represented as $x[k]\delta[n-k]$, is transformed by the system into an output sequence $x[k]h[n-k]$, for $-\infty < n < \infty$, and that, for each k, these sequences are superimposed to form the overall output sequence. This interpretation is illustrated in Figure 2.8, which shows an impulse response, a simple input sequence having three nonzero samples, the individual outputs due to each sample, and the composite output due to all the samples in the input sequence. Specifically, $x[n]$ can be decomposed as the sum of the three sequences $x[-2]\delta[n+2]$, $x[0]\delta[n]$, and $x[3]\delta[n-3]$ representing the three nonzero values in the sequence $x[n]$. The sequences $x[-2]h[n+2]$, $x[0]h[n]$, and $x[3]h[n-3]$ are the system responses to $x[-2]\delta[n+2]$, $x[0]\delta[n]$, and $x[3]\delta[n-3]$, respectively. The response to $x[n]$ is then the sum of these three individual responses.

Although the convolution-sum expression is analogous to the convolution integral of continuous-time linear system theory, the convolution sum should not be thought of as an approximation to the convolution integral. The convolution integral plays mainly a theoretical role in continuous-time linear system theory; we will see that the convolution sum, in addition to its theoretical importance, often serves as an explicit realization of a discrete-time linear system. Thus, it is important to gain some insight into the properties of the convolution sum in actual calculations.

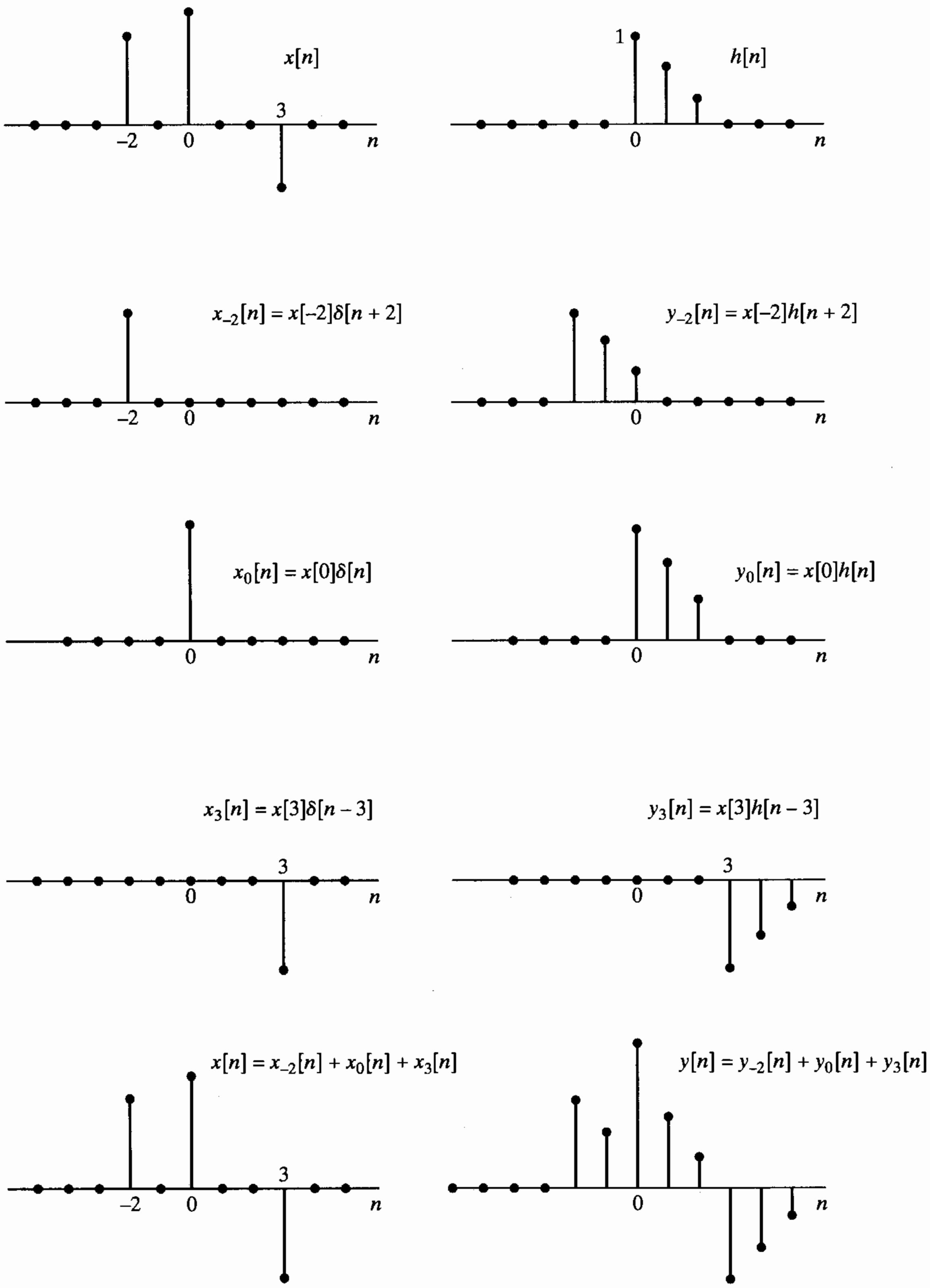


Figure 2.8 Representation of the output of a linear time-invariant system as the superposition of responses to individual samples of the input.

The preceding interpretation of Eq. (2.52) emphasizes that the convolution sum is a direct result of linearity and time invariance. However, a slightly different way of looking at Eq. (2.52) leads to a particularly useful computational interpretation. When viewed as a formula for computing a single value of the output sequence, Eq. (2.52) dictates that $y[n]$ (i.e., the nth value of the output) is obtained by multiplying the input sequence (expressed as a function of k) by the sequence whose values are $h[n-k]$, $-\infty < k < \infty$, and then, for any fixed value of n, summing all the values of the products $x[k]h[n-k]$, with k a counting index in the summation process. Therefore, the operation of convolving two sequences involves doing the computation for all values of n, thus generating the complete output sequence $y[n]$, $-\infty < n < \infty$. The key to carrying out the computations of Eq. (2.52) to obtain $y[n]$ is understanding how to form the sequence $h[n-k]$, $-\infty < k < \infty$, for all values of n that are of interest. To this end, it is useful to note that

$$h[n-k] = h[-(k-n)]. \tag{2.54}$$

The interpretation of Eq. (2.54) is best done with an example.

Example 2.12 Computation of the Convolution Sum

Suppose $h[k]$ is the sequence shown in Figure 2.9(a) and we wish to find $h[n-k] = h[-(k-n)]$. Define $h_1[k]$ to be $h[-k]$, which is shown in Figure 2.9(b). Next, define

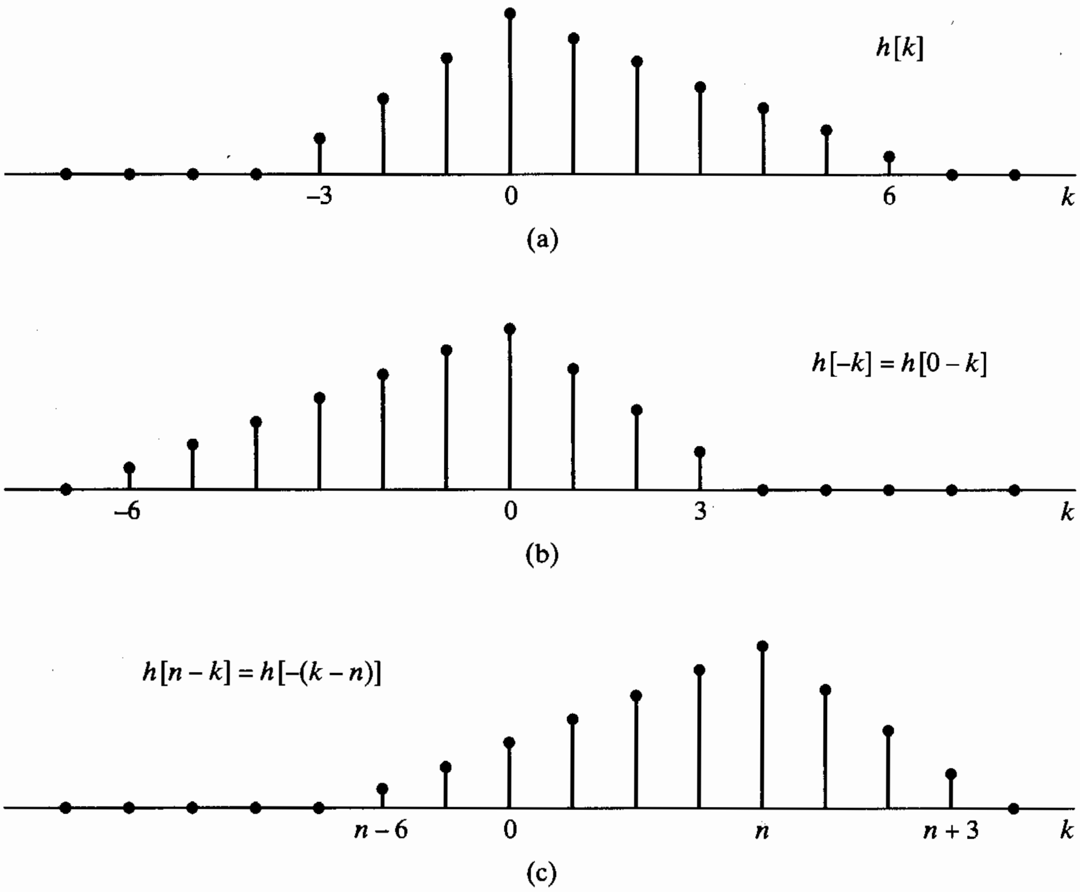


Figure 2.9 Forming the sequence $h[n-k]$. (a) The sequence $h[k]$ as a function of k. (b) The sequence $h[-k]$ as a function of k. (c) The sequence $h[n-k] = h[-(k-n)]$ as a function of k for $n = 4$.

$h_2[k]$ to be $h_1[k]$, delayed, by n samples on the k axis, i.e., $h_2[k] = h_1[k-n]$. Figure 2.9(c) shows the sequence that results from delaying the sequence in Figure 2.9(b) by n samples. Using the relationship between $h_1[k]$ and $h[k]$, we can show that $h_2[k] = h_1[k-n] = h[-(k-n)] = h[n-k]$, and thus, the bottom figure is the desired signal. To summarize, to compute $h[n-k]$ from $h[k]$, we first reverse $h[k]$ in time about $k = 0$ and then delay the time-reversed signal by n samples.

From Example 2.3, it should be clear that, in general, the sequence $h[n-k]$, $-\infty < k < \infty$, is obtained by

1. reflecting $h[k]$ about the origin to obtain $h[-k]$;
2. shifting the origin of the reflected sequence to $k = n$.

To implement discrete-time convolution, the two sequences $x[k]$ and $h[n-k]$ are multiplied together for $-\infty < k < \infty$, and the products are summed to compute the output sample $y[n]$. To obtain another output sample, the origin of the sequence $h[-k]$ is shifted to the new sample position, and the process is repeated. This computational procedure applies whether the computations are carried out numerically on sampled data or analytically with sequences for which the sample values have simple formulas. The following example illustrates discrete-time convolution for the latter case.

Example 2.13 Analytical Evaluation of the Convolution Sum

Consider a system with impulse response

$$h[n] = u[n] - u[n-N] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

The input is

$$x[n] = a^n u[n].$$

To find the output at a particular index n, we must form the sums over all k of the product $x[k]h[n-k]$. In this case, we can find formulas for $y[n]$ for different sets of values of n. For example, Figure 2.10(a) shows the sequences $x[k]$ and $h[n-k]$, plotted for n a negative integer. Clearly, all negative values of n give a similar picture; i.e., the nonzero portions of the sequences $x[k]$ and $h[n-k]$ do not overlap, so

$$y[n] = 0, \qquad n < 0.$$

Figure 2.10(b) illustrates the two sequences when $0 \le n$ and $n - N + 1 \le 0$. These two conditions can be combined into the single condition $0 \le n \le N-1$. By considering Figure 2.10(b), we see that, since

$$x[k]h[n-k] = a^k,$$

it follows that

$$y[n] = \sum_{k=0}^{n} a^k, \qquad \text{for } 0 \le n \le N-1. \tag{2.55}$$

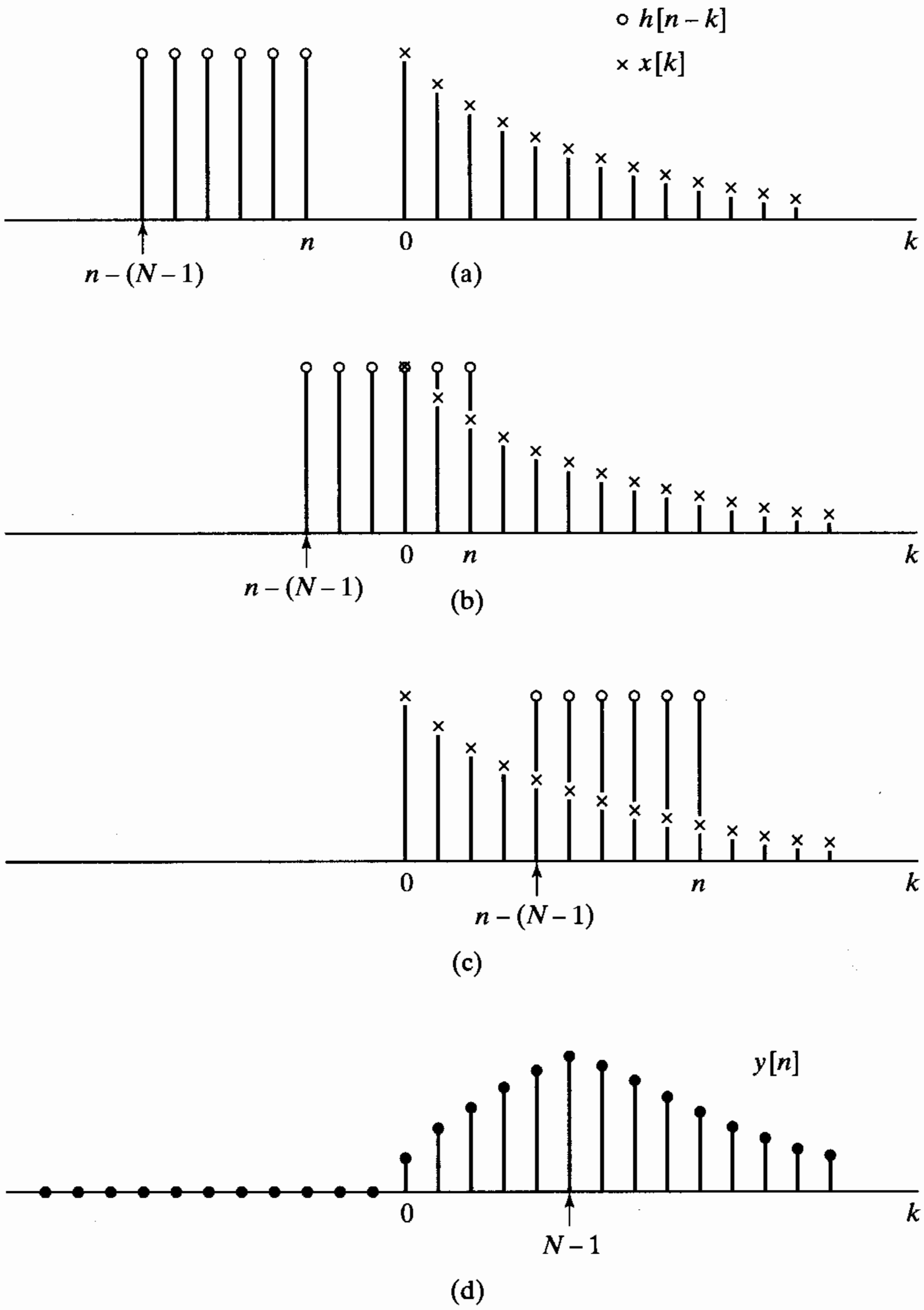


Figure 2.10 Sequence involved in computing a discrete convolution. (a)–(c) The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.) (d) Corresponding output sequence as a function of n.

The limits on the sum are determined directly from Figure 2.10(b). Equation (2.55) shows that $y[n]$ is the sum of $n+1$ terms of a geometric series in which the ratio of terms is a. This sum can be expressed in closed form using the general formula

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \qquad N_2 \geq N_1. \tag{2.56}$$

Applying this formula to Eq. (2.55), we obtain

$$y[n] = \frac{1-a^{n+1}}{1-a}, \qquad 0 \le n \le N-1. \tag{2.57}$$

Finally, Figure 2.10(c) shows the two sequences when $0 < n - N + 1$ or $N - 1 < n$. As before,

$$x[k]h[n-k] = a^k, \qquad n - N + 1 < k \le n,$$

but now the lower limit on the sum is $n - N + 1$, as seen in Figure 2.10(c). Thus,

$$y[n] = \sum_{k=n-N+1}^{n} a^k, \qquad \text{for } N - 1 < n. \tag{2.58}$$

Using Eq. (2.56), we obtain

$$y[n] = \frac{a^{n-N+1} - a^{n+1}}{1-a},$$

or

$$y[n] = a^{n-N+1}\left(\frac{1-a^N}{1-a}\right). \tag{2.59}$$

Thus, because of the piecewise-exponential nature of both the input and the unit sample response, we have been able to obtain the following closed-form expression for $y[n]$ as a function of the index n:

$$y[n] = \begin{cases} 0, & n < 0, \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1, \\ a^{n-N+1}\left(\dfrac{1-a^N}{1-a}\right), & N-1 < n. \end{cases} \tag{2.60}$$

This sequence is shown in Figure 2.10(d).

Example 2.13 illustrates how the convolution sum can be computed analytically when the input and the impulse response are given by simple formulas. In such cases, the sums may have a compact form that may be derived using the formula for the sum of a geometric series or other "closed-form" formulas.[2] When no simple form is available, the convolution sum can still be evaluated numerically using the technique illustrated in Example 2.13 whenever the sums are finite, which will be the case if either the input sequence or the impulse response is of finite length, i.e., has a finite number of nonzero samples.

2.4 PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

Since all linear time-invariant systems are described by the convolution sum of Eq. (2.52), the properties of this class of systems are defined by the properties of discrete-time convolution. Therefore, the impulse response is a complete characterization of the properties of a specific linear time-invariant system.

[2]Such results are discussed, for example, in Grossman (1992).

Some general properties of the class of linear time-invariant systems can be found by considering properties of the convolution operation. For example, the convolution operation is commutative:

$$x[n] * h[n] = h[n] * x[n]. \tag{2.61}$$

This can be shown by applying a substitution of variables to Eq. (2.52). Specifically, with $m = n - k$,

$$y[n] = \sum_{m=\infty}^{-\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n], \tag{2.62}$$

so the roles of $x[n]$ and $h[n]$ in the summation are interchanged. That is, the order of the sequences in a convolution is unimportant, and hence, the system output is the same if the roles of the input and impulse response are reversed. Accordingly, a linear time-invariant system with input $x[n]$ and impulse response $h[n]$ will have the same output as a linear time-invariant system with input $h[n]$ and impulse response $x[n]$. The convolution operation also distributes over addition; i.e.,

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

This follows in a straightforward way from Eq. (2.52) and is a direct result of the linearity and commutativity of convolution.

In a *cascade connection* of systems, the output of the first system is the input to the second, the output of the second is the input to the third, etc. The output of the last system is the overall output. Two linear time-invariant systems in cascade correspond to a linear time-invariant system with an impulse response that is the convolution of the impulse responses of the two systems. This is illustrated in Figure 2.11. In the upper block diagram, the output of the first system will be $h_1[n]$ if $x[n] = \delta[n]$. Thus, the output of the second system (and, by definition, the impulse response of the overall system) will be

$$h[n] = h_1[n] * h_2[n]. \tag{2.63}$$

As a consequence of the commutative property of convolution, the impulse response of a cascade combination of linear time-invariant systems is independent of the order in which they are cascaded. This result is summarized in Figure 2.11, where the three systems all have the same impulse response.

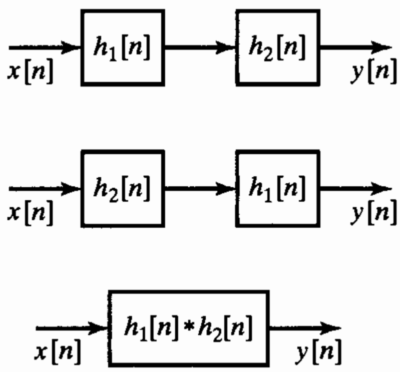


Figure 2.11 Three linear time-invariant systems with identical impulse responses.

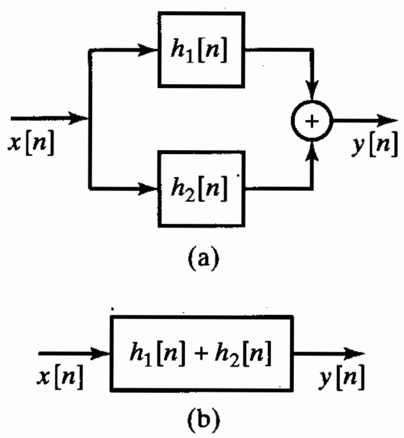


Figure 2.12 (a) Parallel combination of linear time-invariant systems. (b) An equivalent system.

In a *parallel connection,* the systems have the same input, and their outputs are summed to produce an overall output. It follows from the distributive property of convolution that the connection of two linear time-invariant systems in parallel is equivalent to a single system whose impulse response is the sum of the individual impulse responses; i.e.,

$$h[n] = h_1[n] + h_2[n]. \tag{2.64}$$

This is depicted in Figure 2.12.

The constraints of linearity and time invariance define a class of systems with very special properties. Stability and causality represent additional properties, and it is often important to know whether a linear time-invariant system is stable and whether it is causal. Recall from Section 2.2.5 that a stable system is a system for which every bounded input produces a bounded output. Linear time-invariant systems are stable if and only if the impulse response is absolutely summable, i.e., if

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{2.65}$$

This can be shown as follows. From Eq. (2.62),

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|. \tag{2.66}$$

If $x[n]$ is bounded, so that

$$|x[n]| \le B_x,$$

then substituting B_x for $|x[n-k]|$ can only strengthen the inequality. Hence,

$$|y[n]| \le B_x \sum_{k=-\infty}^{\infty} |h[k]|. \tag{2.67}$$

Thus, $y[n]$ is bounded if Eq. (2.65) holds; in other words, Eq. (2.65) is a sufficient condition for stability. To show that it is also a necessary condition, we must show that if $S = \infty$, then a bounded input can be found that will cause an unbounded output. Such an input is the sequence with values

$$x[n] = \begin{cases} \dfrac{h^*[-n]}{|h[-n]|}, & h[n] \ne 0, \\ 0, & h[n] = 0, \end{cases} \tag{2.68}$$

where $h^*[n]$ is the complex conjugate of $h[n]$. The sequence $x[n]$ is clearly bounded by unity. However, the value of the output at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} x[-k]h[k] = \sum_{k=-\infty}^{\infty} \frac{|h[k]|^2}{|h[k]|} = S. \tag{2.69}$$

Therefore, if $S = \infty$, it is possible for a bounded input sequence to produce an unbounded output sequence.

The class of causal systems was defined in Section 2.2.4 as those systems for which the output $y[n_0]$ depends only on the input samples $x[n]$, for $n \leq n_0$. It follows from Eq. (2.52) or Eq. (2.62) that this definition implies the condition

$$h[n] = 0, \qquad n < 0, \tag{2.70}$$

for causality of linear time-invariant systems. (See Problem 2.62.) For this reason, it is sometimes convenient to refer to a sequence that is zero for $n < 0$ as a *causal sequence*, meaning that it could be the impulse response of a causal system.

To illustrate how the properties of linear time-invariant systems are reflected in the impulse response, let us consider again some of the systems defined in Examples 2.3–2.10. First note that only the systems of Examples 2.3, 2.4, 2.6, and 2.10 are linear and time invariant. Although the impulse response of nonlinear or time-varying systems can be found, it is generally of limited interest, since the convolution-sum formula and Eqs. (2.65) and (2.70), expressing stability and causality, do not apply to such systems.

First, let us find the impulse responses of the systems in Examples 2.3, 2.4, 2.6, and 2.10. We can do this by simply computing the response of each system to $\delta[n]$, using the defining relationship for the system. The resulting impulse responses are as follows:

Ideal Delay (Example 2.3)

$$h[n] = \delta[n - n_d], \qquad n_d \text{ a positive fixed integer.} \tag{2.71}$$

Moving Average (Example 2.4)

$$\begin{aligned} h[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k] \\ &= \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.72}$$

Accumulator (Example 2.6)

$$\begin{aligned} h[n] &= \sum_{k=-\infty}^{n} \delta[k] \\ &= \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} \\ &= u[n]. \end{aligned} \tag{2.73}$$

Forward Difference (Example 2.10)

$$h[n] = \delta[n+1] - \delta[n]. \tag{2.74}$$

Backward Difference (Example 2.10)

$$h[n] = \delta[n] - \delta[n-1]. \tag{2.75}$$

Given the impulse responses of these basic systems [Eqs. (2.71)–(2.75)], we can test the stability of each one by computing the sum

$$S = \sum_{n=-\infty}^{\infty} |h[n]|.$$

For the ideal delay, moving-average, forward difference, and backward difference examples, it is clear that $S < \infty$, since the impulse response has only a finite number of nonzero samples. Such systems are called *finite-duration impulse response* (FIR) systems. Clearly, FIR systems will always be stable, as long as each of the impulse response values is finite in magnitude. The accumulator, however, is unstable because

$$S = \sum_{n=0}^{\infty} u[n] = \infty.$$

In Section 2.2.5, we also demonstrated the instability of the accumulator by giving an example of a bounded input (the unit step) for which the output is unbounded.

The impulse response of the accumulator is infinite in duration. This is an example of the class of systems referred to as *infinite-duration impulse response* (IIR) systems. An example of an IIR system that is stable is a system whose impulse response is $h[n] = a^n u[n]$ with $|a| < 1$. In this case,

$$S = \sum_{n=0}^{\infty} |a|^n. \tag{2.76}$$

If $|a| < 1$, the formula for the sum of the terms of an infinite geometric series gives

$$S = \frac{1}{1-|a|} < \infty. \tag{2.77}$$

If, on the other hand, $|a| \geq 1$, the sum is infinite and the system is unstable.

To test causality of the linear time-invariant systems in Examples 2.3, 2.4, 2.6, and 2.10, we can check to see whether $h[n] = 0$ for $n < 0$. As discussed in Section 2.2.4, the ideal delay [$n_d \geq 0$ in Eq. (2.23)] is causal. If $n_d < 0$, the system is noncausal. For the moving average, causality requires that $-M_1 \geq 0$ and $M_2 \geq 0$. The accumulator and backward difference systems are causal, and the forward difference system is noncausal.

The concept of convolution as an operation between two sequences leads to the simplification of many problems involving systems. A particularly useful result can be stated for the ideal delay system. Since the output of the delay system is $y[n] = x[n-n_d]$, and since the delay system has impulse response $h[n] = \delta[n-n_d]$, it follows that

$$x[n] * \delta[n-n_d] = \delta[n-n_d] * x[n] = x[n-n_d]. \tag{2.78}$$

That is, the convolution of a shifted impulse sequence with any signal $x[n]$ is easily evaluated by simply shifting $x[n]$ by the displacement of the impulse.

Since delay is a fundamental operation in the implementation of linear systems, the preceding result is often useful in the analysis and simplification of interconnections of linear time-invariant systems. As an example, consider the system of Figure 2.13(a),

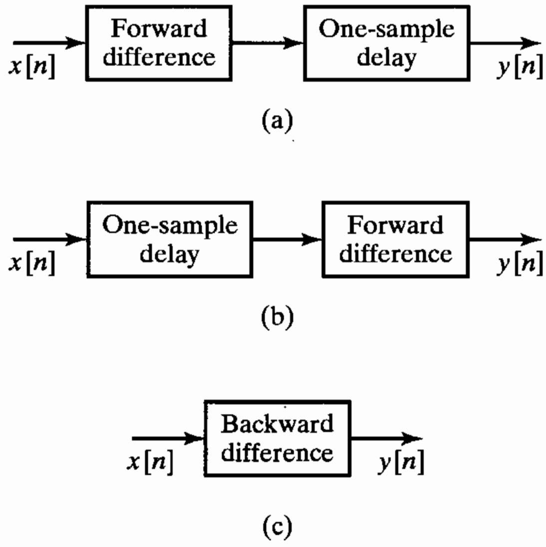


Figure 2.13 Equivalent systems found by using the commutative property of convolution.

which consists of a forward difference system cascaded with an ideal delay of one sample. According to the commutative property of convolution, the order in which systems are cascaded does not matter, as long as they are linear and time invariant. Therefore, we obtain the same result when we compute the forward difference of a sequence and delay the result (Figure 2.13a) as when we delay the sequence first and then compute the forward difference (Figure 2.13b). Also, it follows from Eq. (2.63) that the overall impulse response of each cascade system is the convolution of the individual impulse responses. Consequently,

$$\begin{aligned} h[n] &= (\delta[n+1] - \delta[n]) * \delta[n-1] \\ &= \delta[n-1] * (\delta[n+1] - \delta[n]) \\ &= \delta[n] - \delta[n-1]. \end{aligned} \tag{2.79}$$

Thus, $h[n]$ is identical to the impulse response of the backward difference system; that is, the cascaded systems of Figures 2.13(a) and 2.13(b) can be replaced by a backward difference system, as shown in Figure 2.13(c).

Note that the noncausal forward difference systems in Figures 2.13(a) and (b) have been converted to causal systems by cascading them with a delay. In general, any noncausal FIR system can be made causal by cascading it with a sufficiently long delay.

Another example of cascaded systems introduces the concept of an *inverse system*. Consider the cascade of systems in Figure 2.14. The impulse response of the cascade system is

$$\begin{aligned} h[n] &= u[n] * (\delta[n] - \delta[n-1]) \\ &= u[n] - u[n-1] \\ &= \delta[n]. \end{aligned} \tag{2.80}$$

That is, the cascade combination of an accumulator followed by a backward difference (or vice versa) yields a system whose overall impulse response is the impulse. Thus, the output of the cascade combination will always be equal to the input, since $x[n] * \delta[n] = x[n]$. In this case, the backward difference system compensates exactly for (or inverts) the effect of the accumulator; that is, the backward difference system is the *inverse*

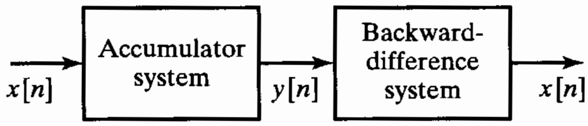


Figure 2.14 An accumulator in cascade with a backward difference. Since the backward difference is the inverse system for the accumulator, the cascade combination is equivalent to the identity system.

system for the accumulator. From the commutative property of convolution, the accumulator is likewise the inverse system for the backward difference system. Note that this example provides a system interpretation of Eqs. (2.8) and (2.10). In general, if a linear time-invariant system has impulse response $h[n]$, then its inverse system, if it exists, has impulse response $h_i[n]$ defined by the relation

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]. \tag{2.81}$$

Inverse systems are useful in many situations in which it is necessary to compensate for the effects of a linear system. In general, it is difficult to solve Eq. (2.81) directly for $h_i[n]$, given $h[n]$. However, in Chapter 3 we will see that the z-transform provides a straightforward method of finding an inverse system.

2.5 LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

An important subclass of linear time-invariant systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]. \tag{2.82}$$

The properties discussed in Section 2.4 and some of the analysis techniques introduced there can be used to find difference equation representations for some of the linear time-invariant systems that we have defined.

Example 2.14 Difference Equation Representation of the Accumulator

An example of the class of linear constant-coefficient difference equations is the accumulator system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]. \tag{2.83}$$

To show that the input and output satisfy a difference equation of the form of Eq. (2.82), note that we can write the output for $n-1$ as

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]. \tag{2.84}$$

By separating the term $x[n]$ from the sum, we can rewrite Eq. (2.83) as

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]. \tag{2.85}$$

Substituting Eq. (2.84) into Eq. (2.85) yields

$$y[n] = x[n] + y[n-1], \tag{2.86}$$

from which the desired form of the difference equation can be obtained by grouping all the input and output terms on separate sides of the equation:

$$y[n] - y[n-1] = x[n]. \tag{2.87}$$

Thus, we have shown that, in addition to satisfying the defining relationship of Eq. (2.83), the input and output satisfy a linear constant-coefficient difference equation of the form Eq. (2.82), with $N = 1, a_0 = 1, a_1 = -1, M = 0$, and $b_0 = 1$.

The difference equation in the form of Eq. (2.86) gives us a better understanding of how we could implement the accumulator system. According to Eq. (2.86), for each value of n, we add the current input value $x[n]$ to the previously accumulated sum $y[n-1]$. This interpretation of the accumulator is represented in block diagram form in Figure 2.15.

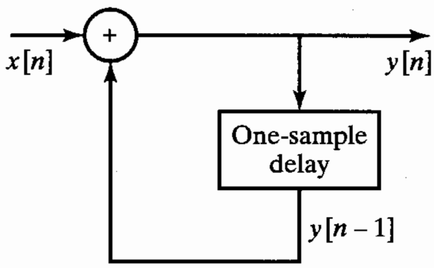


Figure 2.15 Block diagram of a recursive difference equation representing an accumulator.

Equation (2.86) and the block diagram in Figure 2.15 are referred to as a *recursive representation* of the system, since each value is computed using previously computed values. This general notion will be explored in more detail later in the section.

Example 2.15 Difference Equation Representation of the Moving-Average System

Consider the moving-average system of Example 2.4, with $M_1 = 0$ so that the system is causal. In this case, from Eq. (2.72), the impulse response is

$$h[n] = \frac{1}{(M_2+1)}(u[n] - u[n-M_2-1]), \tag{2.88}$$

from which it follows that

$$y[n] = \frac{1}{(M_2+1)}\sum_{k=0}^{M_2} x[n-k], \tag{2.89}$$

which is a special case of Eq. (2.82), with $N = 0, a_0 = 1, M = M_2$, and $b_k = 1/(M_2+1)$ for $0 \le k \le M_2$.

Also, the impulse response can be expressed as

$$h[n] = \frac{1}{(M_2+1)}(\delta[n] - \delta[n-M_2-1]) * u[n], \tag{2.90}$$

which suggests that the causal moving-average system can be represented as the cascade system of Figure 2.16. We can obtain a difference equation for this block diagram

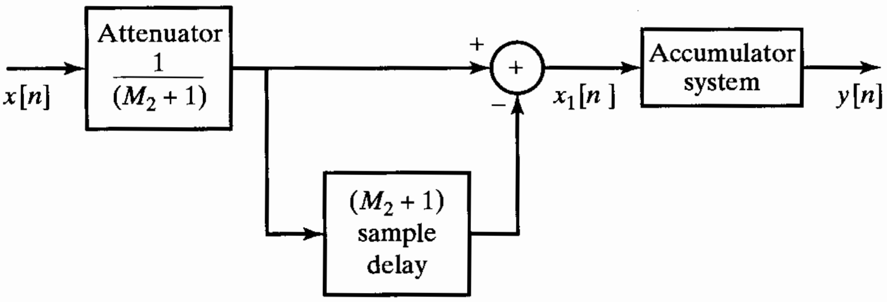


Figure 2.16 Block diagram of the recursive form of a moving-average system.

by noting first that

$$x_1[n] = \frac{1}{(M_2+1)}(x[n] - x[n - M_2 - 1]). \tag{2.91}$$

From Eq. (2.87) of Example 2.14, the output of the accumulator satisfies the difference equation

$$y[n] - y[n-1] = x_1[n],$$

so that

$$y[n] - y[n-1] = \frac{1}{(M_2+1)}(x[n] - x[n - M_2 - 1]). \tag{2.92}$$

Again, we have a difference equation in the form of Eq. (2.82), but this time $N = 1$, $a_0 = 1, a_1 = -1$, $M = M_2$ and $b_0 = -b_{M_2+1} = 1/(M_2+1)$, and $b_k = 0$ otherwise.

In Example 2.15, we showed two different difference equation representations of the moving-average system. In Chapter 6 we will see that an unlimited number of distinct difference equations can be used to represent a given linear time-invariant input–output relation.

Just as in the case of linear constant-coefficient differential equations for continuous-time systems, without additional constraints or information a linear constant-coefficient difference equation for discrete-time systems does not provide a unique specification of the output for a given input. Specifically, suppose that, for a given input $x_p[n]$, we have determined by some means one output sequence $y_p[n]$, so that an equation of the form of Eq. (2.82) is satisfied. Then the same equation with the same input is satisfied by any output of the form

$$y[n] = y_p[n] + y_h[n], \tag{2.93}$$

where $y_h[n]$ is any solution to Eq. (2.82) with $x[n] = 0$, i.e., to the equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0. \tag{2.94}$$

Equation (2.94) is referred to as the *homogeneous equation* and $y_h[n]$ the *homogeneous solution*. The sequence $y_h[n]$ is in fact a member of a family of solutions of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n. \tag{2.95}$$

Substituting Eq. (2.95) into Eq. (2.94) shows that the complex numbers z_m must be roots of the polynomial

$$\sum_{k=0}^{N} a_k z^{-k} = 0. \tag{2.96}$$

Equation (2.95) assumes that all N roots of the polynomial in Eq. (2.96) are distinct. The form of terms associated with multiple roots is slightly different, but there are always N undetermined coefficients. An example of the homogeneous solution with multiple roots is considered in Problem 2.38.

Since $y_h[n]$ has N undetermined coefficients, a set of N auxiliary conditions is required for the unique specification of $y[n]$ for a given $x[n]$. These auxiliary conditions might consist of specifying fixed values of $y[n]$ at specific values of n, such as $y[-1]$, $y[-2], \ldots, y[-N]$, and then solving a set of N linear equations for the N undetermined coefficients.

Alternatively, if the auxiliary conditions are a set of auxiliary values of $y[n]$, the other values of $y[n]$ can be generated by rewriting Eq. (2.82) as a recurrence formula, i.e., in the form

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]. \tag{2.97}$$

If the input $x[n]$, together with a set of auxiliary values, say, $y[-1], y[-2], \ldots, y[-N]$, is specified, then $y[0]$ can be determined from Eq. (2.97). With $y[0], y[-1], \ldots, y[-N+1]$ available, $y[1]$ can then be calculated, and so on. When this procedure is used, $y[n]$ is said to be computed *recursively;* i.e., the output computation involves not only the input sequence, but also previous values of the output sequence.

To generate values of $y[n]$ for $n < -N$ (again assuming that the values $y[-1]$, $y[-2], \ldots, y[-N]$ are given as auxiliary conditions), we can rearrange Eq. (2.82) in the form

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k], \tag{2.98}$$

from which $y[-N-1], y[-N-2], \ldots$ can be computed recursively. The following example illustrates this recursive procedure.

Example 2.16 Recursive Computation of Difference Equations

The difference equation satisfied by the input and output of a system is

$$y[n] = ay[n-1] + x[n]. \tag{2.99}$$

Consider the input $x[n] = K\delta[n]$, where K is an arbitrary number, and the auxiliary condition $y[-1] = c$. Beginning with this value, the output for $n > -1$ can be computed recursively as follows:

$$y[0] = ac + K,$$

$$y[1] = ay[0] + 0 = a(ac + K) = a^2c + aK,$$

$$y[2] = ay[1] + 0 = a(a^2c + aK) = a^3c + a^2K,$$
$$y[3] = ay[2] + 0 = a(a^3c + a^2K) = a^4c + a^3K,$$
$$\vdots \qquad\qquad \vdots$$

For this simple case, we can see that for $n \geq 0$,

$$y[n] = a^{n+1}c + a^nK, \qquad \text{for } n \geq 0. \tag{2.100}$$

To determine the output for $n < 0$, we express the difference equation in the form

$$y[n-1] = a^{-1}(y[n] - x[n]), \tag{2.101a}$$

or

$$y[n] = a^{-1}(y[n+1] - x[n+1]). \tag{2.101b}$$

Using the auxiliary condition $y[-1] = c$, we can compute $y[n]$ for $n < -1$ as follows:

$$y[-2] = a^{-1}(y[-1] - x[-1]) = a^{-1}c,$$
$$y[-3] = a^{-1}(y[-2] - x[-2]) = a^{-1}a^{-1}c = a^{-2}c,$$
$$y[-4] = a^{-1}(y[-3] - x[-3]) = a^{-1}a^{-2}c = a^{-3}c,$$

It then follows that

$$y[n] = a^{n+1}c \qquad \text{for } n \leq -1. \tag{2.102}$$

In sum, combining Eqs. (2.100) and (2.102), we obtain, as the result of the recursive computation,

$$y[n] = a^{n+1}c + Ka^nu[n], \qquad \text{for all } n. \tag{2.103}$$

Several important points are illustrated by the solution of Example 2.16. First, note that we implemented the system by recursively computing the output in both the positive and the negative direction, beginning with $n = -1$. Clearly, this procedure is noncausal. Also, note that when $K = 0$, the input is zero, but $y[n] = a^{n+1}c$. A linear system requires that the output be zero for all time when the input is zero for all time. (See Problem 2.21.) Consequently, this system is not linear. Furthermore, if the input were shifted by n_0 samples, i.e., $x_1[n] = K\delta[n - n_0]$, the output would be

$$y_1[n] = a^{n+1}c + Ka^{n-n_0}u[n - n_0], \tag{2.104}$$

and the system is therefore not time invariant.

Our principal interest in this text is in systems that are linear and time invariant, in which case the auxiliary conditions must be consistent with these additional requirements. In Chapter 3, when we discuss the solution of difference equations using the z-transform, we implicitly incorporate conditions of linearity and time invariance. As we will see in that discussion, even with the additional constraints of linearity and time invariance, the solution to the difference equation, and therefore the system, is not uniquely specified. In particular, there are, in general, both causal and noncausal linear time-invariant systems consistent with a given difference equation.

If a system is characterized by a linear constant-coefficient difference equation and is further specified to be linear, time invariant, and causal, the solution is unique. In this case, the auxiliary conditions are often stated as *initial-rest conditions*. In other words, the auxiliary information is that if the input $x[n]$ is zero for n less than some time

n_0, then the output $y[n]$ is constrained to be zero for n less than n_0. This then provides sufficient initial conditions to obtain $y[n]$ for $n \geq n_0$ recursively using Eq. (2.97).

To summarize, for a system for which the input and output satisfy a linear constant-coefficient difference equation:

- The output for a given input is not uniquely specified. Auxiliary information or conditions are required.
- If the auxiliary information is in the form of N sequential values of the output, later values can be obtained by rearranging the difference equation as a recursive relation running forward in n, and prior values can be obtained by rearranging the difference equation as a recursive relation running backward in n.
- Linearity, time invariance, and causality of the system will depend on the auxiliary conditions. If an additional condition is that the system is initially at rest, then the system will be linear, time invariant, and causal.

With the preceding discussion in mind, let us now consider again Example 2.16, but with initial-rest conditions. With $x[n] = K\delta[n]$, $y[-1] = 0$, since $x[n] = 0$, $n < 0$. Consequently, from Eq. (2.103),

$$y[n] = Ka^n u[n]. \tag{2.105}$$

If the input is instead $K\delta[n - n_0]$, again with initial-rest conditions, then the recursive solution is carried out using the initial condition $y[n] = 0, n < n_0$. Note that for $n_0 < 0$, initial rest implies that $y[-1] \neq 0$. That is, initial rest does not always mean $y[-1] = \cdots = y[-N] = 0$. It does mean that $y[n_0 - 1] = \cdots = y[n_0 - N] = 0$ if $x[n] = 0$ for $n < n_0$. Note also that the impulse response for the example is $h[n] = a^n u[n]$; i.e., $h[n]$ is zero for $n < 0$, consistent with the causality imposed by the assumption of initial rest.

The preceding discussion assumed that $N \geq 1$ in Eq. (2.82). If, instead, $N = 0$, no recursion is required to use the difference equation to compute the output, and therefore, no auxiliary conditions are required. That is,

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]. \tag{2.106}$$

Equation (2.106) is in the form of a convolution, and by setting $x[n] = \delta[n]$, we see that the impulse response is

$$h[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) \delta[n-k],$$

or

$$h[n] = \begin{cases} \left(\dfrac{b_n}{a_0}\right), & 0 \leq n \leq M, \\ 0, & \text{otherwise.} \end{cases} \tag{2.107}$$

The impulse response is obviously finite in duration. Indeed, the output of any FIR system can be computed nonrecursively using the difference equation of Eq. (2.106), where the coefficients are the values of the impulse response sequence. The moving-average system of Example 2.15 with $M_1 = 0$ is an example of a causal FIR system. An interesting feature of that system was that we also found a recursive equation for the output. In

Chapter 6 we will show that there are many possible ways of implementing a desired signal transformation using difference equations. Advantages of one method over another depend on practical considerations such as numerical accuracy, data storage, and the number of multiplications and additions required to compute each sample of the output.

2.6 FREQUENCY-DOMAIN REPRESENTATION OF DISCRETE-TIME SIGNALS AND SYSTEMS

In the previous sections, we have introduced some of the fundamental concepts of the theory of discrete-time signals and systems. For linear time-invariant systems, we saw that a representation of the input sequence as a weighted sum of delayed impulses leads to a representation of the output as a weighted sum of delayed impulse responses. As with continuous-time signals, discrete-time signals may be represented in a number of different ways. For example, sinusoidal and complex exponential sequences play a particularly important role in representing discrete-time signals. This is because complex exponential sequences are eigenfunctions of linear time-invariant systems and the response to a sinusoidal input is sinusoidal with the same frequency as the input and with amplitude and phase determined by the system. This fundamental property of linear time-invariant systems makes representations of signals in terms of sinusoids or complex exponentials (i.e., Fourier representations) very useful in linear system theory.

2.6.1 Eigenfunctions for Linear Time-Invariant Systems

To demonstrate the eigenfunction property of complex exponentials for discrete-time systems, consider an input sequence $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$, i.e., a complex exponential of radian frequency ω. From Eq. (2.62), the corresponding output of a linear time-invariant system with impulse response $h[n]$ is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} \\ &= e^{j\omega n} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right). \end{aligned} \tag{2.108}$$

If we define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}, \tag{2.109}$$

Eq. (2.108) becomes

$$y[n] = H(e^{j\omega}) e^{j\omega n}. \tag{2.110}$$

Consequently, $e^{j\omega n}$ is an eigenfunction of the system, and the associated eigenvalue is $H(e^{j\omega})$. From Eq. (2.110), we see that $H(e^{j\omega})$ describes the change in complex amplitude of a complex exponential input signal as a function of the frequency ω. The eigenvalue $H(e^{j\omega})$ is called the *frequency response* of the system. In general, $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary parts as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j H_I(e^{j\omega}) \tag{2.111}$$

or in terms of magnitude and phase as

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\sphericalangle H(e^{j\omega})}. \tag{2.112}$$

Example 2.17 Frequency Response of the Ideal Delay System

As a simple example of how we can find the frequency response of a linear time-invariant system, consider the ideal delay system defined by

$$y[n] = x[n - n_d], \tag{2.113}$$

where n_d is a fixed integer. If we consider $x[n] = e^{j\omega n}$ as input to this system, then, from Eq. (2.113), we have

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d}e^{j\omega n}.$$

Thus, for any given value of ω, we obtain an output that is the input multiplied by a complex constant, the value of which depends on the frequency ω and the delay n_d. The frequency response of the ideal delay is therefore

$$H(e^{j\omega}) = e^{-j\omega n_d}. \tag{2.114}$$

As an alternative method of obtaining the frequency response, recall that $h[n] = \delta[n - n_d]$ for the ideal delay system. Using Eq. (2.109), we obtain

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d]e^{-j\omega n} = e^{-j\omega n_d}.$$

From the Euler relation, the real and imaginary parts of the frequency response are

$$H_R(e^{j\omega}) = \cos(\omega n_d), \tag{2.115a}$$

$$H_I(e^{j\omega}) = -\sin(\omega n_d). \tag{2.115b}$$

The magnitude and phase are

$$|H(e^{j\omega})| = 1, \tag{2.116a}$$

$$\sphericalangle H(e^{j\omega}) = -\omega n_d. \tag{2.116b}$$

In Section 2.7 we will show that a broad class of signals can be represented as a linear combination of complex exponentials in the form

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}. \tag{2.117}$$

From the principle of superposition, the corresponding output of a linear time-invariant system is

$$y[n] = \sum_k \alpha_k H(e^{j\omega_k})e^{j\omega_k n}. \tag{2.118}$$

Thus, if we can find a representation of $x[n]$ as a superposition of complex exponential sequences, as in Eq. (2.117), then we can find the output using Eq. (2.118) if we know the frequency response of the system. The following simple example illustrates this fundamental property of linear time-invariant systems.

Example 2.18 Sinusoidal Response of LTI Systems

Since it is simple to express a sinusoid as a linear combination of complex exponentials, let us consider a sinusoidal input

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{2.119}$$

From Eq. (2.110), the response to $x_1[n] = (A/2)e^{j\phi}e^{j\omega_0 n}$ is

$$y_1[n] = H(e^{j\omega_0})\frac{A}{2}e^{j\phi}e^{j\omega_0 n}. \tag{2.120a}$$

The response to $x_2[n] = (A/2)e^{-j\phi}e^{-j\omega_0 n}$ is

$$y_2[n] = H(e^{-j\omega_0})\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{2.120b}$$

Thus, the total response is

$$y[n] = \frac{A}{2}[H(e^{j\omega_0})e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})e^{-j\phi}e^{-j\omega_0 n}]. \tag{2.121}$$

If $h[n]$ is real, it can be shown (see Problem 2.71) that $H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$. Consequently,

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta), \tag{2.122}$$

where $\theta = \sphericalangle H(e^{j\omega_0})$ is the phase of the system function at frequency ω_0.

For the simple example of the ideal delay, $|H(e^{j\omega_0})| = 1$ and $\theta = -\omega_0 n_d$, as we determined in Example 2.17. Therefore,

$$\begin{aligned} y[n] &= A\cos(\omega_0 n + \phi - \omega_0 n_d) \\ &= A\cos[\omega_0(n - n_d) + \phi], \end{aligned} \tag{2.123}$$

which is consistent with what we would obtain directly using the definition of the ideal delay system.

The concept of the frequency response of linear time-invariant systems is essentially the same for continuous-time and discrete-time systems. However, an important distinction arises because the frequency response of discrete-time linear time-invariant systems is *always* a periodic function of the frequency variable ω with period 2π. To show this, we substitute $\omega + 2\pi$ into Eq. (2.109) to obtain

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n}. \tag{2.124}$$

Using the fact that $e^{\pm j2\pi n} = 1$ for n an integer, we have

$$e^{-j(\omega+2\pi)n} = e^{-j\omega n}e^{-j2\pi n} = e^{-j\omega n}.$$

Therefore,

$$H(e^{j(\omega+2\pi)}) = H(e^{j\omega}), \tag{2.125}$$

and, more generally,

$$H(e^{j(\omega+2\pi r)}) = H(e^{j\omega}), \qquad \text{for } r \text{ an integer}. \tag{2.126}$$

That is, $H(e^{j\omega})$ is periodic with period 2π. Note that this is obviously true for the ideal delay system, since $e^{-j(\omega+2\pi)n_d} = e^{-j\omega n_d}$ when n_d is an integer.

The reason for this periodicity is related directly to our earlier observation that the sequence

$$\{e^{j\omega n}\}, \qquad -\infty < n < \infty,$$

is indistinguishable from the sequence

$$\{e^{j(\omega+2\pi)n}\}, \qquad -\infty < n < \infty.$$

Because these two sequences have identical values for all n, the system must respond identically to both input sequences. This condition requires that Eq. (2.125) hold.

Since $H(e^{j\omega})$ is periodic with period 2π, and since the frequencies ω and $\omega + 2\pi$ are indistinguishable, it follows that we need only specify $H(e^{j\omega})$ over an interval of length 2π, e.g., $0 \leq \omega \leq 2\pi$ or $-\pi < \omega \leq \pi$. The inherent periodicity defines the frequency response everywhere outside the chosen interval. For simplicity and for consistency with the continuous-time case, it is generally convenient to specify $H(e^{j\omega})$ over the interval $-\pi < \omega \leq \pi$. With respect to this interval, the "low frequencies" are frequencies close to zero, while the "high frequencies" are frequencies close to $\pm\pi$. Recalling that frequencies differing by an integer multiple of 2π are indistinguishable, we might generalize the preceding statement as follows: The "low frequencies" are those that are close to an even multiple of π, while the "high frequencies" are those that are close to an odd multiple of π, consistent with our earlier discussion in Section 2.1.

Example 2.19 Ideal Frequency-Selective Filters

An important class of linear time-invariant systems includes those systems for which the frequency response is unity over a certain range of frequencies and is zero at the remaining frequencies. These correspond to *ideal frequency-selective filters*. The frequency response of an ideal lowpass filter is shown in Figure 2.17(a). Because of the inherent periodicity of the discrete-time frequency response, it has the appearance of

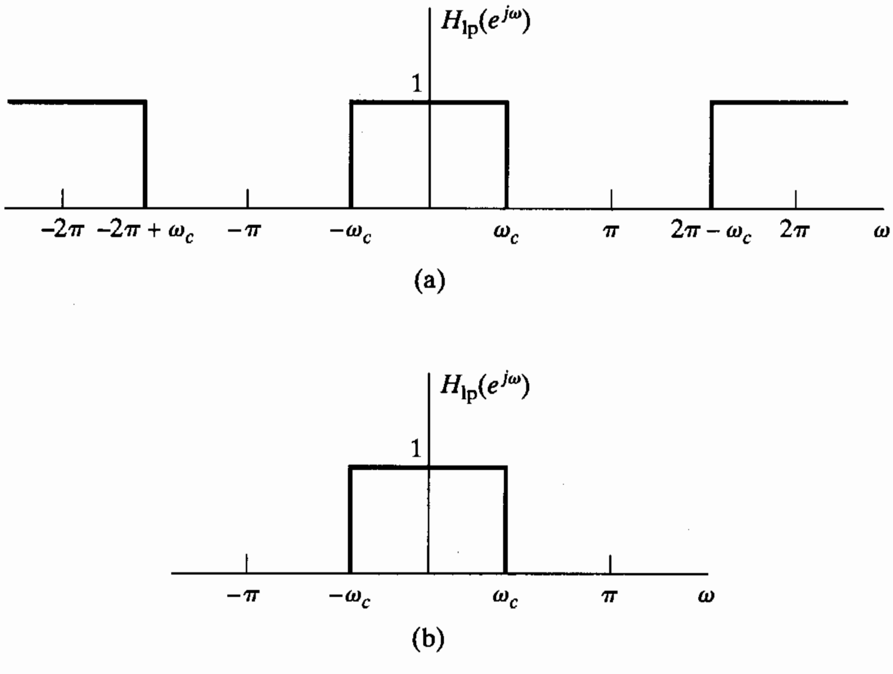


Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

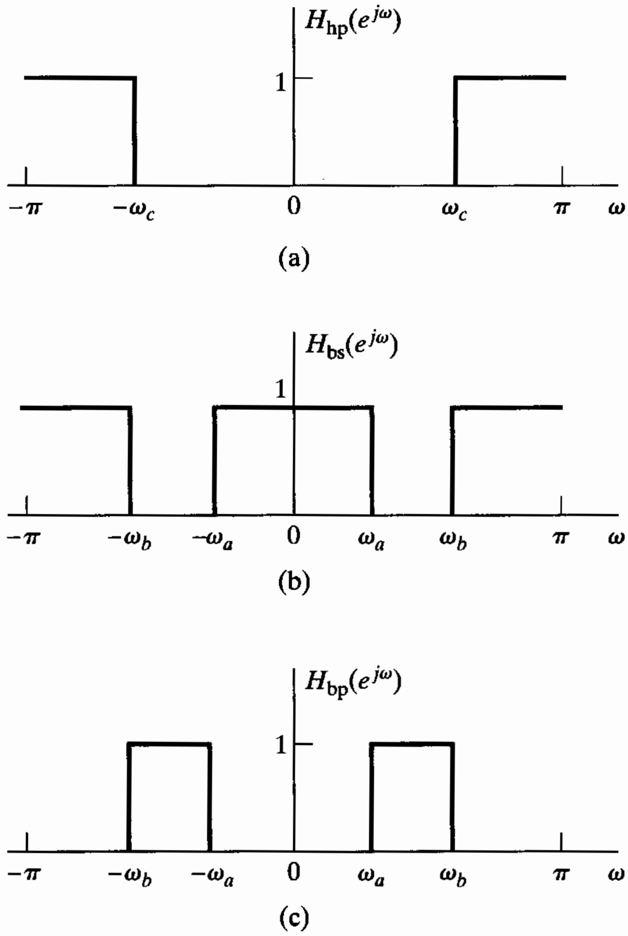


Figure 2.18 Ideal frequency-selective filters. (a) Highpass filter. (b) Bandstop filter. (c) Bandpass filter. In each case, the frequency response is periodic with period 2π. Only one period is shown.

a multiband filter, since frequencies around $\omega = 2\pi$ are indistinguishable from frequencies around $\omega = 0$. In effect, however, the frequency response passes only low frequencies and rejects high frequencies. Since the frequency response is completely specified by its behavior over the interval $-\pi < \omega \leq \pi$, the ideal lowpass filter frequency response is more typically shown only in the interval $-\pi < \omega \leq \pi$, as in Figure 2.17(b). It is understood that the frequency response repeats periodically with period 2π outside the plotted interval. The frequency responses for ideal highpass, bandstop, and bandpass filters are shown in Figures 2.18(a), (b), and (c), respectively.

Example 2.20 Frequency Response of the Moving-Average System

The impulse response of the moving-average system of Example 2.4 is

$$h[n] = \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the frequency response is

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}. \tag{2.127}$$

Equation (2.127) can be expressed in closed form by using Eq. (2.56), so that

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{1 - e^{-j\omega}} e^{-j\omega(M_2-M_1+1)/2} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{e^{j\omega/2} - e^{-j\omega/2}} e^{-j\omega(M_2-M_1)/2} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-j\omega(M_2-M_1)/2}. \end{aligned} \tag{2.128}$$

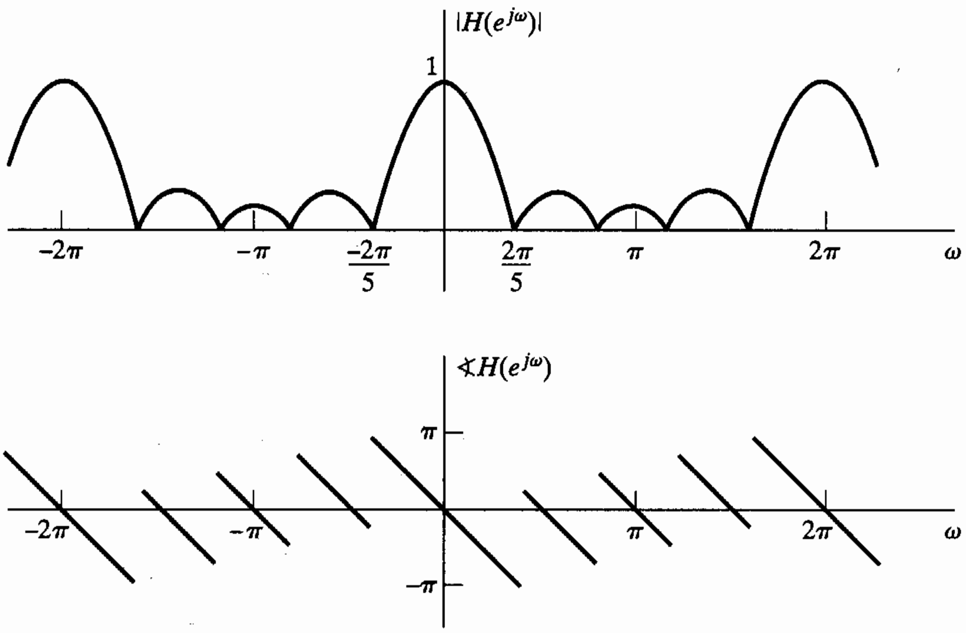


Figure 2.19 (a) Magnitude and (b) phase of the frequency response of the moving-average system for the case $M_1 = 0$ and $M_2 = 4$.

The magnitude and phase of $H(e^{j\omega})$ are plotted in Figure 2.19 for $M_1 = 0$ and $M_2 = 4$. Note that $H(e^{j\omega})$ is periodic, as is required of the frequency response of a discrete-time system. Note also that $|H(e^{j\omega})|$ falls off at "high frequencies" and $\sphericalangle H(e^{j\omega})$, i.e., the phase of $H(e^{j\omega})$, varies linearly with ω. This attenuation of the high frequencies suggests that the system will smooth out rapid variations in the input sequence; in other words, the system is a rough approximation to a lowpass filter. This is consistent with what we would intuitively expect about the behavior of the moving-average system.

2.6.2 Suddenly Applied Complex Exponential Inputs

We have seen that complex exponential inputs of the form $e^{j\omega n}$ for $-\infty < n < \infty$ produce outputs of the form $H(e^{j\omega})e^{j\omega n}$ for linear time-invariant systems. Such inputs, nonzero over a doubly infinite domain, may seem to be impractical models of signals; however, as we will see in the next section, models of this kind are crucial to the mathematical representation of a wide range of signals, even those that exist only over a finite domain. Even so, we can gain additional insight into linear time-invariant systems by considering more practical-appearing inputs of the form

$$x[n] = e^{j\omega n}u[n],$$

i.e., complex exponentials that are suddenly applied at an arbitrary time, which for convenience here we choose as $n = 0$. Using the convolution sum in Eq. (2.62), the corresponding output of a causal linear time-invariant system with impulse response $h[n]$ is

$$y[n] = \begin{cases} 0, & n < 0, \\ \left(\sum_{k=0}^{n} h[k]e^{-j\omega k}\right) e^{j\omega n}, & n \geq 0. \end{cases}$$

If we consider the output for $n \geq 0$, we can write

$$y[n] = \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} \tag{2.129}$$

$$= H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}. \tag{2.130}$$

From Eq. (2.130), we see that the output consists of the sum of two terms, i.e., $y[n] = y_{ss}[n] + y_t[n]$. The first term,

$$y_{ss}[n] = H(e^{j\omega})e^{j\omega n},$$

is called the *steady-state response*. It is identical to the response of the system when the input is $e^{j\omega n}$ for all n. In a sense, the second term,

$$y_t[n] = -\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}e^{j\omega n},$$

is the amount by which the output differs from the eigenfunction result. This part is called the *transient response*, because it is clear that in some cases it may approach zero. To see the conditions for which this is true, let us consider the size of the second term. Its magnitude is bounded as follows:

$$|y_t[n]| = \left|\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}e^{j\omega n}\right| \leq \sum_{k=n+1}^{\infty} |h[k]|. \tag{2.131}$$

From Eq. (2.131), it should be clear that if the impulse response has finite length, so that $h[n] = 0$ except for $0 \leq n \leq M$, then the term $y_t[n] = 0$ for $n + 1 > M$, or $n > M - 1$. In this case,

$$y[n] = y_{ss}[n] = H(e^{j\omega})e^{j\omega n}, \qquad \text{for } n > M - 1.$$

When the impulse response has infinite duration, the transient response does not disappear abruptly, but if the samples of the impulse response approach zero with increasing n, then $y_t[n]$ will approach zero. Note that Eq. (2.131) can be written

$$|y_t[n]| = \left| \sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}e^{j\omega n} \right| \leq \sum_{k=n+1}^{\infty} |h[k]| \leq \sum_{k=0}^{\infty} |h[k]|. \tag{2.132}$$

That is, the transient response is bounded by the sum of the absolute values of *all* of the impulse response samples. If the right-hand side of Eq. (2.132) is bounded, so that

$$\sum_{k=0}^{\infty} |h[k]| < \infty,$$

then the system is stable. From Eq. (2.132), it follows that, for stable systems, the transient response must become increasingly smaller as $n \to \infty$. Thus, a sufficient condition for the transient response to die out is that the system be stable.

Figure 2.20 shows the real part of a complex exponential signal with frequency $\omega = 2\pi/10$. The solid dots indicate the samples $x[k]$ of the suddenly applied complex exponential, while the open circles indicate the samples of the complex exponential that are "missing." The shaded dots indicate the samples of the impulse response $h[n-k]$ as a function of k for $n = 8$. In the finite-length case shown in Figure 2.20(a), it is clear that the output would consist only of the steady-state component for $n \geq 8$, while in the infinite-length case, it is clear that the missing samples have less and less effect as n increases, due to the decaying nature of the impulse response.

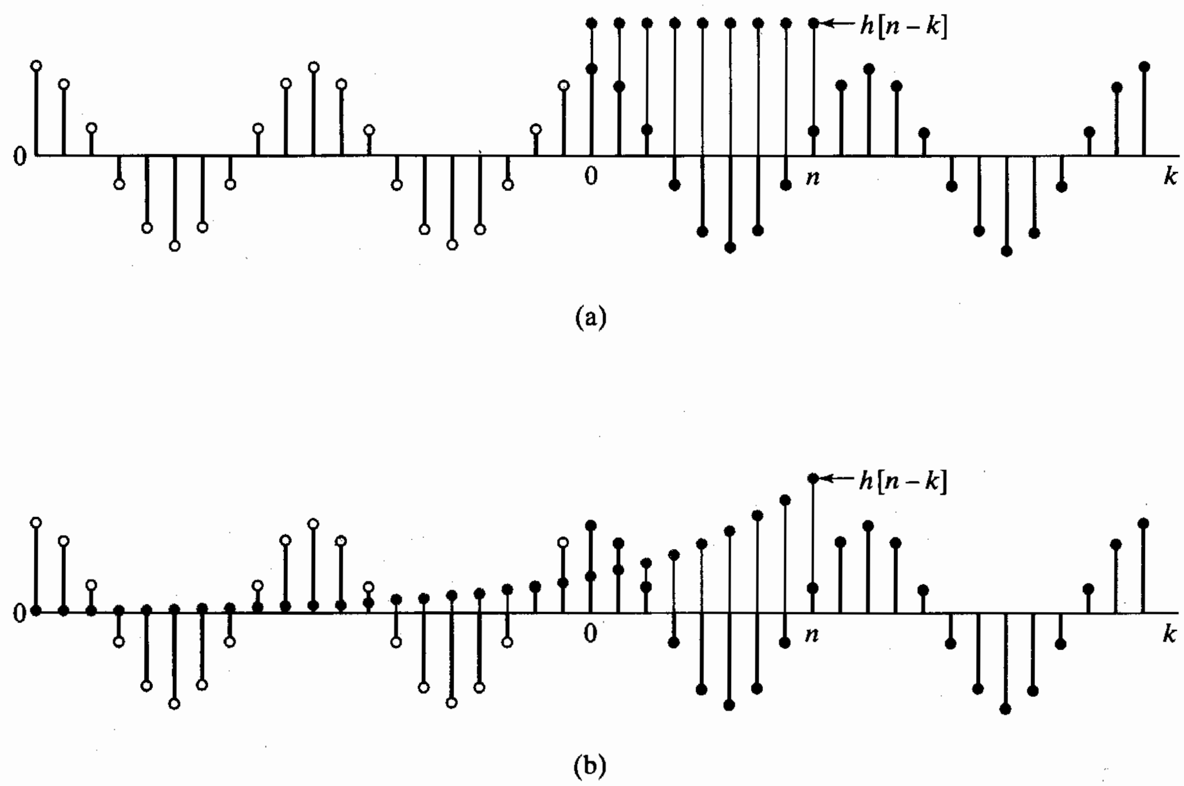


Figure 2.20 Illustration of real part of suddenly applied complex exponential input with (a) finite-length impulse response and (b) infinite-length impulse response.

The condition for stability is also a sufficient condition for the existence of the frequency response function. To see this, note that, in general,

$$|H(e^{j\omega})| = \left|\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right| \le \sum_{k=-\infty}^{\infty} |h[k]e^{-j\omega k}| \le \sum_{k=-\infty}^{\infty} |h[k]|,$$

so the general condition

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

ensures that $H(e^{j\omega})$ exists. It is no surprise that the condition for existence of the frequency response is the same as the condition for dominance of the steady-state solution. Indeed, a complex exponential that exists for all n can be thought of as one that is applied at $n = -\infty$. The eigenfunction property of complex exponentials depends on stability of the system, since at finite n, the transient response must have become zero, so that we only see the steady-state response $H(e^{j\omega})e^{j\omega n}$ for all finite n.

2.7 REPRESENTATION OF SEQUENCES BY FOURIER TRANSFORMS

One of the advantages of the frequency-response representation of a linear time-invariant system is that interpretations of system behavior such as the one we made in Example 2.20 often follow easily. We will elaborate on this point in considerably more detail in Chapter 5. At this point, however, let us return to the question of how we may find representations of the form of Eq. (2.117) for an arbitrary input sequence.

Many sequences can be represented by a Fourier integral of the form

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad (2.133)$$

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \qquad (2.134)$$

Equations (2.133) and (2.134) together form a *Fourier representation* for the sequence. Equation (2.133), the *inverse Fourier transform,* is a *synthesis* formula. That is, it represents $x[n]$ as a superposition of infinitesimally small complex sinusoids of the form

$$\frac{1}{2\pi}X(e^{j\omega})e^{j\omega n}d\omega,$$

with ω ranging over an interval of length 2π and with $X(e^{j\omega})$ determining the relative amount of each complex sinusoidal component. Although, in writing Eq. (2.133), we have chosen the range of values for ω between $-\pi$ and $+\pi$, any interval of length 2π can be used. Equation (2.134), the *Fourier transform,*[3] is an expression for computing $X(e^{j\omega})$ from the sequence $x[n]$, i.e., for *analyzing* the sequence $x[n]$ to determine how much of each frequency component is required to synthesize $x[n]$ using Eq. (2.133).

[3]Sometimes we will refer to Eq. (2.134) more explicitly as the discrete-time Fourier transform, or DTFT, particularly when it is important to distinguish it from the continuous-time Fourier transform.

In general, the Fourier transform is a complex-valued function of ω. As with the frequency response, we may either express $X(e^{j\omega})$ in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \tag{2.135a}$$

or in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}. \tag{2.135b}$$

The quantities $|X(e^{j\omega})|$ and $\sphericalangle X(e^{j\omega})$ are the *magnitude* and *phase,* respectively, of the Fourier transform. The Fourier transform is sometimes referred to as the *Fourier spectrum* or, simply, the *spectrum*. Also, the terminology *magnitude spectrum* or *amplitude spectrum* is sometimes used to refer to $|X(e^{j\omega})|$, and the angle or phase $\sphericalangle X(e^{j\omega})$ is sometimes called the *phase spectrum*.

The phase $\sphericalangle X(e^{j\omega})$ is not uniquely specified by Eq. (2.135b), since any integer multiple of 2π may be added to $\sphericalangle X(e^{j\omega})$ at any value of ω without affecting the result of the complex exponentiation. When we specifically want to refer to the principal value, i.e., $\sphericalangle X(e^{j\omega})$ restricted to the range of values between $-\pi$ and $+\pi$, we will denote this as ARG$[X(e^{j\omega})]$. If we want to refer to a phase function that is a continuous function of ω for $0 < \omega < \pi$, we will use the notation arg$[X(e^{j\omega})]$.

By comparing Eqs. (2.109) and (2.134), we can see that the frequency response of a linear time-invariant system is simply the Fourier transform of the impulse response and that, therefore, the impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral; i.e.,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega. \tag{2.136}$$

As discussed previously, the frequency response is a periodic function. Likewise, the Fourier transform is periodic with period 2π. Indeed, Eq. (2.134) is of the form of a Fourier series for the continuous-variable periodic function $X(e^{j\omega})$, and Eq. (2.133), which expresses the sequence values $x[n]$ in terms of the periodic function $X(e^{j\omega})$, is of the form of the integral that would be used to obtain the coefficients in the Fourier series. Our use of Eqs. (2.133) and (2.134) focuses on the representation of the sequence $x[n]$. Nevertheless, it is useful to be aware of the equivalence between the Fourier series representation of continuous-variable periodic functions and the Fourier transform representation of discrete-time signals, since all the familiar properties of Fourier series can be applied, with appropriate interpretation of variables, to the Fourier transform representation of a sequence.

We have not yet shown explicitly that Eqs. (2.133) and (2.134) are inverses of each other, nor have we considered the question of how broad a class of signals can be represented in the form of Eq. (2.133). To demonstrate that Eq. (2.133) is the inverse of Eq. (2.134), we can find $X(e^{j\omega})$ using Eq. (2.134) and then substitute the result into

Eq. (2.133). Specifically, consider

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{j\omega n}d\omega = \hat{x}[n], \tag{2.137}$$

where we have tentatively used $\hat{x}[n]$ to denote the result of the Fourier synthesis. We wish to show that $\hat{x}[n] = x[n]$ if $X(e^{j\omega})$ can be found using Eq. (2.134). Note that the "dummy index" of summation has been changed to m to distinguish it from n, the variable index in Eq. (2.133). If the infinite sum converges uniformly for all ω, then we can interchange the order of integration and summation to obtain

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega\right). \tag{2.138}$$

Evaluating the integral within the parentheses gives

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega &= \frac{\sin\pi(n-m)}{\pi(n-m)}\\ &= \begin{cases}1, & m=n,\\ 0, & m\neq n,\end{cases}\\ &= \delta[n-m].\end{aligned}$$

Thus,

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] = x[n],$$

which is what we set out to show.

Determining the class of signals that can be represented by Eq. (2.133) is equivalent to considering the convergence of the infinite sum in Eq. (2.134). That is, we are concerned with the conditions that must be satisfied by the terms in the sum in Eq. (2.134) such that

$$|X(e^{j\omega})| < \infty \qquad \text{for all } \omega,$$

where $X(e^{j\omega})$ is the limit as $M \to \infty$ of the finite sum

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}. \tag{2.139}$$

A sufficient condition for convergence can be found as follows:

$$\begin{aligned}|X(e^{j\omega})| &= \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|\\ &\le \sum_{n=-\infty}^{\infty} |x[n]|\,|e^{-j\omega n}|\\ &\le \sum_{n=-\infty}^{\infty} |x[n]| < \infty.\end{aligned}$$

Thus, if $x[n]$ is *absolutely summable,* then $X(e^{j\omega})$ exists. Furthermore, in this case, the series can be shown to converge uniformly to a continuous function of ω.

Since a stable sequence is, by definition, absolutely summable, all stable sequences have Fourier transforms. It also follows, then, that any stable *system* will have a finite and continuous frequency response.

Absolute summability is a sufficient condition for the existence of a Fourier transform representation. In Examples 2.17 and 2.20, we computed the Fourier transforms of the sequences $\delta[n-n_d]$ and $[1/(M_1+M_2+1)](u[n+M_1]-u[n-M_2-1])$. These sequences are absolutely summable, since they are finite in length. Clearly, any finite-length sequence is absolutely summable and thus will have a Fourier transform representation. In the context of linear time-invariant systems, any FIR system will be stable and therefore will have a finite, continuous frequency response. When a sequence has infinite length, we must be concerned about convergence of the infinite sum. The following example illustrates this case.

Example 2.21 Absolute Summability for a Suddenly-Applied Exponential

Let $x[n]=a^n u[n]$. The Fourier transform of this sequence is

$$X(e^{j\omega})=\sum_{n=0}^{\infty}a^n e^{-j\omega n}=\sum_{n=0}^{\infty}(ae^{-j\omega})^n$$

$$=\frac{1}{1-ae^{-j\omega}}\qquad \text{if } |ae^{-j\omega}|<1 \quad\text{or}\quad |a|<1.$$

Clearly, the condition $|a|<1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty}|a|^n=\frac{1}{1-|a|}<\infty \qquad \text{if } |a|<1. \tag{2.140}$$

Absolute summability is a *sufficient* condition for the existence of a Fourier transform representation, and it also guarantees uniform convergence. Some sequences are not absolutely summable, but are square summable, i.e.,

$$\sum_{n=-\infty}^{\infty}|x[n]|^2<\infty. \tag{2.141}$$

Such sequences can be represented by a Fourier transform if we are willing to relax the condition of uniform convergence of the infinite sum defining $X(e^{j\omega})$. Specifically, in this case we have mean-square convergence; that is, with

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n} \tag{2.142a}$$

and

$$X_M(e^{j\omega})=\sum_{n=-M}^{M}x[n]e^{-j\omega n}, \tag{2.142b}$$

it follows that

$$\lim_{M\to\infty}\int_{-\pi}^{\pi}|X(e^{j\omega})-X_M(e^{j\omega})|^2d\omega=0. \tag{2.143}$$

In other words, the error $|X(e^{j\omega})-X_M(e^{j\omega})|$ may not approach zero at each value of ω as $M \to \infty$, but the total "energy" in the error does. Example 2.22 illustrates this case.

Example 2.22 Square-Summability for the Ideal Lowpass Filter

Let us determine the impulse response of the ideal lowpass filter discussed in Example 2.19. The frequency response is

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{2.144}$$

with periodicity 2π also understood. The impulse response $h_{\text{lp}}[n]$ can be found using the Fourier transform synthesis equation (2.133):

$$\begin{aligned} h_{\text{lp}}[n] &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi jn}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}(e^{j\omega_c n} - e^{-j\omega_c n}) \\ &= \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \end{aligned} \tag{2.145}$$

We note that, since $h_{\text{lp}}[n]$ is nonzero for $n < 0$, the ideal lowpass filter is noncausal. Also, $h_{\text{lp}}[n]$ is *not* absolutely summable. The sequence values approach zero as $n \to \infty$, but only as $1/n$. This is because $H_{\text{lp}}(e^{j\omega})$ is discontinuous at $\omega = \omega_c$. Since $h_{\text{lp}}[n]$ is not absolutely summable, the infinite sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

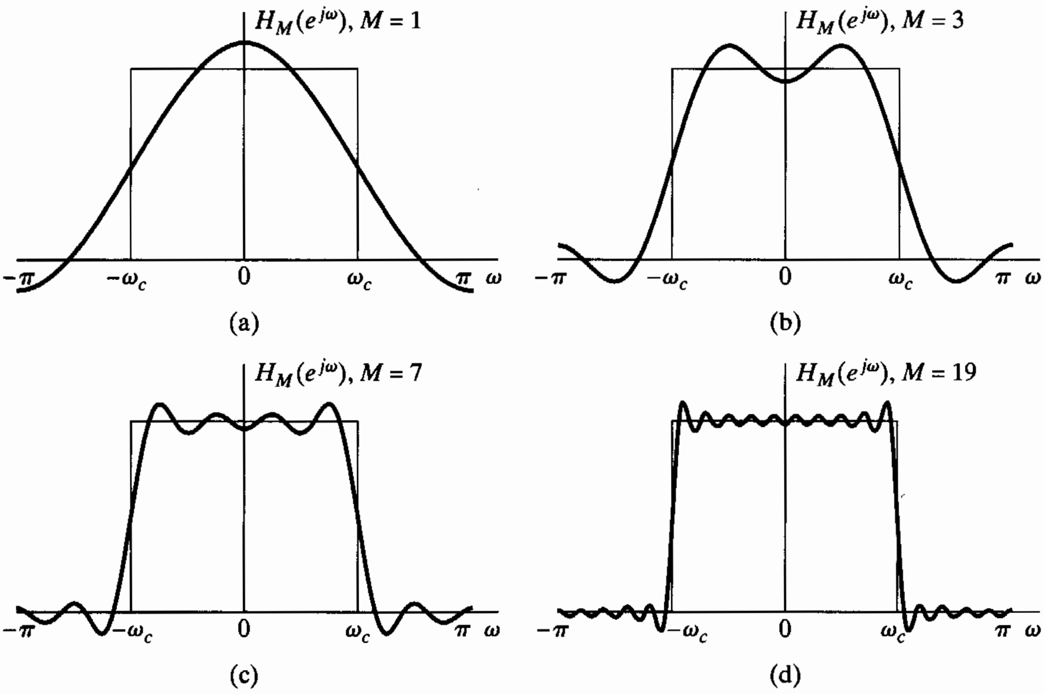


Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

does not converge uniformly for all values of ω. To obtain an intuitive feeling for this, let us consider $H_M(e^{j\omega})$ as the sum of a finite number of terms:

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}. \tag{2.146}$$

We can show that $H_M(e^{j\omega})$ can be expressed as

$$H_M(e^{j\omega}) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{\sin[(2M+1)(\omega-\theta)/2]}{\sin[(\omega-\theta)]/2} d\theta.$$

The function $H_M(e^{j\omega})$ is evaluated in Figure 2.21 for several values of M. Note that as M increases, the oscillatory behavior at $\omega = \omega_c$ (often referred to as the Gibbs phenomenon) is more rapid, but the size of the ripples does not decrease. In fact, it can be shown that as $M \to \infty$, the maximum amplitude of the oscillations does not approach zero, but the oscillations converge in location toward the point $\omega = \omega_c$. Thus, the infinite sum does not converge uniformly to the discontinuous function $H_{lp}(e^{j\omega})$ of Eq. (2.144). However, $h_{lp}[n]$, as given in Eq. (2.145), is square summable, and correspondingly, $H_M(e^{j\omega})$ converges in the mean-square sense to $H_{lp}(e^{j\omega})$; i.e.,

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |H_{lp}(e^{j\omega}) - H_M(e^{j\omega})|^2 d\omega = 0.$$

Although the error between $\lim_{M\to\infty} H_M(e^{j\omega})$ and $H_{lp}(e^{j\omega})$ might seem unimportant because the two functions differ only at $\omega = \omega_c$, we will see in Chapter 7 that the behavior of finite sums has important implications in the design of discrete-time systems for filtering.

It is sometimes useful to have a Fourier transform representation for certain sequences that are neither absolutely summable nor square summable. We illustrate several of these in the following examples.

Example 2.23 Fourier Transform of a Constant

Consider the sequence $x[n] = 1$ for all n. This sequence is neither absolutely summable nor square summable, and Eq. (2.134) does not converge in either the uniform or mean-square sense for this case. However, it is possible and useful to define the Fourier transform of the sequence $x[n]$ to be the periodic impulse train[4]

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r). \tag{2.147}$$

The impulses in this case are functions of a continuous variable and therefore are of "infinite height, zero width, and unit area," consistent with the fact that Eq. (2.134) does not converge. The use of Eq. (2.147) as a Fourier representation of the sequence $x[n] = 1$ is justified principally because formal substitution of Eq. (2.147) into Eq. (2.133) leads to the correct result. Example 2.24 represents a generalization of this example.

[4]The impulse function is defined as that "function" that has the following properties: $\delta(\omega) = 0$ for $\omega \neq 0$; $X(e^{j\omega})\delta(\omega) = X(e^{j0})\delta(\omega)$; $\int_{-\infty}^{\infty} \delta(\omega)d\omega = 1$; and $\delta(\omega) * X(e^{j\omega}) = X(e^{j\omega})$, where $*$ denotes continuous-variable convolution. See Oppenheim and Willsky (1997) for a discussion of the impulse function.

Example 2.24 Fourier Transform of Complex Exponential Sequences

Consider a sequence $x[n]$ whose Fourier transform is the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r). \tag{2.148}$$

We show in this example that $x[n]$ is the complex exponential sequence $e^{j\omega_0 n}$.

We can safely assume that $-\pi < \omega_0 \leq \pi$ in this problem. If the chosen value of ω_0 does not satisfy this requirement, there is a choice of ω_0 in the interval which produces the same $X(e^{j\omega})$, since the impulses repeat periodically every 2π. Thus, we can redefine ω_0 to be the frequency of the impulse in the summation of Eq. (2.148), which falls in the interval between $-\pi$ and π without any change in the spectrum $X(e^{j\omega})$.

We can determine $x[n]$ by substituting $X(e^{j\omega})$ into the inverse Fourier transform integral of Eq. (2.133). Because the integration of $X(e^{j\omega})$ extends only over one period, from $-\pi < \omega < \pi$, we need include only the $r = 0$ term from Eq. (2.148). Consequently, we can write

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega. \tag{2.149}$$

From the definition of the impulse function, it follows that

$$x[n] = e^{j\omega_0 n} \qquad \text{for any } n.$$

For $\omega_0 = 0$, this reduces to the sequence considered in Example 2.23.

Clearly, $x[n]$ in Example 2.24 is not absolutely summable, nor is it square summable, and $|X(e^{j\omega})|$ is not finite for all ω. Thus, the mathematical statement

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n}e^{-j\omega n} = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r) \tag{2.150}$$

must be interpreted in a special way. Such an interpretation is provided by the theory of generalized functions (Lighthill, 1958). Using that theory, we can rigorously extend the concept of a Fourier transform representation to the class of sequences that can be expressed as a sum of discrete frequency components, such as

$$x[n] = \sum_k a_k e^{j\omega_k n}, \qquad -\infty < n < \infty. \tag{2.151}$$

From the result of Example 2.24, it follows that

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\sum_k 2\pi a_k\delta(\omega - \omega_k + 2\pi r) \tag{2.152}$$

is a consistent Fourier transform representation of $x[n]$ in Eq. (2.151).

Another sequence that is neither absolutely summable nor square summable is the unit step sequence $u[n]$. Although it is not completely straightforward to show, this sequence can be represented by the following Fourier transform:

$$U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega + 2\pi r). \tag{2.153}$$

2.8 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

In using Fourier transforms, it is useful to have a detailed knowledge of the way that properties of the sequence manifest themselves in the Fourier transform and vice versa. In this section and Section 1.9, we discuss and summarize a number of such properties.

Symmetry properties of the Fourier transform are often very useful for simplifying the solution of problems. The following discussion presents these properties, and the proofs are considered in Problems 1.72 and 1.73. Before presenting the properties, however, we begin with some definitions.

A *conjugate-symmetric sequence* $x_e[n]$ is defined as a sequence for which $x_e[n] = x_e^*[-n]$, and a *conjugate-antisymmetric sequence* $x_o[n]$ is defined as a sequence for which $x_o[n] = -x_o^*[-n]$, where * denotes complex conjugation. Any sequence $x[n]$ can be expressed as a sum of a conjugate-symmetric and conjugate-antisymmetric sequence. Specifically,

$$x[n] = x_e[n] + x_o[n], \tag{2.154a}$$

where

$$x_e[n] = \tfrac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n] \tag{2.154b}$$

and

$$x_o[n] = \tfrac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]. \tag{2.154c}$$

A real sequence that is conjugate symmetric such that $x_e[n] = x_e[-n]$ is called an *even sequence*, and a real sequence that is conjugate antisymmetric such that $x_o[n] = -x_o[-n]$ is called an *odd sequence*.

A Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate-symmetric and conjugate-antisymmetric functions as

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}), \tag{2.155a}$$

where

$$X_e(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})] \tag{2.155b}$$

and

$$X_o(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})]. \tag{2.155c}$$

By substituting $-\omega$ for ω in Eqs. (2.155b) and (2.155c), it follows that $X_e(e^{j\omega})$ is conjugate symmetric and $X_o(e^{j\omega})$ is conjugate antisymmetric; i.e.,

$$X_e(e^{j\omega}) = X_e^*(e^{-j\omega}) \tag{2.156a}$$

and

$$X_o(e^{j\omega}) = -X_o^*(e^{-j\omega}). \tag{2.156b}$$

If a real function of a continuous variable is conjugate symmetric, it is referred to as an *even function*, and a real conjugate-antisymmetric function of a continuous variable is referred to as an *odd function*.

The symmetry properties of the Fourier transform are summarized in Table 1.1. The first six properties apply for a general complex sequence $x[n]$ with Fourier transform $X(e^{j\omega})$. Properties 1 and 2 are considered in Problem 1.72. Property 3 follows from

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

properties 1 and 2, together with the fact that the Fourier transform of the sum of two sequences is the sum of their Fourier transforms. Specifically, the Fourier transform of $\mathcal{R}e\{x[n]\} = \frac{1}{2}(x[n]+x^*[n])$ is the conjugate-symmetric part of $X(e^{j\omega})$, or $X_e(e^{j\omega})$. Similarly, $j\mathcal{I}m\{x[n]\} = \frac{1}{2}(x[n]-x^*[n])$, or equivalently, $j\mathcal{I}m\{x[n]\}$ has a Fourier transform that is the conjugate-antisymmetric component $X_o(e^{j\omega})$ corresponding to property 4. By considering the Fourier transform of $x_e[n]$ and $x_o[n]$, the conjugate-symmetric and conjugate-antisymmetric components, respectively, of $x[n]$, it can be shown that properties 5 and 6 follow.

If $x[n]$ is a real sequence, these symmetry properties become particularly straightforward and useful. Specifically, for a real sequence, the Fourier transform is conjugate symmetric; i.e., $X(e^{j\omega}) = X^*(e^{-j\omega})$ (property 7). Expressing $X(e^{j\omega})$ in terms of its real and imaginary parts as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}), \tag{2.157}$$

we can derive properties 8 and 9—specifically,

$$X_R(e^{j\omega}) = X_R(e^{-j\omega}) \tag{2.158a}$$

and

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega}). \tag{2.158b}$$

In other words, the real part of the Fourier transform is an even function, and the imaginary part is an odd function, if the sequence is real. In a similar manner, by expressing $X(e^{j\omega})$ in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}, \tag{2.159}$$

we can show that, for a real sequence $x[n]$, the magnitude of the Fourier transform, $|X(e^{j\omega})|$, is an even function of ω and the phase, $\sphericalangle X(e^{j\omega})$, can be chosen to be an odd function of ω (properties 10 and 11). Also, for a real sequence, the even part of $x[n]$ transforms to $X_R(e^{j\omega})$, and the odd part of $x[n]$ transforms to $jX_I(e^{j\omega})$ (properties 12 and 13).

Example 2.25 Illustration of Symmetry Properties

Let us return to the sequence of Example 2.21, where we showed that the Fourier transform of the real sequence $x[n] = a^n u[n]$ is

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |a| < 1. \tag{2.160}$$

Then, from the properties of complex numbers, it follows that

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = X^*(e^{-j\omega}) \quad \text{(property 7)},$$

$$X_R(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R(e^{-j\omega}) \quad \text{(property 8)},$$

$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I(e^{-j\omega}) \quad \text{(property 9)},$$

$$|X(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} = |X(e^{-j\omega})| \quad \text{(property 10)},$$

$$\sphericalangle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = -\sphericalangle X(e^{-j\omega}) \quad \text{(property 11)}.$$

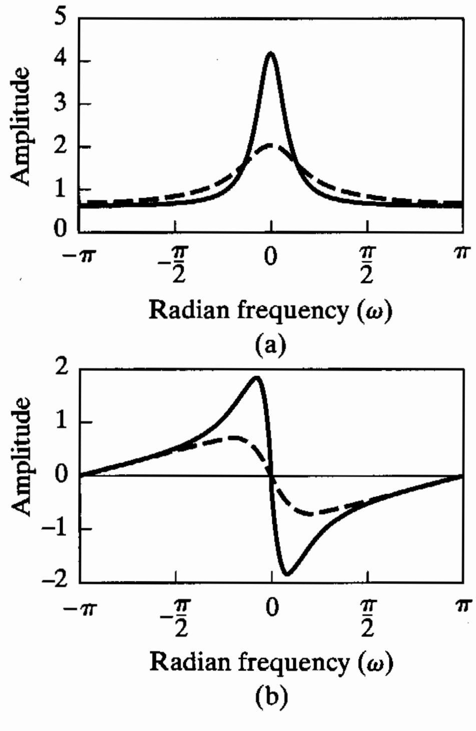


Figure 2.22 Frequency response for a system with impulse response $h[n] = a^n u[n]$. (a) Real part. $a > 0$; $a = 0.9$ (solid curve) and $a = 0.5$ (dashed curve). (b) Imaginary part.

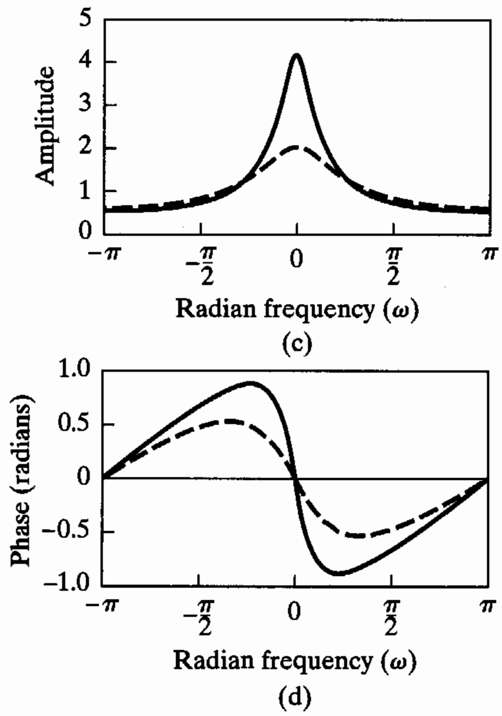


Figure 2.22 (*Continued*) (c) Magnitude. $a > 0$; $a = 0.9$ (solid curve) and $a = 0.5$ (dashed curve). (d) Phase.

These functions are plotted in Figure 2.22 for $a > 0$, specifically, $a = 0.9$ (solid curve) and $a = 0.5$ (dashed curve). In Problem 2.43, we consider the corresponding plots for $a < 0$.

2.9 FOURIER TRANSFORM THEOREMS

In addition to the symmetry properties, a variety of theorems (presented in Sections 2.9.1–2.9.7) relate operations on the sequence to operations on the Fourier transform. We will see that these theorems are quite similar in most cases to corresponding theorems for continuous-time signals and their Fourier transforms. To facilitate the statement of the theorems, we introduce the following operator notation:

$$\begin{aligned} X(e^{j\omega}) &= \mathcal{F}\{x[n]\}, \\ x[n] &= \mathcal{F}^{-1}\{X(e^{j\omega})\}, \\ x[n] &\overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}). \end{aligned}$$

That is, $\mathcal{F}$ denotes the operation of "taking the Fourier transform of $x[n]$," and $\mathcal{F}^{-1}$ is the inverse of that operation. Most of the theorems will be stated without proof. The proofs, which are left as exercises (Problem 2.74), generally involve only simple manipulations of variables of summation or integration. The theorems in this section are summarized in Table 2.2.

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2d\omega$	
9. $\sum_{n=-\infty}^{\infty}x[n]y^*[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

2.9.1 Linearity of the Fourier Transform

If

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega}),$$

then it follows by substitution into the definition of the discrete-time Fourier transform that

$$ax_1[n]+bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega})+bX_2(e^{j\omega}). \tag{2.161}$$

2.9.2 Time Shifting and Frequency Shifting

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then, for the time-shifted sequence, a simple transformation of the index of summation in the discrete-time Fourier transform yields

$$x[n-n_d] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d}X(e^{j\omega}). \tag{2.162}$$

Direct substitution proves the following result for the frequency-shifted Fourier transform:

$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}). \tag{2.163}$$

2.9.3 Time Reversal

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then if the sequence is time reversed,

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega}). \tag{2.164}$$

If $x[n]$ is real, this theorem becomes

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(e^{j\omega}). \tag{2.165}$$

2.9.4 Differentiation in Frequency

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then, by differentiating the discrete-time Fourier transform, it is seen that

$$nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}. \tag{2.166}$$

2.9.5 Parseval's Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega. \tag{2.167}$$

The function $|X(e^{j\omega})|^2$ is called the *energy density spectrum,* since it determines how the energy is distributed in the frequency domain. Necessarily, the energy density spectrum is defined only for finite-energy signals. A more general form of Parseval's theorem is shown in Problem 2.77.

2.9.6 The Convolution Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

and

$$h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega}),$$

and if

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n], \tag{2.168}$$

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}). \tag{2.169}$$

Thus, convolution of sequences implies multiplication of the corresponding Fourier transforms. Note that the time-shifting property is a special case of the convolution property, since

$$\delta[n-n_d] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} \tag{2.170}$$

and if $h[n] = \delta[n-n_d]$, then $y[n] = x[n] * \delta[n-n_d] = x[n-n_d]$. Therefore,

$$H(e^{j\omega}) = e^{-j\omega n_d} \qquad \text{and} \qquad Y(e^{j\omega}) = e^{-j\omega n_d} X(e^{j\omega}).$$

A formal derivation of the convolution theorem is easily achieved by applying the definition of the Fourier transform to $y[n]$ as expressed in Eq. (2.168). This theorem can also be interpreted as a direct consequence of the eigenfunction property of complex exponentials for linear time-invariant systems. Recall that $H(e^{j\omega})$ is the frequency response of the linear time-invariant system whose impulse response is $h[n]$. Recall also that if

$$x[n] = e^{j\omega n},$$

then

$$y[n] = H(e^{j\omega})e^{j\omega n}.$$

That is, complex exponentials are *eigenfunctions* of linear time-invariant systems, where $H(e^{j\omega})$, the Fourier transform of $h[n]$, is the eigenvalue. From the definition of integration, the Fourier transform synthesis equation corresponds to the representation of a sequence $x[n]$ as a superposition of complex exponentials of infinitesimal size; that is,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \lim_{\Delta\omega\to 0}\frac{1}{2\pi}\sum_k X(e^{jk\Delta\omega})e^{jk\Delta\omega n}\Delta\omega.$$

By the eigenfunction property of linear systems and by the principle of superposition, the corresponding output will be

$$y[n] = \lim_{\Delta\omega\to 0}\frac{1}{2\pi}\sum_k H(e^{jk\Delta\omega})X(e^{jk\Delta\omega})e^{jk\Delta\omega n}\Delta\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega.$$

Thus, we conclude that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}),$$

as in Eq. (2.169).

2.9.7 The Modulation or Windowing Theorem

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

and

$$w[n] \overset{\mathcal{F}}{\longleftrightarrow} W(e^{j\omega}),$$

and if

$$y[n] = x[n]w[n], \tag{2.171}$$

then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta. \tag{2.172}$$

Equation (2.172) is a periodic convolution, i.e., a convolution of two periodic functions with the limits of integration extending over only one period. The duality inherent in most Fourier transform theorems is evident when we compare the convolution and modulation theorems. However, in contrast to the continuous-time case, where this duality is complete, in the discrete-time case fundamental differences arise because the Fourier transform is a sum while the inverse transform is an integral with a periodic integrand. Although for continuous time we can state that convolution in the time domain is represented by multiplication in the frequency domain and vice versa, in discrete time this statement must be modified somewhat. Specifically, discrete-time convolution of sequences (the convolution sum) is equivalent to multiplication of corresponding periodic Fourier transforms, and multiplication of sequences is equivalent to *periodic* convolution of corresponding Fourier transforms.

The theorems of this section and a number of fundamental Fourier transform pairs are summarized in Tables 2.2 and 2.3, respectively. One of the ways that knowledge of

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k) + \pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

Fourier transform theorems and properties is useful is in determining Fourier transforms or inverse transforms. Often, by using the theorems and known transform pairs, it is possible to represent a sequence in terms of operations on other sequences for which the transform is known, thereby simplifying an otherwise difficult or tedious problem. Examples 2.26–2.30 illustrate this approach.

Example 2.26 Determining a Fourier Transform using Tables 2.2 and 2.3

Suppose we wish to find the Fourier transform of the sequence $x[n] = a^n u[n-5]$. This transform can be computed by exploiting Theorems 1 and 2 of Table 2.2 and transform pair 4 of Table 2.3. Let $x_1[n] = a^n u[n]$. We start with this signal because it is the most similar signal to $x[n]$ in Table 2.3. The table states that

$$X_1(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{2.173}$$

To obtain $x[n]$ from $x_1[n]$, we first delay $x_1[n]$ by 5 samples, i.e., $x_2[n] = x_1[n-5]$. Theorem 2 of Table 2.2 gives the corresponding frequency-domain relationship, $X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega})$, so

$$X_2(e^{j\omega}) = \frac{e^{-j5\omega}}{1 - ae^{-j\omega}}. \tag{2.174}$$

In order to get from $x_2[n]$ to the desired $x[n]$, we need only multiply by the constant a^5, i.e., $x[n] = a^5 x_2[n]$. The linearity property of the Fourier transform, Theorem 1 of Table 2.2, then yields the desired Fourier transform,

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}. \tag{2.175}$$

Example 2.27 Determining an Inverse Fourier Transform Using Tables 2.2 and 2.3

Suppose that

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - be^{-j\omega})}. \tag{2.176}$$

Direct substitution of $X(e^{j\omega})$ into Eq. (2.133) leads to an integral that is difficult to evaluate by ordinary real integration techniques. However, using the technique of partial fraction expansion, which we discuss in detail in Chapter 3, we can expand $X(e^{j\omega})$ into the form

$$X(e^{j\omega}) = \frac{a/(a-b)}{1 - ae^{-j\omega}} - \frac{b/(a-b)}{1 - be^{-j\omega}}. \tag{2.177}$$

From Theorem 1 of Table 2.2 and transform pair 4 of Table 2.3, it follows that

$$x[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]. \tag{2.178}$$

Example 2.28 Determining the Impulse Response from the Frequency Response

The frequency response of a highpass filter with delay is

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_c < |\omega| < \pi, \\ 0, & |\omega| < \omega_c, \end{cases} \tag{2.179}$$

where a period of 2π is understood. This frequency response can be expressed as

$$H(e^{j\omega}) = e^{-j\omega n_d}(1 - H_{\text{lp}}(e^{j\omega})) = e^{-j\omega n_d} - e^{-j\omega n_d} H_{\text{lp}}(e^{j\omega}),$$

where $H_{\text{lp}}(e^{j\omega})$ is periodic with period 2π and

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi. \end{cases}$$

Using the result of Example 2.22 to obtain the inverse transform of $H_{\text{lp}}(e^{j\omega})$, together with properties 1 and 2 of Table 2.2, we have

$$\begin{aligned} h[n] &= \delta[n - n_d] - r[n - n_d] \\ &= \delta[n - n_d] - \frac{\sin \omega_c(n - n_d)}{\pi(n - n_d)}. \end{aligned}$$

Example 2.29 Determining the Impulse Response for a Difference Equation

In this example we determine the impulse response for a stable linear time-invariant system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] - \tfrac{1}{4}x[n-1]. \tag{2.180}$$

In Chapter 3 we will see that the z-transform is more useful than the Fourier transform for dealing with difference equations. However, this example offers a hint of the utility of transform methods in the analysis of linear systems. To find the impulse response, we set $x[n] = \delta[n]$; with $h[n]$ denoting the impulse response, Eq. (2.180) becomes

$$h[n] - \tfrac{1}{2}h[n-1] = \delta[n] - \tfrac{1}{4}\delta[n-1]. \tag{2.181}$$

Applying the Fourier transform to both sides of Eq. (2.181) and using properties 1 and 2 of Table 2.2, we obtain

$$H(e^{j\omega}) - \tfrac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \tfrac{1}{4}e^{-j\omega}, \tag{2.182}$$

or

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{2.183}$$

To obtain $h[n]$, we want to determine the inverse Fourier transform of $H(e^{j\omega})$. Toward this end, we rewrite Eq. (2.183) as

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{2.184}$$

From transform 4 of Table 2.3,

$$\left(\tfrac{1}{2}\right)^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

Combining this transform with property 3 of Table 2.2, we obtain

$$-\left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1}u[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} -\frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}. \quad (2.185)$$

Based on property 1 of Table 2.2, then,

$$h[n] = \left(\tfrac{1}{2}\right)^{n}u[n] - \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1}u[n-1]. \quad (2.186)$$

2.10 DISCRETE-TIME RANDOM SIGNALS

The preceding sections have focused on mathematical representations of discrete-time signals and systems and the insights that derive from such mathematical representations. We have seen that discrete-time signals and systems have both a time-domain and a frequency-domain representation, each with an important place in the theory and design of discrete-time signal-processing systems. Until now, we have assumed that the signals are deterministic, i.e., that each value of a sequence is uniquely determined by a mathematical expression, a table of data, or a rule of some type.

In many situations, the processes that generate signals are so complex as to make precise description of a signal extremely difficult or undesirable, if not impossible. In such cases, modeling the signal as a stochastic process is analytically useful. As an example, we will see in Chapter 6 that many of the effects encountered in implementing digital signal-processing algorithms with finite register length can be represented by additive noise, i.e., a stochastic sequence. Many mechanical systems generate acoustic or vibratory signals that can be processed to diagnose potential failure; again, signals of this type are often best modeled in terms of stochastic signals. Speech signals to be processed for automatic recognition or bandwidth compression and music to be processed for quality enhancement are two more of many examples.

A stochastic signal is considered to be a member of an ensemble of discrete-time signals that is characterized by a set of probability density functions. More specifically, for a particular signal at a particular time, the amplitude of the signal sample at that time is assumed to have been determined by an underlying scheme of probabilities. That is, each individual sample $x[n]$ of a particular signal is assumed to be an outcome of some underlying random variable $\mathbf{x}_n$. The entire signal is represented by a collection of such random variables, one for each sample time, $-\infty < n < \infty$. This collection of random variables is called a *random process,* and we assume that a particular sequence of samples $x[n]$ for $-\infty < n < \infty$ has been generated by the random process that underlies the signal. To completely describe the random process, we need to specify the individual and joint probability distributions of all the random variables.

The key to obtaining useful results from such models of signals lies in their description in terms of averages that can be computed from assumed probability laws or estimated from specific signals. While stochastic signals are not absolutely summable or square summable and, consequently, do not directly have Fourier transforms, many (but not all) of the properties of such signals can be summarized in terms of averages such as the *autocorrelation* or *autocovariance sequence,* for which the Fourier transform often exists. As we will discuss in this section, the Fourier transform of the autocovariance sequence has a useful interpretation in terms of the frequency distribution of the power in the signal. The use of the autocovariance sequence and its transform has another important advantage: The effect of processing stochastic signals with a discrete-time linear system can be conveniently described in terms of the effect of the system on the autocovariance sequence.

In the following discussion, we assume that the reader is familiar with the basic concepts of stochastic processes, such as averages, correlation and covariance functions, and the power spectrum. A brief review and summary of notation and concepts is provided in Appendix A. A more detailed presentation of the theory of random signals can be found in a variety of excellent texts, such as Davenport (1970) and Papoulis (1984).

Our primary objective in this section is to present a specific set of results that will be useful in subsequent chapters. Therefore, we focus on wide-sense stationary random signals and their representation in the context of processing with linear time-invariant systems. Although, for simplicity, we assume that $x[n]$ and $h[n]$ are real valued, the results can be generalized to the complex case.

Consider a stable linear time-invariant system with real impulse response $h[n]$. Let $x[n]$ be a real-valued sequence that is a sample sequence of a wide-sense stationary discrete-time random process. Then the output of the linear system is also a sample function of a random process related to the input process by the linear transformation

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

As we have shown, since the system is stable, $y[n]$ will be bounded if $x[n]$ is bounded. We will see shortly that if the input is stationary,[5] then so is the output. The input signal may be characterized by its mean m_x and its autocorrelation function $\phi_{xx}[m]$, or we may also have additional information about first- or even second-order probability distributions. In characterizing the output random process $y[n]$ we desire similar information. For many applications, it is sufficient to characterize both the input and output in terms of simple averages, such as the mean, variance, and autocorrelation. Therefore, we will derive input–output relationships for these quantities.

The means of the input and output processes are, respectively,

$$m_{x_n} = \mathcal{E}\{\mathbf{x}_n\}, \qquad m_{y_n} = \mathcal{E}\{\mathbf{y}_n\}, \tag{2.187}$$

where $\mathcal{E}\{\cdot\}$ denotes the expected value. In most of our discussion, it will not be necessary to carefully distinguish between the random variables $\mathbf{x}_n$ and $\mathbf{y}_n$ and their specific values $x[n]$ and $y[n]$. This will simplify the mathematical notation significantly. For example, Eqs. (2.187) will alternatively be written

$$m_x[n] = \mathcal{E}\{x[n]\}, \qquad m_y[n] = \mathcal{E}\{y[n]\}. \tag{2.188}$$

If $x[n]$ is stationary, then $m_x[n]$ is independent of n and will be written as m_x, with similar notation for $m_y[n]$ if $y[n]$ is stationary.

[5] In the remainder of the text, we will use the term *stationary* to mean "wide-sense stationary."

The mean of the output process is

$$m_y[n] = \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\},$$

where we have used the fact that the expected value of a sum is the sum of the expected values. Since the input is stationary, $m_x[n-k] = m_x$, and consequently,

$$m_y[n] = m_x \sum_{k=-\infty}^{\infty} h[k]. \tag{2.189}$$

From Eq. (2.189), we see that the mean of the output is also constant. An equivalent expression to Eq. (2.189) in terms of the frequency response is

$$m_y = H(e^{j0})m_x. \tag{2.190}$$

Assuming temporarily that the output is nonstationary, the autocorrelation function of the output process for a real input is

$$\begin{aligned}\phi_{yy}[n, n+m] &= \mathcal{E}\{y[n]y[n+m]\}\\ &= \mathcal{E}\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right\}\\ &= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\mathcal{E}\{x[n-k]x[n+m-r]\}.\end{aligned}$$

Since $x[n]$ is assumed to be stationary, $\mathcal{E}\{x[n-k]x[n+m-r]\}$ depends only on the time difference $m+k-r$. Therefore,

$$\phi_{yy}[n, n+m] = \sum_{k=-\infty}^{\infty} h[r] \sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] = \phi_{yy}[m]. \tag{2.191}$$

That is, the output autocorrelation sequence also depends only on the time difference m. Thus, for a linear time-invariant system having a wide-sense stationary input, the output is also wide-sense stationary.

By making the substitution $\ell = r - k$, we can express Eq. (2.191) as

$$\begin{aligned}\phi_{yy}[m] &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell] \sum_{k=-\infty}^{\infty} h[k]h[\ell+k]\\ &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell],\end{aligned} \tag{2.192}$$

where we have defined

$$c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]h[\ell+k]. \tag{2.193}$$

A sequence of the form of $c_{hh}[\ell]$ is called a *deterministic autocorrelation sequence* or, simply, the *autocorrelation sequence of* $h[n]$. It should be emphasized that $c_{hh}[\ell]$ is the autocorrelation of an aperiodic—i.e., finite-energy—sequence and should not be confused with the autocorrelation of an infinite-energy random sequence. Indeed, it can be seen that $c_{hh}[\ell]$ is simply the discrete convolution of $h[n]$ with $h[-n]$. Equation (2.192), then, can be interpreted to mean that the autocorrelation of the output of a linear system is the convolution of the autocorrelation of the input with the aperiodic autocorrelation of the system impulse response.

Equation (2.192) suggests that Fourier transforms may be useful in characterizing the response of a linear time-invariant system to a stochastic input. Assume, for convenience, that $m_x = 0$; i.e., the autocorrelation and autocovariance sequences are identical. Then, with $\Phi_{xx}(e^{j\omega})$, $\Phi_{yy}(e^{j\omega})$, and $C_{hh}(e^{j\omega})$ denoting the Fourier transforms of $\phi_{xx}[m]$, $\phi_{yy}[m]$, and $c_{hh}[\ell]$, respectively, from Eq. (2.192),

$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega}). \tag{2.194}$$

Also, from Eq. (2.193),

$$\begin{aligned} C_{hh}(e^{j\omega}) &= H(e^{j\omega})H^*(e^{j\omega}) \\ &= |H(e^{j\omega})|^2, \end{aligned}$$

so

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega}). \tag{2.195}$$

Equation (2.195) provides the motivation for the term *power density spectrum*. Specifically,

$$\begin{aligned} \mathcal{E}\{y^2[n]\} = \phi_{yy}[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})\,d\omega \\ &= \text{total average power in output}. \end{aligned} \tag{2.196}$$

Substituting Eq. (2.195) into Eq. (2.196), we have

$$\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega})\,d\omega. \tag{2.197}$$

Suppose that $H(e^{j\omega})$ is an ideal bandpass filter, as shown in Figure 2.18(c). We recall that $\phi_{xx}[m]$ is an even sequence, so

$$\Phi_{xx}(e^{j\omega}) = \Phi_{xx}(e^{-j\omega}).$$

Likewise, $|H(e^{j\omega})|^2$ is an even function of ω. Therefore, we can write

$$\begin{aligned} \phi_{yy}[0] &= \text{average power in output} \\ &= \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \Phi_{xx}(e^{j\omega})\,d\omega + \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} \Phi_{xx}(e^{j\omega})\,d\omega. \end{aligned} \tag{2.198}$$

Thus, the area under $\Phi_{xx}(e^{j\omega})$ for $\omega_a \le |\omega| \le \omega_b$ can be taken to represent the mean-square value of the input in that frequency band. We observe that the output power must remain nonnegative, so

$$\lim_{(\omega_b-\omega_a)\to 0} \phi_{yy}[0] \ge 0.$$

This result, together with Eq. (2.198) and the fact that the band $\omega_a \le \omega \le \omega_b$ can be arbitrarily small, implies that

$$\Phi_{xx}(e^{j\omega}) \ge 0 \qquad \text{for all } \omega. \tag{2.199}$$

Hence, we note that the power density function of a real signal is real, even, and nonnegative.

Example 2.30 White Noise

The concept of white noise is exceedingly useful in quantization error analysis. A white-noise signal is a signal for which $\phi_{xx}[m] = \sigma_x^2\delta[m]$. We assume in this example that the signal has zero mean. The power spectrum of a white noise signal is a constant, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2 \qquad \text{for all } \omega.$$

The average power of a white-noise signal is therefore

$$\phi_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_x^2\,d\omega = \sigma_x^2.$$

The concept of white noise is also useful in the representation of random signals whose power spectra are not constant with frequency. For example, a random signal $y[n]$ with power spectrum $\Phi_{yy}(e^{j\omega})$ can be assumed to be the output of a linear time-invariant system with a white-noise input. That is, we use Eq. (2.195) to define a system with frequency response $H(e^{j\omega})$ to satisfy the equation

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_x^2,$$

where σ_x^2 is the average power of the assumed white-noise input signal. We adjust the average power of this input signal to give the correct average power for $y[n]$. For example, suppose that $h[n] = a^n u[n]$. Then

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}},$$

and we can represent all random signals whose power spectra are of the form

$$\Phi_{yy}(e^{j\omega}) = \left|\frac{1}{1 - ae^{-j\omega}}\right|^2 \sigma_x^2 = \frac{\sigma_x^2}{1 + a^2 - 2a\cos\omega}.$$

Another important result concerns the cross-correlation between the input and output of a linear time-invariant system:

$$\begin{aligned}\phi_{xy}[m] &= \mathcal{E}\{x[n]y[n+m]\}\\ &= \mathcal{E}\left\{x[n]\sum_{k=-\infty}^{\infty} h[k]x[n+m-k]\right\} \qquad (2.200)\\ &= \sum_{k=-\infty}^{\infty} h[k]\phi_{xx}[m-k].\end{aligned}$$

In this case, we note that the cross-correlation between input and output is the convolution of the impulse response with the input autocorrelation sequence.

The Fourier transform of Eq. (2.200) is

$$\Phi_{xy}(e^{j\omega}) = H(e^{j\omega})\Phi_{xx}(e^{j\omega}). \qquad (2.201)$$

This result has a useful application when the input is white noise, i.e., when $\phi_{xx}[m] = \sigma_x^2\delta[m]$. Substituting into Eq. (2.198), we note that

$$\phi_{xy}[m] = \sigma_x^2 h[m]. \qquad (2.202)$$

That is, for a zero-mean white-noise input, the cross-correlation between input and output of a linear system is proportional to the impulse response of the system. Similarly, the power spectrum of a white-noise input is

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2, \qquad -\pi \leq \omega \leq \pi. \tag{2.203}$$

Thus, from Eq. (2.201),

$$\Phi_{xy}(e^{j\omega}) = \sigma_x^2 H(e^{j\omega}). \tag{2.204}$$

In other words, the cross power spectrum is in this case proportional to the frequency response of the system. Equations (2.202) and (2.204) may serve as the basis for estimating the impulse response or frequency response of a linear time-invariant system if it is possible to observe the output of the system in response to a white-noise input.

2.11 SUMMARY

In this chapter, we have considered a number of basic definitions relating to discrete-time signals and systems. We considered the definition of a set of basic sequences, the definition and representation of linear time-invariant systems in terms of the convolution sum, and some implications of stability and causality. The class of systems for which the input and output satisfy a linear constant-coefficient difference equation with initial rest conditions was shown to be an important subclass of linear time-invariant systems. The recursive solution of such difference equations was discussed and the classes of FIR and IIR systems defined.

An important means for the analysis and representation of linear time-invariant systems lies in their frequency-domain representation. The response of a system to a complex exponential input was considered, leading to the definition of the frequency response. The relation between impulse response and frequency response was then interpreted as a Fourier transform pair.

We called attention to many properties of Fourier transform representations and discussed a variety of useful Fourier transform pairs. Tables 2.1 and 2.2 summarize the properties and theorems, and Table 2.3 contains some useful Fourier transform pairs.

The chapter concludes with an introduction to discrete-time random signals. These basic ideas and results will be developed further and used in later chapters.

Although the material in this chapter was presented without direct reference to continuous-time signals, an important class of discrete-time signal-processing problems arises from sampling such signals. In Chapter 4 we consider the relationship between continuous-time signals and sequences obtained by periodic sampling.

PROBLEMS

Basic Problems with Answers

2.1. For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, (4) time invariant, and (5) memoryless:

(a) $T(x[n]) = g[n]x[n]$ with $g[n]$ given
(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$
(c) $T(x[n]) = \sum_{k=n-n_0}^{n+n_0} x[k]$
(d) $T(x[n]) = x[n - n_0]$
(e) $T(x[n]) = e^{x[n]}$
(f) $T(x[n]) = ax[n] + b$
(g) $T(x[n]) = x[-n]$
(h) $T(x[n]) = x[n] + 3u[n + 1]$

2.2. **(a)** The impulse response $h[n]$ of a linear time-invariant system is known to be zero, except in the interval $N_0 \leq n \leq N_1$. The input $x[n]$ is known to be zero, except in the interval $N_2 \leq n \leq N_3$. As a result, the output is constrained to be zero, except in some interval $N_4 \leq n \leq N_5$. Determine N_4 and N_5 in terms of N_0, N_1, N_2, and N_3.
(b) If $x[n]$ is zero, except for N consecutive points, and $h[n]$ is zero, except for M consecutive points, what is the maximum number of consecutive points for which $y[n]$ can be nonzero?

2.3. By direct evaluation of the convolution sum, determine the step response of a linear time-invariant system whose impulse response is

$$h[n] = a^{-n}u[-n], \qquad 0 < a < 1.$$

2.4. Consider the linear constant-coefficient difference equation

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = 2x[n-1].$$

Determine $y[n]$ for $n \geq 0$ when $x[n] = \delta[n]$ and $y[n] = 0, n < 0$.

2.5. A causal linear time-invariant system is described by the difference equation

$$y[n] - 5y[n-1] + 6y[n-2] = 2x[n-1].$$

(a) Determine the homogeneous response of the system, i.e., the possible outputs if $x[n] = 0$ for all n.
(b) Determine the impulse response of the system.
(c) Determine the step response of the system.

2.6. **(a)** Find the frequency response $H(e^{j\omega})$ of the linear time-invariant system whose input and output satisfy the difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].$$

(b) Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}.$$

2.7. Determine whether each of the following signals is periodic. If the signal is periodic, state its period.
(a) $x[n] = e^{j(\pi n/6)}$
(b) $x[n] = e^{j(3\pi n/4)}$
(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$
(d) $x[n] = e^{j\pi n/\sqrt{2}}$

2.8. An LTI system has impulse response $h[n] = 5(-1/2)^n u[n]$. Use the Fourier transform to find the output of this system when the input is $x[n] = (1/3)^n u[n]$.

2.9. Consider the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = \frac{1}{3}x[n-1].$$

(a) What are the impulse response, frequency response, and step response for the causal LTI system satisfying this difference equation.

(b) What is the general form of the homogeneous solution of the difference equation?

(c) Consider a different system satisfying the difference equation that is neither causal nor LTI, but that has $y[0] = y[1] = 1$. Find the response of this system to $x[n] = \delta[n]$.

2.10. Determine the output of a linear time-invariant system if the impulse response $h[n]$ and the input $x[n]$ are as follows:

(a) $x[n] = u[n]$ and $h[n] = a^n u[-n-1]$, with $a > 1$.

(b) $x[n] = u[n-4]$ and $h[n] = 2^n u[-n-1]$.

(c) $x[n] = u[n]$ and $h[n] = (0.5)2^n u[-n]$.

(d) $h[n] = 2^n u[-n-1]$ and $x[n] = u[n] - u[n-10]$

Use your knowledge of linearity and time invariance to minimize the work in Parts (b)–(d).

2.11. Consider an LTI system with frequency response

$$H(e^{j\omega}) = \frac{1 - e^{-j2\omega}}{1 + \frac{1}{2}e^{-j4\omega}}, \qquad -\pi < \omega \leq \pi.$$

Determine the output $y[n]$ for all n if the input $x[n]$ for all n is

$$x[n] = \sin\left(\frac{\pi n}{4}\right).$$

2.12. Consider a system with input $x[n]$ and output $y[n]$ that satisfy the difference equation

$$y[n] = ny[n-1] + x[n].$$

The system is causal and satisfies initial-rest conditions; i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$.

(a) If $x[n] = \delta[n]$, determine $y[n]$ for all n.

(b) Is the system linear? Justify your answer.

(c) Is the system time invariant? Justify your answer.

2.13. Indicate which of the following discrete-time signals are eigenfunctions of stable, linear time-invariant discrete-time systems:

(a) $e^{j2\pi n/3}$

(b) 3^n

(c) $2^n u[-n-1]$

(d) $\cos(\omega_0 n)$

(e) $(1/4)^n$

(f) $(1/4)^n u[n] + 4^n u[-n-1]$

2.14. A single input–output relationship is given for each of the following three systems:

(a) System A: $x[n] = (1/3)^n$, $y[n] = 2(1/3)^n$.

(b) System B: $x[n] = (1/2)^n$, $y[n] = (1/4)^n$.

(c) System C: $x[n] = (2/3)^n u[n]$, $y[n] = 4(2/3)^n u[n] - 3(1/2)^n u[n]$.

Based on this information, pick the strongest possible conclusion that you can make about each system from the following list of statements:

(i) The system cannot possibly be LTI.

(ii) The system must be LTI.

(iii) The system can be LTI, and there is only one LTI system that satisfies this input–output constraint.

(iv) The system can be LTI, but cannot be uniquely determined from the information in this input–output constraint.

If you chose option (iii) from this list, specify either the impulse response $h[n]$ or the frequency response $H(e^{j\omega})$ for the LTI system.

2.15. Consider the system illustrated in Figure P2.15-1. The ouput of an LTI system with an impulse response $h[n] = \left(\frac{1}{4}\right)^n u[n+10]$ is multiplied by a unit step function $u[n]$ to yield the output of the overall system. Answer each of the following questions, and briefly justify your answers:

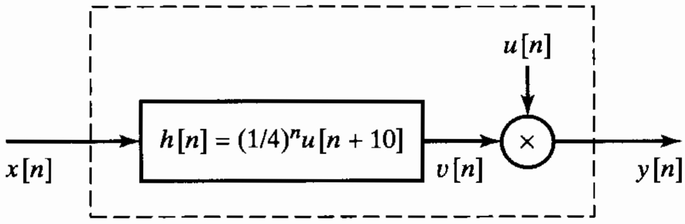


Figure P2.15-1

(a) Is the overall system LTI?
(b) Is the overall system causal?
(c) Is the overall system stable in the BIBO sense?

2.16. Consider the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n].$$

(a) Determine the general form of the homogeneous solution to this difference equation.
(b) Both a causal and an anticausal LTI system are characterized by this difference equation. Find the impulse responses of the two systems.
(c) Show that the causal LTI system is stable and the anticausal LTI system is unstable.
(d) Find a particular solution to the difference equation when $x[n] = (1/2)^n u[n]$.

2.17. (a) Determine the Fourier transform of the sequence

$$r[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Consider the sequence

$$w[n] = \begin{cases} \dfrac{1}{2}\left[1 - \cos\left(\dfrac{2\pi n}{M}\right)\right], & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $w[n]$ and express $W(e^{j\omega})$, the Fourier transform of $w[n]$, in terms of $R(e^{j\omega})$, the Fourier transform of $r[n]$. (*Hint:* First express $w[n]$ in terms of $r[n]$ and the complex exponentials $e^{j(2\pi n/M)}$ and $e^{-j(2\pi n/M)}$.)
(c) Sketch the magnitude of $R(e^{j\omega})$ and $W(e^{j\omega})$ for the case when $M = 4$.

2.18. For each of the following impulse responses of LTI systems, indicate whether or not the system is causal:
(a) $h[n] = (1/2)^n u[n]$
(b) $h[n] = (1/2)^n u[n-1]$
(c) $h[n] = (1/2)^{|n|}$
(d) $h[n] = u[n+2] - u[n-2]$
(e) $h[n] = (1/3)^n u[n] + 3^n u[-n-1]$

2.19. For each of the following impulse responses of LTI systems, indicate whether or not the system is stable:
(a) $h[n] = 4^n u[n]$
(b) $h[n] = u[n] - u[n-10]$
(c) $h[n] = 3^n u[-n-1]$

(d) $h[n] = \sin(\pi n/3)u[n]$
(e) $h[n] = (3/4)^{|n|}\cos(\pi n/4 + \pi/4)$
(f) $h[n] = 2u[n+5] - u[n] - u[n-5]$

2.20. Consider the difference equation representing a causal LTI system

$$y[n] + (1/a)y[n-1] = x[n-1].$$

(a) Find the impulse response of the system, $h[n]$, as a function of the constant a.
(b) For what range of values of a will the system be stable?

Basic Problems

2.21. Consider an arbitrary linear system with input $x[n]$ and output $y[n]$. Show that if $x[n] = 0$ for all n, then $y[n]$ must also be zero for all n.

2.22. For each of the pairs of sequences in Figure P2.22-1, use discrete convolution to find the response to the input $x[n]$ of the linear time-invariant system with impulse response $h[n]$.

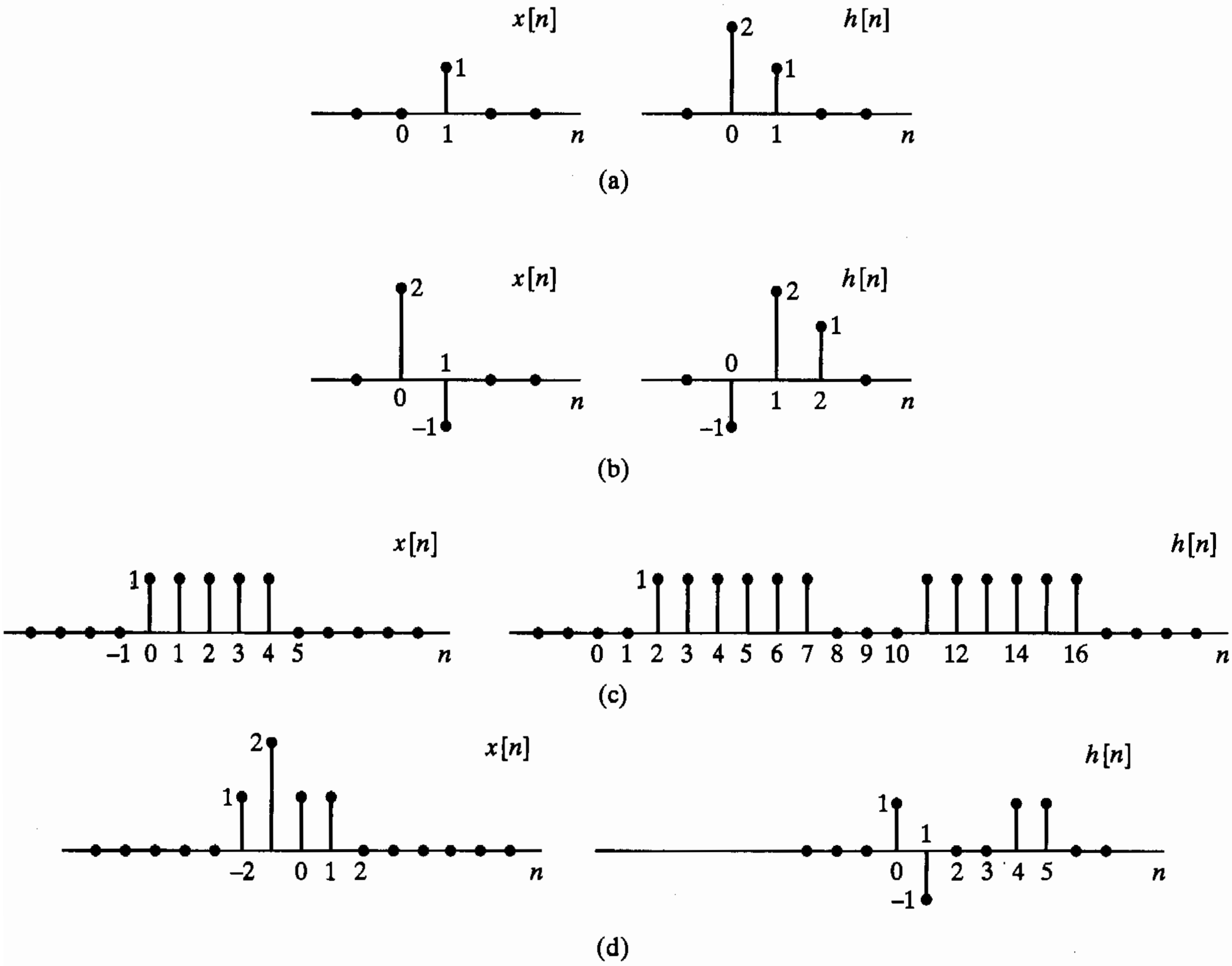


Figure P2.22-1

2.23. Using the definition of linearity (Eqs. (2.26a)–(2.26b)), show that the ideal delay system (Example 2.3) and the moving-average system (Example 2.4) are both linear systems.

2.24. The impulse response of a linear time-invariant system is shown in Figure P2.24-1. Determine and carefully sketch the response of this system to the input $x[n] = u[n-4]$.

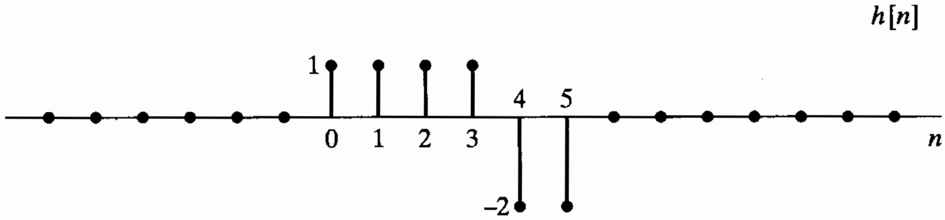


Figure P2.24-1

2.25. A linear time-invariant system has impulse response $h[n] = u[n]$. Determine the response of this system to the input $x[n]$ shown in Figure P2.25-1 and described as

$$x[n] = \begin{cases} 0, & n < 0, \\ a^n, & 0 \le n \le N_1, \\ 0, & N_1 < n < N_2, \\ a^{n-N_2}, & N_2 \le n \le N_2 + N_1, \\ 0, & N_2 + N_1 < n, \end{cases}$$

where $0 < a < 1$.

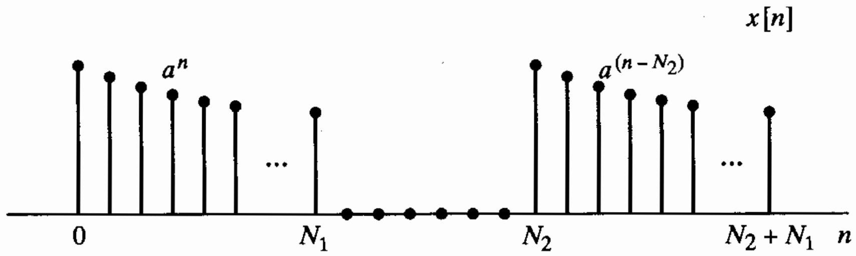


Figure P2.25-1

2.26. Which of the following discrete-time signals could be eigenfunctions of any stable LTI system?

(a) $5^n u[n]$
(b) $e^{j2\omega n}$
(c) $e^{j\omega n} + e^{j2\omega n}$
(d) 5^n
(e) $5^n \cdot e^{j2\omega n}$

2.27. Three systems A, B, and C have the inputs and outputs indicated in Figure P2.27-1. Determine whether each system could be LTI. If your answer is yes, specify whether there could be more than one LTI system with the given input–output pair. Explain your answer.

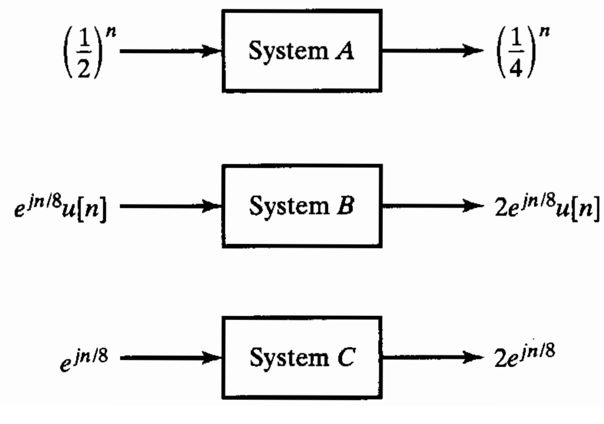


Figure P2.27-1

2.28. Determine which of the following signals is periodic. If a signal is periodic, determine its period.

(a) $x[n] = e^{j(2\pi n/5)}$

(b) $x[n] = \sin(\pi n/19)$

(c) $x[n] = ne^{j\pi n}$

(d) $x[n] = e^{jn}$

2.29. A discrete-time signal $x[n]$ is shown in Figure P2.29-1.

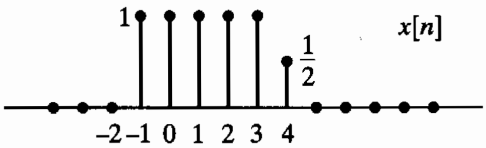


Figure P2.29-1

Sketch and label carefully each of the following signals:

(a) $x[n-2]$

(b) $x[4-n]$

(c) $x[2n]$

(d) $x[n]u[2-n]$

(e) $x[n-1]\delta[n-3]$

2.30. For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, and (4) time invariant.

(a) $T(x[n]) = (\cos \pi n)x[n]$

(b) $T(x[n]) = x[n^2]$

(c) $T(x[n]) = x[n]\sum_{k=0}^{\infty} \delta[n-k]$

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

2.31. Consider the difference equation

$$y[n] + \frac{1}{15}y[n-1] - \frac{2}{5}y[n-2] = x[n].$$

(a) Determine the general form of the homogeneous solution to this equation.

(b) Both a causal and an anticausal LTI system are characterized by the given difference equation. Find the impulse responses of the two systems.

(c) Show that the causal LTI system is stable and the anticausal LTI system is unstable.

(d) Find a particular solution to the difference equation when $x[n] = (3/5)^n u[n]$.

2.32. Consider an LTI system with frequency response

$$H(e^{j\omega}) = e^{-j(\omega - \frac{\pi}{4})}\left(\frac{1 + e^{-j2\omega} + 4e^{-j4\omega}}{1 + \frac{1}{2}e^{-j2\omega}}\right), \qquad -\pi < \omega \le \pi.$$

Determine the output $y[n]$ for all n if the input for all n is

$$x[n] = \cos\left(\frac{\pi n}{2}\right).$$

2.33. Consider an LTI system with $|H(e^{j\omega})| = 1$, and let $\arg[H(e^{j\omega})]$ be as shown in Figure P2.33-1. If the input is

$$x[n] = \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right),$$

determine the output $y[n]$.

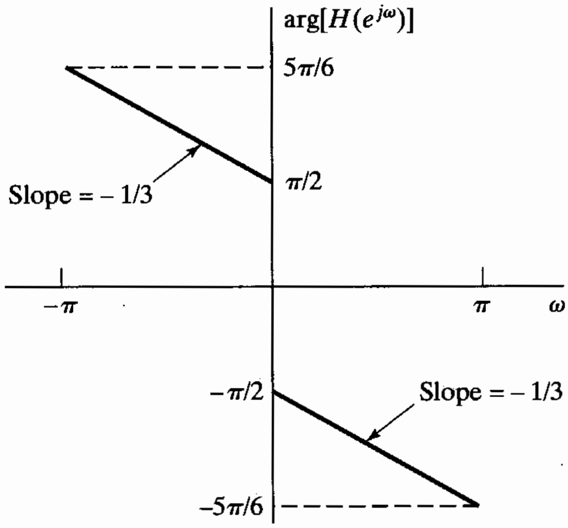


Figure P2.33-1

2.34. The input–output pair shown in Figure P2.34-1 is given for a stable LTI system.

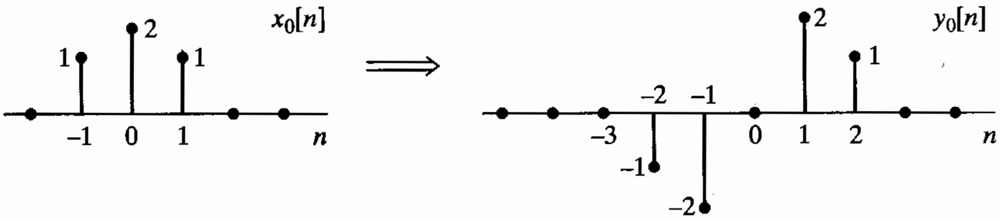


Figure P2.34-1

(a) Determine the response to the input $x_1[n]$ in Figure P2.34-2.

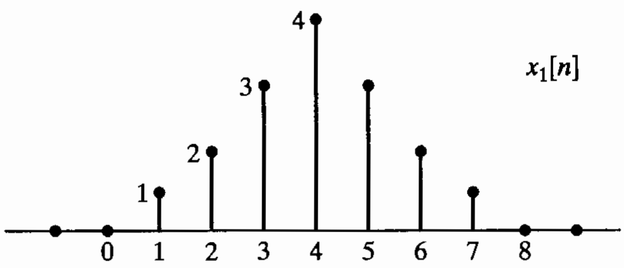


Figure P2.34-2

(b) Determine the impulse response of the system.

Advanced Problems

2.35. The system T in Figure P2.35-1 is known to be *time invariant.* When the inputs to the system are $x_1[n]$, $x_2[n]$, and $x_3[n]$, the responses of the system are $y_1[n]$, $y_2[n]$, and $y_3[n]$, as shown.

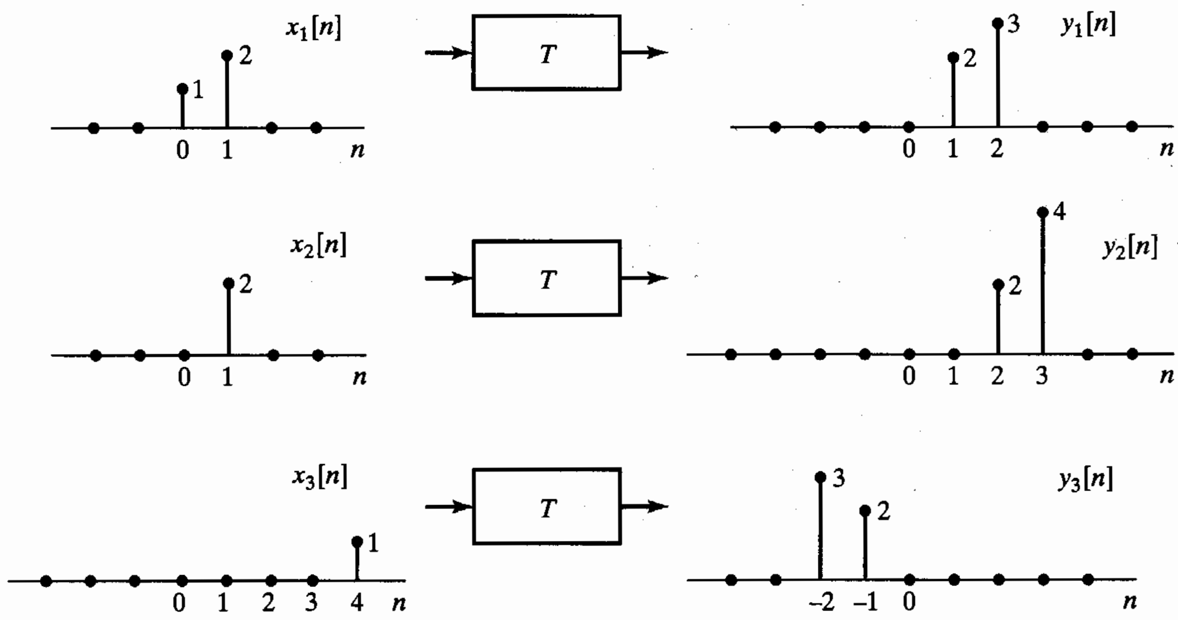


Figure P2.35-1

(a) Determine whether the system T could be linear.
(b) If the input $x[n]$ to the system T is $\delta[n]$, what is the system response $y[n]$?
(c) What are all possible inputs $x[n]$ for which the response of the system T can be determined from the given information alone?

2.36. The system L in Figure P2.36-1 is known to be *linear*. Shown are three output signals $y_1[n]$, $y_2[n]$, and $y_3[n]$ in response to the input signals $x_1[n]$, $x_2[n]$, and $x_3[n]$, respectively.

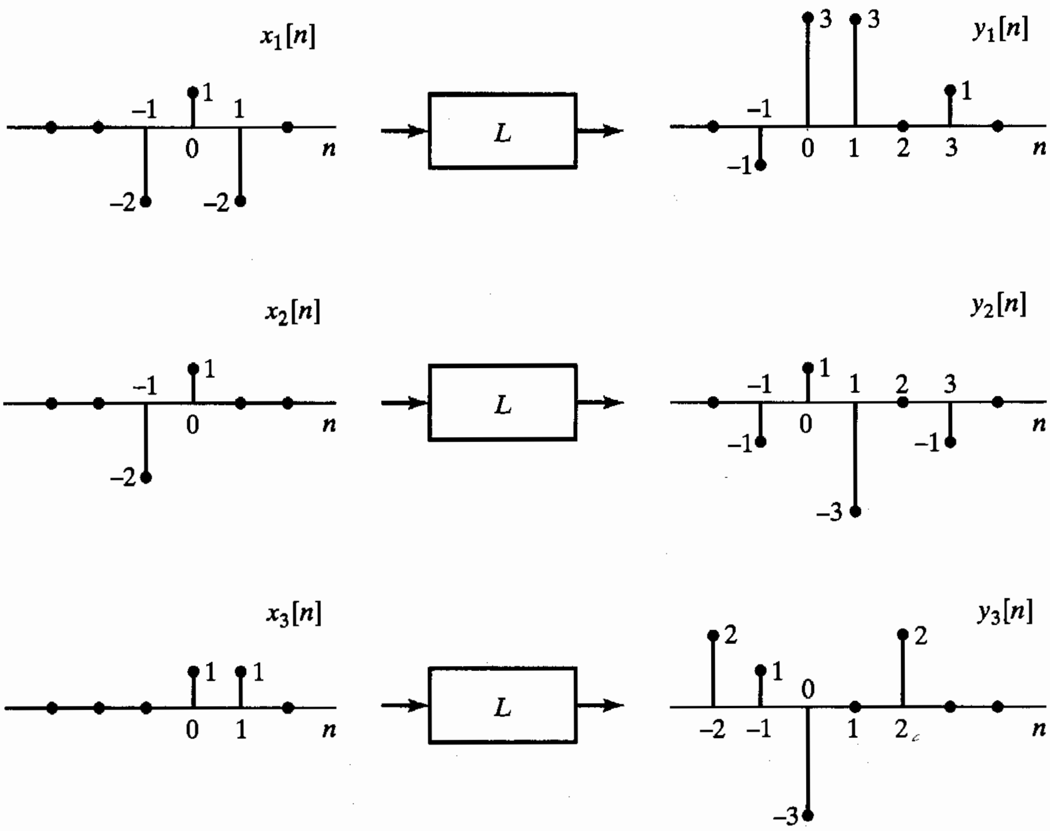


Figure P2.36-1

(a) Determine whether the system L could be time invariant.
(b) If the input $x[n]$ to the system L is $\delta[n]$, what is the system response $y[n]$?

2.37. Consider a discrete-time linear time-invariant system with impulse response $h[n]$. If the input $x[n]$ is a periodic sequence with period N (i.e., if $x[n] = x[n+N]$), show that the output $y[n]$ is also a periodic sequence with period N.

2.38. In Section 2.5, we stated that the solution to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0 \qquad \text{(P2.38-1)}$$

is of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n, \qquad \text{(P2.38-2)}$$

with the A_m's arbitrary and the z_m's the N roots of the polynomial

$$\sum_{k=0}^{N} a_k z^{-k} = 0; \qquad \text{(P2.38-3)}$$

i.e.,

$$\sum_{k=0}^{N} a_k z^{-k} = \prod_{m=1}^{N} (1 - z_m z^{-1}). \qquad \text{(P2.38-4)}$$

(a) Determine the general form of the homogeneous solution to the difference equation

$$y[n] - \tfrac{3}{4} y[n-1] + \tfrac{1}{8} y[n-2] = 2x[n-1]. \qquad \text{(P2.38-5)}$$

(b) Determine the coefficients A_m in the homogeneous solution if $y[-1] = 1$ and $y[0] = 0$.
(c) Now consider the difference equation

$$y[n] - y[n-1] + \tfrac{1}{4} y[n-2] = 2y[n-1]. \qquad \text{(P2.38-6)}$$

If the homogeneous solution contains only terms of the form of Eq. (P2.38-2), show that the initial conditions $y[-1] = 1$ and $y[0] = 0$ cannot be satisfied.

(d) If Eq. (P2.38-3) has two roots that are identical, then, in place of Eq. (P2.38-2), $y_h[n]$ will take the form

$$y_h[n] = \sum_{m=1}^{N-1} A_m z_m^n + nB_1 z_1^n, \qquad \text{(P2.38-7)}$$

where we have assumed that the double root is z_1. Using Eq. (P2.38-7), determine the general form of $y_h[n]$ for Eq. (P2.38-6). Verify explicitly that your answer satisfies Eq. (P2.38-6) with $x[n] = 0$.

(e) Determine the coefficients A_1 and B_1 in the homogeneous solution obtained in Part (d) if $y[-1] = 1$ and $y[0] = 0$.

2.39. Consider a system with input $x[n]$ and output $y[n]$. The input–output relation for the system is defined by the following two properties:

1. $y[n] - ay[n-1] = x[n]$,
2. $y[0] = 1$.

(a) Determine whether the system is time invariant.
(b) Determine whether the system is linear.

(c) Assume that the difference equation (property 1) remains the same, but the value $y[0]$ is specified to be zero. Does this change your answer to either Part (a) or Part (b)?

2.40. Consider the linear time-invariant system with impulse response

$$h[n] = \left(\frac{j}{2}\right)^n u[n], \qquad \text{where } j = \sqrt{-1}.$$

Determine the steady-state response, i.e., the response for large n, to the excitation

$$x[n] = \cos(\pi n)u[n].$$

2.41. A linear time-invariant system has frequency response

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega 3}, & |\omega| < \frac{2\pi}{16}\left(\frac{3}{2}\right), \\ 0, & \frac{2\pi}{16}\left(\frac{3}{2}\right) \le |\omega| \le \pi. \end{cases}$$

The input to the system is a periodic unit-impulse train with period $N = 16$; i.e.,

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n + 16k].$$

Find the output of the system.

2.42. Consider the system in Figure P2.42-1.

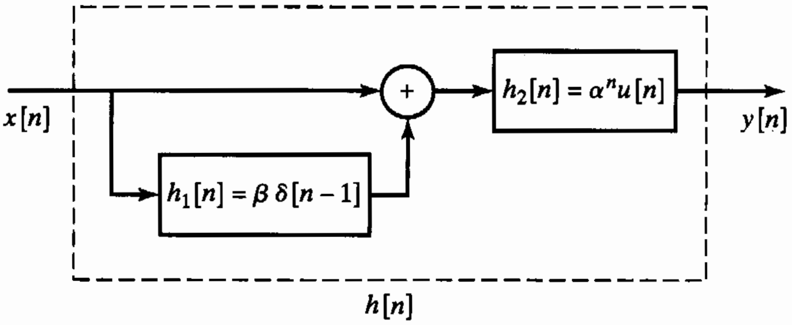


Figure P2.42-1

(a) Find the impulse response $h[n]$ of the overall system.
(b) Find the frequency response of the overall system.
(c) Specify a difference equation that relates the output $y[n]$ to the input $x[n]$.
(d) Is this system causal? Under what condition would the system be stable?

2.43. For $X(e^{j\omega}) = 1/(1 - ae^{-j\omega})$, with $-1 < a < 0$, determine and sketch the following as a function of ω:
(a) $\mathcal{R}e\{X(e^{j\omega})\}$
(b) $\mathcal{I}m\{X(e^{j\omega})\}$
(c) $|X(e^{j\omega})|$
(d) $\sphericalangle X(e^{j\omega})$

2.44. Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ shown in Figure P2.44-1. Perform the following calculations without explicitly evaluating $X(e^{j\omega})$:

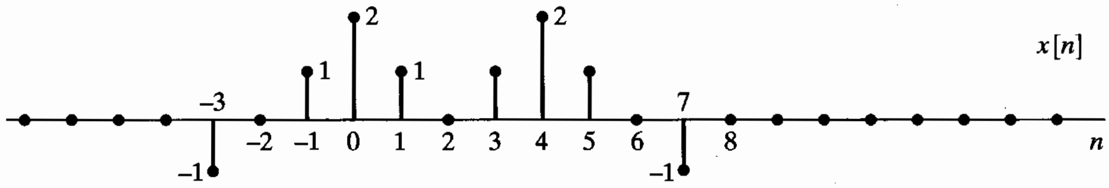


Figure P2.44-1

(a) Evaluate $X(e^{j\omega})|_{\omega=0}$.
(b) Evaluate $X(e^{j\omega})|_{\omega=\pi}$.
(c) Find $\sphericalangle X(e^{j\omega})$.
(d) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.
(e) Determine and sketch the signal whose Fourier transform is $X(e^{-j\omega})$.
(f) Determine and sketch the signal whose Fourier transform is $\mathcal{R}e\{X(e^{j\omega})\}$.

2.45. For the system in Figure P2.45-1, determine the output $y[n]$ when the input $x[n]$ is $\delta[n]$ and $H(e^{j\omega})$ is an ideal lowpass filter as indicated, i.e.,

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/2, \\ 0, & \pi/2 < |\omega| \le \pi. \end{cases}$$

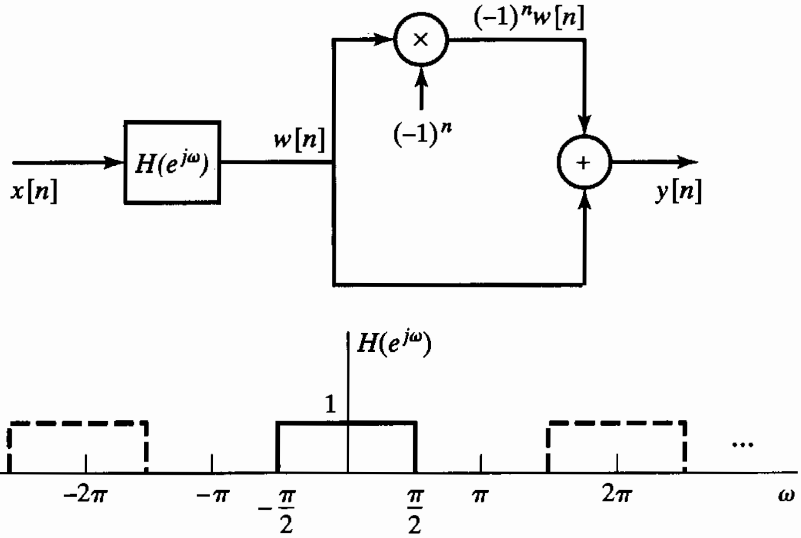


Figure P2.45-1

2.46. A sequence has the discrete-time Fourier transform

$$X(e^{j\omega}) = \frac{1-a^2}{(1-ae^{-j\omega})(1-ae^{j\omega})}, \qquad |a| < 1.$$

(a) Find the sequence $x[n]$.
(b) Calculate $\int_{-\pi}^{\pi} X(e^{j\omega})\cos(\omega)d\omega/2\pi$.

2.47. A linear time-invariant system is described by the input–output relation

$$y[n] = x[n] + 2x[n-1] + x[n-2].$$

(a) Determine $h[n]$, the impulse response of the system.
(b) Is this a stable system?
(c) Determine $H(e^{j\omega})$, the frequency response of the system. Use trigonometric identities to obtain a simple expression for $H(e^{j\omega})$.

(d) Plot the magnitude and phase of the frequency response.
(e) Now consider a new system whose frequency response is $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$. Determine $h_1[n]$, the impulse response of the new system.

2.48. Let the real discrete-time signal $x[n]$ with Fourier transform $X(e^{j\omega})$ be the input to a system with the output defined by

$$y[n] = \begin{cases} x[n], & \text{if } n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the discrete-time signal $s[n] = 1 + \cos(\pi n)$ and its (generalized) Fourier transform $S(e^{j\omega})$.
(b) Express $Y(e^{j\omega})$, the Fourier transform of the output, as a function of $X(e^{j\omega})$ and $S(e^{j\omega})$.
(c) You would like to approximate $x[n]$ by the interpolated signal $w[n] = y[n] + (1/2)(y[n+1] + y[n-1])$. Determine the Fourier transform $W(e^{j\omega})$ as a function of $Y(e^{j\omega})$.
(d) Sketch $X(e^{j\omega})$, $Y(e^{j\omega})$, and $W(e^{j\omega})$ for the case when $x[n] = \sin(\pi n/a)/(\pi n/a)$ and $a > 1$. Under what conditions is the proposed interpolated signal $w[n]$ a good approximation for the original $x[n]$.

2.49. Consider a discrete-time LTI system with frequency response $H(e^{j\omega})$ and corresponding impulse response $h[n]$.
(a) We are first given the following three clues about the system:
(i) The system is causal.
(ii) $H(e^{j\omega}) = H^*(e^{-j\omega})$.
(iii) The DTFT of the sequence $h[n+1]$ is real.
Using these three clues, show that the system has an impulse response of finite duration.
(b) In addition to the preceding three clues, we are now given two more clues:
(iv) $\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})d\omega = 2.$
(v) $H(e^{j\pi}) = 0.$
Is there enough information to identify the system uniquely? If so, determine the impulse response $h[n]$. If not, specify as much as you can about the sequence $h[n]$.

2.50. Consider the three sequences

$$\begin{aligned} v[n] &= u[n] - u[n-6], \\ w[n] &= \delta[n] + 2\delta[n-2] + \delta[n-4], \\ q[n] &= v[n] * w[n]. \end{aligned}$$

(a) Find and sketch the sequence $q[n]$.
(b) Find and sketch the sequence $r[n]$ such that $r[n] * v[n] = \sum_{k=-\infty}^{n-1} q[k]$.
(c) Is $q[-n] = v[-n] * w[-n]$? Justify your answer.

2.51. A linear time-invariant system has impulse response $h[n] = a^n u[n]$.
(a) Determine $y_1[n]$, the response of the system to the input $x_1[n] = e^{j(\pi/2)n}$.
(b) Use the result of Part (a) to help to determine $y_2[n]$, the response of the system to the input $x_2[n] = \cos(\pi n/2)$.
(c) Determine $y_3[n]$, the response of the system to the input $x_3[n] = e^{j(\pi/2)n}u[n]$.
(d) Compare $y_3[n]$ with $y_1[n]$ for large n.

2.52. The frequency response of an LTI system is

$$H(e^{j\omega}) = e^{-j\omega/4}, \qquad -\pi < \omega \le \pi.$$

Determine the output of the system, $y[n]$, when the input is $x[n] = \cos(5\pi n/2)$. Express your answer in as simple a form as you can.

2.53. Consider the cascade of LTI discrete-time systems shown in Figure P2.53-1.

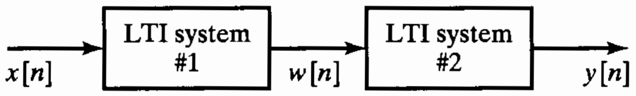


Figure P2.53-1

The first system is described by the equation

$$H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < 0.5\pi, \\ 0, & 0.5\pi \le |\omega| < \pi, \end{cases}$$

and the second system is described by the equation

$$y[n] = w[n] - w[n-1].$$

The input to this system is

$$x[n] = \cos(0.6\pi n) + 3\delta[n-5] + 2.$$

Determine the output $y[n]$. With careful thought, you will be able to use the properties of LTI systems to write down the answer by inspection.

2.54. Consider an LTI system with frequency response

$$H(e^{j\omega}) = e^{-j[(\omega/2)+(\pi/4)]}, \qquad -\pi < \omega \le \pi.$$

Determine $y[n]$, the output of this system, if the input is

$$x[n] = \cos\left(\frac{15\pi n}{4} - \frac{\pi}{3}\right)$$

for all n.

2.55. For the system shown in Figure P2.55-1, System 1 is a memoryless nonlinear system. System 2 determines the value of A according to the relation

$$A = \sum_{n=0}^{100} y[n].$$

Figure P2.55-1

Specifically, consider the class of inputs of the form $x[n] = \cos(\omega n)$, with ω a real finite number. Varying the value of ω at the input will change A; i.e., A will be a function of ω. In general, will A be periodic in ω? Justify your answer.

2.56. Consider a system S with input $x[n]$ and output $y[n]$ related according to the block diagram in Figure P2.56-1.

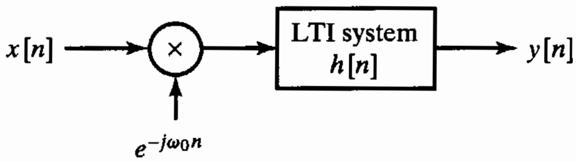


Figure P2.56-1

The input $x[n]$ is multiplied by $e^{-j\omega_0 n}$, and the product is passed through a stable LTI system with impulse response $h[n]$.

(a) Is the system S linear? Justify your answer.

(b) Is the system S time invariant? Justify your answer.

(c) Is the system S stable? Justify your answer.

(d) Specify a system C such that the block diagram in Figure P2.56-2 represents an alternative way of expressing the input–output relationship of the system S. (*Note:* The system C does not have to be an LTI system.)

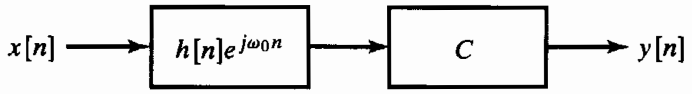


Figure P2.56-2

2.57. An ideal lowpass filter with zero delay has impulse response $h_{\text{lp}}[n]$ and frequency response

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < 0.2\pi, \\ 0, & 0.2\pi \le |\omega| \le \pi. \end{cases}$$

(a) A new filter is defined by the equation $h_1[n] = (-1)^n h_{\text{lp}}[n] = e^{j\pi n} h_{\text{lp}}[n]$. Determine an equation for the frequency response of $H_1(e^{j\omega})$, and plot the equation for $|\omega| < \pi$. What kind of filter is this?

(b) A second filter is defined by the equation $h_2[n] = 2h_{\text{lp}}[n]\cos(0.5\pi n)$. Determine the equation for the frequency response $H_2(e^{j\omega})$, and plot the equation for $|\omega| < \pi$. What kind of filter is this?

(c) A third filter is defined by the equation

$$h_3[n] = \frac{\sin(0.1\pi n)}{\pi n} h_{\text{lp}}[n].$$

Determine the equation for the frequency response $H_3(e^{j\omega})$, and plot the equation for $|\omega| < \pi$. What kind of filter is this?

2.58. The LTI system

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi, \\ j, & -\pi < \omega < 0, \end{cases}$$

is referred to as a 90° phase shifter and is used to generate what is referred to as an analytic signal $w[n]$ as shown in Figure P2.58-1. Specifically, the analytic signal $w[n]$ is a complex-valued signal for which

$$\mathcal{R}e\{w[n]\} = x[n],$$
$$\mathcal{I}m\{w[n]\} = y[n].$$

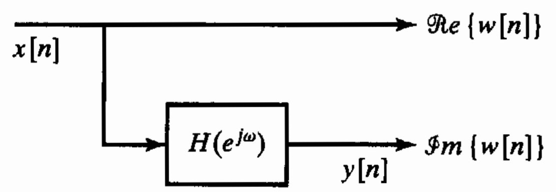


Figure P2.58-1

If $X(e^{j\omega})$ is as shown in Figure P2.58-2, determine and sketch $W(e^{j\omega})$, the Fourier transform of the analytic signal $w[n] = x[n] + jy[n]$.

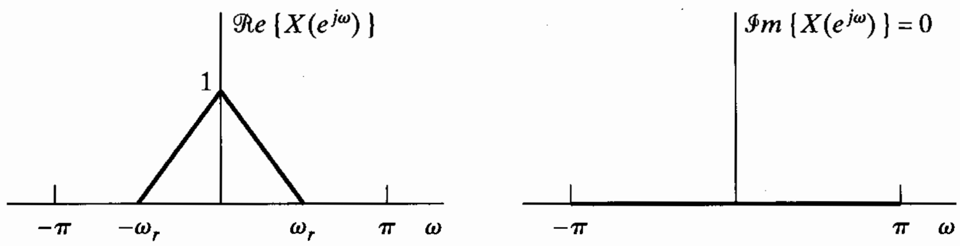


Figure P2.58-2

2.59. The autocorrelation sequence of a signal $x[n]$ is defined as

$$R_x[n] = \sum_{k=-\infty}^{\infty} x^*[k]x[n+k].$$

(a) Show that for an appropriate choice of the signal $g[n]$, $R_x[n] = x[n] * g[n]$, and identify the proper choice for $g[n]$.

(b) Show that the Fourier transform of $R_x[n]$ is equal to $|X(e^{j\omega})|^2$.

2.60. The signals $x[n]$ and $y[n]$ shown in Figure P2.60-1 are the input and corresponding output for an LTI system.

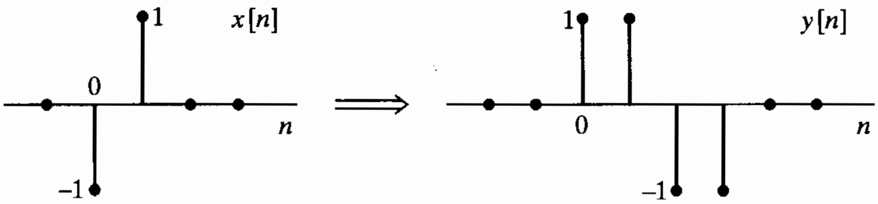


Figure P2.60-1

(a) Find the response of the system to the sequence $x_2[n]$ in Figure P2.60-2.

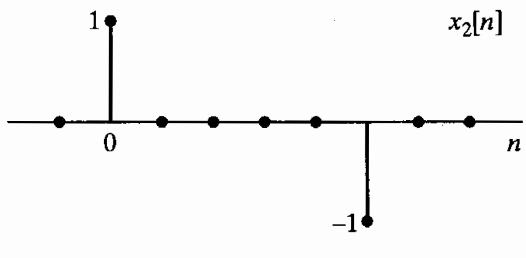


Figure P2.60-2

(b) Find the impulse response $h[n]$ for this LTI system.

2.61. Consider a system for which the input $x[n]$ and output $y[n]$ satisfy the difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n]$$

and for which $y[-1]$ is constrained to be zero for every input. Determine whether or not the system is stable. If you conclude that the system is stable, show your reasoning. If you conclude that the system is **not** stable, give an example of a bounded input that results in an unbounded output.

Extension Problems

2.62. The causality of a system was defined in Section 2.2.4. From this definition, show that, for a linear time-invariant system, causality implies that the impulse response $h[n]$ is zero for $n < 0$. One approach is to show that if $h[n]$ is *not* zero for $n < 0$, then the system *cannot* be causal. Show also that if the impulse response is zero for $n < 0$, then the system will necessarily be causal.

2.63. Consider a discrete-time system with input $x[n]$ and output $y[n]$. When the input is

$$x[n] = \left(\frac{1}{4}\right)^n u[n],$$

the output is

$$y[n] = \left(\frac{1}{2}\right)^n \qquad \text{for all } n.$$

Determine which of the following statements is correct:

- The system must be LTI.
- The system could be LTI.
- The system cannot be LTI.

If your answer is that the system must or could be LTI, give a possible impulse response. If your answer is that the system could not be LTI, explain clearly why not.

2.64. Consider an LTI system whose frequency response is

$$H(e^{j\omega}) = e^{-j\omega/2}, \quad |\omega| < \pi.$$

Determine whether or not the system is causal. Show your reasoning.

2.65. In Figure P2.65-1, two sequences $x_1[n]$ and $x_2[n]$ are shown. Both sequences are zero for all n outside the regions shown. The Fourier transforms of these sequences are $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, which, in general, can be expected to be complex and can be written in the form

$$X_1(e^{j\omega}) = A_1(\omega)e^{j\theta_1(\omega)},$$
$$X_2(e^{j\omega}) = A_2(\omega)e^{j\theta_2(\omega)},$$

where $A_1(\omega)$, $\theta_1(\omega)$, $A_2(\omega)$, and $\theta_2(\omega)$ are all real functions chosen so that both $A_1(\omega)$ and $A_2(\omega)$ are nonnegative at $\omega = 0$, but otherwise can take on both positive and negative values. Determine appropriate choices for $\theta_1(\omega)$ and $\theta_2(\omega)$, and sketch these two phase functions in the range $0 < \omega < 2\pi$.

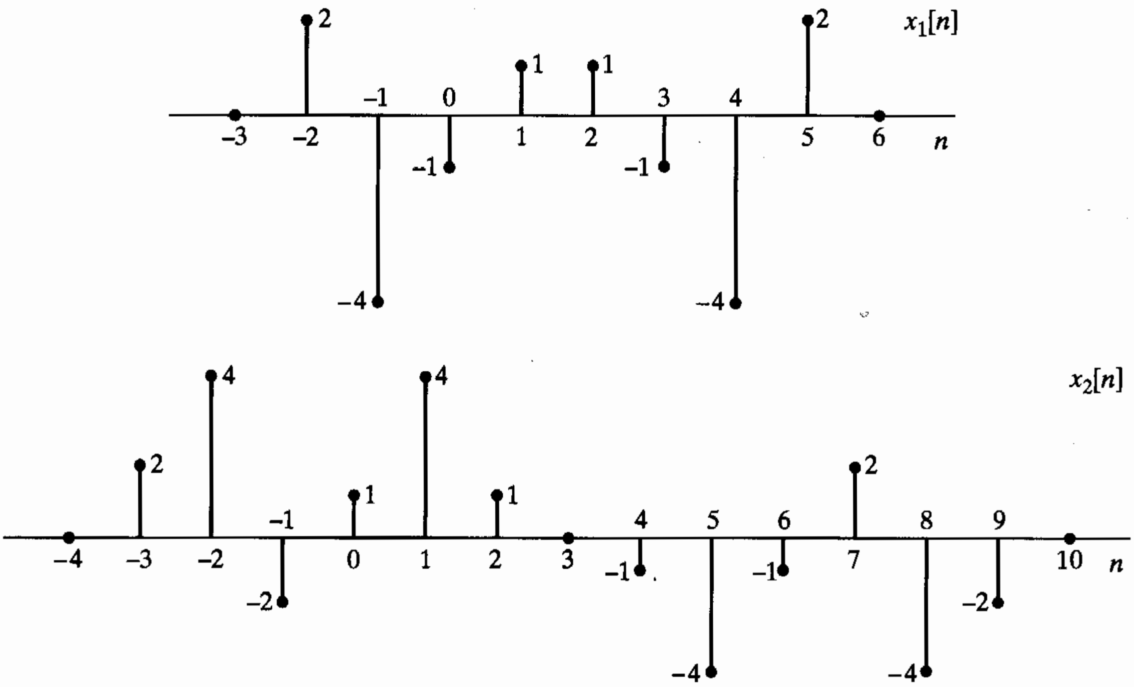


Figure P2.65-1

2.66. Consider the cascade of discrete-time systems in Figure P1.66-1. The time-reversal systems are defined by the equations $f[n] = e[-n]$ and $y[n] = g[-n]$. Assume throughout the problem that $x[n]$ and $h_1[n]$ are real sequences.

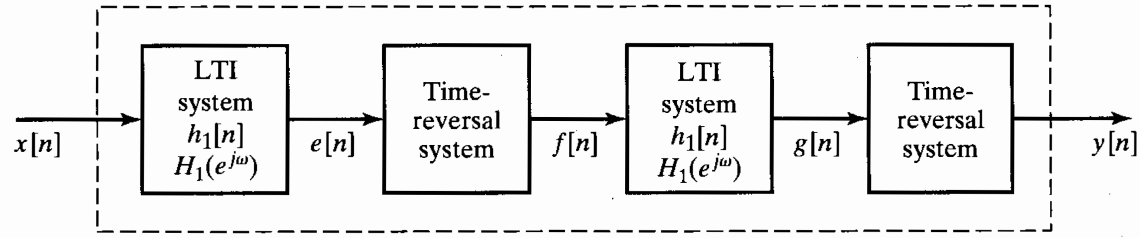


Figure P2.66-1

(a) Express $E(e^{j\omega})$, $F(e^{j\omega})$, $G(e^{j\omega})$, and $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$ and $H_1(e^{j\omega})$.

(b) The result from Part (a) should convince you that the overall system is LTI. Find the frequency response $H(e^{j\omega})$ of the overall system.

(c) Determine an expression for the impulse response $h[n]$ of the overall system in terms of $h_1[n]$.

2.67. The overall system in the dotted box in Figure P1.67-1 can be shown to be linear and time invariant.

(a) Determine an expression for $H(e^{j\omega})$, the frequency response of the overall system from the input $x[n]$ to the output $y[n]$, in terms of $H_1(e^{j\omega})$, the frequency response of the internal LTI system. Remember that $(-1)^n = e^{j\pi n}$.

(b) Plot $H(e^{j\omega})$ for the case when the frequency response of the internal LTI system is

$$H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases}$$

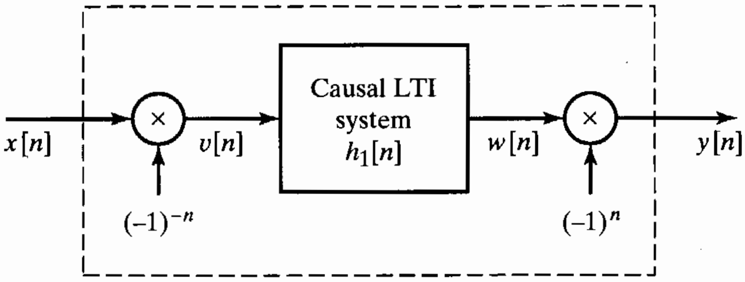


Figure P2.67-1

2.68. Figure P1.68-1 shows the input–output relationships of Systems A and B, while Figure P1.68-2 contains two possible cascade combinations of these systems.

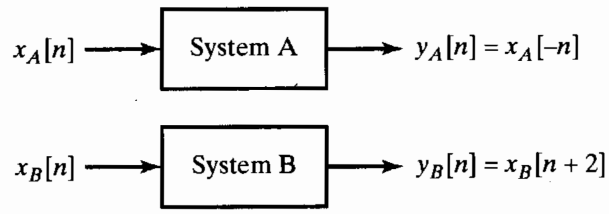


Figure P2.68-1

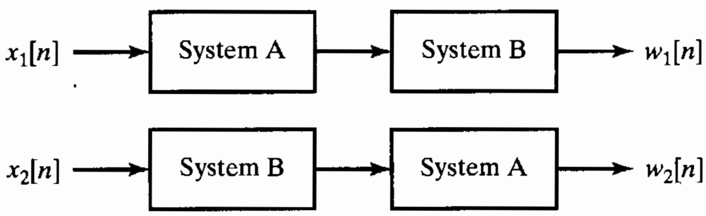


Figure P2.68-2

If $x_1[n] = x_2[n]$, will $w_1[n]$ and $w_2[n]$ necessarily be equal? If your answer is *yes*, clearly and concisely explain why and demonstrate with an example. If your answer is *not necessarily*, demonstrate with a counterexample.

2.69. Consider the system in Figure P2.69-1, where the subsystems S_1 and S_2 are LTI.

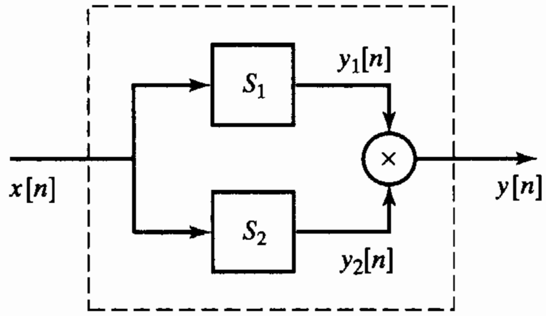


Figure P2.69-1

(a) Is the overall system enclosed by the dashed box, with input $x[n]$ and output $y[n]$ equal to the product of $y_1[n]$ and $y_2[n]$, guaranteed to be an LTI system? If so, explain your reasoning. If not, provide a counterexample.

(b) Suppose S_1 and S_2 have frequency responses $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ that are known to be zero over certain regions. Let

$$H_1(e^{j\omega}) = \begin{cases} 0, & |\omega| \le 0.2\pi, \\ \text{unspecified}, & 0.2\pi < |\omega| \le \pi, \end{cases}$$

$$H_2(e^{j\omega}) = \begin{cases} \text{unspecified}, & |\omega| \le 0.4\pi, \\ 0, & 0.4\pi < |\omega| \le \pi. \end{cases}$$

Suppose also that the input $x[n]$ is known to be bandlimited to 0.3π, i.e.,

$$X(e^{j\omega}) = \begin{cases} \text{unspecified}, & |\omega| < 0.3\pi, \\ 0, & 0.3\pi \le |\omega| \le \pi. \end{cases}$$

Over what region of $-\pi \le \omega < \pi$ is $Y(e^{j\omega})$, the DTFT of $y[n]$, guaranteed to be zero?

2.70. A commonly used numerical operation called the *first backward difference* is defined as

$$y[n] = \nabla(x[n]) = x[n] - x[n-1],$$

where $x[n]$ is the input and $y[n]$ is the output of the first-backward-difference system.

(a) Show that this system is linear and time invariant.

(b) Find the impulse response of the system.

(c) Find and sketch the frequency response (magnitude and phase).

(d) Show that if

$$x[n] = f[n] * g[n],$$

then

$$\nabla(x[n]) = \nabla(f[n]) * g[n] = f[n] * \nabla(g[n]),$$

where $*$ denotes discrete convolution.

(e) Find the impulse response of a system that could be cascaded with the first-difference system to recover the input; i.e., find $h_i[n]$, where

$$h_i[n] * \nabla(x[n]) = x[n].$$

2.71. Let $H(e^{j\omega})$ denote the frequency response of an LTI system with impulse response $h[n]$, where $h[n]$ is, in general, complex.

(a) Using Eq. (2.109), show that $H^*(e^{-j\omega})$ is the frequency response of a system with impulse response $h^*[n]$, where * denotes complex conjugation.

(b) Show that if $h[n]$ is real, the frequency response is conjugate symmetric, i.e., $H(e^{-j\omega}) = H^*(e^{j\omega})$.

2.72. Let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$. Using the Fourier transform synthesis or analysis equations (Eqs. (2.133) and (2.134)), show that

(a) the Fourier transform of $x^*[n]$ is $X^*(e^{-j\omega})$,

(b) the Fourier transform of $x^*[-n]$ is $X^*(e^{j\omega})$.

2.73. Show that for $x[n]$ real, property 7 in Table 2.1 follows from property 1 and that properties 8–11 follow from property 7.

2.74. In Section 2.9, we stated a number of Fourier transform theorems without proof. Using the Fourier synthesis or analysis equations (Eqs. (2.133) and (2.134)), demonstrate the validity of Theorems 1–5 in Table 2.2.

2.75. In Section 2.9.6, it was argued intuitively that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \qquad \text{(P2.75-1)}$$

when $Y(e^{j\omega})$, $H(e^{j\omega})$, and $X(e^{j\omega})$ are, respectively, the Fourier transforms of the output $y[n]$, impulse response $h[n]$, and input $x[n]$ of a linear time-invariant system; i.e.,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \qquad \text{(P2.75-2)}$$

Verify Eq. (P2.75-1) by applying the Fourier transform to the convolution sum given in Eq. (P2.75-2).

2.76. By applying the Fourier synthesis equation (Eq. (2.133)) to Eq. (2.172) and using Theorem 3 in Table 2.2, demonstrate the validity of the modulation theorem (Theorem 7, Table 2.2).

2.77. Let $x[n]$ and $y[n]$ denote complex sequences and $X(e^{j\omega})$ and $Y(e^{j\omega})$ their respective Fourier transforms.

(a) By using the convolution theorem (Theorem 6 in Table 2.2) and appropriate properties from Table 2.2, determine, in terms of $x[n]$ and $y[n]$, the sequence whose Fourier transform is $X(e^{j\omega})Y^*(e^{j\omega})$.

(b) Using the result in Part (a), show that

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega. \qquad \text{(P2.77-1)}$$

Equation (P2.77-1) is a more general form of Parseval's theorem, as given in Section 2.9.5.

(c) Using Eq. (P2.77-1), determine the numerical value of the sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi n/4)}{2\pi n}\frac{\sin(\pi n/6)}{5\pi n}.$$

2.78. Let $x[n]$ and $X(e^{j\omega})$ represent a sequence and its Fourier transform, respectively. Determine, in terms of $X(e^{j\omega})$, the transforms of $y_s[n]$, $y_d[n]$, and $y_e[n]$. In each case, sketch $Y(e^{j\omega})$ for $X(e^{j\omega})$ as shown in Figure P2.78-1.

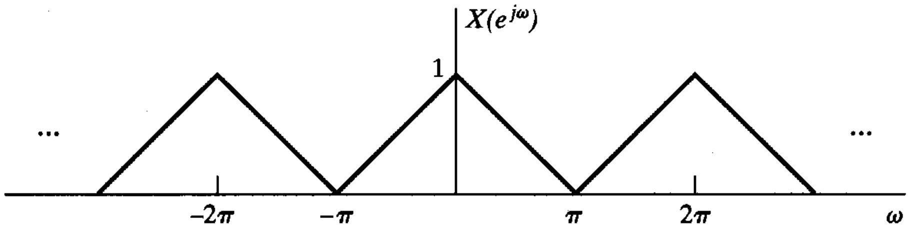


Figure P2.78-1

(a) Sampler:

$$y_s[n] = \begin{cases} x[n], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Note that $y_s[n] = \frac{1}{2}\{x[n] + (-1)^n x[n]\}$ and $-1 = e^{j\pi}$.

(b) Compressor:

$$y_d[n] = x[2n].$$

(c) Expander:

$$y_e[n] = \begin{cases} x[n/2], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

2.79. The two-frequency correlation function $\Phi_x(N, \omega)$ is often used in radar and sonar to evaluate the frequency and travel-time resolution of a signal. For discrete-time signals, we define

$$\Phi_x(N, \omega) = \sum_{n=-\infty}^{\infty} x[n+N]x^*[n-N]e^{-j\omega n}.$$

(a) Show that

$$\Phi_x(-N, -\omega) = \Phi_x^*(N, \omega).$$

(b) If

$$x[n] = Aa^n u[n], \qquad 0 < a < 1,$$

find $\Phi_x(N, \omega)$. (Assume that $N \geq 0$.)

(c) The function $\Phi_x(N, \omega)$ has a frequency domain dual. Show that

$$\Phi_x(N, \omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j[v+(\omega/2)]}\right) X^*\left(e^{j[v-(\omega/2)]}\right) e^{j2vN} dv.$$

2.80. Let $x[n]$ and $y[n]$ be stationary, uncorrelated random signals. Show that if

$$w[n] = x[n] + y[n],$$

then

$$m_w = m_x + m_y \quad \text{and} \quad \sigma_w^2 = \sigma_x^2 + \sigma_y^2.$$

2.81. Let $e[n]$ denote a white-noise sequence, and let $s[n]$ denote a sequence that is uncorrelated with $e[n]$. Show that the sequence

$$y[n] = s[n]e[n]$$

is white, i.e., that

$$E\{y[n]y[n+m]\} = A\delta[m],$$

where A is a constant.

2.82. Consider a random signal $x[n] = s[n] + e[n]$, where both $s[n]$ and $e[n]$ are independent zero-mean stationary random signals with autocorrelation functions $\phi_{ss}[m]$ and $\phi_{ee}[m]$ respectively.

(a) Determine expressions for $\phi_{xx}[m]$ and $\Phi_{xx}(e^{j\omega})$.

(b) Determine expressions for $\phi_{xe}[m]$ and $\Phi_{xe}(e^{j\omega})$.

(c) Determine expressions for $\phi_{xs}[m]$ and $\Phi_{xs}(e^{j\omega})$.

2.83. Consider an LTI system with impulse response $h[n] = a^n u[n]$ with $|a| < 1$.

(a) Compute the deterministic autocorrelation function $\phi_{hh}[m]$ for this impulse response.

(b) Determine the energy density function $|H(e^{j\omega})|^2$ for the system.

(c) Use Parseval's theorem to evaluate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega$$

for the system.

2.84. The input to the first-backward-difference system (Example 2.10) is a zero-mean white-noise signal whose autocorrelation function is $\phi_{xx}[m] = \sigma_x^2\delta[m]$.

(a) Determine and plot the autocorrelation function and the power spectrum of the corresponding output of the system.

(b) What is the average power of the output of the system?

(c) What does this problem tell you about the first backward difference of a noisy signal?

2.85. Let $x[n]$ be a real, stationary, white-noise process, with zero mean and variance σ_x^2. Let $y[n]$ be the corresponding output when $x[n]$ is the input to a linear time-invariant system with impulse response $h[n]$. Show that

(a) $E\{x[n]y[n]\} = h[0]\sigma_x^2$,

(b) $\sigma_y^2 = \sigma_x^2 \sum_{n=-\infty}^{\infty} h^2[n]$.

2.86. Let $x[n]$ be a real stationary white-noise sequence, with zero mean and variance σ_x^2. Let $x[n]$ be the input to the cascade of two causal linear time-invariant discrete-time systems, as shown in Figure P1.86-1.

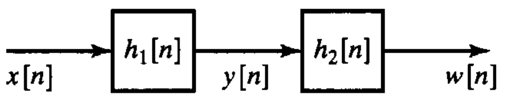


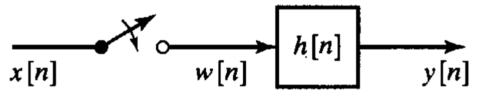
Figure P2.86-1

(a) Is $\sigma_y^2 = \sigma_x^2 \sum_{k=0}^{\infty} h_1^2[k]$?

(b) Is $\sigma_w^2 = \sigma_y^2 \sum_{k=0}^{\infty} h_2^2[k]$?

(c) Let $h_1[n] = a^n u[n]$ and $h_2[n] = b^n u[n]$. Determine the impulse response of the overall system in Figure P1.86-1, and, from this, determine σ_w^2. Are your answers to parts (b) and (c) consistent?

2.87. Sometimes we are interested in the statistical behavior of a linear time-invariant system when the input is a suddenly applied random signal. Such a situation is depicted in Figure P1.87-1.



(switch closed at $n = 0$)

Figure P2.87–1

Let $x[n]$ be a stationary white-noise process. The input to the system, $w[n]$, given by

$$w[n] = \begin{cases} x[n], & n \geq 0, \\ 0, & n < 0, \end{cases}$$

is a nonstationary process, as is the output $y[n]$.

(a) Derive an expression for the mean of the output in terms of the mean of the input.
(b) Derive an expression for the autocorrelation sequence $\phi_{yy}[n_1, n_2]$ of the output.
(c) Show that, for large n, the formulas derived in parts (a) and (b) approach the results for stationary inputs.
(d) Assume that $h[n] = a^n u[n]$. Find the mean and mean-square values of the output in terms of the mean and mean-square values of the input. Sketch these parameters as a function of n.

2.88. Let $x[n]$ and $y[n]$ respectively denote the input and output of a system. The input–output relation of a system sometimes used for the purpose of noise reduction in images is given by

$$y[n] = \frac{\sigma_s^2[n]}{\sigma_x^2[n]}(x[n] - m_x[n]) + m_x[n],$$

where

$$\sigma_x^2[n] = \frac{1}{3}\sum_{k=n-1}^{n+1}(x[k] - m_x[n])^2,$$

$$m_x[n] = \frac{1}{3}\sum_{k=n-1}^{n+1} x[k],$$

$$\sigma_s^2[n] = \begin{cases} \sigma_x^2[n] - \sigma_w^2, & \sigma_x^2[n] \geq \sigma_w^2, \\ 0, & \text{otherwise}, \end{cases}$$

and σ_w^2 is a known constant proportional to the noise power.
(a) Is the system linear?
(b) Is the system shift invariant?
(c) Is the system stable?
(d) Is the system causal?
(e) For a fixed $x[n]$, determine $y[n]$ when σ_w^2 is very large (large noise power) and when σ_w^2 is very small (small noise power). Does $y[n]$ make sense for these extreme cases?

2.89. Consider a random process $x[n]$ that is the response of the linear time-invariant system shown in Figure P2.89-1. In the figure, $w[n]$ represents a real zero-mean stationary white-noise process with $E\{w^2[n]\} = \sigma_w^2$.

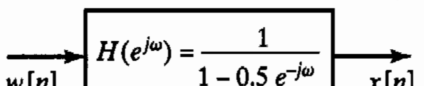


Figure P2.89-1

(a) Express $\mathcal{E}\{x^2[n]\}$ in terms of $\phi_{xx}[n]$ or $\Phi_{xx}(e^{j\omega})$.
(b) Determine $\Phi_{xx}(e^{j\omega})$, the power density spectrum of $x[n]$.
(c) Determine $\phi_{xx}[n]$, the correlation function of $x[n]$.

2.90. Consider a linear time-invariant system whose impulse response is real and is given by $h[n]$. Suppose the responses of the system to the two inputs $x[n]$ and $v[n]$ are, respectively, $y[n]$ and $z[n]$, as shown in Figure P2.90-1.

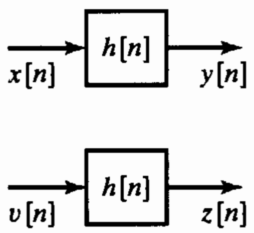


Figure P2.90-1

The inputs $x[n]$ and $v[n]$ in the figure represent real zero-mean stationary random processes with autocorrelation functions $\phi_{xx}[n]$ and $\phi_{vv}[n]$, cross-correlation function $\phi_{xv}[n]$, power spectra $\Phi_{xx}(e^{j\omega})$ and $\Phi_{vv}(e^{j\omega})$, and cross power spectrum $\Phi_{xv}(e^{j\omega})$.

(a) Given $\phi_{xx}[n]$, $\phi_{vv}[n]$, $\phi_{xv}[n]$, $\Phi_{xx}(e^{j\omega})$, $\Phi_{vv}(e^{j\omega})$, and $\Phi_{xv}(e^{j\omega})$, determine $\Phi_{yz}(e^{j\omega})$, the cross power spectrum of $y[n]$ and $z[n]$, where $\Phi_{yz}(e^{j\omega})$ is defined by

$$\phi_{yz}[n] \overset{\mathcal{F}}{\longleftrightarrow} \Phi_{yz}(e^{j\omega}),$$

with $\phi_{yz}[n] = E\{y[k]z[k-n]\}$.

(b) Is the cross power spectrum $\Phi_{xv}(e^{j\omega})$ always nonnegative; i.e., is $\Phi_{xv}(e^{j\omega}) \geq 0$ for all ω? Justify your answer.

2.91. Consider the LTI system shown in Figure P2.91-1. The input to this system, $e[n]$, is a stationary zero-mean white-noise signal with average power σ_e^2. The first system is a backward-difference system as defined in Eq. 2.45 with $f[n] = e[n] - e[n-1]$. The second system is an ideal lowpass filter with frequency response

$$H_2(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases}$$

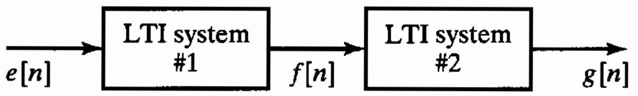


Figure P2.91-1

(a) Determine an expression for $\Phi_{ff}(e^{j\omega})$, the power spectrum of $f[n]$, and plot this expression for $-2\pi < \omega < 2\pi$.

(b) Determine an expression for $\phi_{ff}[m]$, the autocorrelation function of $f[n]$.

(c) Determine an expression for $\Phi_{gg}(e^{j\omega})$, the power spectrum of $g[n]$, and plot this expression for $-2\pi < \omega < 2\pi$.

(d) Determine an expression for σ_g^2, the average power of the output.