5

TRANSFORM ANALYSIS OF LINEAR TIME-INVARIANT SYSTEMS

5.0 INTRODUCTION

In Chapter 2 we developed the Fourier transform representation of discrete-time signals and systems, and in Chapter 3 we extended that representation to the z-transform. In both chapters, the emphasis was on the transforms and their properties, with only a brief preview of the details of their use in the analysis of linear time-invariant (LTI) systems. In this chapter, we develop in more detail the representation and analysis of LTI systems using the Fourier and z-transforms. The material is essential background for our discussion in Chapter 6 of the implementation of LTI systems and in Chapter 7 of the design of such systems.

As developed in Chapter 2, an LTI system can be completely characterized in the time domain by its impulse response $h[n]$, with the output $y[n]$ due to a given input $x[n]$ specified through the convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{5.1}$$

Alternatively, as discussed in Section 2.7, since the frequency response and impulse response are directly related through the Fourier transform, the frequency response, assuming it exists (i.e., converges), provides an equally complete characterization of LTI systems. In Chapter 3 we developed the z-transform as a generalization of the Fourier transform, and we showed that $Y(z)$, the z-transform of the output of an LTI system, is related to $X(z)$, the z-transform of the input, and $H(z)$, the z-transform of the system impulse response, by

$$Y(z) = H(z)X(z), \tag{5.2}$$

with an appropriate region of convergence. $H(z)$ is referred to as the *system function.* Since the z-transform and a sequence form a unique pair, it follows that any LTI system is completely characterized by its system function, again assuming convergence.

As we will see in this chapter, both the frequency response and the system function are extremely useful in the analysis and representation of LTI systems, because we can readily infer many properties of the system response from them.

5.1 THE FREQUENCY RESPONSE OF LTI SYSTEMS

The frequency response $H(e^{j\omega})$ of an LTI system was defined in Section 2.6 as the complex gain (eigenvalue) that the system applies to the complex exponential input (eigenfunction) $e^{j\omega n}$. Furthermore, in Section 2.9.6 we developed the fact that, since the Fourier transform of a sequence represents a decomposition as a linear combination of complex exponentials, the Fourier transforms of the system input and output are related by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \tag{5.3}$$

where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of the system input and output, respectively. With the frequency response expressed in polar form, the magnitude and phase of the Fourier transforms of the system input and output are related by

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|, \tag{5.4a}$$

$$\sphericalangle Y(e^{j\omega}) = \sphericalangle H(e^{j\omega}) + \sphericalangle X(e^{j\omega}). \tag{5.4b}$$

$|H(e^{j\omega})|$ is referred to as the *magnitude* response or the *gain* of the system, and $\sphericalangle H(e^{j\omega})$ is referred to as the *phase response* or *phase shift* of the system.

The magnitude and phase effects represented by Eqs. (5.4a) and (5.4b) can be either desirable, if the input signal is modified in a useful way, or undesirable, if the input signal is changed in a deleterious manner. In the latter case, we often refer to the effects of an LTI system on a signal, as represented by Eqs. (5.4a) and (5.4b), as *magnitude* and *phase distortions,* respectively.

5.1.1 Ideal Frequency-Selective Filters

An important implication of Eq. (5.4a) is that frequency components of the input are suppressed in the output if $|H(e^{j\omega})|$ is small at those frequencies. Whether this suppression of Fourier components is viewed as desirable or undesirable depends on the specific problem. Example 2.19 formalized the general notion of frequency-selective filters through the definition of certain ideal frequency responses. For example, the ideal lowpass filter was defined as the discrete-time linear time-invariant system whose frequency response is

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{5.5}$$

and, of course, $H_{lp}(e^{j\omega})$ is also periodic with period 2π. The ideal lowpass filter selects the low-frequency components of the signal and rejects the high-frequency components. The corresponding impulse response was shown in Example 2.22 to be

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \tag{5.6}$$

Analogously, the *ideal highpass filter* is defined as

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c < |\omega| \leq \pi, \end{cases} \tag{5.7}$$

and since $H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$, its impulse response is

$$h_{hp}[n] = \delta[n] - h_{lp}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}. \tag{5.8}$$

The ideal highpass filter passes the frequency band $\omega_c < \omega \leq \pi$ undistorted and rejects frequencies below ω_c. Other ideal frequency-selective filters were defined in Example 2.19.

The ideal lowpass filters are noncausal, and their impulse responses extend from $-\infty$ to $+\infty$. Therefore, it is not possible to compute the output of either the ideal lowpass or the ideal highpass filter either recursively or nonrecursively; i.e., the systems are not *computationally realizable.*

Another important property of the ideal lowpass filter as defined in Eq. (5.5) is that the phase response is specified to be zero. If it were not zero, the low-frequency band selected by the filter would also have phase distortion. It will become clear later in this chapter that causal approximations to ideal frequency-selective filters must have a nonzero phase response.

5.1.2 Phase Distortion and Delay

To understand the effect of the phase of a linear system, let us first consider the ideal delay system. The impulse response is

$$h_{id}[n] = \delta[n - n_d], \tag{5.9}$$

and the frequency response is

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}, \tag{5.10}$$

or

$$|H_{id}(e^{j\omega})| = 1, \tag{5.11a}$$

$$\sphericalangle H_{id}(e^{j\omega}) = -\omega n_d, \qquad |\omega| < \pi, \tag{5.11b}$$

with periodicity 2π in ω assumed. For now, we will assume that n_d is an integer.

In many applications, delay distortion would be considered a rather mild form of phase distortion, since its effect is only to shift the sequence in time. Often this would be inconsequential, or it could easily be compensated for by introducing delay in other parts of a larger system. Thus, in designing approximations to ideal filters and other linear time-invariant systems, we frequently are willing to accept a linear phase response rather than a zero phase response as our ideal. For example, an ideal lowpass filter with linear phase would be defined as

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases} \tag{5.12}$$

Its impulse response is

$$h_{\text{lp}}[n] = \frac{\sin \omega_c(n - n_d)}{\pi(n - n_d)}, \qquad -\infty < n < \infty. \tag{5.13}$$

In a similar manner, we could define other ideal frequency-selective filters with linear phase. These filters would have the desired effect of isolating a band of frequencies in the input signal, as well as the additional effect of delaying the output by n_d. Note, however, that no matter how large we make n_d, the ideal lowpass filter is always noncausal.

A convenient measure of the linearity of the phase is the *group delay.* The basic concept of group delay relates to the effect of the phase on a narrowband signal. Specifically, consider the output of a system with frequency response $H(e^{j\omega})$ for a narrowband input of the form $x[n] = s[n]\cos(\omega_0 n)$. Since it is assumed that $X(e^{j\omega})$ is nonzero only around $\omega = \omega_0$, the effect of the phase of the system can be approximated around $\omega = \omega_0$ as the linear approximation

$$\sphericalangle H(e^{j\omega}) \simeq -\phi_0 - \omega n_d. \tag{5.14}$$

With this approximation, it can be shown (see Problem 5.57) that the response $y[n]$ to $x[n] = s[n]\cos(\omega_0 n)$ is approximately $y[n] = |H(e^{j\omega_0})|s[n - n_d]\cos(\omega_0 n - \phi_0 - \omega_0 n_d)$. Consequently, the time delay of the envelope $s[n]$ of the narrowband signal $x[n]$ with Fourier transform centered at ω_0 is given by the negative of the slope of the phase at ω_0. In considering the linear approximation to $\sphericalangle H(e^{j\omega})$ around $\omega = \omega_0$, as given in Eq. (5.14), we must consider the phase response as a continuous function of ω. The phase response specified in this way will be denoted as $\arg[H(e^{j\omega})]$ and is referred to as the *continuous phase of* $H(e^{j\omega})$.

With phase specified as a continuous function of ω, the group delay of a system is defined as

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}. \tag{5.15}$$

The deviation of the group delay from a constant indicates the degree of nonlinearity of the phase.

Example 5.1 Effects of Attenuation and Group Delay

As an illustration of the effect of group delay, consider a filter with frequency response magnitude and group delay shown in Figure 5.1. In Figure 5.2, we show an input signal and its spectrum. In Figure 5.3 is the resulting output signal. Note that the input signal consists of three consecutive narrowband pulses, at frequencies $\omega = 0.85\pi$, $\omega = 0.25\pi$, and $\omega = 0.5\pi$.

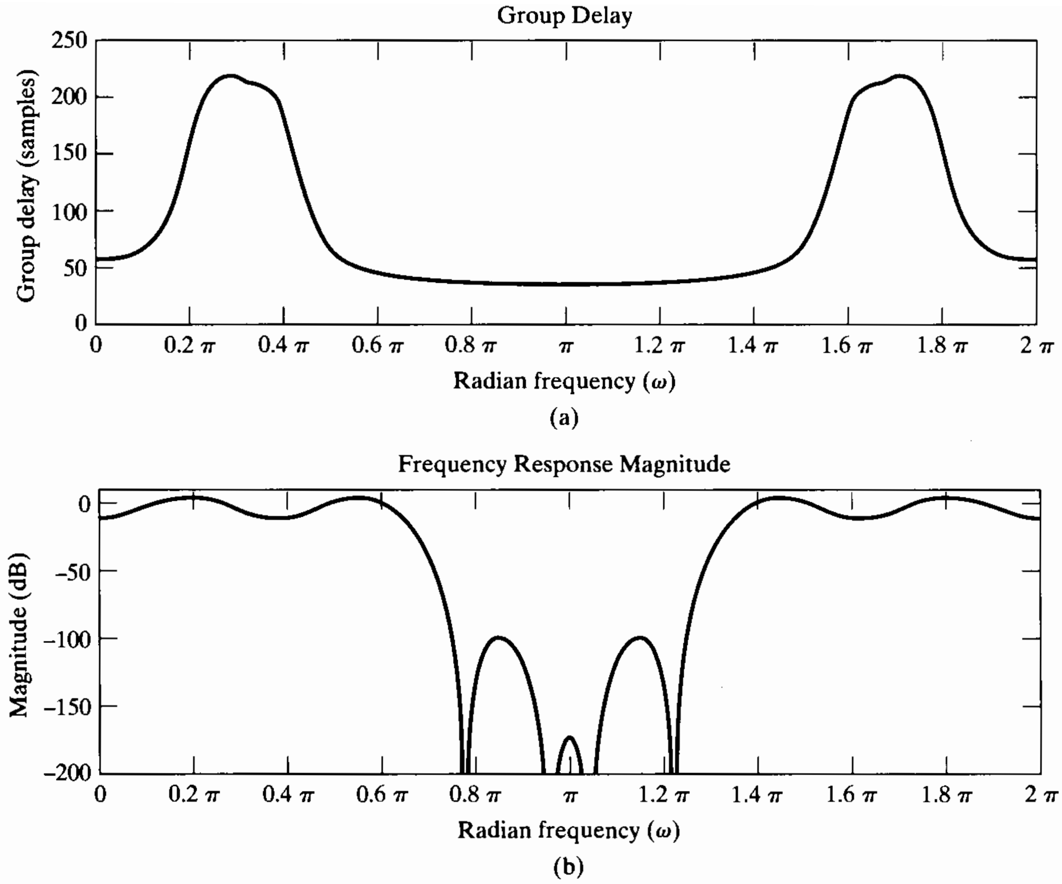


Figure 5.1 Frequency response magnitude and group delay for the filter in Example 5.1.

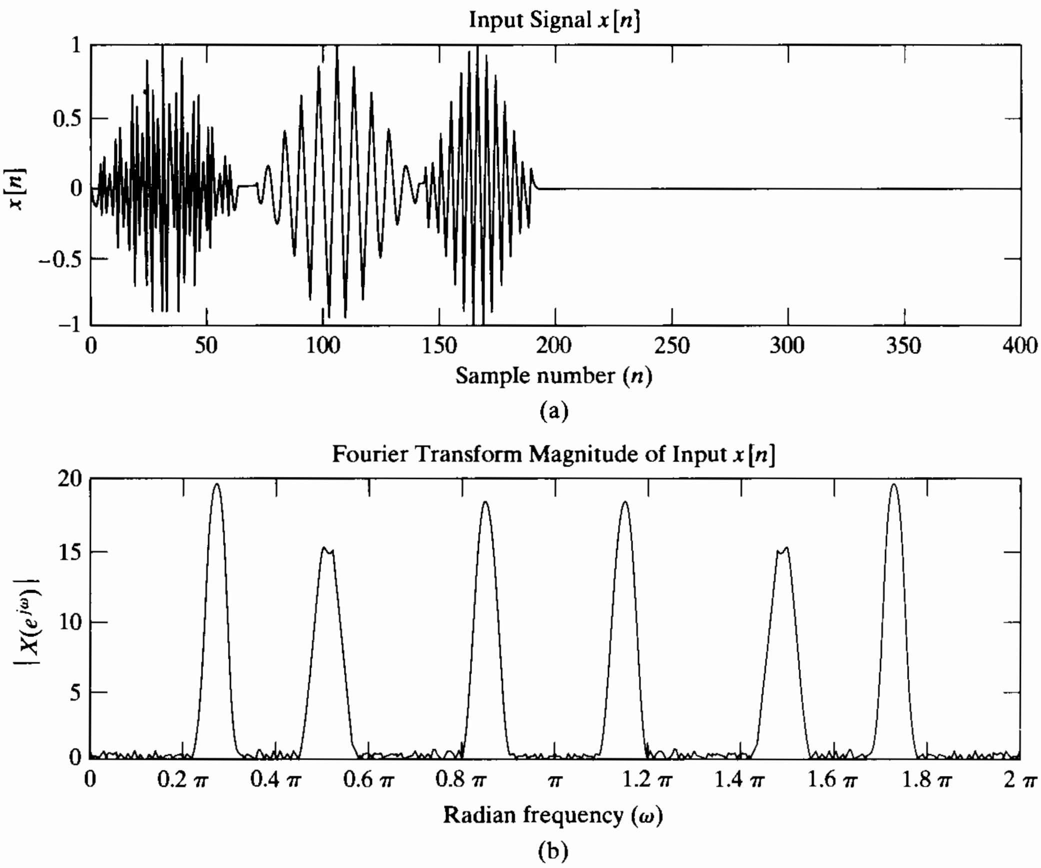


Figure 5.2 Input signal and associated Fourier transform magnitude for Example 5.1.

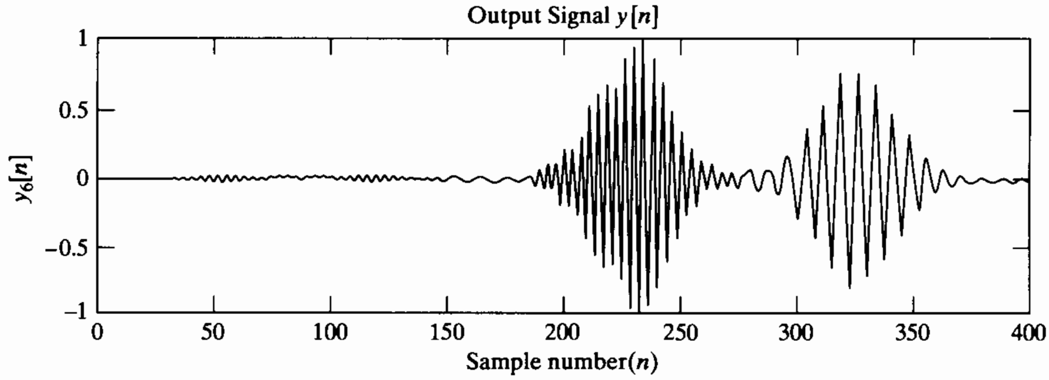


Figure 5.3 Output signal for Example 5.1.

Since the filter has considerable attenuation at $\omega = 0.85\pi$, the pulse at that frequency is not clearly present in the output. Also, since the group delay at $\omega = 0.25\pi$ is approximately 200 samples and at $\omega = 0.5\pi$ is approximately 50 samples, the second pulse in $x[n]$ will be delayed by about 200 samples and the third pulse by 50 samples, as we see is the case in Figure 5.3.

5.2 SYSTEM FUNCTIONS FOR SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

While ideal frequency-selective filters are useful conceptually, they cannot be implemented with finite computation. Therefore, it is of interest to consider a class of systems that can be implemented as approximations to ideal frequency-selective filters.

In Section 2.5, we considered the class of systems whose input and output satisfy a linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{5.16}$$

We showed that if we further assume that the system is causal, the difference equation can be used to compute the output recursively. If the auxiliary conditions correspond to initial rest, the system will be causal, linear, and time invariant.

The properties and characteristics of LTI systems for which the input and output satisfy a linear constant-coefficient difference equation are best developed through the z-transform. Applying the z-transform to both sides of Eq. (5.16) and using the linearity property (Section 3.4.1) and the time-shifting property (Section 3.4.2), we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

or equivalently,

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z). \tag{5.17}$$

From Eq. (5.2) and Eq. (5.17), it follows that, for a system whose input and output satisfy a difference equation of the form of Eq. (5.16), the system function has the algebraic form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{5.18}$$

$H(z)$ in Eq. (5.18) is a ratio of polynomials in z^{-1}, because Eq. (5.16) consists of a linear combination of delay terms. Although Eq. (5.18) can, of course, be rewritten so that the polynomials are expressed as powers of z rather than of z^{-1}, it is common practice not to do so. Also, it is often convenient to express Eq. (5.18) in factored form as

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}. \tag{5.19}$$

Each of the factors $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$. Similarly, each of the factors $(1 - d_k z^{-1})$ in the denominator contributes a zero at $z = 0$ and a pole at $z = d_k$.

There is a straightforward relationship between the difference equation and the corresponding algebraic expression for the system function. Specifically, the numerator polynomial in Eq. (5.18) has the same coefficients and algebraic structure as the right-hand side of Eq. (5.16) (the terms of the form $b_k z^{-k}$ correspond to $b_k x[n-k]$), while the denominator polynomial in Eq. (5.18) has the same coefficients and algebraic structure as the left-hand side of Eq. (5.16) (the terms of the form $a_k z^{-k}$ correspond to $a_k y[n-k]$). Thus, given either the system function in the form of Eq. (5.18) or the difference equation in the form of Eq. (5.16), it is straightforward to obtain the other.

Example 5.2 Second-Order System

Suppose that the system function of a linear time-invariant system is

$$H(z) = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{3}{4}z^{-1}\right)}. \tag{5.20}$$

To find the difference equation that is satisfied by the input and output of this system, we express $H(z)$ in the form of Eq. (5.18) by multiplying the numerator and denominator factors to obtain the ratio of polynomials

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}. \tag{5.21}$$

Thus,

$$\left(1 + \tfrac{1}{4}z^{-1} - \tfrac{3}{8}z^{-2}\right) Y(z) = (1 + 2z^{-1} + z^{-2})X(z),$$

and the difference equation is

$$y[n] + \tfrac{1}{4}y[n-1] - \tfrac{3}{8}y[n-2] = x[n] + 2x[n-1] + x[n-2]. \tag{5.22}$$

Note that once the correspondence is well understood, it is possible to proceed directly from Eq. (5.21) to Eq. (5.22) without the intervening algebra (and vice versa).

5.2.1 Stability and Causality

To obtain Eq. (5.18) from Eq. (5.16), we assumed that the system was linear and time invariant, so that Eq. (5.2) applied, but we made no further assumption about stability or causality. Correspondingly, from the difference equation, we can obtain the algebraic expression for the system function, but not the region of convergence. Specifically, the region of convergence of $H(z)$ is not determined from the derivation leading to Eq. (5.18), since all that is required for Eq. (5.17) to hold is that $X(z)$ and $Y(x)$ have overlapping regions of convergence. This is consistent with the fact that, as we saw in Chapter 2, the difference equation does not uniquely specify the impulse response of a linear time-invariant system. For the system function of Eq. (5.18) or (5.19), there are a number of choices for the region of convergence. For a given ratio of polynomials, each possible choice for the region of convergence will lead to a different impulse response, but they will all correspond to the same difference equation. However, if we assume that the system is causal, it follows that $h[n]$ must be a right-sided sequence, and therefore, the region of convergence of $H(z)$ must be outside the outermost pole. Alternatively, if we assume that the system is stable, then, from the discussion in Section 2.4, the impulse response must be absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{5.23}$$

Since Eq. (5.23) is identical to the condition that

$$\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty \tag{5.24}$$

for $|z| = 1$, the condition for stability is equivalent to the condition that the ROC of $H(z)$ include the unit circle.

Example 5.3 Determining the ROC

Consider the LTI system with input and output related through the difference equation

$$y[n] - \tfrac{5}{2}y[n-1] + y[n-2] = x[n]. \tag{5.25}$$

From the previous discussions, $H(z)$ is given by

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})}. \tag{5.26}$$

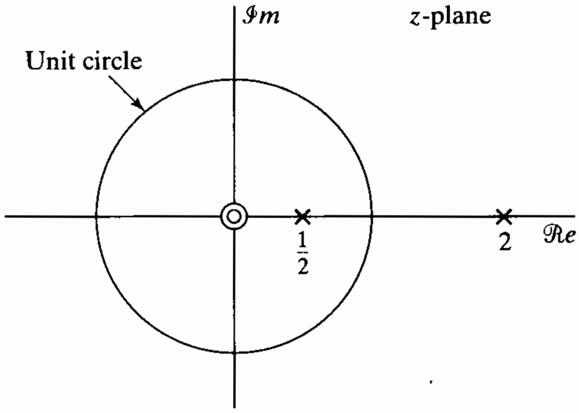


Figure 5.4 Pole–zero plot for Example 5.3.

The pole–zero plot for $H(z)$ is indicated in Figure 5.4. There are three possible choices for the ROC. If the system is assumed to be causal, then the ROC is outside the outermost pole, i.e., $|z| > 2$. In this case the system will not be stable, since the ROC does not include the unit circle. If we assume that the system is stable, then the ROC will be $\frac{1}{2} < |z| < 2$. For the third possible choice of ROC, $|z| < \frac{1}{2}$, the system will be neither stable nor causal.

As Example 5.3 suggests, causality and stability are not necessarily compatible requirements. In order for a linear time-invariant system whose input and output satisfy a difference equation of the form of Eq. (5.16) to be both causal and stable, the ROC of the corresponding system function must be outside the outermost pole *and* include the unit circle. Clearly, this requires that all the poles of the system function be inside the unit circle.

5.2.2 Inverse Systems

For a given linear time-invariant system with system function $H(z)$, the corresponding inverse system is defined to be the system with system function $H_i(z)$ such that if it is cascaded with $H(z)$, the overall effective system function is unity; i.e.,

$$G(z) = H(z)H_i(z) = 1. \tag{5.27}$$

This implies that

$$H_i(z) = \frac{1}{H(z)}. \tag{5.28}$$

The time-domain condition equivalent to Eq. (5.27) is

$$g[n] = h[n] * h_i[n] = \delta[n]. \tag{5.29}$$

From Eq. (5.28), the frequency response of the inverse system, if it exists, is

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}; \tag{5.30}$$

i.e., $H_i(e^{j\omega})$ is the reciprocal of $H(e^{j\omega})$. Equivalently, the log magnitude, phase, and group delay of the inverse system are negatives of the corresponding functions for the original system. Not all systems have an inverse. For example, the ideal lowpass filter does not. There is no way to recover the frequency components above the cutoff frequency that are set to zero by such a filter.

Many systems do have inverses, and the class of systems with rational system functions provides a very useful and interesting example. Specifically, consider

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \tag{5.31}$$

with zeros at $z = c_k$ and poles at $z = d_k$, in addition to possible zeros and/or poles at $z = 0$ and $z = \infty$. Then

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}; \tag{5.32}$$

i.e., the poles of $H_i(z)$ are the zeros of $H(z)$ and vice versa. The question arises as to what region of convergence to associate with $H_i(z)$. The answer is provided by the convolution theorem, expressed in this case by Eq. (5.29). For Eq. (5.29) to hold, the regions of convergence of $H(z)$ and $H_i(z)$ must overlap. If $H(z)$ is causal, its region of convergence is

$$|z| > \max_k |d_k|. \tag{5.33}$$

Thus, any appropriate region of convergence for $H_i(z)$ that overlaps with the region specified by Eq. (5.33) is a valid region of convergence for $H_i(z)$. Some simple examples will illustrate some of the possibilities.

Example 5.4 Inverse System for First-Order System

Let $H(z)$ be

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$$

with ROC $|z| > 0.9$. Then $H_i(z)$ is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}.$$

Since $H_i(z)$ has only one pole, there are only two possibilities for its ROC, and the only choice for the ROC of $H_i(z)$ that overlaps with $|z| > 0.9$ is $|z| > 0.5$. Therefore, the impulse response of the inverse system is

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1}u[n-1].$$

In this case, the inverse system is both causal and stable.

Example 5.5 Inverse for System with a Zero in the ROC

Suppose that $H(z)$ is

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}}, \qquad |z| > 0.9.$$

The inverse system function is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}.$$

As before, there are two possible regions of convergence: $|z| < 2$ and $|z| > 2$. In this case, however, *both* regions overlap with $|z| > 0.9$, so both are valid inverse systems. The corresponding impulse response for an ROC $|z| < 2$ is

$$h_{i1}[n] = 2(2)^n u[-n-1] - 1.8(2)^{n-1}u[-n]$$

and, for an ROC $|z| > 2$, is

$$h_{i2}[n] = -2(2)^n u[n] + 1.8(2)^{n-1}u[n-1].$$

We see that $h_{i1}[n]$ is stable and noncausal, while $h_{i2}[n]$ is unstable and causal.

A generalization from Examples 5.4 and 5.5 is that if $H(z)$ is a causal system with zeros at c_k, $k = 1, \ldots, M$, then its inverse system will be causal if and only if we associate the region of convergence,

$$|z| > \max_k |c_k|,$$

with $H_i(z)$. If we also require that the inverse system be stable, then the region of convergence of $H_i(z)$ must include the unit circle. Therefore, it must be true that

$$\max_k |c_k| < 1;$$

i.e., all the zeros of $H(z)$ must be inside the unit circle. Thus, a linear time-invariant system is stable and causal and also has a stable and causal inverse if and only if both the poles and the zeros of $H(z)$ are inside the unit circle. Such systems are referred to as *minimum-phase* systems and will be discussed in more detail in Section 5.6.

5.2.3 Impulse Response for Rational System Functions

The discussion of the partial fraction expansion technique for finding inverse z-transforms (Section 3.3.2) can be applied to the system function $H(z)$ to obtain a general expression for the impulse response of a system that has a rational system function as in Eq. (5.19). Recall that any rational function of z^{-1} with only first-order poles can be expressed in the form

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \tag{5.34}$$

where the terms in the first summation would be obtained by long division of the denominator into the numerator and would be present only if $M \geq N$. The coefficients A_k in the second set of terms are obtained using Eq. (3.41). If $H(z)$ has a multiple-order pole, its partial fraction expansion would have the form of Eq. (3.44). If the system is assumed to be causal, then the ROC is outside all of the poles in Eq. (5.34), and it follows that

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n], \tag{5.35}$$

where the first summation is included only if $M \geq N$.

In discussing LTI systems, it is useful to identify two classes. In the first class, at least one nonzero pole of $H(z)$ is not canceled by a zero. In this case there will be at least one term of the form $A_k(d_k)^n u[n]$, and $h[n]$ will not be of finite length, i.e., will not be zero outside a finite interval. Systems of this class are therefore called *infinite impulse response* (IIR) systems. A simple IIR system is discussed in the following example.

Example 5.6 A First-Order IIR System

Consider a causal system whose input and output satisfy the difference equation

$$y[n] - ay[n-1] = x[n]. \tag{5.36}$$

The system function is (by inspection)

$$H(z) = \frac{1}{1 - az^{-1}}. \tag{5.37}$$

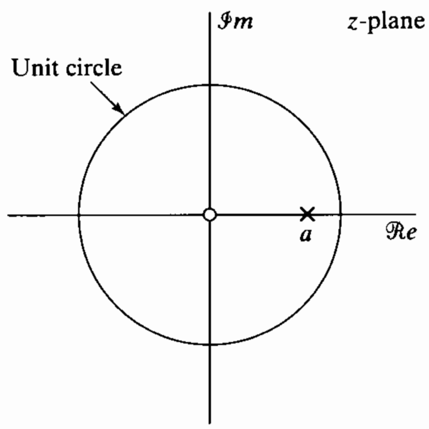


Figure 5.5 Pole–zero plot for Example 5.6.

Figure 5.5 shows the pole–zero plot of $H(z)$. The region of convergence is $|z| > |a|$, and the condition for stability is $|a| < 1$. The inverse z-transform of $H(z)$ is

$$h[n] = a^n u[n]. \tag{5.38}$$

For the second class of systems, $H(z)$ has no poles except at $z = 0$; i.e., $N = 0$ in Eqs. (5.16) and (5.18). Thus, a partial fraction expansion is not possible, and $H(z)$ is simply a polynomial in z^{-1} of the form

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}. \tag{5.39}$$

(We assume, without loss of generality, that $a_0 = 1$.) In this case, $H(z)$ is determined to within a constant multiplier by its zeros. From Eq. (5.39), $h[n]$ is seen by inspection to be

$$h[n] = \sum_{k=0}^{M} b_k \, \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases} \tag{5.40}$$

In this case, the impulse response is finite in length; i.e., it is zero outside a finite interval. Consequently, these systems are called *finite impulse response* (FIR) systems. Note that for FIR systems, the difference equation of Eq. (5.16) is identical to the convolution sum, i.e.,

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]. \tag{5.41}$$

Example 5.7 gives a simple example of an FIR system.

Example 5.7 A Simple FIR System

Consider an impulse response that is a truncation of the impulse response of Example 5.6:

$$h[n] = \begin{cases} a^n, & 0 \le n \le M, \\ 0 & \text{otherwise.} \end{cases}$$

Then the system function is

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}. \tag{5.42}$$

Since the zeros of the numerator are at

$$z_k = a e^{j2\pi k/(M+1)}, \qquad k = 0, 1, \ldots, M, \tag{5.43}$$

where a is assumed real and positive, the pole at $z = a$ is canceled by a zero. The pole–zero plot for the case $M = 7$ is shown in Figure 5.6.

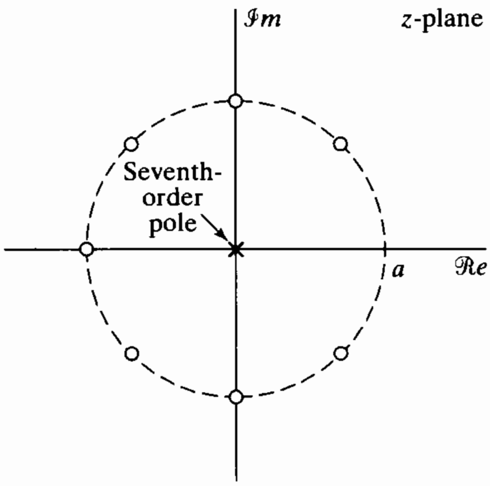


Figure 5.6 Pole–zero plot for Example 5.7.

The difference equation satisfied by the input and output of the linear time-invariant system is the discrete convolution

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]. \tag{5.44}$$

However, Eq. (5.42) suggests that the input and output also satisfy the difference equation

$$y[n] - ay[n-1] = x[n] - a^{M+1}x[n-M-1]. \tag{5.45}$$

These two equivalent difference equations result from the two equivalent forms of $H(z)$ in Eq. (5.42).

5.3 FREQUENCY RESPONSE FOR RATIONAL SYSTEM FUNCTIONS

If a stable linear time-invariant system has a rational system function (i.e., if its input and output satisfy a difference equation of the form of Eq. (5.16), then its frequency response (the system function of Eq. (5.18) evaluated on the unit circle) has the form

$$H(e^{j\omega}) = \frac{\displaystyle\sum_{k=0}^{M} b_k e^{-j\omega k}}{\displaystyle\sum_{k=0}^{N} a_k e^{-j\omega k}}. \tag{5.46}$$

That is, $H(e^{j\omega})$ is a ratio of polynomials in the variable $e^{-j\omega}$. To determine the magnitude, phase, and group delay associated with the frequency response of such systems, it is useful to express $H(e^{j\omega})$ in terms of the poles and zeros of $H(z)$. Such an expression results from substituting $z = e^{j\omega}$ into Eq. (5.19):

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}. \tag{5.47}$$

From Eq. (5.47), it follows that

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N}|1 - d_k e^{-j\omega}|}. \tag{5.48}$$

Sometimes it is convenient to consider the magnitude squared, rather than the magnitude, of the system function. The magnitude-squared function is

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}),$$

where * denotes complex conjugation, and for $H(e^{j\omega})$ as in Eq. (5.47),

$$|H(e^{j\omega})|^2 = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})(1 - c_k^* e^{j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})(1 - d_k^* e^{j\omega})}. \tag{5.49}$$

From Eq. (5.48), we see that $|H(e^{j\omega})|$ is the product of the magnitudes of all the zero factors of $H(z)$ evaluated on the unit circle, divided by the product of the magnitudes of all the pole factors evaluated on the unit circle. It is common practice to transform these products into a corresponding sum of terms by considering $20\log_{10}|H(e^{j\omega})|$ instead of $|H(e^{j\omega})|$. The logarithm of Eq. (5.48) is

$$\begin{aligned} 20\log_{10}|H(e^{j\omega})| &= 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}|1 - c_k e^{-j\omega}| \\ &\quad - \sum_{k=1}^{N} 20\log_{10}|1 - d_k e^{-j\omega}|. \end{aligned} \tag{5.50}$$

The function $20\log_{10}|H(e^{j\omega})|$ is referred to as the *log magnitude* of $H(e^{j\omega})$ and is expressed in *decibels* (dB). Sometimes this quantity is called the *gain in dB;* i.e.,

$$\text{Gain in dB} = 20\log_{10}|H(e^{j\omega})|. \tag{5.51}$$

Note that zero dB corresponds to a value of $|H(e^{j\omega})| = 1$, while $|H(e^{j\omega})| = 10^m$ is $20m$ dB. Also, $|H(e^{j\omega})| = 2^m$ is approximately $6m$ dB. When $|H(e^{j\omega})| < 1$, the quantity $20\log_{10}|H(e^{j\omega})|$ is negative. This would be the case, for example, in the stopband of a frequency-selective filter. It is common practice to define

$$\begin{aligned}\text{Attenuation in dB} &= -20\log_{10}|H(e^{j\omega})| \\ &= -\text{Gain in dB}.\end{aligned} \tag{5.52}$$

The attenuation is therefore a positive number when the magnitude response is less than unity. For example, a 60-dB attenuation at a given frequency ω means that at that frequency $|H(e^{j\omega})| = 0.001$.

Another advantage to expressing the magnitude in decibels stems from Eq. (5.4a), which, after taking logarithms of both sides, becomes

$$20\log_{10}|Y(e^{j\omega})| = 20\log_{10}|H(e^{j\omega})| + 20\log_{10}|X(e^{j\omega})|, \tag{5.53}$$

so the frequency response in dB is added to the log magnitude of the input Fourier transform to find the log magnitude of the output Fourier transform. If Eq. (5.53) replaces Eq. (5.4a) in Eqs. (5.4), then the effects of both magnitude and phase are additive.

The phase response for a rational system function has the form

$$\sphericalangle H(e^{j\omega}) = \sphericalangle\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M}\sphericalangle[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N}\sphericalangle[1 - d_k e^{-j\omega}]. \tag{5.54}$$

As in Eq. (5.50), the zero factors contribute with a plus sign and the pole factors contribute with a minus sign.

The corresponding group delay for a rational system function is

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N}\frac{d}{d\omega}(\arg[1 - d_k e^{-j\omega}]) - \sum_{k=1}^{M}\frac{d}{d\omega}(\arg[1 - c_k e^{-j\omega}]), \tag{5.55}$$

where arg[ ] represents the continuous phase. An equivalent expression is

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N}\frac{|d_k|^2 - \mathcal{R}e\{d_k e^{-j\omega}\}}{1 + |d_k|^2 - 2\mathcal{R}e\{d_k e^{-j\omega}\}} - \sum_{k=1}^{M}\frac{|c_k|^2 - \mathcal{R}e\{c_k e^{-j\omega}\}}{1 + |c_k|^2 - 2\mathcal{R}e\{c_k e^{-j\omega}\}}. \tag{5.56}$$

In Eq. (5.54), as written, the phase of each of the terms is ambiguous; i.e., any integer multiple of 2π can be added to each term at each value of ω without changing the value of the complex number. The expression for the group delay, on the other hand, involves differentiating the continuous phase.

When the angle of a complex number is computed, with the use of an arctangent subroutine on a calculator or with a computer system subroutine, the principal value is obtained. The principal value of the phase of $H(e^{j\omega})$ is denoted as ARG$[H(e^{j\omega})]$, where

$$-\pi < \text{ARG}[H(e^{j\omega})] \le \pi. \tag{5.57}$$

Any other angle that gives the correct complex value of the function $H(e^{j\omega})$ can be represented in terms of the principal value as

$$\sphericalangle H(e^{j\omega}) = \text{ARG}[H(e^{j\omega})] + 2\pi r(\omega), \tag{5.58}$$

where $r(\omega)$ is a positive or negative integer that can be different at each value of ω. Similarly, in calculating any of the individual terms in Eq. (5.54), we would typically obtain the principal value.

If the principal value is used to compute the phase response as a function of ω, then $\text{ARG}[H(e^{j\omega})]$ may be a discontinuous function. The discontinuities introduced by taking the principal value will be jumps of 2π radians. This is illustrated in Figure 5.7(a), which shows a continuous-phase function $\arg[H(e^{j\omega})]$ and its principal value $\text{ARG}[H(e^{j\omega})]$ plotted over the range $0 \le \omega \le \pi$. The phase function plotted in Figure 5.7(a) exceeds the range $-\pi$ to $+\pi$. The principal value, shown in Figure 5.7(b),

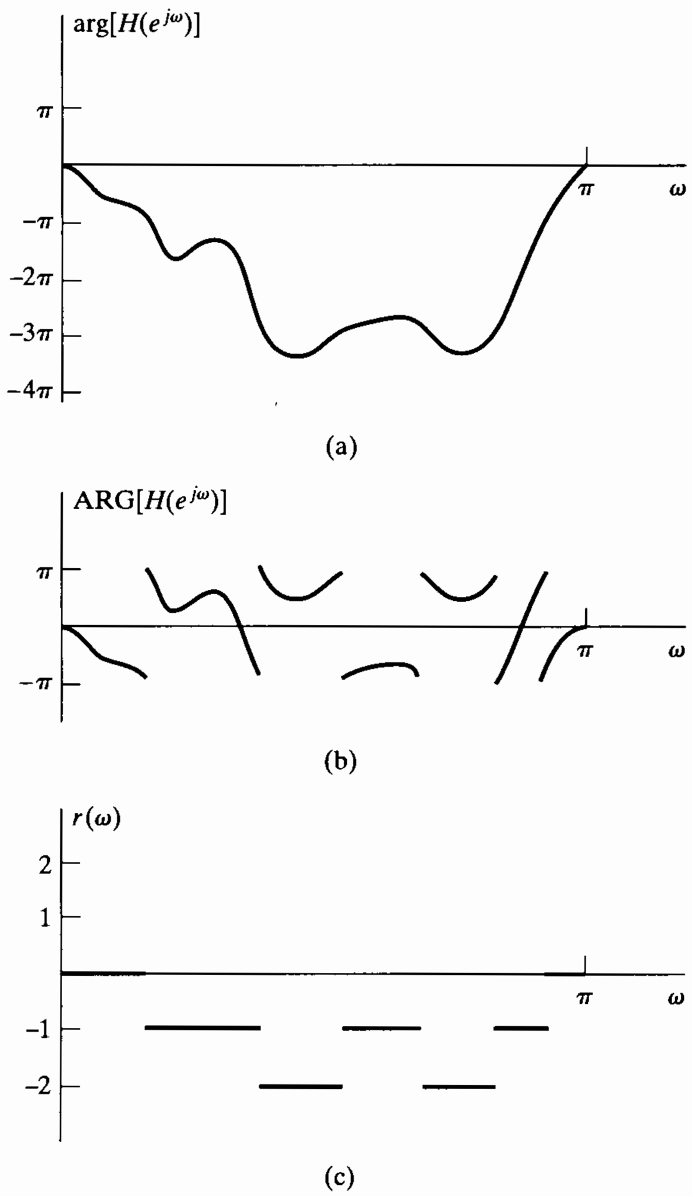


Figure 5.7 (a) Continuous-phase curve for a system function evaluated on the unit circle. (b) Principal value of the phase curve in part (a). (c) Integer multiples of 2π to be added to $\text{ARG}[H(e^{j\omega})]$ to obtain $\arg[H(e^{j\omega})]$.

has jumps of 2π due to the integer multiples of 2π that must be subtracted in certain regions to bring the phase curve within the range of the principal value. Figure 5.7(c) shows the corresponding value of $r(\omega)$ in Eq. (5.58).

Let us consider Eq. (5.54) when the principal value is used to compute the individual contributions to the phase. It is not difficult to see that

$$\begin{aligned}\text{ARG}[H(e^{j\omega})] = \text{ARG}\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \text{ARG}[1 - c_k e^{-j\omega}] \\ - \sum_{k=1}^{N} \text{ARG}[1 - d_k e^{-j\omega}] + 2\pi r(\omega),\end{aligned} \tag{5.59}$$

where $r(\omega)$ is an integer that can be different at each value of ω. The last term, $+2\pi r$, is required because, in general, the principal value of a sum of angles is *not* equal to the sum of the principal values of the individual angles. This is of considerable importance in the theory of cepstral analysis and homomorphic systems. (See Oppenheim, Schafer, and Stockham, 1968 and Tribolet, 1977.) However, it presents no problem in plotting phase functions, since the principal value can be used to compute the phase function for each pole and zero, and an appropriate multiple of 2π can then be added or subtracted as in Eq. (5.59) to obtain the principal value of the total phase function.

The principal value of the phase function can be computed using Eq. (5.59). Alternatively, we can use the relation

$$\text{ARG}[H(e^{j\omega})] = \arctan\left[\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right], \tag{5.60}$$

where $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ are the real and imaginary parts, respectively, of $H(e^{j\omega})$. However, in computing the group delay function of Eq. (5.15), it is the derivative of the continuous phase function, i.e., $\arg[H(e^{j\omega})]$, in which we are interested:

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}. \tag{5.61}$$

Except at the discontinuities of $\text{ARG}[H(e^{j\omega})]$ corresponding to jumps between $+\pi$ and $-\pi$,

$$\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \frac{d}{d\omega}\{\text{ARG}[H(e^{j\omega})]\}. \tag{5.62}$$

Consequently, the group delay can be obtained from the principal value by differentiating, except at the discontinuities. Similarly, we can express the group delay in terms of the ambiguous phase $\sphericalangle H(e^{j\omega})$ as

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}[\sphericalangle H(e^{j\omega})], \tag{5.63}$$

with the interpretation that impulses caused by discontinuities of size 2π in $\sphericalangle H(e^{j\omega})$ are ignored.

5.3.1 Frequency Response of a Single Zero or Pole

Equations (5.50), (5.54), and (5.56) represent the magnitude in dB, the phase, and the group delay, respectively, as a sum of contributions from each of the poles and zeros of the system function. To obtain further insight into the properties of frequency responses of stable linear time-invariant systems with rational system functions, it is worthwhile to first examine the properties of a single factor of the form $(1 - re^{j\theta}e^{-j\omega})$, where r is the radius and θ is the angle of the pole or zero in the z-plane. This factor is typical of either a pole or a zero at a radius r and angle θ in the z-plane.

The square of the magnitude of such a factor is

$$|1-re^{j\theta}e^{-j\omega}|^2 = (1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{j\omega}) = 1+r^2-2r\cos(\omega-\theta). \quad (5.64)$$

Since, for any complex quantity C,

$$10\log_{10}|C|^2 = 20\log_{10}|C|,$$

the log magnitude in dB is

$$20\log_{10}|1 - re^{j\theta}e^{-j\omega}| = 10\log_{10}[1+r^2-2r\cos(\omega-\theta)]. \quad (5.65)$$

The principal value phase for such a factor is

$$\text{ARG}[1 - re^{j\theta}e^{-j\omega}] = \arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]. \quad (5.66)$$

Differentiating the right-hand side of Eq. (5.66) (except at discontinuities) gives the group delay of the factor as

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)} = \frac{r^2 - r\cos(\omega-\theta)}{|1-re^{j\theta}e^{-j\omega}|^2}. \quad (5.67)$$

The functions in Eqs. (5.64)–(5.67) are, of course, periodic in ω with period 2π. Figure 5.8(a) shows a plot of Eq. (5.65) as a function of ω over one period $(0 \le \omega < 2\pi)$ for several values of θ with $r = 0.9$. Note that the function dips sharply in the vicinity of $\omega = \theta$. Note also, from Eq. (5.65), that when r is fixed, the log magnitude is a function of $(\omega - \theta)$, so as θ changes, the dip is shifted in frequency. In general, the maximum value of Eq. (5.65) occurs at $(\omega - \theta) = \pi$ and is

$$10\log_{10}(1+r^2+2r) = 20\log_{10}(1+r),$$

which, for $r = 0.9$, is equal to 5.57 dB. Similarly, the minimum value of Eq. (5.65), occurring at $\omega = \theta$, is

$$10\log_{10}(1+r^2-2r) = 20\log_{10}|1-r|,$$

which is equal to −20 dB for $r = 0.9$. Note that the plot of the magnitude-squared function in Eq. (5.64) would look similar to Figure 5.8(a), except that it would have a much wider relative range of values. Hence, its plot would appear much sharper for the same value of r.

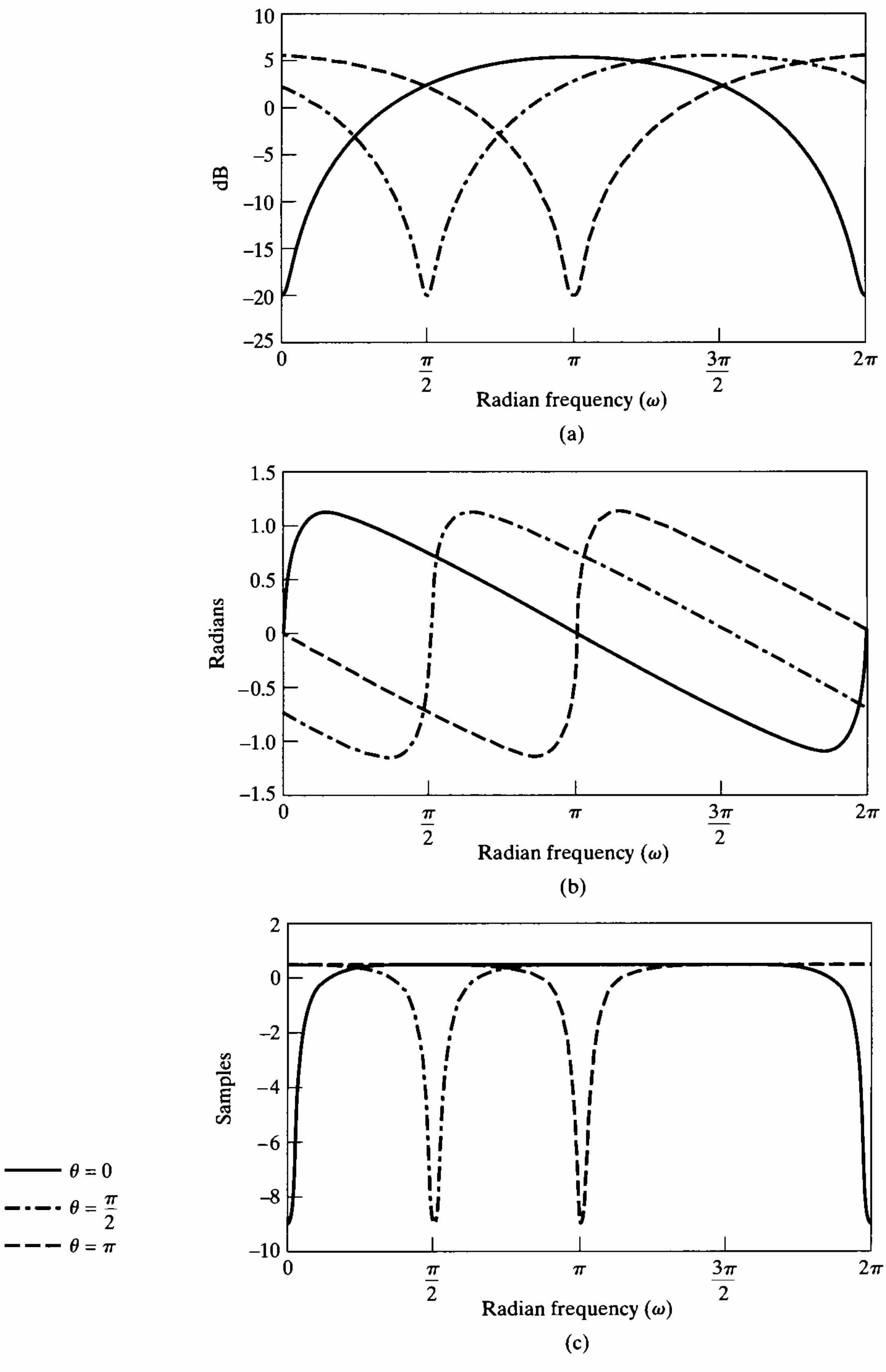


Figure 5.8 Frequency response for a single zero, with $r = 0.9$ and the three values of θ shown. (a) Log magnitude. (b) Phase. (c) Group delay.

Figure 5.8(b) shows the phase function in Eq. (5.66) as a function of ω for $r = 0.9$ and several values of θ. Note that the phase is zero at $\omega = \theta$ and that, for fixed r, the function simply shifts with θ. Figure 5.8(c) shows the group delay function in Eq. (5.67) for the same conditions on r and θ. Note that the high positive slope of the phase around $\omega = \theta$ corresponds to a large negative peak in the group delay function at $\omega = \theta$.

A simple geometric construction is often very useful in approximate sketching of frequency-response functions directly from the pole–zero plot. The procedure is based on the facts that the frequency response corresponds to the system function evaluated on the unit circle in the z-plane and that the complex value of each pole and zero factor can be represented by a vector in the z-plane from the pole or zero to a point on the unit circle. Let us first illustrate the procedure for a first-order system function of the form

$$H(z) = (1 - re^{j\theta}z^{-1}) = \frac{(z - re^{j\theta})}{z}, \qquad r < 1. \tag{5.68}$$

In Section 5.3.2, we will consider higher order examples. Such a factor has a pole at $z = 0$ and a zero at $z = re^{j\theta}$, as illustrated in Figure 5.9. Also indicated in this figure are the vectors v_1, v_2, and $v_3 = v_1 - v_2$, representing the complex numbers $e^{j\omega}$, $re^{j\theta}$, and $(e^{j\omega} - re^{j\theta})$, respectively. In terms of these vectors, the magnitude of the complex number

$$\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}$$

is the ratio of the magnitudes of the vectors v_3 and v_1, i.e.,

$$|1 - re^{j\theta}e^{-j\omega}| = \left|\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}\right| = \frac{|v_3|}{|v_1|}, \tag{5.69}$$

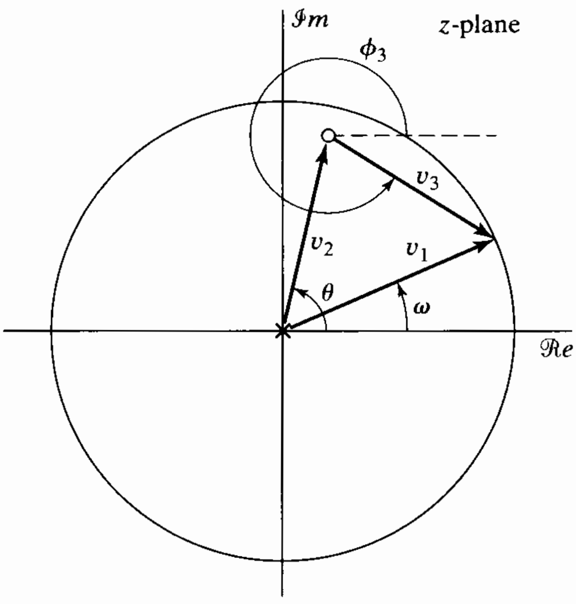


Figure 5.9 z-plane vectors for a first-order system function evaluated on the unit circle, with $r < 1$.

or, since $|v_1| = 1$, Eq. (5.69) is just equal to $|v_3|$. The corresponding phase is

$$\begin{aligned}\sphericalangle(1 - re^{j\theta}e^{-j\omega}) &= \sphericalangle(e^{j\omega} - re^{j\theta}) - \sphericalangle(e^{j\omega}) = \sphericalangle(v_3) - \sphericalangle(v_1) \\ &= \phi_3 - \phi_1 = \phi_3 - \omega.\end{aligned} \tag{5.70}$$

Typically, a vector such as v_3 from a zero to the unit circle is referred to as a zero vector, and a vector from a pole to the unit circle is called a pole vector. Thus, the contribution of a single zero factor $(1-re^{j\theta}z^{-1})$ to the magnitude function at frequency ω is the length of the zero vector v_3 from the zero to the point $z = e^{j\omega}$ on the unit circle. The vector has minimum length when $\omega = \theta$. This accounts for the sharp dip in the magnitude function at $\omega = \theta$ in Figure 5.8(a). Note that the pole vector v_1 from the pole at $z = 0$ to $z = e^{j\omega}$ always has unit length. Thus, it does not have any effect on the magnitude response. Equation (5.70) states that the phase function is equal to the difference between the angle of the zero vector from the zero at $re^{j\theta}$ to the point $z = e^{j\omega}$ and the angle of the pole vector from the pole at $z = 0$ to the point $z = e^{j\omega}$.

The pole–zero plot for the case $\theta = \pi$ is shown in Figure 5.10, and the pole and zero vectors are shown for two different values of ω. Clearly, as ω increases from zero, the magnitude of the vector v_3 decreases until it reaches a minimum at $\omega = \pi$, thereby accounting for the shape of the curve corresponding to $\theta = \pi$ in Figure 5.8(a). The angle of vector v_3 in Figure 5.10 increases more slowly than ω at first, so that the phase curve starts out negative; then, when ω is close to π, the angle of vector v_3 increases more rapidly than ω, thereby accounting for the steep positive slope of the phase function around $\omega = \pi$. Note that when $\omega = \pi$, the angles of vectors v_3 and v_1 are equal, so the net phase is zero.

The dependence of the frequency-response contributions of a single factor $(1 - re^{j\theta}e^{-j\omega})$ on the radius r is shown in Figure 5.11 for $\theta = \pi$ and several values of r. Note that the log magnitude function plotted in Figure 5.11(a) dips more sharply as r becomes closer to 1; indeed, the magnitude in dB approaches $-\infty$ at $\omega = \theta$ as r approaches 1. The phase function plotted in Figure 5.11(b) has positive slope around $\omega = \theta$, which becomes infinite as r approaches 1. Thus, for $r = 1$, the phase function is discontinuous, with a jump of π radians at $\omega = \theta$. Away from $\omega = \theta$, the slope of the phase function is negative. Since the group delay is the negative of the slope of the phase curve, the group

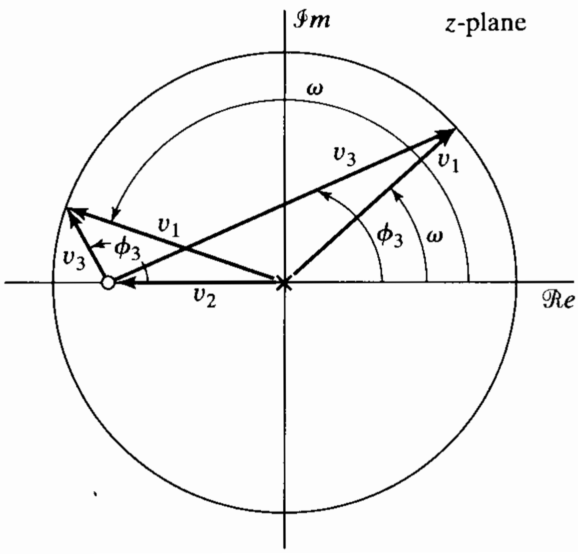


Figure 5.10 z-plane vectors for a first-order system function evaluated on the unit circle, with $\theta = \pi$, $r < 1$. The pole vector v_1 and the zero vector v_3 are shown for two different values of ω.

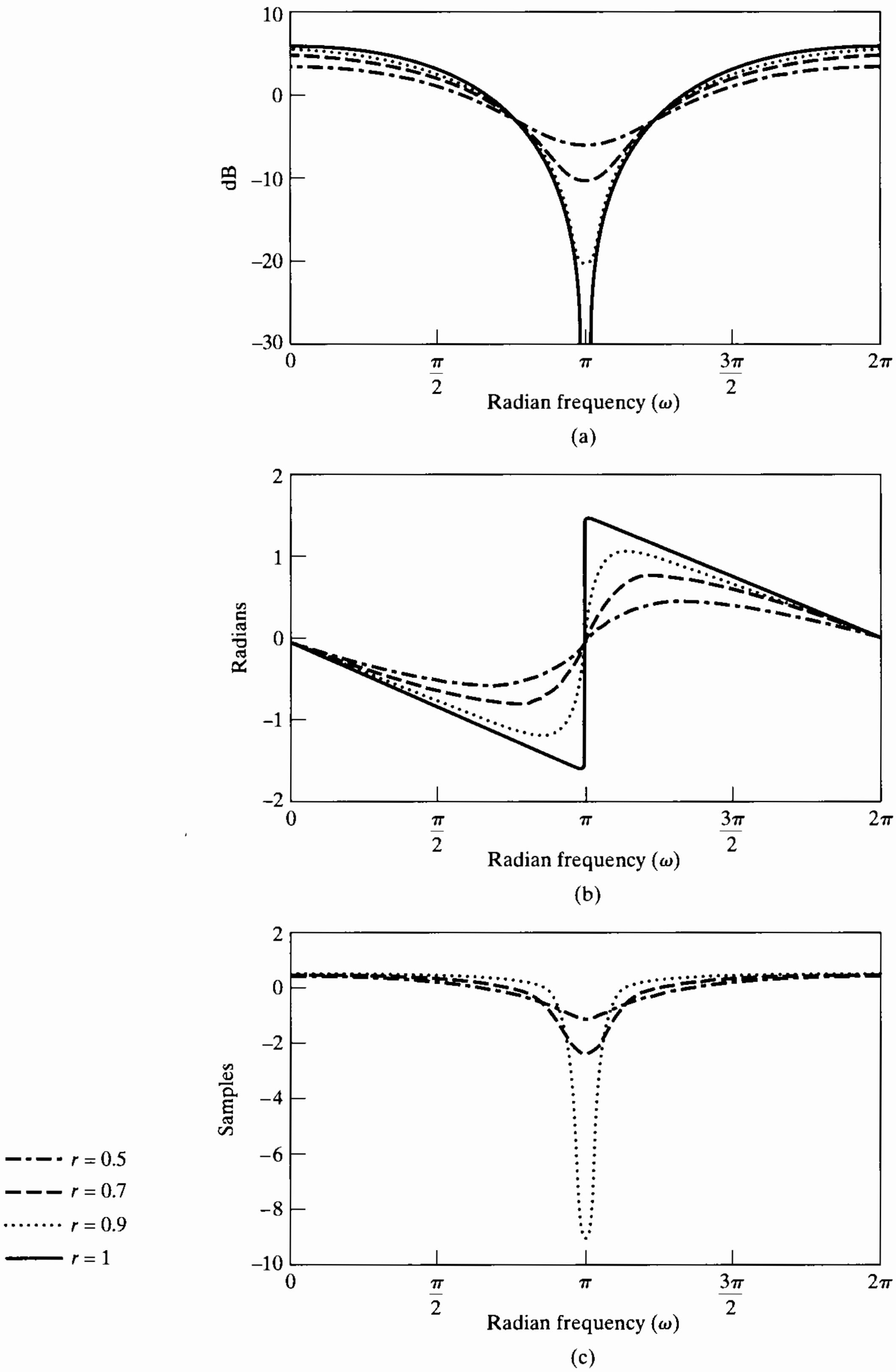


Figure 5.11 Frequency response for a single zero, with $\theta = \pi$, $r = 1$, 0.9, 0.7, and 0.5. (a) Log magnitude. (b) Phase. (c) Group delay for $r = 0.9$, 0.7, and 0.5.

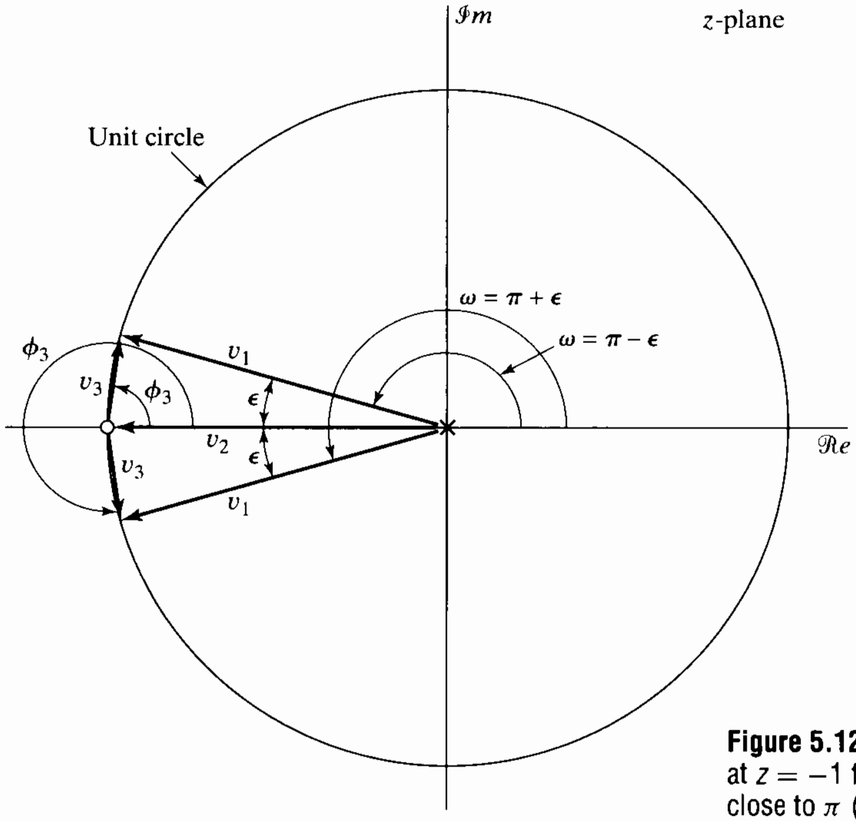


Figure 5.12 z-plane vectors for a zero at $z = -1$ for two different frequencies close to π ($\omega = \pi - \varepsilon$ and $\pi + \varepsilon$).

delay is negative around $\omega = \theta$, and it dips sharply as r approaches 1. Figure 5.11(c) shows that as we move away from $\omega = \theta$, the group delay becomes positive and relatively flat. When $r = 1$, the group delay is equal to $\frac{1}{2}$ everywhere, except at $\omega = \theta$, where it is undefined.

The geometric construction for a zero on the unit circle at $z = -1$ is shown in Figure 5.12. Indicated are vectors for two different frequencies, $\omega = (\pi - \varepsilon)$ and $\omega = (\pi + \varepsilon)$, where ε is small. Two observations can be made. First, the length of the vector v_3 approaches zero as ω approaches the angle of the zero vector ($\varepsilon \to 0$). Therefore, the multiplicative contribution to the frequency response is zero ($-\infty$ dB). Second, the vector v_3 changes its angle discontinuously by π radians as ω goes from $(\pi - \varepsilon)$ to $(\pi + \varepsilon)$.

Figures 5.8 and 5.11 were restricted to $r \le 1$. If $r > 1$, the log magnitude function behaves similarly to the case $r < 1$; i.e., it dips more sharply as $r \to 1$, as shown in Figure 5.13(a). The phase function in Figure 5.13(b) shows a discontinuity of 2π radians at $\omega = \theta$ for all values of $r > 1$. The source of this discontinuity can be seen from Figure 5.14, which shows vectors for $\omega = (\pi - \varepsilon)$ and $\omega = (\pi + \varepsilon)$. Note that the pole vector v_1 has an angle of ω, which varies continuously from $\omega = 0$ to $\omega = 2\pi$. The angle of the zero vector v_3 is labeled ϕ_3 in the figure. If this angle is measured positively in the counterclockwise direction, the figure shows that ϕ_3 jumps from zero to 2π radians as ω goes from $(\pi - \varepsilon)$ to $(\pi + \varepsilon)$. This jump of 2π radians is evident in Figure 5.13(b). The discontinuity of 2π radians can be interpreted as being due to computing the principal-value phase function. The angle ϕ_3 can also be seen to be positive for $\omega = (\pi - \varepsilon)$ and negative for $\omega = (\pi + \varepsilon)$. With this interpretation, the angle is continuous at $\omega = \theta$. However, since the total angle of the factor $(1 - re^{j\theta}e^{-j\omega})$ is less than $-\pi$ radians at $\omega = (\pi + \varepsilon)$, the principal value would appear as in Figure 5.13(b).

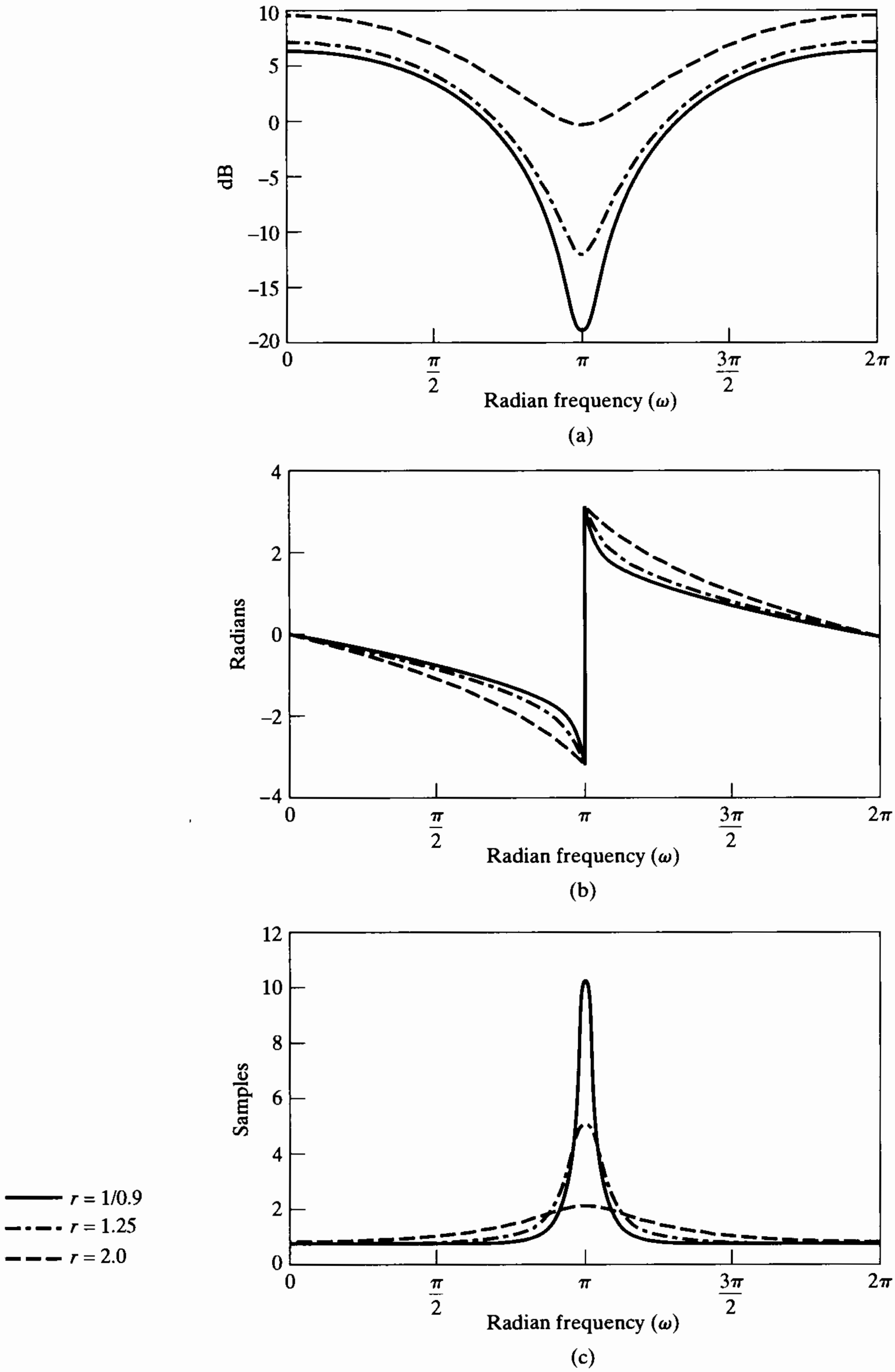


Figure 5.13 Frequency response for a single real zero outside the unit circle, with $\theta = \pi$, $r = 1/0.9$, 1.25, 2. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

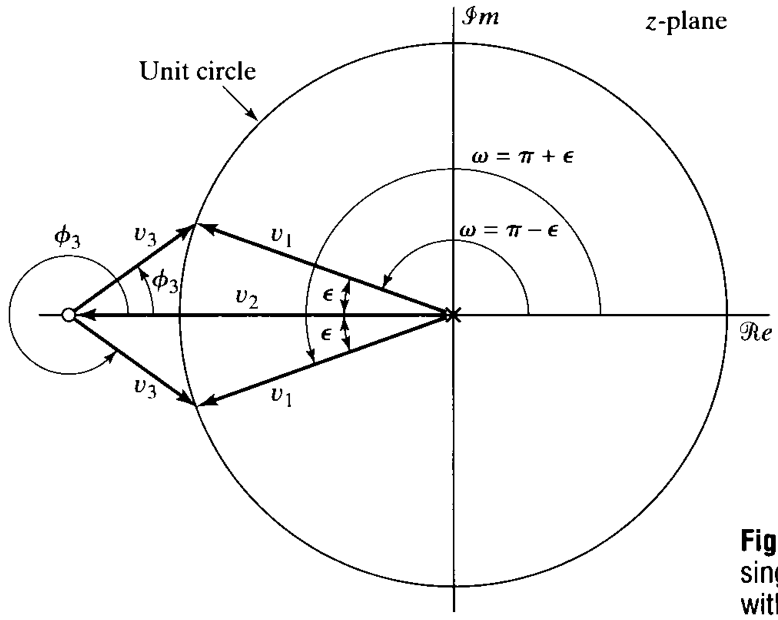


Figure 5.14 z-plane vectors for a single zero evaluated on the unit circle, with $\theta = \pi$, $r > 1$.

The phase curves in Figure 5.13(b) all have negative slope. Therefore, the group delay function for $r > 1$ is positive for all ω. This is also easily seen by considering Eq. (5.67) for $r > 1$.

The preceding discussion and Figures 5.8, 5.11, and 5.13 all pertain to a single factor of the form $(1 - re^{j\theta}e^{-j\omega})$. If the factor represents a zero of $H(z)$, then the curves of Figures 5.8, 5.11, and 5.13 will contribute to the frequency-response functions with positive algebraic sign. If the factor represents a pole of $H(z)$, then all the contributions will enter with opposite sign. Thus, the contribution of a pole $z = re^{j\theta}$ would be the negative of the curves in Figures 5.8 and 5.11. Instead of dipping toward zero ($-\infty$ dB), the magnitude function would peak around $\omega = \theta$. The dependence on r would be the same as for a zero; i.e., the closer r is to 1, the more peaked will be the contribution to the magnitude function. For stable and causal systems, there will, of course, be no poles outside the unit circle; i.e., r will always be less than 1.

5.3.2 Examples with Multiple Poles and Zeros

In this section, we illustrate the use of the results of Section 5.3.1 to determine the frequency response of systems with rational system functions.

Example 5.8 Second-Order IIR System

Consider the second-order system

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2z^{-2}}. \quad (5.71)$$

The difference equation satisfied by the input and output of the system is

$$y[n] - 2r\cos\theta y[n-1] + r^2y[n-2] = x[n].$$

Using the partial fraction expansion technique, we can show that the impulse response

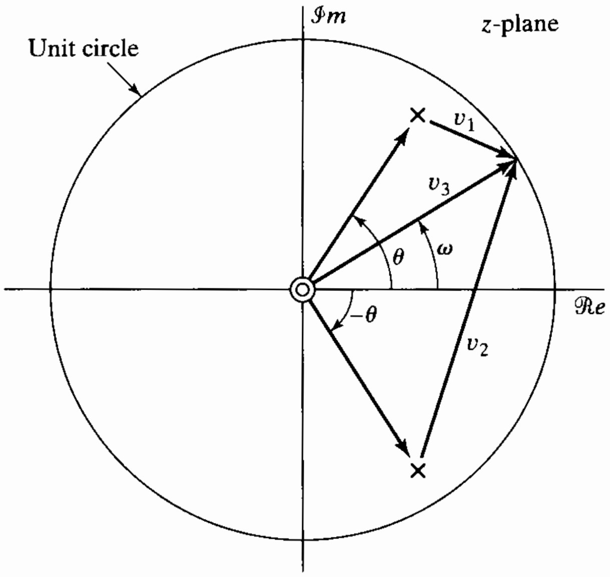


Figure 5.15 Pole–zero plot for Example 5.8.

of a causal system with this system function is

$$h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u[n]. \tag{5.72}$$

The system function in Eq. (5.71) has a pole at $z = re^{j\theta}$ and at the conjugate location, $z = re^{-j\theta}$, and two zeros at $z = 0$. The pole–zero plot is shown in Figure 5.15. From our discussion in Section 5.3.1,

$$\begin{aligned} 20\log_{10}|H(e^{j\omega})| = &- 10\log_{10}[1 + r^2 - 2r\cos(\omega - \theta)] \\ &- 10\log_{10}[1 + r^2 - 2r\cos(\omega + \theta)], \end{aligned} \tag{5.73a}$$

$$\sphericalangle H(e^{j\omega}) = -\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right] - \arctan\left[\frac{r\sin(\omega + \theta)}{1 - r\cos(\omega + \theta)}\right], \tag{5.73b}$$

and

$$\text{grd}[H(e^{j\omega})] = -\frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} - \frac{r^2 - r\cos(\omega + \theta)}{1 + r^2 - 2r\cos(\omega + \theta)}. \tag{5.73c}$$

These functions are plotted in Figure 5.16 for $r = 0.9$ and $\theta = \pi/4$.

Figure 5.15 shows the pole and zero vectors v_1, v_2, and v_3. The magnitude response is the product of the lengths of the zero vectors (which in this case are always unity), divided by the product of the lengths of the pole vectors. That is,

$$|H(e^{j\omega})| = \frac{|v_3|^2}{|v_1|\cdot|v_2|} = \frac{1}{|v_1|\cdot|v_2|}. \tag{5.74}$$

When $\omega \approx \theta$, the length of the vector $v_1 = e^{j\omega} - re^{j\theta}$ becomes small and changes significantly as ω varies about θ, while the length of the vector $v_2 = e^{j\omega} - re^{-j\theta}$ changes only slightly as ω varies around $\omega = \theta$. Thus, the pole at angle θ dominates the frequency response around $\omega = \theta$, as is evident from Figure 5.16. By symmetry, the pole at angle $-\theta$ dominates the frequency response around $\omega = -\theta$.

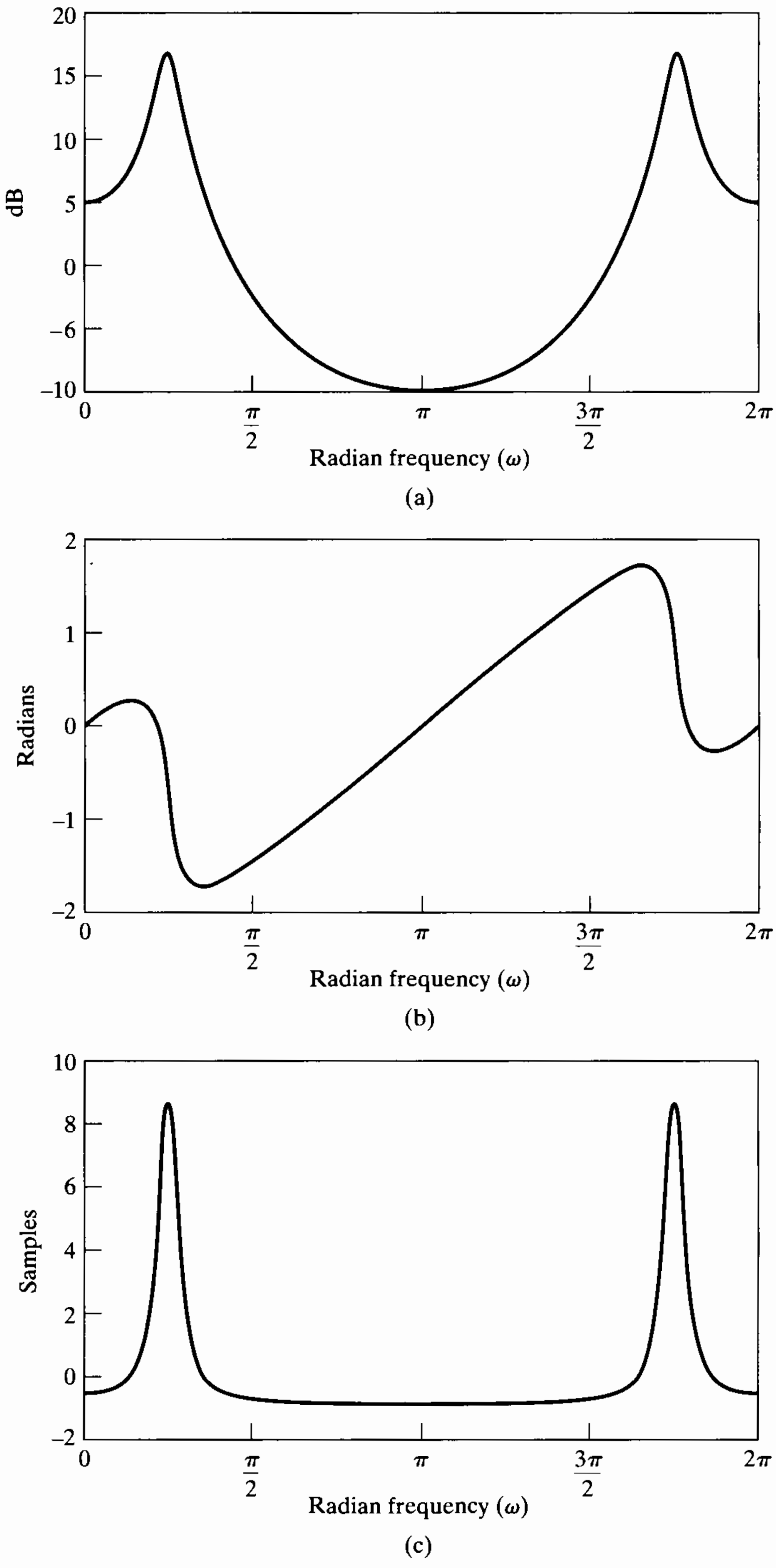


Figure 5.16 Frequency response for a complex-conjugate pair of poles as in Example 5.8, with $r = 0.9, \pi/4$. (a) Log magnitude. (b) Phase. (c) Group delay.

Example 5.9 Second-Order FIR System

Consider an FIR system whose impulse response is

$$h[n] = \delta[n] - 2r\cos\theta\delta[n-1] + r^2\delta[n-2]. \tag{5.75}$$

The corresponding system function is

$$H(z) = 1 - 2r\cos\theta z^{-1} + r^2 z^{-2}, \tag{5.76}$$

which is the reciprocal of the system function in Example 5.8. Therefore, the frequency-response plots for this FIR system are simply the negative of the plots in Figure 5.16. Note that the pole and zero locations are interchanged in the reciprocal.

Example 5.10 Third-Order IIR System

In this example, we consider a lowpass filter designed using one of the approximation methods to be described in Chapter 7. The system function to be considered is

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}, \tag{5.77}$$

and the system is specified to be stable. The zeros of this system function are at the following locations:

Radius	Angle
1	π rad
1	± 1.0376 rad (59.45°)

The poles are at the following locations:

Radius	Angle
0.683	0
0.892	± 0.6257 rad (35.85°)

The pole–zero plot for this system is shown in Figure 5.17. Figure 5.18 shows the

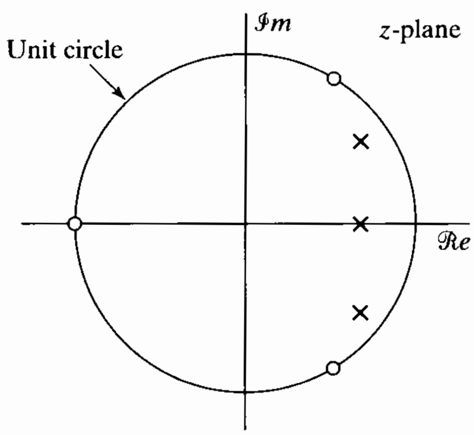


Figure 5.17 Pole–zero plot for the lowpass filter of Example 5.10.

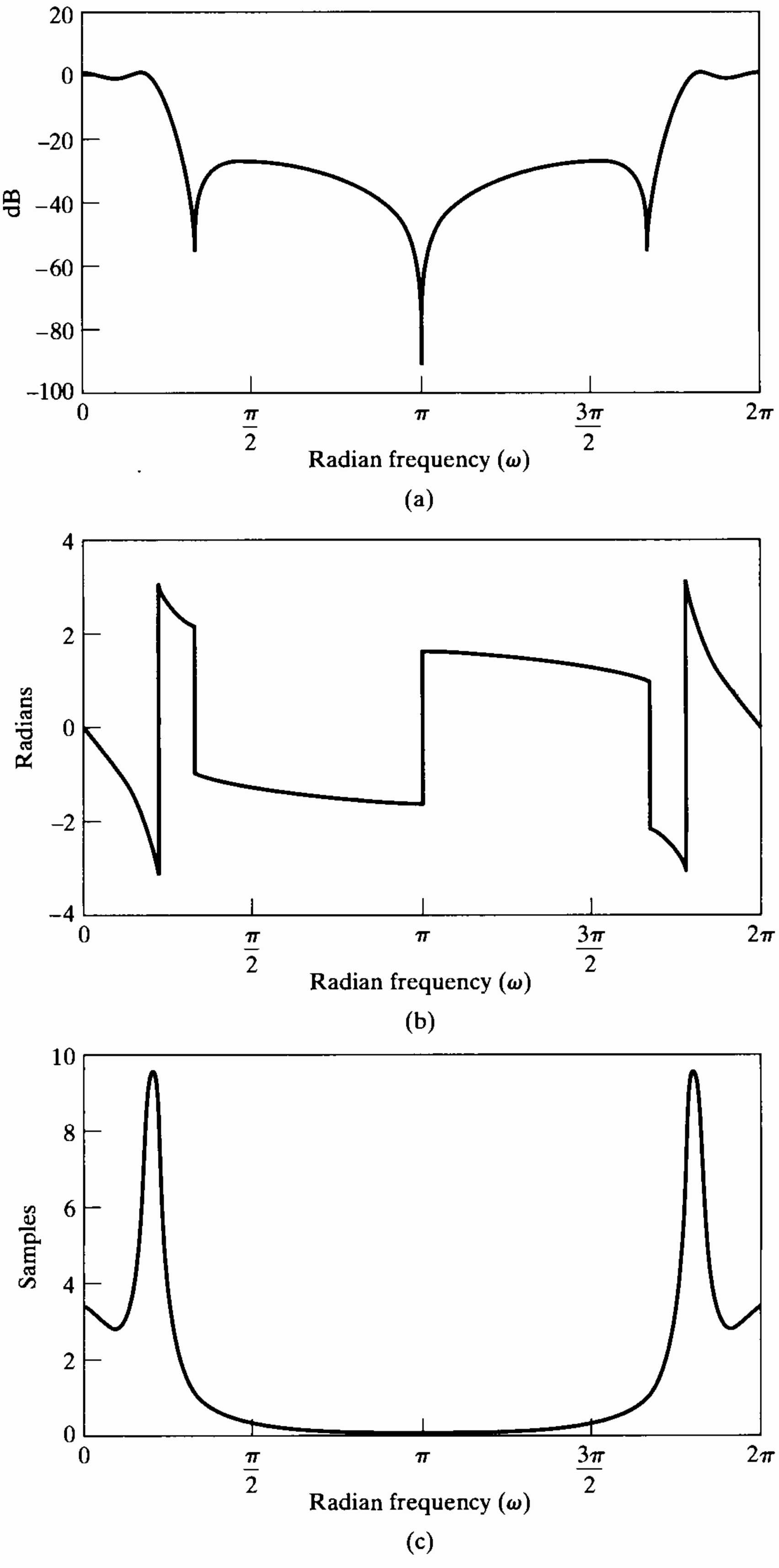


Figure 5.18 Frequency response for the lowpass filter of Example 5.10. (a) Log magnitude. (b) Phase. (c) Group delay.

log magnitude, phase, and group delay of the system. The effect of the zeros that are on the unit circle at $\omega = \pm 1.0376$ and π is clearly evident. However, the poles are placed so that, rather than peaking for frequencies close to their angles, the total log magnitude remains close to 0 dB over a band from $\omega = 0$ to $\omega = 0.2\pi$ (and, by symmetry, from $\omega = 1.8\pi$ to $\omega = 2\pi$), and then it drops abruptly and remains below −25 dB from about $\omega = 0.3\pi$ to 1.7π. As suggested by this example, useful approximations to frequency-selective filter responses can be achieved using poles to build up the magnitude response and zeros to suppress it.

In this example, we see both types of discontinuities in the plotted phase curve. At $\omega \approx 0.22\pi$, there is a discontinuity of 2π due to the use of the principal value in plotting. At $\omega = \pm 1.0376$ and $\omega = \pi$, the discontinuities of π are due to the zeros on the unit circle.

5.4 RELATIONSHIP BETWEEN MAGNITUDE AND PHASE

The frequency response of a linear time-invariant system is the Fourier transform of the impulse response. In general, knowledge about the magnitude provides no information about the phase, and vice versa. However, for systems described by linear constant-coefficient difference equations, i.e., rational system functions, there is some constraint between magnitude and phase. In particular, as we discuss in this section, if the magnitude of the frequency response and the number of poles and zeros are known, then there are only a finite number of choices for the associated phase. Similarly, if the number of poles and zeros and the phase are known, then, to within a scale factor, there are only a finite number of choices for the magnitude. Furthermore, under a constraint referred to as minimum phase, the frequency-response magnitude specifies the phase uniquely, and the frequency-response phase specifies the magnitude to within a scale factor.

To explore the possible choices of system function, given the square of the magnitude of the system frequency response, we consider $|H(e^{j\omega})|^2$ expressed as

$$\begin{aligned} |H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) \\ &= H(z)H^*(1/z^*)|_{z=e^{j\omega}}. \end{aligned} \tag{5.78}$$

Restricting the system function $H(z)$ to be rational in the form of Eq. (5.19), i.e.,

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \tag{5.79}$$

we see that $H^*(1/z^*)$ in Eq. (5.78) is

$$H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k^* z)}, \tag{5.80}$$

where we have assumed that a_0 and b_0 are real. Therefore, Eq. (5.78) states that the square of the magnitude of the frequency response is the evaluation on the unit circle of the z-transform

$$C(z) = H(z)H^*(1/z^*) \tag{5.81}$$

$$= \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}. \tag{5.82}$$

If we are given $|H(e^{j\omega})|^2$, then by replacing $e^{j\omega}$ by z, we can construct $C(z)$. From $C(z)$, we would like to infer as much as possible about $H(z)$. We first note that for each pole d_k of $H(z)$, there is a pole of $C(z)$ at d_k and $(d_k^*)^{-1}$. Similarly, for each zero c_k of $H(z)$, there is a zero of $C(z)$ at c_k and $(c_k^*)^{-1}$. Consequently, the poles and zeros of $C(z)$ occur in conjugate reciprocal pairs, with one element of each pair associated with $H(z)$ and one element of each pair associated with $H^*(1/z^*)$. Furthermore, if one element of each pair is inside the unit circle, then the other (i.e., the conjugate reciprocal) will be outside the unit circle. The only other alternative is for both to be on the unit circle in the same location.

If $H(z)$ is assumed to correspond to a causal, stable system, then all its poles must lie inside the unit circle. With this constraint, the poles of $H(z)$ can be identified from the poles of $C(z)$. However, with this constraint alone, the zeros of $H(z)$ cannot be uniquely identified from the zeros of $C(z)$. This can be seen from the following example.

Example 5.11 Systems with the Same $C(z)$

Consider two stable systems with system functions

$$H_1(z) = \frac{2(1 - z^{-1})(1 + 0.5z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} \tag{5.83}$$

and

$$H_2(z) = \frac{(1 - z^{-1})(1 + 2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}. \tag{5.84}$$

The pole–zero plots for these systems are shown in Figures 5.19(a) and 5.19(b), respectively.

Now,

$$C_1(z) = H_1(z)H_1^*(1/z^*)$$
$$= \frac{2(1 - z^{-1})(1 + 0.5z^{-1})2(1 - z)(1 + 0.5z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)} \tag{5.85}$$

and

$$C_2(z) = H_2(z)H_2^*(1/z^*)$$
$$= \frac{(1 - z^{-1})(1 + 2z^{-1})(1 - z)(1 + 2z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)}. \tag{5.86}$$

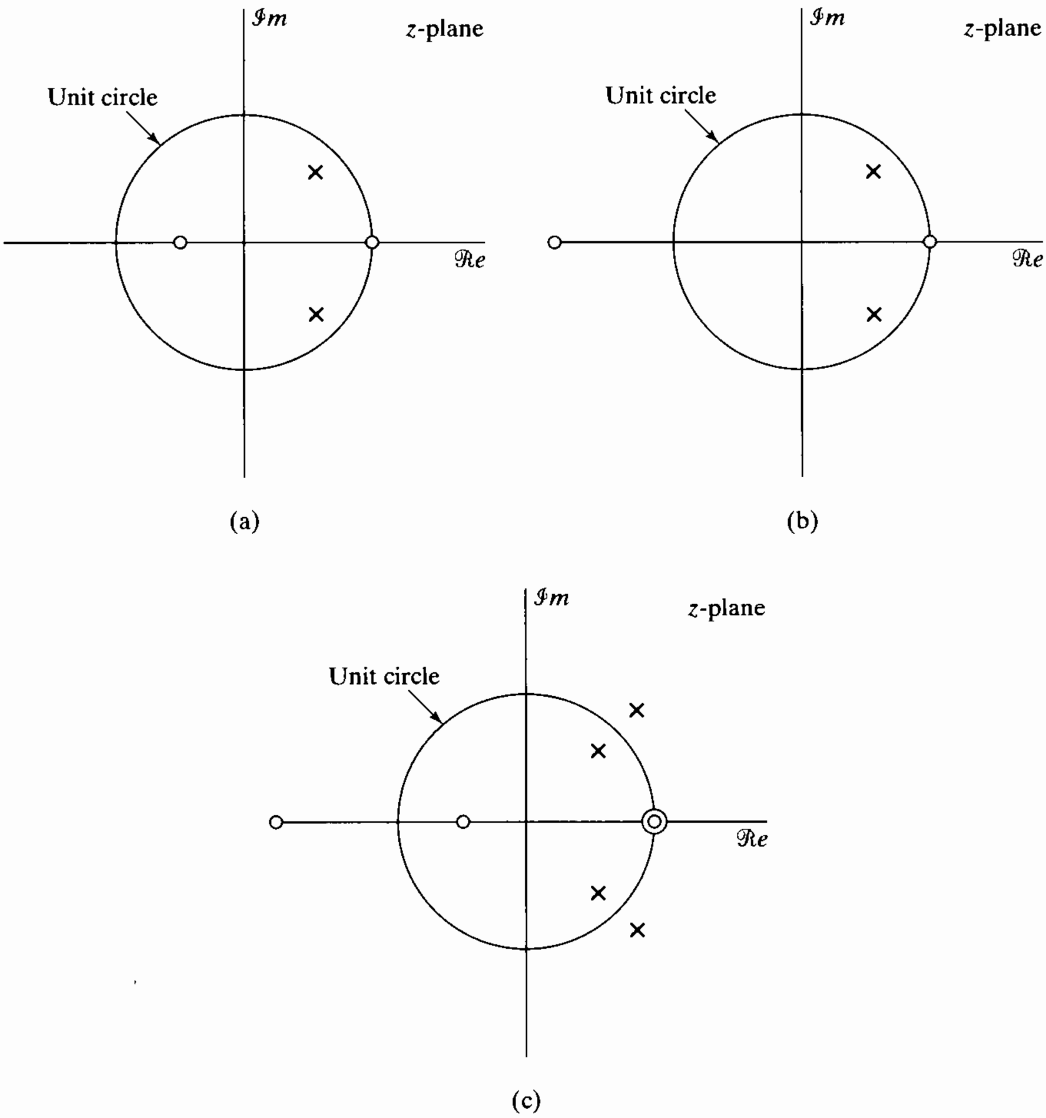


Figure 5.19 Pole–zero plots for two system functions and their common magnitude-squared function. (a) $H_1(z)$. (b) $H_2(z)$. (c) $C_1(z)$, $C_2(z)$.

Using the fact that

$$4(1 + 0.5z^{-1})(1 + 0.5z) = (1 + 2z^{-1})(1 + 2z), \tag{5.87}$$

we see that $C_1(z) = C_2(z)$. The pole–zero plot for $C_1(z)$ and $C_2(z)$ is shown in Figure 5.19(c).

The system functions $H_1(z)$ and $H_2(z)$ in Example 5.11 differ only in the location of the zeros. In the example, the factor $2(1 + 0.5z^{-1}) = (z^{-1} + 2)$ contributes the same to the square of the magnitude of the frequency response as the factor $(1 + 2z^{-1})$, and consequently, $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$ are equal. However, the phase functions for these two frequency responses are different.

Example 5.12

Suppose we are given the pole–zero plot for $C(z)$ in Figure 5.20 and want to determine the poles and zeros to associate with $H(z)$. The conjugate reciprocal pairs of poles and zeros for which one element of each is associated with $H(z)$ and one with $H^*(1/z^*)$ are as follows:

$$\begin{aligned}
\text{Pole pair 1}: &\quad (P_1, P_4) \\
\text{Pole pair 2}: &\quad (P_2, P_5) \\
\text{Pole pair 3}: &\quad (P_3, P_6) \\
\text{Zero pair 1}: &\quad (Z_1, Z_4) \\
\text{Zero pair 2}: &\quad (Z_2, Z_5) \\
\text{Zero pair 3}: &\quad (Z_3, Z_6)
\end{aligned}$$

Knowing that $H(z)$ corresponds to a stable, causal system, we must choose the poles from each pair that are inside the unit circle, i.e., P_1, P_2, and P_3. No such constraint is imposed on the zeros. However, if we assume that the coefficients a_k and b_k are real in Eqs. (5.16) and (5.18), the zeros (and poles) either are real or occur in complex conjugate pairs. Consequently, the zeros to associate with $H(z)$ are

$$Z_3 \quad \text{or} \quad Z_6$$

and

$$(Z_1, Z_2) \quad \text{or} \quad (Z_4, Z_5).$$

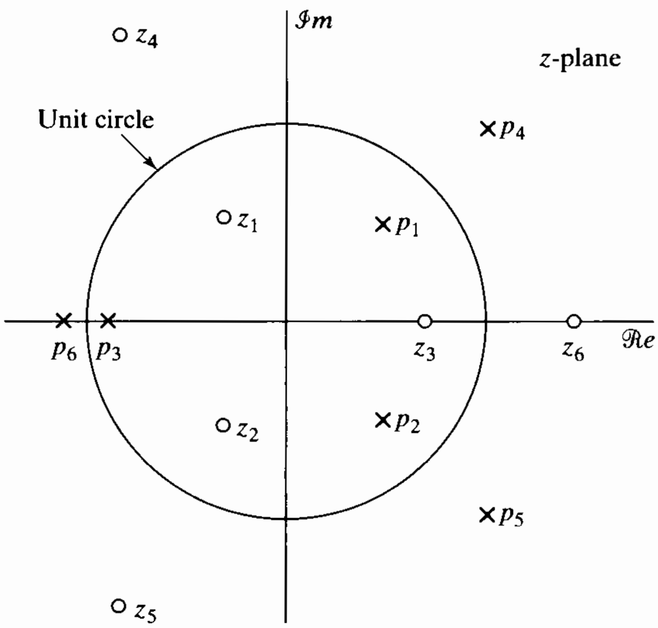


Figure 5.20 Pole–zero plot for the magnitude-squared function in Example 5.12.

Therefore, there are a total of four different stable, causal systems with three poles and three zeros for which the pole–zero plot of $C(z)$ is that shown in Figure 5.20 and, equivalently, for which the frequency-response magnitude is the same. If we had not assumed that the coefficients a_k and b_k were real, the number of choices would be greater. Furthermore, if the number of poles and zeros of $H(z)$ were not restricted, the number of choices for $H(z)$ would be unlimited. To see this, assume that $H(z)$ has a factor of the form

$$\frac{z^{-1} - a^*}{1 - az^{-1}},$$

i.e.,

$$H(z) = H_1(z)\frac{z^{-1} - a^*}{1 - az^{-1}}. \tag{5.88}$$

Factors of this form are referred to as *all-pass factors*, since they have unity magnitude on the unit circle; they are discussed in more detail in Section 5.5. It is easily verified that

$$C(z) = H(z)H^*(1/z^*) = H_1(z)H_1^*(1/z^*); \tag{5.89}$$

i.e., all-pass factors cancel in $C(z)$ and therefore would not be identifiable from the pole–zero plot of $C(z)$. Consequently, if the number of poles and zeros of $H(z)$ is unspecified, then, given $C(z)$, any choice for $H(z)$ can be cascaded with an arbitrary number of all-pass factors with poles inside the unit circle (i.e., $|a| < 1$).

5.5 ALL-PASS SYSTEMS

As indicated in the discussion of Example 5.12, a stable system function of the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \tag{5.90}$$

has a frequency-response magnitude that is independent of ω. This can be seen by writing $H_{ap}(e^{j\omega})$ in the form

$$\begin{aligned} H_{ap}(e^{j\omega}) &= \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \\ &= e^{-j\omega}\frac{1 - a^*e^{j\omega}}{1 - ae^{-j\omega}}. \end{aligned} \tag{5.91}$$

In Eq. (5.91), the term $e^{-j\omega}$ has unity magnitude, and the remaining numerator and denominator factors are complex conjugates of each other and therefore have the same magnitude. Consequently, $|H_{ap}(e^{j\omega})| = 1$. A system for which the frequency-response magnitude is a constant is called an *all-pass system,* since the system passes all of the frequency components of its input with constant gain or attenuation. The most general form for the system function of an all-pass system with a real-valued impulse response is a product of factors like Eq. (5.90), with complex poles being paired with their conjugates; i.e.,

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1} - d_k}{1 - d_kz^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_kz^{-1})(1 - e_k^*z^{-1})}, \tag{5.92}$$

where A is a positive constant and the d_k's are the real poles, and the e_k's the complex poles, of $H_{ap}(z)$. For causal and stable all-pass systems, $|d_k| < 1$ and $|e_k| < 1$. In terms

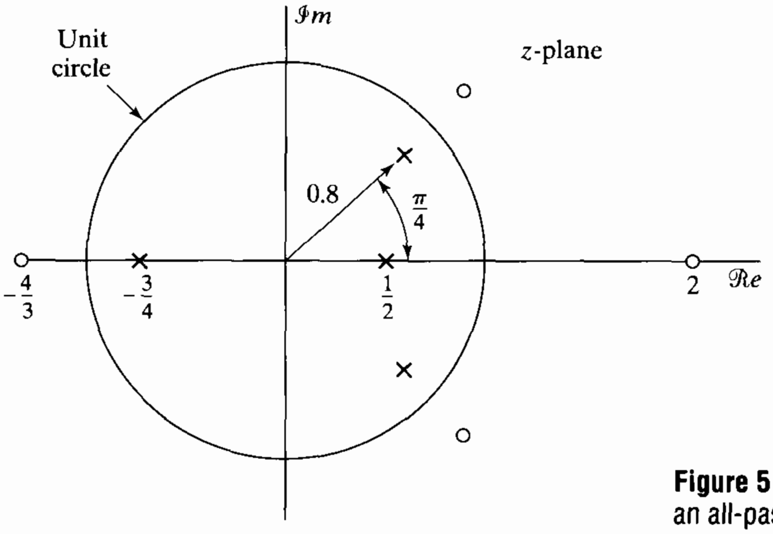


Figure 5.21 Typical pole–zero plot for an all-pass system.

of our general notation for system functions, all-pass systems have $M = N = 2M_c + M_r$ poles and zeros. Figure 5.21 shows a typical pole–zero plot for an all-pass system. In this case $M_r = 2$ and $M_c = 1$. Note that each pole of $H_{\text{ap}}(z)$ is paired with a conjugate reciprocal zero.

The frequency response for a general all-pass system can be expressed in terms of the frequency responses of first-order all-pass systems like that specified in Eq. (5.90). For a causal all-pass system, each of these terms consists of a single pole inside the unit circle and a zero at the conjugate reciprocal location. The magnitude response for such a term is, as we have shown, unity. Thus, the log magnitude in dB is zero. With a expressed in polar form as $a = re^{j\theta}$, the phase function for Eq. (5.90) is

$$\sphericalangle\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = -\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]. \tag{5.93}$$

Likewise, the phase of a second-order all-pass system with poles at $z = re^{j\theta}$ and $z = re^{-j\theta}$ is

$$\begin{aligned}\sphericalangle\left[\frac{(e^{-j\omega} - re^{-j\theta})(e^{-j\omega} - re^{j\theta})}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}\right] &= -2\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right] \\ &\quad -2\arctan\left[\frac{r\sin(\omega+\theta)}{1 - r\cos(\omega+\theta)}\right].\end{aligned} \tag{5.94}$$

Example 5.13 First- and Second-Order All-Pass Systems

Figure 5.22 shows plots of the log magnitude, phase, and group delay for two first-order all-pass systems, one with a real pole at $z = 0.9$ ($\theta = 0, r = 0.9$) and another with a pole at $z = -0.9$ ($\theta = \pi, r = 0.9$). For both systems, the radii of the poles are $r = 0.9$. Likewise, Figure 5.23 shows the same functions for a second-order all-pass system with poles at $z = 0.9e^{j\pi/4}$ and $z = 0.9e^{-j\pi/4}$.

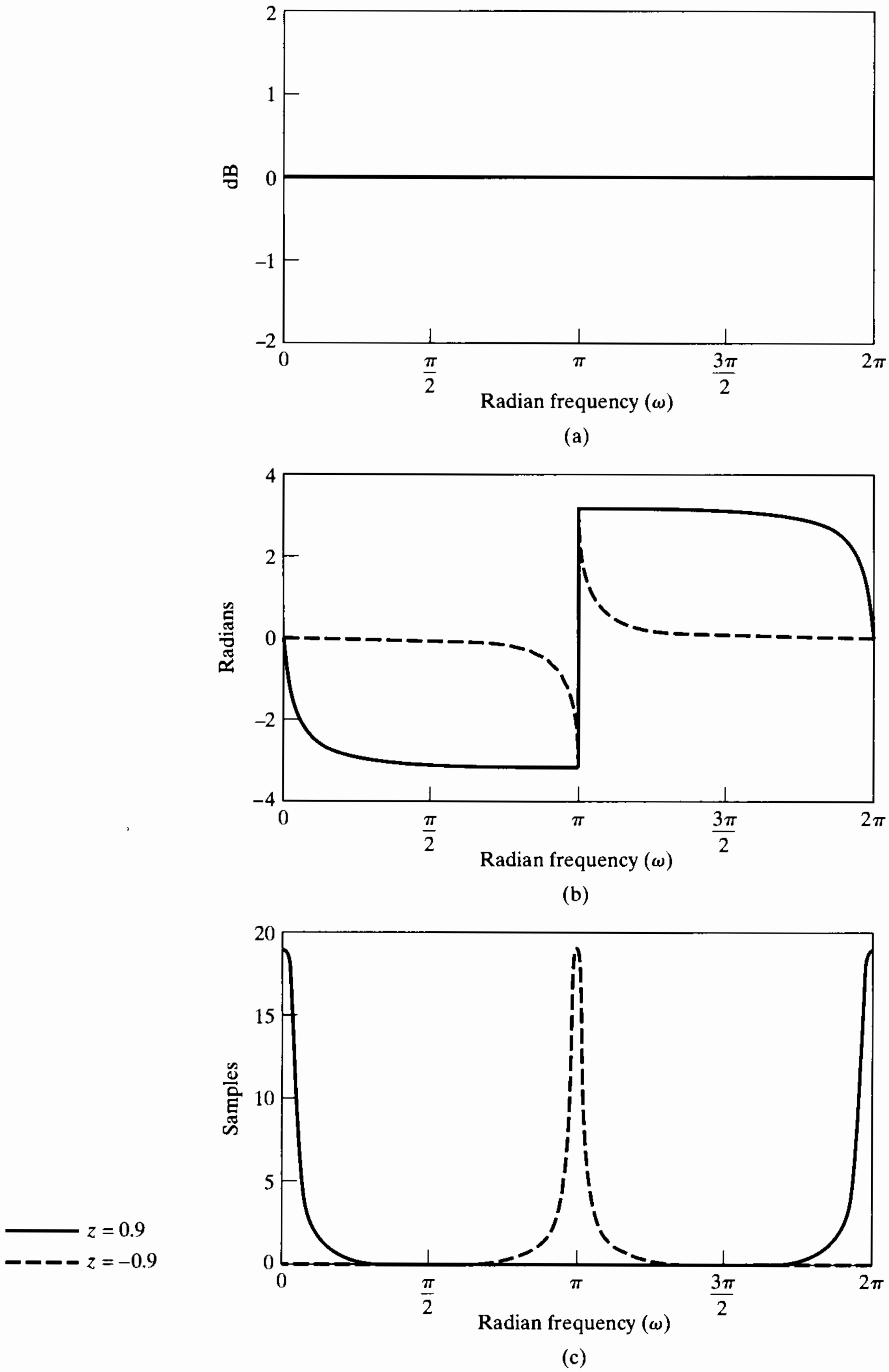


Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

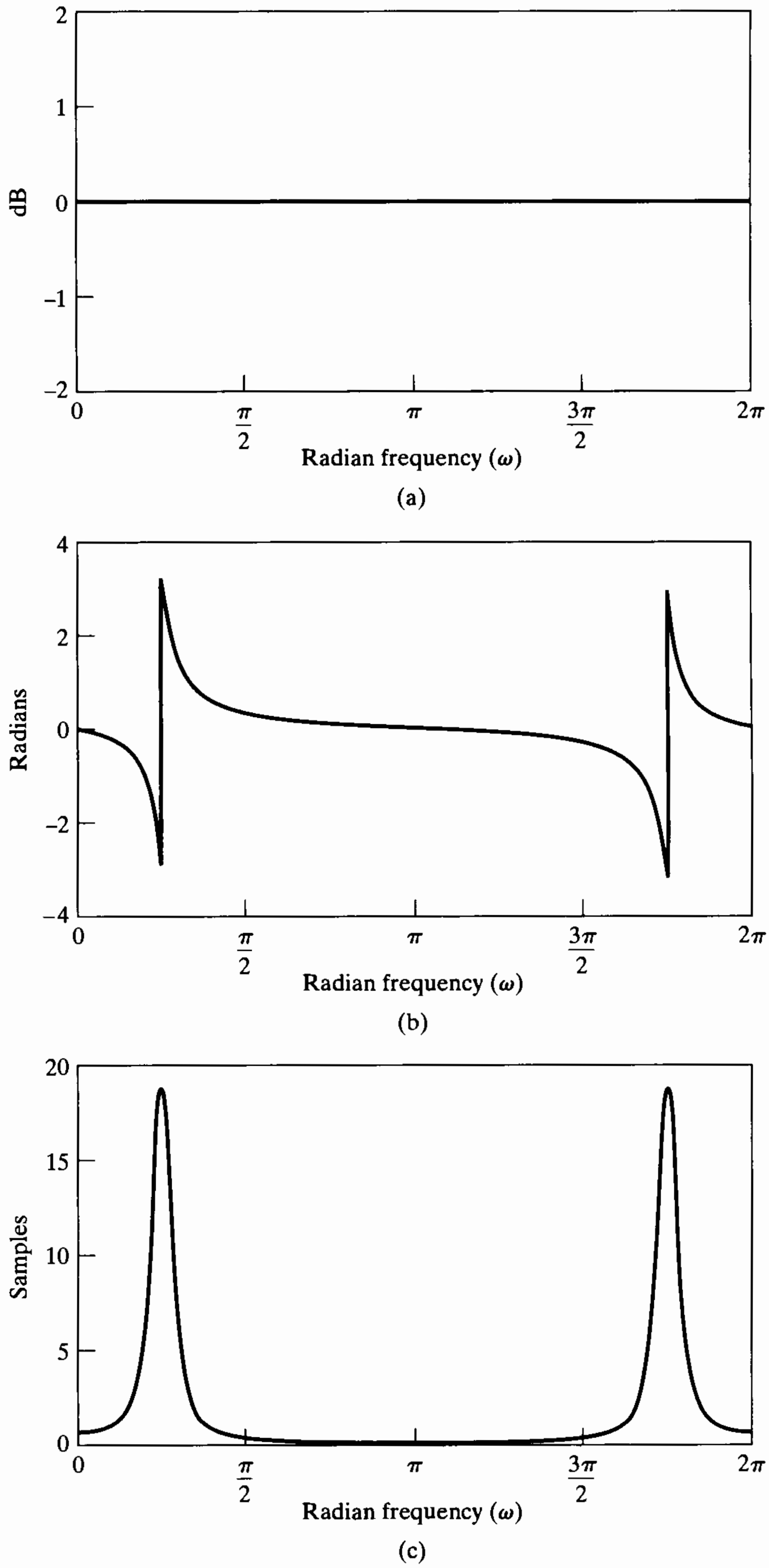


Figure 5.23 Frequency response of second-order all-pass system with poles at $z = 0.9e^{\pm j\pi/4}$. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

Example 5.13 illustrates a general property of causal all-pass systems. In Figure 5.22(b), we see that the phase is nonpositive for $0 < \omega < \pi$. Similarly, in Figure 5.23(b), if the discontinuity of 2π resulting from the computation of the principal value is removed, the resulting continuous-phase curve is nonpositive for $0 < \omega < \pi$. Since the more general all-pass system given by Eq. (5.92) is just a product of such first- and second-order factors, we can conclude that the (continuous) phase, $\arg[H_{\text{ap}}(e^{j\omega})]$, of a causal all-pass system is always nonpositive for $0 < \omega < \pi$. This may not appear to be true if the principal value is plotted, as is illustrated in Figure 5.24, which shows the log magnitude, phase, and group delay for an all-pass system with poles and zeros as in Figure 5.21. However, we can establish this result by first considering the group delay.

The group delay of the simple one-pole all-pass system of Eq. (5.90) is the negative derivative of the phase given by Eq. (5.93). With a small amount of algebra, it can be shown that

$$\text{grd}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{1 - r^2}{|1 - re^{j\theta}e^{-j\omega}|^2}. \tag{5.95}$$

Since $r < 1$ for a stable and causal all-pass system, from Eq. (5.95) the group delay contributed by a single causal all-pass factor is always positive. Since the group delay of a higher order all-pass system will be a sum of positive terms, as in Eq. (5.95), it is true in general that the group delay of a causal rational all-pass system is always positive. This is confirmed by Figures 5.22(c), 5.23(c), and 5.24(c), which show the group delay for first-order, second-order, and third-order all-pass systems, respectively.

The positivity of the group delay of a causal all-pass system is the basis for a simple proof of the negativity of the phase of such a system. First, note that

$$\arg[H_{\text{ap}}(e^{j\omega})] = -\int_0^{\omega} \text{grd}[H_{\text{ap}}(e^{j\phi})]d\phi + \arg[H_{\text{ap}}(e^{j0})] \tag{5.96}$$

for $0 \le \omega \le \pi$. From Eq. (5.92), it follows that

$$H_{\text{ap}}(e^{j0}) = A\prod_{k=1}^{M_r}\frac{1 - d_k}{1 - d_k}\prod_{k=1}^{M_c}\frac{|1 - e_k|^2}{|1 - e_k|^2} = A. \tag{5.97}$$

Therefore, $\arg[H_{\text{ap}}(e^{j0})] = 0$, and since

$$\text{grd}[H_{\text{ap}}(e^{j\omega})] \ge 0, \tag{5.98}$$

it follows from Eq. (5.96) that

$$\arg[H_{\text{ap}}(e^{j\omega})] \le 0 \qquad \text{for } 0 \le \omega < \pi. \tag{5.99}$$

The positivity of the group delay and the nonpositivity of the continuous phase are important properties of causal all-pass systems.

All-pass systems have many uses. They can be used as compensators for phase (or group delay) distortion, as we will see in Chapter 7, and they are useful in the theory of minimum-phase systems, as we will see in Section 5.6. They are also useful in transforming frequency-selective lowpass filters into other frequency-selective forms and in obtaining variable-cutoff frequency-selective filters. These applications are discussed in Chapter 7 and applied in the problems in that chapter.

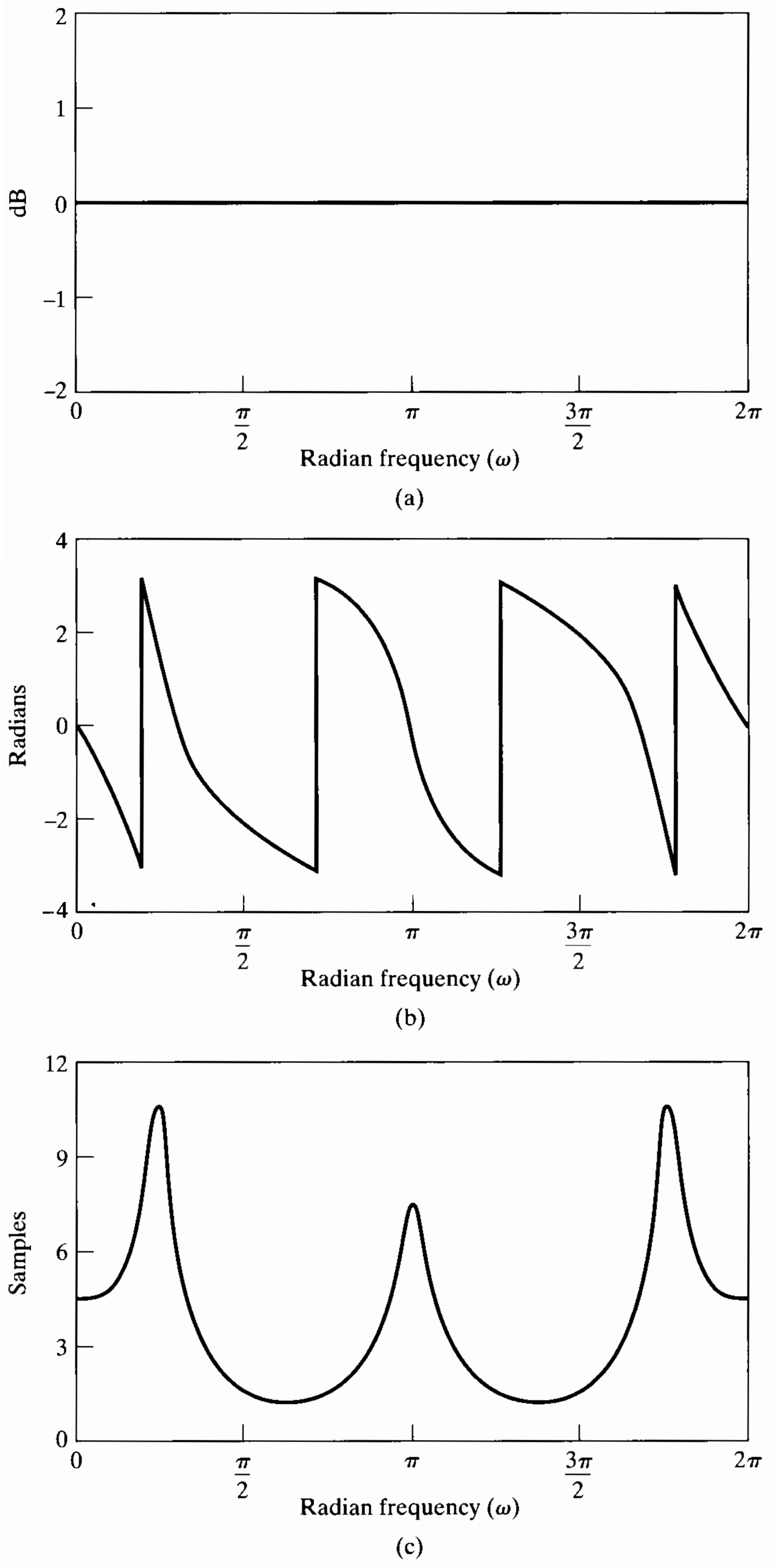


Figure 5.24 Frequency response for an all-pass system with the pole–zero plot in Figure 5.21. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

5.6 MINIMUM-PHASE SYSTEMS

In Section 5.4, we showed that the frequency-response magnitude for an LTI system with rational system function does not uniquely characterize the system. If the system is stable and causal, the poles must be inside the unit circle, but stability and causality place no such restriction on the zeros. For certain classes of problems, it is useful to impose the additional restriction that the inverse system (one with system function $1/H(z)$) also be stable and causal. As discussed in Section 5.2.2, this then restricts the zeros, as well as the poles, to be inside the unit circle, since the poles of $1/H(z)$ are the zeros of $H(z)$. Such systems are commonly referred to as *minimum-phase* systems. The name *minimum-phase* comes from a property of the phase response, which is not obvious from the preceding definition. This and other fundamental properties that we discuss are unique to this class of systems, and therefore, any one of them could be taken as the definition of the class. These properties are developed in Section 5.6.3.

If we are given a magnitude-squared function in the form of Eq. (5.82) and we know that the system is a minimum-phase system, then $H(z)$ is uniquely determined and will consist of all the poles and zeros of $C(z) = H(z)H^*(1/z^*)$ that lie inside the unit circle.[1] This approach is often followed in filter design when only the magnitude response is determined by the design method used. (See Chapter 7.)

5.6.1 Minimum-Phase and All-Pass Decomposition

In Section 5.4 we showed that, from the square of the magnitude of the frequency response alone, we could not uniquely determine the system function $H(z)$, since any choice that had the given frequency-response magnitude could be cascaded with arbitrary all-pass factors without affecting the magnitude. A related observation is that any rational system function[2] can be expressed as

$$H(z) = H_{\text{min}}(z)H_{\text{ap}}(z), \tag{5.100}$$

where $H_{\text{min}}(z)$ is a minimum-phase system and $H_{\text{ap}}(z)$ is an all-pass system.

To show this, suppose that $H(z)$ has one zero outside the unit circle at $z = 1/c^*$, where $|c| < 1$, and the remaining poles and zeros are inside the unit circle. Then $H(z)$ can be expressed as

$$H(z) = H_1(z)(z^{-1} - c^*), \tag{5.101}$$

where, by definition, $H_1(z)$ is minimum phase. An equivalent expression for $H(z)$ is

$$H(z) = H_1(z)(1 - cz^{-1})\frac{z^{-1} - c^*}{1 - cz^{-1}}. \tag{5.102}$$

Since $|c| < 1$, the factor $H_1(z)(1 - cz^{-1})$ also is minimum phase, and it differs from $H(z)$ only in that the zero of $H(z)$ that was outside the unit circle at $z = 1/c^*$ is reflected inside

[1] We have assumed that $C(z)$ has no poles or zeros on the unit circle. Strictly speaking, systems with poles on the unit circle are unstable and are generally to be avoided in practice. Zeros on the unit circle, however, often occur in practical filter designs. By our definition, such systems are nonminimum phase, but many of the properties of minimum-phase systems hold even in this case.

[2] Somewhat for convenience, we will restrict the discussion to stable, causal systems, although the observation applies more generally.

the unit circle to the conjugate reciprocal location $z = c$. The term $(z^{-1} - c^*)/(1 - cz^{-1})$ is all-pass. This example can be generalized in a straightforward way to include more zeros outside the unit circle, thereby showing that, in general, any system function can be expressed as

$$H(z) = H_{\text{min}}(z)H_{\text{ap}}(z), \tag{5.103}$$

where $H_{\text{min}}(z)$ contains the poles and zeros of $H(z)$ that lie inside the unit circle, plus zeros that are the conjugate reciprocals of the zeros of $H(z)$ that lie outside the unit circle. $H_{\text{ap}}(z)$ is comprised of all the zeros of $H(z)$ that lie outside the unit circle, together with poles to cancel the reflected conjugate reciprocal zeros in $H_{\text{min}}(z)$.

Using Eq. (5.103), we can form a nonminimum-phase system from a minimum-phase system by reflecting one or more zeros lying inside the unit circle to their conjugate reciprocal locations outside the unit circle, or, conversely, we can form a minimum-phase system from a nonminimum-phase system by reflecting all the zeros lying outside the unit circle to their conjugate reciprocal locations inside. In either case, both the minimum-phase and the nonminimum-phase systems will have the same frequency-response magnitude.

Example 5.14 Minimum-Phase/All-Pass Decomposition

To illustrate the decomposition of a stable, causal system into the cascade of a minimum-phase and an all-pass system, consider the two stable, causal systems specified by the system functions

$$H_1(z) = \frac{(1 + 3z^{-1})}{1 + \frac{1}{2}z^{-1}}$$

and

$$H_2(z) = \frac{\left(1 + \frac{3}{2}e^{+j\pi/4}z^{-1}\right)\left(1 + \frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)}.$$

The first system function, $H_1(z)$, has a pole inside the unit circle at $z = -\frac{1}{2}$, but a zero outside at $z = -3$. We will need to choose the appropriate all-pass system to reflect this zero inside the unit circle. From Eq. (5.101), we have $c = -\frac{1}{3}$. Therefore, from Eqs. (5.102) and (5.103), the all-pass component will be

$$H_{\text{ap}}(z) = \frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{3}z^{-1}},$$

and the minimum-phase component will be

$$H_{\text{min}}(z) = 3\frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1}};$$

i.e.,

$$H_1(z) = \left(3\frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{3}z^{-1}}\right).$$

The second system function, $H_2(z)$, has two complex zeros outside the unit circle and a real pole inside. We can express $H_2(z)$ in the form of Eq. (5.101) by factoring $\frac{3}{2}e^{j\pi/4}$ and $\frac{3}{2}e^{-j\pi/4}$ out of the numerator terms to get

$$H_2(z) = \frac{9}{4}\frac{\left(z^{-1} + \frac{2}{3}e^{-j\pi/4}\right)\left(z^{-1} + \frac{2}{3}e^{j\pi/4}\right)}{1 - \frac{1}{3}z^{-1}}.$$

Factoring as in Eq. (5.102) yields

$$H_2(z) = \left[\frac{9}{4}\frac{\left(1 + \frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1 + \frac{2}{3}e^{j\pi/4}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}}\right]$$
$$\times\left[\frac{\left(z^{-1} + \frac{2}{3}e^{-j\pi/4}\right)\left(z^{-1} + \frac{2}{3}e^{j\pi/4}\right)}{\left(1 + \frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1 + \frac{2}{3}e^{j\pi/4}z^{-1}\right)}\right].$$

The first term in square brackets is a minimum-phase system, while the second term is an all-pass system.

5.6.2 Frequency-Response Compensation

In many signal-processing contexts, a signal has been distorted by an LTI system with an undesirable frequency response. It may then be of interest to process the distorted signal with a compensating system, as indicated in Figure 5.25. This situation may arise, for example, in transmitting signals over a communication channel. If perfect compensation is achieved, then $s_c[n] = s[n]$, i.e., $H_c(z)$ is the inverse of $H_d(z)$. However, if we assume that the distorting system is stable and causal and require the compensating system to be stable and causal, then perfect compensation is possible only if $H_d(z)$ is a minimum-phase system, so that it has a stable, causal inverse.

Based on the previous discussions, assuming that $H_d(z)$ is known or approximated as a rational system function, we can form a minimum-phase system $H_{d\min}(z)$ by reflecting all the zeros of $H_d(z)$ that are outside the unit circle to their conjugate reciprocal locations inside the unit circle. $H_d(z)$ and $H_{d\min}(z)$ have the same frequency-response magnitude and are related through an all-pass system $H_{\text{ap}}(z)$, i.e.,

$$H_d(z) = H_{d\min}(z)H_{\text{ap}}(z). \tag{5.104}$$

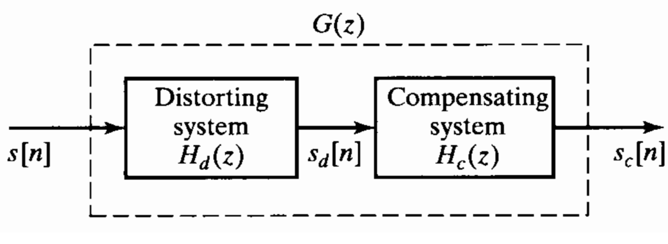


Figure 5.25 Illustration of distortion compensation by linear filtering.

Choosing the compensating filter to be

$$H_c(z) = \frac{1}{H_{d\,\min}(z)}, \tag{5.105}$$

we find that the overall system function relating $s[n]$ and $s_c[n]$ is

$$G(z) = H_d(z)H_c(z) = H_{\text{ap}}(z); \tag{5.106}$$

i.e., $G(z)$ corresponds to an all-pass system. Consequently, the frequency-response magnitude is exactly compensated for, while the phase response is modified to $\sphericalangle H_{\text{ap}}(e^{j\omega})$.

The following example illustrates compensation of the frequency response magnitude when the system to be compensated for is a nonminimum-phase FIR system.

Example 5.15 Compensation of an FIR System

Consider the distorting system function to be

$$\begin{aligned} H_d(z) &= (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1}) \\ &\quad \times (1 - 1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1}). \end{aligned} \tag{5.107}$$

The pole–zero plot is shown in Figure 5.26. Since $H_d(z)$ has only zeros (all poles are at $z = 0$), it follows that the system has a finite-duration impulse response. Therefore the system is stable; and since $H_d(z)$ is a polynomial with only negative powers of z, the system is causal. However, since two of the zeros are outside the unit circle, the system is nonminimum phase. Figure 5.27 shows the log magnitude, phase, and group delay for $H_d(e^{j\omega})$.

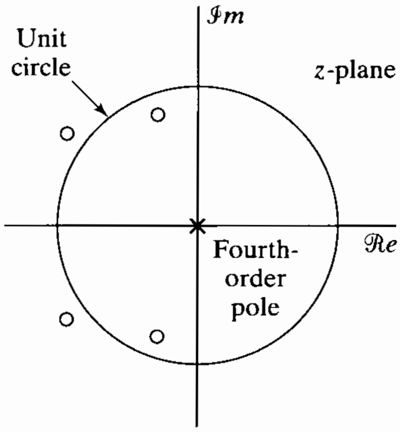


Figure 5.26 Pole–zero plot of FIR system in Example 5.15.

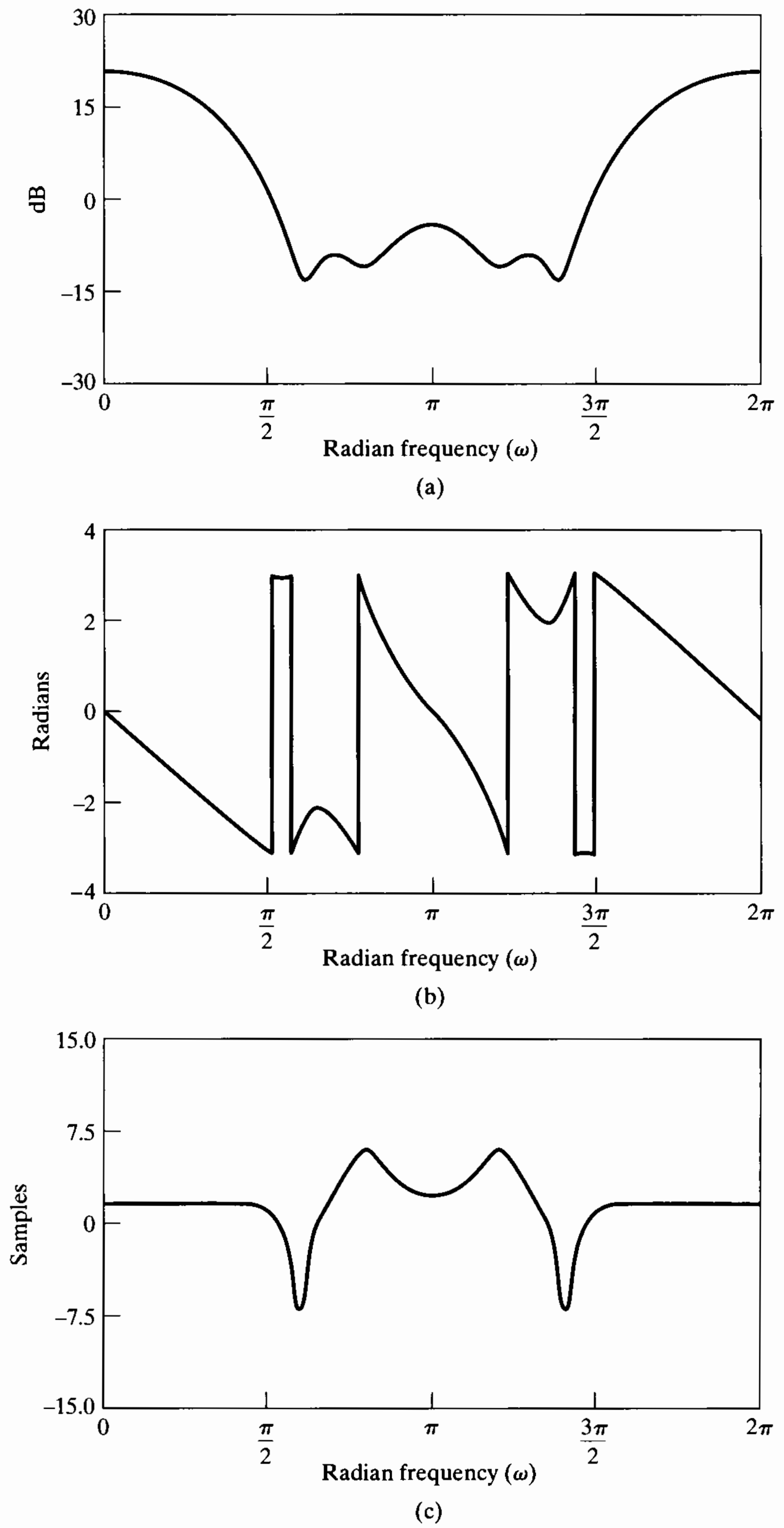


Figure 5.27 Frequency response for FIR system with pole–zero plot in Figure 5.26. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

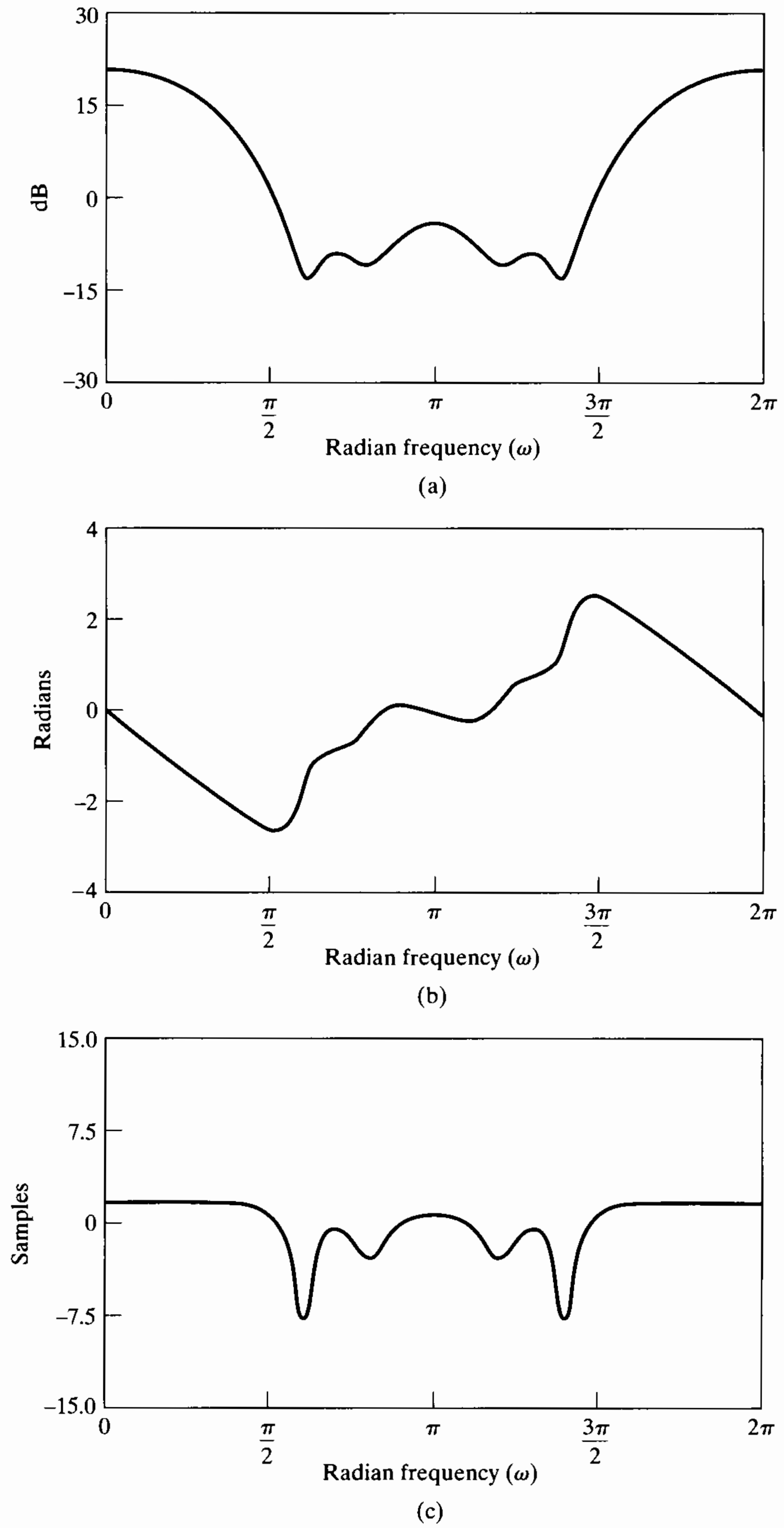


Figure 5.28 Frequency response for minimum-phase system in Example 5.15. (a) Log magnitude. (b) Phase. (c) Group delay.

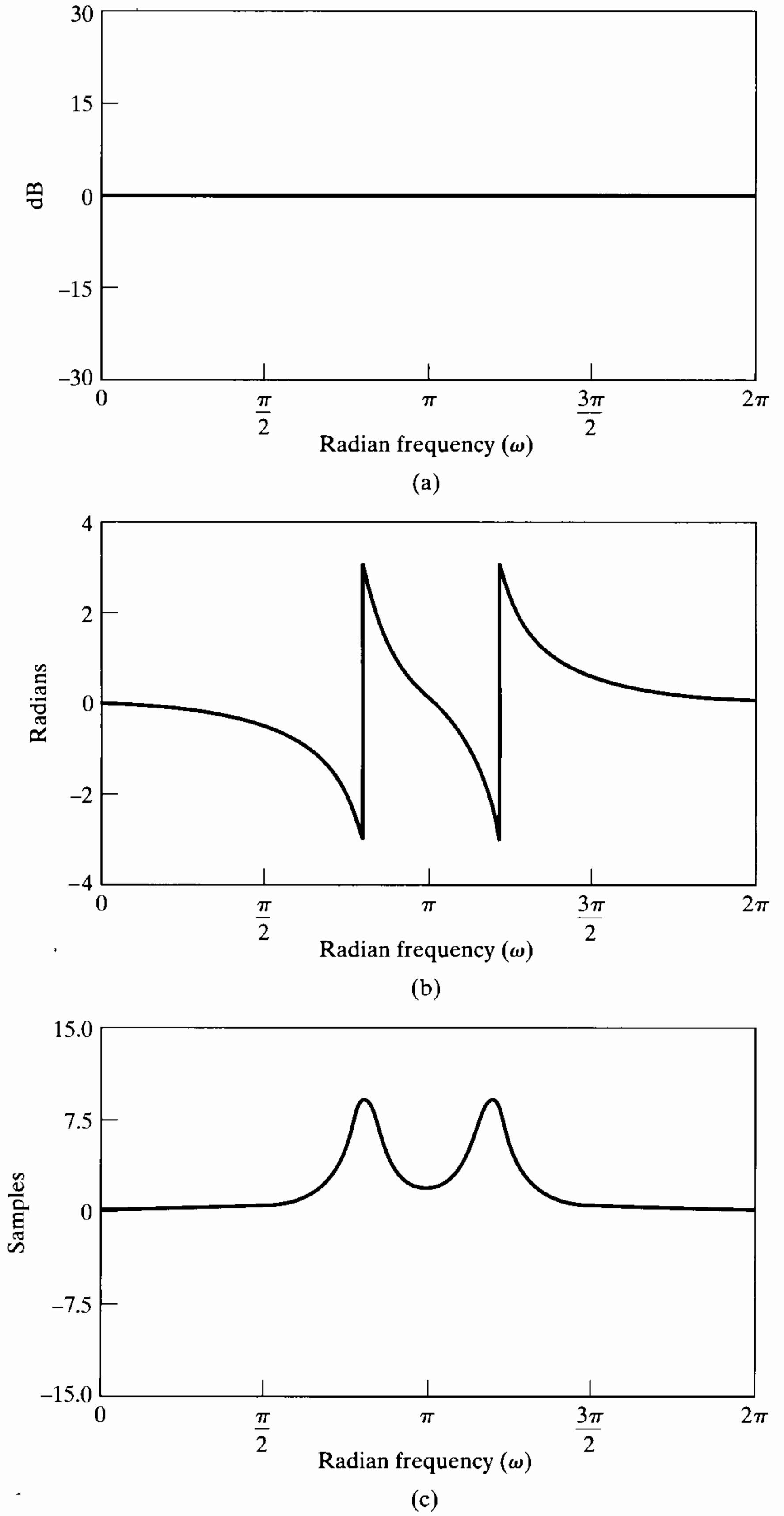


Figure 5.29 Frequency response of all-pass system of Example 5.15. (The sum of corresponding curves in Figures 5.28 and 5.29 equals the corresponding curve in Figure 5.27 with the sum of the phase curves taken modulo 2π.) (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

The corresponding minimum-phase system is obtained by reflecting the zeros that occur at $z = 1.25e^{\pm j0.8\pi}$ to their conjugate reciprocal locations inside the unit circle. If we express $H_d(z)$ as

$$\begin{aligned} H_d(z) = {} & (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1})(1.25)^2 \\ & \times (z^{-1} - 0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi}), \end{aligned} \tag{5.108}$$

then

$$\begin{aligned} H_{\min}(z) = {} & (1.25)^2(1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1}) \\ & \times (1 - 0.8e^{-j0.8\pi}z^{-1})(1 - 0.8e^{j0.8\pi}z^{-1}), \end{aligned} \tag{5.109}$$

and the all-pass system that relates $H_{\min}(z)$ and $H_d(z)$ is

$$H_{ap}(z) = \frac{(z^{-1} - 0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi})}{(1 - 0.8e^{j0.8\pi}z^{-1})(1 - 0.8e^{-j0.8\pi}z^{-1})}. \tag{5.110}$$

The log magnitude, phase, and group delay of $H_{\min}(e^{j\omega})$ are shown in Figure 5.28. Figures 5.27(a) and 5.28(a) are, of course, identical. The log magnitude, phase, and group delay for $H_{ap}(e^{j\omega})$ are plotted in Figure 5.29.

Note that the inverse system for $H_d(z)$ would have poles at $z = 1.25e^{\pm j0.8\pi}$ and at $z = 0.9e^{\pm j0.6\pi}$, and thus, the causal inverse would be unstable. The minimum-phase inverse would be the reciprocal of $H_{\min}(z)$, as given by Eq. (5.109), and if this inverse were used in the cascade system of Figure 5.25, the overall effective system function would be $H_{ap}(z)$, as given in Eq. (5.110).

5.6.3 Properties of Minimum-Phase Systems

We have been using the term "minimum phase" to refer to systems that are causal and stable and that have a causal and stable inverse. This choice of name is motivated by a property of the phase function that, while not obvious, follows from our chosen definition. In this section, we develop a number of interesting and important properties of minimum-phase systems relative to all other systems that have the same frequency-response magnitude.

The Minimum Phase-Lag Property

The use of the terminology "minimum phase" as a descriptive name for a system having all its poles and zeros inside the unit circle is suggested by Example 5.15. Recall that, as a consequence of Eq. (5.100), the continuous phase, i.e., $\arg[H(e^{j\omega})]$, of any nonminimum-phase system can be expressed as

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]. \tag{5.111}$$

Therefore, the continuous phase that would correspond to the principal-value phase of Figure 5.27(b) is the sum of the continuous phase associated with the minimum-phase function of Figure 5.28(b) and the continuous phase of the all-pass system associated with the principal-value phase shown in Figure 5.29(b). As was shown in Section 5.5, and as indicated by the principal-value phase curves of Figures 5.22(b), 5.23(b), 5.24(b), and 5.29(b), the continuous-phase curve of an all-pass system is negative for

$0 \leq \omega \leq \pi$. Thus, the reflection of zeros of $H_{\min}(z)$ from inside the unit circle to conjugate reciprocal locations outside always *decreases* the (continuous) phase or *increases* the negative of the phase, which is called the *phase-lag* function. Hence, the causal, stable system that has $|H_{\min}(e^{j\omega})|$ as its magnitude response and also has all its zeros (and, of course, poles) inside the unit circle has the minimum phase-lag function (for $0 \leq \omega < \pi$) of all the systems having that same magnitude response. Therefore, a more precise terminology is *minimum phase-lag* system, but *minimum phase* is historically the established terminology.

To make the interpretation of minimum phase-lag systems more precise, it is necessary to impose the additional constraint that $H(e^{j\omega})$ be positive at $\omega = 0$, i.e.,

$$H(e^{j0}) = \sum_{n=-\infty}^{\infty} h[n] > 0. \tag{5.112}$$

Note that $H(e^{j0})$ will be real if we restrict $h[n]$ to be real. The condition of Eq. (5.112) is necessary because a system with impulse response $-h[n]$ has the same poles and zeros for its system function as a system with impulse response $h[n]$. However, multiplying by -1 would alter the phase by π radians. Thus, to remove this ambiguity, we must impose the condition of Eq. (5.112) to ensure that a system with all its poles and zeros inside the unit circle also has the minimum phase-lag property. However, this constraint is often of little significance, and our definition at the beginning of Section 5.6, which does not include it, is the generally accepted definition of the class of minimum-phase systems.

The Minimum Group-Delay Property

Example 5.15 illustrates another property of systems whose poles and zeros are all inside the unit circle. First note that the group delay for the systems that have the same magnitude response is

$$\text{grd}[H(e^{j\omega})] = \text{grd}[H_{\min}(e^{j\omega})] + \text{grd}[H_{\text{ap}}(e^{j\omega})]. \tag{5.113}$$

The group delay for the minimum-phase system shown in Figure 5.28(c) is always less than the group delay for the nonminimum-phase system shown in Figure 5.27(c). This is because, as Figure 5.29(c) shows, the all-pass system that converts the minimum-phase system into the nonminimum-phase system has a positive group delay. In Section 5.5, we showed this to be a general property of all-pass systems; they always have positive group delay for all ω. Thus, if we again consider all the systems that have a given magnitude response $|H_{\min}(e^{j\omega})|$, the one that has all its poles and zeros inside the unit circle has the minimum group delay. An equally appropriate name for such systems would therefore be *minimum group-delay* systems, but this terminology is not generally used.

The Minimum Energy-Delay Property

In Example 5.15, there are a total of four causal FIR systems with real impulse responses that have the same frequency-response magnitude as the system in Eq. (5.107). The associated pole–zero plots are shown in Figure 5.30, where Figure 5.30(d) corresponds to Eq. (5.107) and Figure 5.30(a) to the minimum-phase system of Eq. (5.109). The impulse responses for these four cases are plotted in Figure 5.31. If we compare the

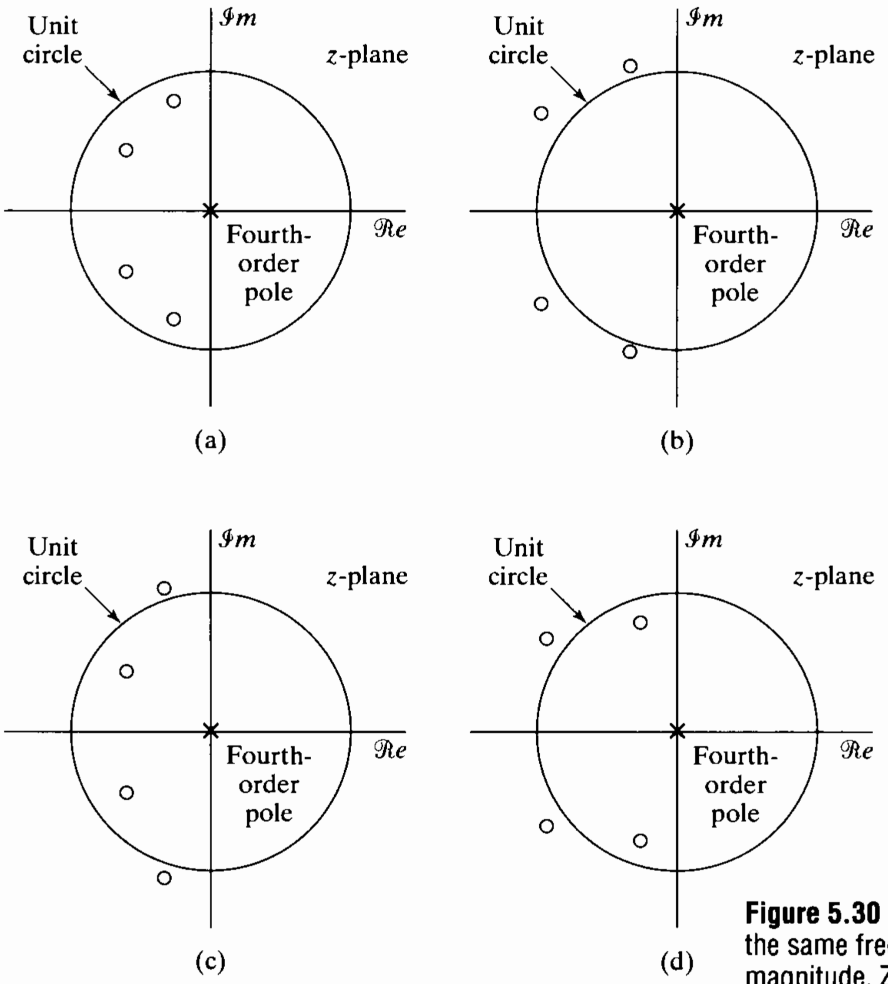


Figure 5.30 Four systems, all having the same frequency-response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals.

four sequences in this figure, we observe that the minimum-phase sequence appears to have larger samples at its left-hand end than do all the other sequences. Indeed, it is true for this example and, in general, that

$$|h[0]| \leq |h_{\min}[0]| \tag{5.114}$$

for any causal, stable sequence $h[n]$ for which

$$|H(e^{j\omega})| = |H_{\min}(e^{j\omega})|. \tag{5.115}$$

A proof of this property is suggested in Problem 5.65.

All the impulse responses whose frequency-response magnitude is equal to $|H_{\min}(e^{j\omega})|$ have the same total energy as $h_{\min}[n]$, since, by Parseval's theorem,

$$\begin{aligned}\sum_{n=0}^{\infty}|h[n]|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}|H_{\min}(e^{j\omega})|^2 d\omega \\ &= \sum_{n=0}^{\infty}|h_{\min}[n]|^2.\end{aligned} \tag{5.116}$$

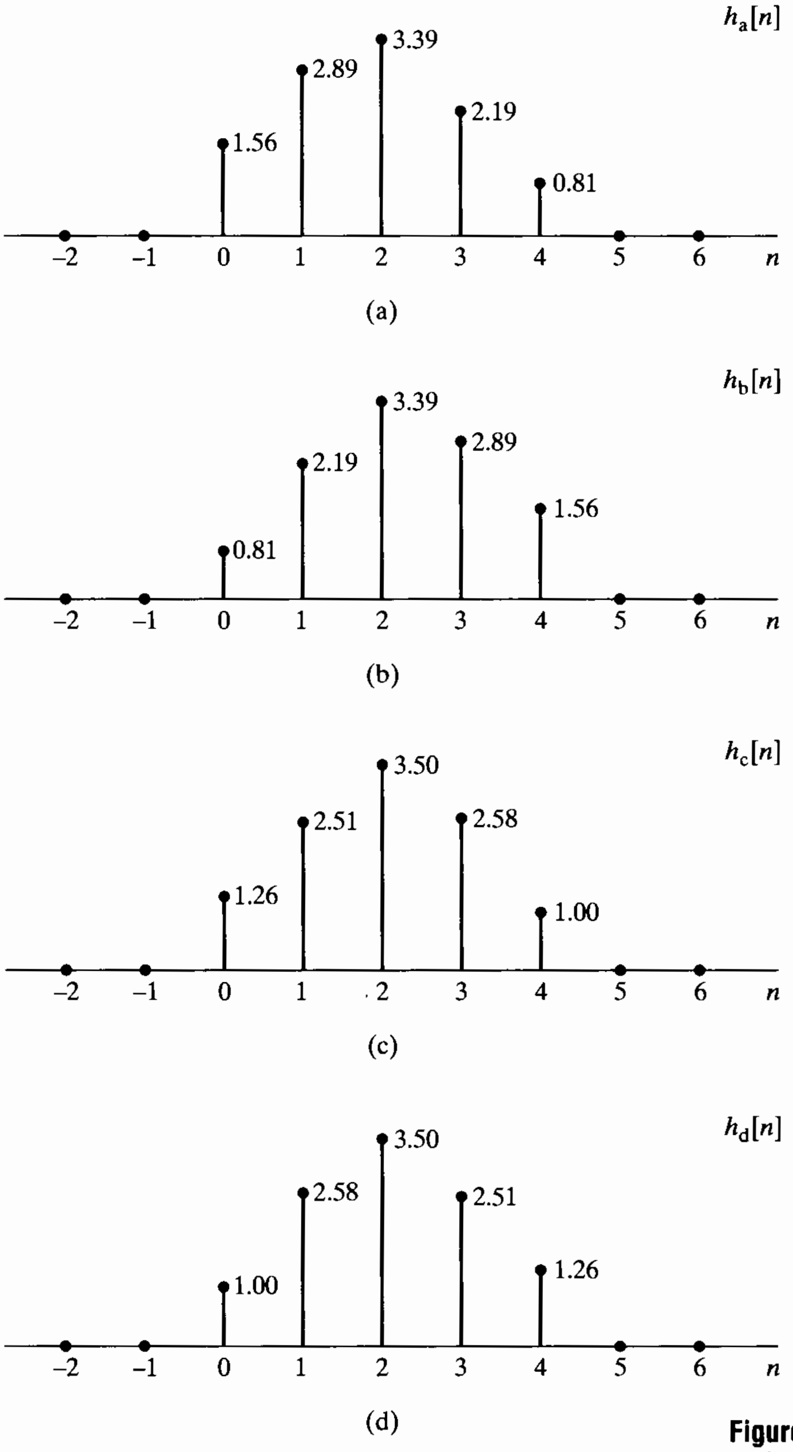


Figure 5.31 Sequences corresponding to the pole–zero plots of Figure 5.30.

If we define the *partial energy* of the impulse response as

$$E[n] = \sum_{m=0}^{n} |h[m]|^2, \tag{5.117}$$

then it can be shown that (see Problem 5.66)

$$\sum_{m=0}^{n} |h[m]|^2 \leq \sum_{m=0}^{n} |h_{\min}[m]|^2 \tag{5.118}$$

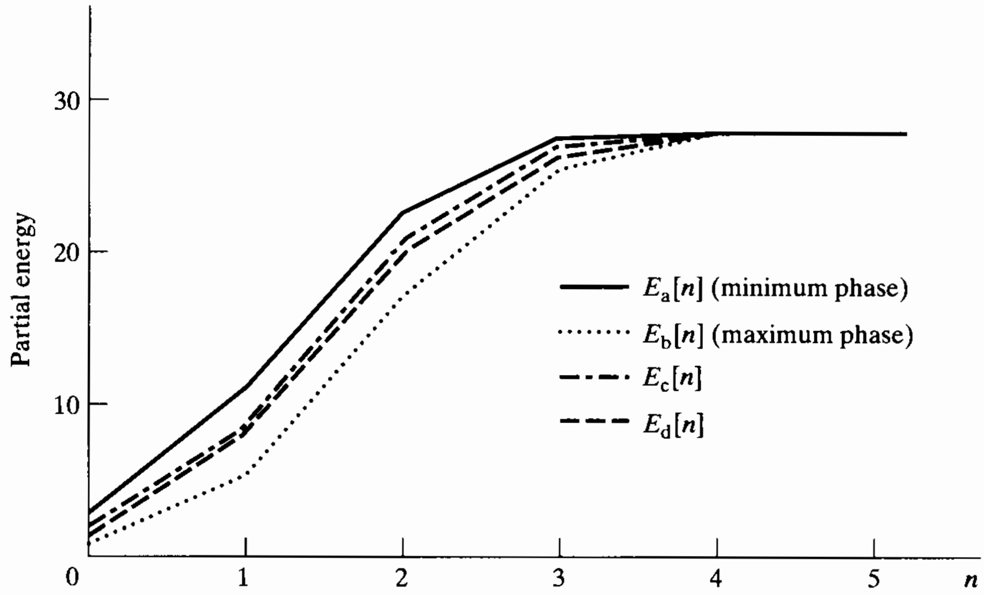


Figure 5.32 Partial energies for the four sequences of Figure 5.31. (Note that $E_a[n]$ is for the minimum-phase sequence $h_a[n]$ and $E_b[n]$ is for the maximum-phase sequence $h_b[n]$.)

for all impulse responses $h[n]$ belonging to the family of systems that have magnitude response given by Eq. (5.115). According to Eq. (5.118), the partial energy of the minimum-phase system is most concentrated around $n = 0$; i.e., the energy of the minimum-phase system is delayed the least of all systems having the same magnitude response function. For this reason, minimum-phase (lag) systems are also called *minimum energy-delay systems*, or simply, *minimum-delay systems*. This delay property is illustrated by Figure 5.32, which shows plots of the partial energy for the four sequences in Figure 5.31. We note for this example, and it is true in general, that the minimum energy delay occurs for the system that has all its zeros *inside* the unit circle (i.e., the minimum-phase system) and the maximum energy delay occurs for the system that has all its zeros *outside* the unit circle. Maximum energy-delay systems are also often called *maximum-phase systems*.

5.7 LINEAR SYSTEMS WITH GENERALIZED LINEAR PHASE

In designing filters and other signal-processing systems that pass some portion of the frequency band undistorted, it is desirable to have approximately constant frequency-response magnitude and zero phase in that band. For causal systems, zero phase is not attainable, and consequently, some phase distortion must be allowed. As we saw in Section 5.1.2, the effect of linear phase with integer slope is a simple time shift. A nonlinear phase, on the other hand, can have a major effect on the shape of a signal, even when the frequency-response magnitude is constant. Thus, in many situations it is particularly desirable to design systems to have exactly or approximately linear phase. In this section, we consider a formalization and generalization of the notions of linear phase and ideal time delay by considering the class of systems that have constant group delay. We begin by reconsidering the concept of delay in a discrete-time system.

5.7.1 Systems with Linear Phase

Consider an LTI system whose frequency response over one period is

$$H_{\text{id}}(e^{j\omega}) = e^{-j\omega\alpha}, \qquad |\omega| < \pi, \tag{5.119}$$

where α is a real number, not necessarily an integer. Such a system is an "ideal delay" system, where α is the delay introduced by the system. Note that this system has constant magnitude response, linear phase, and constant group delay; i.e.,

$$|H_{\text{id}}(e^{j\omega})| = 1, \tag{5.120a}$$

$$\sphericalangle H_{\text{id}}(e^{j\omega}) = -\omega\alpha, \tag{5.120b}$$

$$\text{grd}[H_{\text{id}}(e^{j\omega})] = \alpha. \tag{5.120c}$$

The inverse Fourier transform of $H_{\text{id}}(e^{j\omega})$ is the impulse response

$$h_{\text{id}}[n] = \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)}, \qquad -\infty < n < \infty. \tag{5.121}$$

The output of this system for an input $x[n]$ is

$$y[n] = x[n] * \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)} = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin \pi(n-k-\alpha)}{\pi(n-k-\alpha)}. \tag{5.122}$$

If $\alpha = n_d$, where n_d is an integer, then, as mentioned in Section 5.1.2,

$$h_{\text{id}}[n] = \delta[n-n_d] \tag{5.123}$$

and

$$y[n] = x[n] * \delta[n-n_d] = x[n-n_d]. \tag{5.124}$$

That is, if $\alpha = n_d$ is an integer, the system with linear phase and unity gain in Eq. (5.119) simply shifts the input sequence by n_d samples. If α is not an integer, the most straightforward interpretation is the one developed in Example 4.9 in Chapter 4. Specifically, a representation of the system of Eq. (5.119) is that shown in Figure 5.33, with $h_c(t) = \delta(t - \alpha T)$ and $H_c(j\Omega) = e^{-j\Omega\alpha T}$, so that

$$H(e^{j\omega}) = e^{-j\omega\alpha}, \qquad |\omega| < \pi. \tag{5.125}$$

In this representation, the choice of T is irrelevant and could simply be normalized to unity. It is important to stress again that the representation is valid whether or not $x[n]$ was originally obtained by sampling a continuous-time signal. According to the representation in Figure 5.33, $y[n]$ is the sequence of samples of the time-shifted, band-limited interpolation of the input sequence $x[n]$; i.e., $y[n] = x_c(nT - \alpha T)$. The system of Eq. (5.119) is said to have a time shift of α samples, even if α is not an integer. If the group delay α is positive, the time shift is a time delay. If α is negative, the time shift is a time advance.

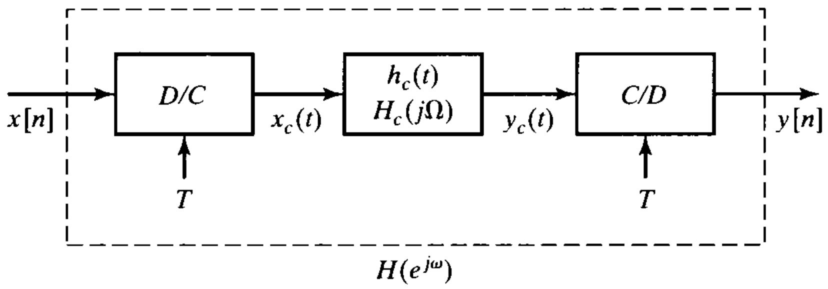


Figure 5.33 Interpretation of noninteger delay in discrete-time systems.

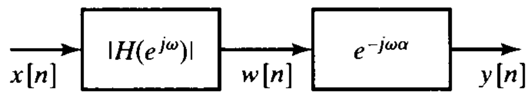


Figure 5.34 Representation of a linear-phase LTI system as a cascade of a magnitude filter and a time shift.

This discussion also provides a useful interpretation of linear phase when it is associated with a nonconstant magnitude response. For example, consider a more general frequency response with linear phase, i.e.,

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}, \qquad |\omega| < \pi. \tag{5.126}$$

Equation (5.126) suggests the interpretation of Figure 5.34. The signal $x[n]$ is filtered by the zero-phase frequency response $|H(e^{j\omega})|$, and the filtered output is then "time shifted" by the (integer or noninteger) amount α. Suppose, for example, that $H(e^{j\omega})$ is the linear-phase ideal lowpass filter

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases} \tag{5.127}$$

The corresponding impulse response is

$$h_{\text{lp}}[n] = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}. \tag{5.128}$$

Note that Eq. (5.121) is obtained if $\omega_c = \pi$.

Example 5.16 Ideal Lowpass with Linear Phase

The impulse response of the ideal lowpass filter illustrates some interesting properties of linear-phase systems. Figure 5.35(a) shows $h_{\text{lp}}[n]$ for $\omega_c = 0.4\pi$ and $\alpha = n_d = 5$. Note that when α is an integer, the impulse response is symmetric about $n = n_d$; i.e.,

$$\begin{aligned} h_{\text{lp}}[2n_d - n] &= \frac{\sin \omega_c(2n_d - n - n_d)}{\pi(2n_d - n - n_d)} \\ &= \frac{\sin \omega_c(n_d - n)}{\pi(n_d - n)} \\ &= h_{\text{lp}}[n]. \end{aligned} \tag{5.129}$$

In this case we could define a *zero-phase system*

$$\hat{H}_{\text{lp}}(e^{j\omega}) = H_{\text{lp}}(e^{j\omega})e^{j\omega n_d} = |H_{\text{lp}}(e^{j\omega})|, \tag{5.130}$$

where the impulse response is shifted to the left by n_d samples, yielding an even sequence

$$\hat{h}_{\text{lp}}[n] = \frac{\sin \omega_c n}{\pi n} = \hat{h}_{\text{lp}}[-n]. \tag{5.131}$$

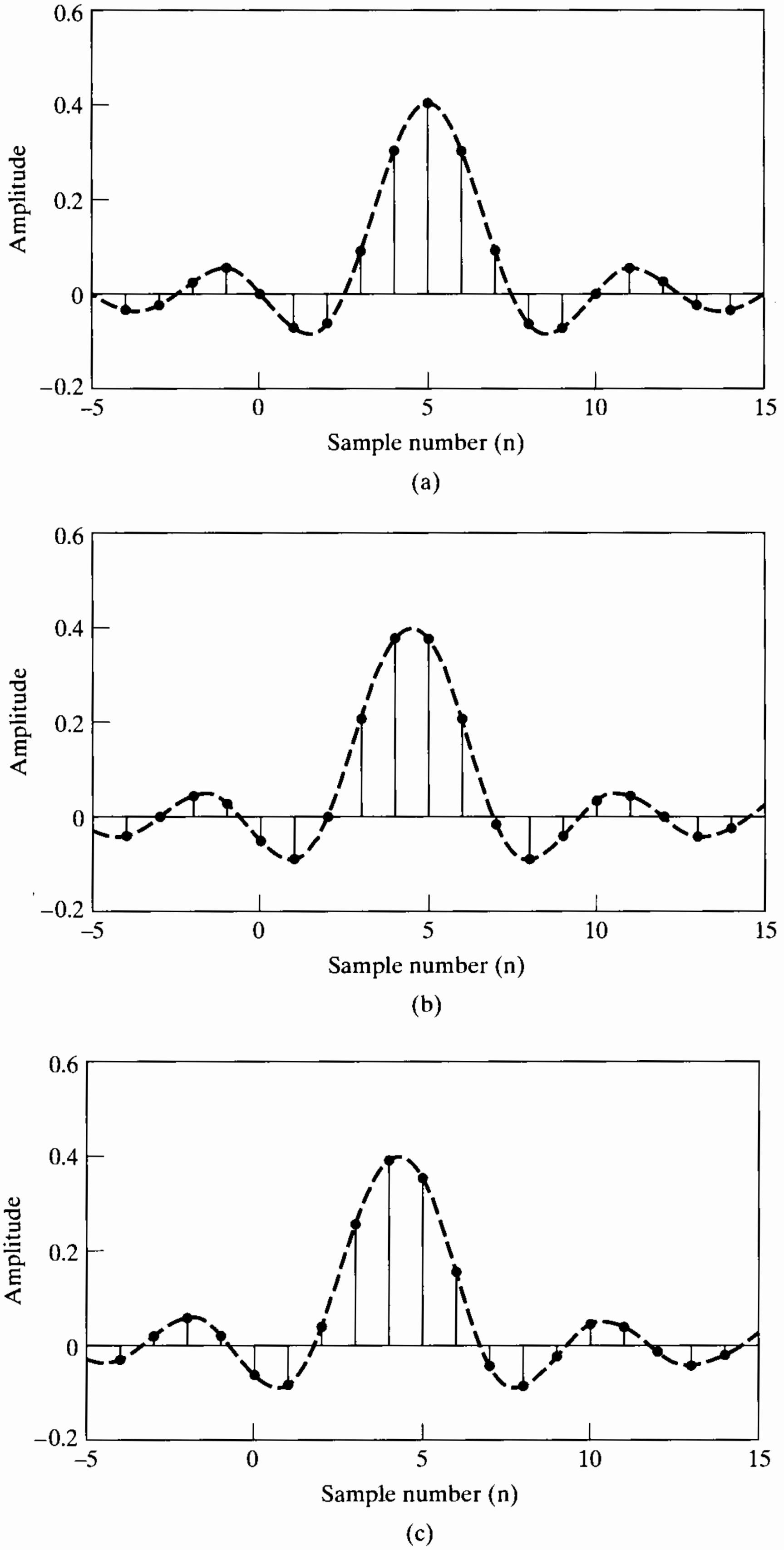


Figure 5.35 Ideal lowpass filter impulse responses, with $\omega_c = 0.4\pi$. (a) Delay $= \alpha = 5$. (b) Delay $= \alpha = 4.5$. (c) Delay $= \alpha = 4.3$.

Figure 5.35(b) shows $h_{\text{lp}}[n]$ for $\omega_c = 0.4\pi$ and $\alpha = 4.5$. This is typical of the case when the linear phase corresponds to an integer plus one-half. As in the case of the integer delay, it is easily shown that if α is an integer plus one-half (or 2α is an integer), then

$$h_{\text{lp}}[2\alpha - n] = h_{\text{lp}}[n]. \tag{5.132}$$

In this case, the point of symmetry is α, which is not an integer. Therefore, since the symmetry is not about a point of the sequence, it is not possible to shift the sequence to obtain an even sequence that has zero phase. This is similar to the case of Example 4.10 with M odd.

Figure 5.35(c) represents a third case, in which there is no symmetry at all. In this case, $\omega_c = 0.4\pi$ and $\alpha = 4.3$.

In general a linear-phase system has frequency response

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}. \tag{5.133}$$

As illustrated in Example 5.16, if 2α is an integer (i.e., if α is an integer or an integer plus one-half), the corresponding impulse response has even symmetry about α; i.e.,

$$h[2\alpha - n] = h[n]. \tag{5.134}$$

If 2α is not an integer, then the impulse response will not have symmetry. This is illustrated in Figure 5.35(c), which shows an impulse response that is not symmetric, but that has linear phase, or equivalently, constant group delay.

5.7.2 Generalized Linear Phase

In the discussion in Section 5.7.1, we considered a class of systems whose frequency response is of the form of Eq. (5.126), i.e., a real-valued nonnegative function of ω multiplied by a linear phase term $e^{-j\omega\alpha}$. For a frequency response of this form, the phase of $H(e^{j\omega})$ is entirely associated with the linear phase factor $e^{-j\omega\alpha}$, i.e., $\sphericalangle H(e^{j\omega}) = -\omega\alpha$, and consequently, systems in this class are referred to as linear-phase systems. In the moving average of Example 4.10, the frequency response in Eq. (4.67) is a real-valued function of ω multiplied by a linear-phase term, but the system is not, strictly speaking, a linear-phase system, since, at frequencies for which the factor

$$\frac{1}{M+1}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}$$

is negative, this term contributes an additional phase of π radians to the total phase.

Many of the advantages of linear-phase systems apply to systems with frequency response having the form of Eq. (4.67) as well, and consequently, it is useful to generalize somewhat the definition and concept of linear phase. Specifically, a system is referred to as a *generalized linear-phase system* if its frequency response can be expressed in the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}, \tag{5.135}$$

where α and β are constants and $A(e^{j\omega})$ is a real (possibly bipolar) function of ω. For the linear-phase system of Eq. (5.127) and the moving-average filter of Example 4.10, $\beta = 0$. We see, however, that the bandlimited differentiator of Example 4.5 has the form of Eq. (5.135) with $\alpha = 0$, $\beta = \pi/2$, and $A(e^{j\omega}) = \omega/T$.

A system whose frequency response has the form of Eq. (5.135) is called a generalized linear-phase system because the phase of such a system consists of constant terms added to the linear function $-\omega\alpha$; i.e. $-\omega\alpha + \beta$ is the equation of a straight line. However, if we ignore any discontinuities that result from the addition of constant phase over all or part of the band $|\omega| < \pi$, then such a system can be characterized by constant group delay. That is, the class of systems such that

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \alpha \tag{5.136}$$

have linear phase of the more general form

$$\arg[H(e^{j\omega})] = \beta - \omega\alpha, \qquad 0 < \omega < \pi, \tag{5.137}$$

where β and α are both real constants.

Recall that we showed in Section 5.7.1 that the impulse responses of linear-phase systems may have symmetry about α if 2α is an integer. To see the implication of this for generalized linear-phase systems, it is useful to derive an equation that must be satisfied by $h[n]$, α, and β for constant group-delay systems. This equation is derived by noting that, for such systems, the frequency response can be expressed as

$$\begin{aligned} H(e^{j\omega}) &= A(e^{j\omega})e^{j(\beta-\alpha\omega)} \\ &= A(e^{j\omega})\cos(\beta - \omega\alpha) + jA(e^{j\omega})\sin(\beta - \omega\alpha), \end{aligned} \tag{5.138}$$

or equivalently, as

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]\cos\omega n - j\sum_{n=-\infty}^{\infty} h[n]\sin\omega n, \end{aligned} \tag{5.139}$$

where we have assumed that $h[n]$ is real. The tangent of the phase angle of $H(e^{j\omega})$ can be expressed as

$$\tan(\beta - \omega\alpha) = \frac{\sin(\beta - \omega\alpha)}{\cos(\beta - \omega\alpha)} = \frac{-\sum_{n=-\infty}^{\infty} h[n]\sin\omega n}{\sum_{n=-\infty}^{\infty} h[n]\cos\omega n}.$$

Cross multiplying and combining terms with a trigonometric identity leads to the equation

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha)+\beta] = 0 \quad \text{for all } \omega. \tag{5.140}$$

This equation is a necessary condition on $h[n]$, α, and β for the system to have constant group delay. It is not a sufficient condition, however, and, due to its implicit nature, it does not tell us how to find a linear-phase system. For example, it can be shown that one set of conditions that satisfies Eq. (5.140) is

$$\beta = 0 \quad \text{or} \quad \pi, \tag{5.141a}$$

$$2\alpha = M = \text{an integer}, \tag{5.141b}$$

$$h[2\alpha - n] = h[n]. \tag{5.141c}$$

With $\beta = 0$ or π, Eq. (5.140) becomes

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha)] = 0, \tag{5.142}$$

from which it can be shown that if 2α is an integer, terms in Eq. (5.142) can be paired so that each pair of terms is identically zero for all ω. These conditions in turn imply that the corresponding frequency response has the form of Eq. (5.135) with $\beta = 0$ or π and $A(e^{j\omega})$ an even (and, of course, real) function of ω.

Alternatively, if $\beta = \pi/2$ or $3\pi/2$, then Eq. (5.140) becomes

$$\sum_{n=-\infty}^{\infty} h[n]\cos[\omega(n-\alpha)] = 0, \tag{5.143}$$

and it can be shown that

$$\beta = \pi/2 \quad \text{or} \quad 3\pi/2, \tag{5.144a}$$

$$2\alpha = M = \text{an integer}, \tag{5.144b}$$

and

$$h[2\alpha - n] = -h[n] \tag{5.144c}$$

satisfy Eq. (5.143) for all ω. Equations (5.144) imply that the frequency response has the form of Eq. (5.135) with $\beta = \pi/2$ and $A(e^{j\omega})$ an odd function of ω.

Note that Eqs. (5.141) and (5.144) give two sets of conditions that guarantee generalized linear phase or constant group delay, but as we have already seen in Figure 5.35(c), there are other systems that satisfy Eq. (5.135) without these symmetry conditions.

5.7.3 Causal Generalized Linear-Phase Systems

If the system is causal, then Eq. (5.140) becomes

$$\sum_{n=0}^{\infty} h[n]\sin[\omega(n-\alpha)+\beta] = 0 \qquad \text{for all } \omega. \tag{5.145}$$

Causality and the conditions in Eqs. (5.141) and (5.144) imply that

$$h[n] = 0, \qquad n < 0 \quad \text{and} \quad n > M;$$

i.e., causal FIR systems have generalized linear phase if they have impulse response length $(M+1)$ and satisfy either Eq. (5.141c) or (5.144c). Specifically, it can be shown that if

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases} \tag{5.146a}$$

then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}, \tag{5.146b}$$

where $A_e(e^{j\omega})$ is a real, even, periodic function of ω. Similarly, if

$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases} \tag{5.147a}$$

then it follows that

$$H(e^{j\omega}) = jA_o(e^{j\omega})e^{-j\omega M/2} = A_o(e^{j\omega})e^{-j\omega M/2 + j\pi/2}, \tag{5.147b}$$

where $A_o(e^{j\omega})$ is a real, odd, periodic function of ω. Note that in both cases the length of the impulse response is $(M+1)$ samples.

The conditions in Eqs. (5.146a) and (5.147a) are sufficient to guarantee a causal system with generalized linear phase. However, they are not necessary conditions. Clements and Pease (1989) have shown that causal infinite-duration impulse responses can also have Fourier transforms with generalized linear phase. The corresponding system functions, however, are not rational, and thus, the systems cannot be implemented with difference equations.

Expressions for the frequency response of FIR linear-phase systems are useful in filter design and in understanding some of the properties of such systems. In deriving these expressions, it turns out that significantly different expressions result, depending on the type of symmetry and whether M is an even or odd integer. For this reason, it is generally useful to define four types of FIR generalized linear-phase systems.

Type I FIR Linear-Phase Systems

A type I system is defined as a system that has a symmetric impulse response

$$h[n] = h[M-n], \qquad 0 \le n \le M, \tag{5.148}$$

with M an even integer. The delay $M/2$ is an integer. The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}. \tag{5.149}$$

By applying the symmetry condition, Eq. (5.148), the sum in Eq. (5.149) can be rewritten in the form

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right), \tag{5.150a}$$

where

$$a[0] = h[M/2], \tag{5.150b}$$

$$a[k] = 2h[(M/2)-k], \qquad k = 1, 2, \dots, M/2. \tag{5.150c}$$

Thus, from Eq. (5.150a), we see that $H(e^{j\omega})$ has the form of Eq. (5.146b), and in particular, β in Eq. (5.135) is either 0 or π.

Type II FIR Linear-Phase Systems

A type II system has a symmetric impulse response as in Eq. (5.148), with M an odd integer. $H(e^{j\omega})$ for this case can be expressed as

$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right\}, \qquad (5.151a)$$

where

$$b[k] = 2h[(M+1)/2 - k], \qquad k = 1, 2, \ldots, (M+1)/2. \qquad (5.151b)$$

Again, $H(e^{j\omega})$ has the form of Eq. (5.146b) with a time delay of $M/2$, which in this case is an integer plus one-half, and β in Eq. (5.135) is either 0 or π.

Type III FIR Linear-Phase Systems

If the system has an antisymmetric impulse response

$$h[n] = -h[M-n], \qquad 0 \le n \le M, \qquad (5.152)$$

with M an even integer, then $H(e^{j\omega})$ has the form

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin\omega k\right], \qquad (5.153a)$$

where

$$c[k] = 2h[(M/2) - k], \qquad k = 1, 2, \ldots, M/2. \qquad (5.153b)$$

In this case, $H(e^{j\omega})$ has the form of Eq. (5.147b) with a delay of $M/2$, which is an integer, and β in Eq. (5.135) is $\pi/2$ or $3\pi/2$.

Type IV FIR Linear-Phase Systems

If the impulse response is antisymmetric as in Eq. (5.152) and M is odd, then

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right], \qquad (5.154a)$$

where

$$d[k] = 2h[(M+1)/2 - k], \qquad k = 1, 2, \ldots, (M+1)/2. \qquad (5.154b)$$

As in the case of type III systems, $H(e^{j\omega})$ has the form of Eq. (5.147b) with delay $M/2$, which is an integer plus one-half, and β in Eq. (5.135) is $\pi/2$ or $3\pi/2$.

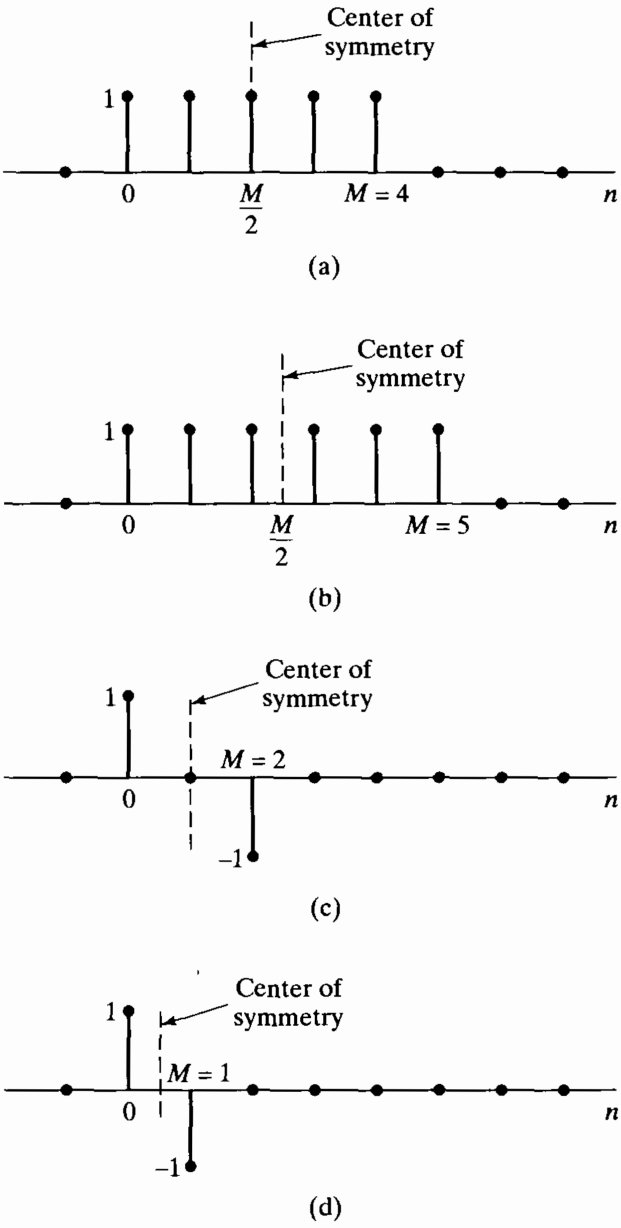


Figure 5.36 Examples of FIR linear-phase systems. (a) Type I, M even, $h[n] = h[M - n]$. (b) Type II, M odd, $h[n] = h[M - n]$. (c) Type III, M even, $h[n] = -h[M - n]$. (d) Type IV, M odd, $h[n] = -h[M - n]$.

Examples of FIR Linear-Phase Systems

Figure 5.36 shows an example of each of the four types of FIR linear-phase systems. The associated frequency responses are given in Examples 5.17–5.20.

Example 5.17 Type I Linear-Phase System

If the impulse response is

$$h[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise}, \end{cases} \tag{5.155}$$

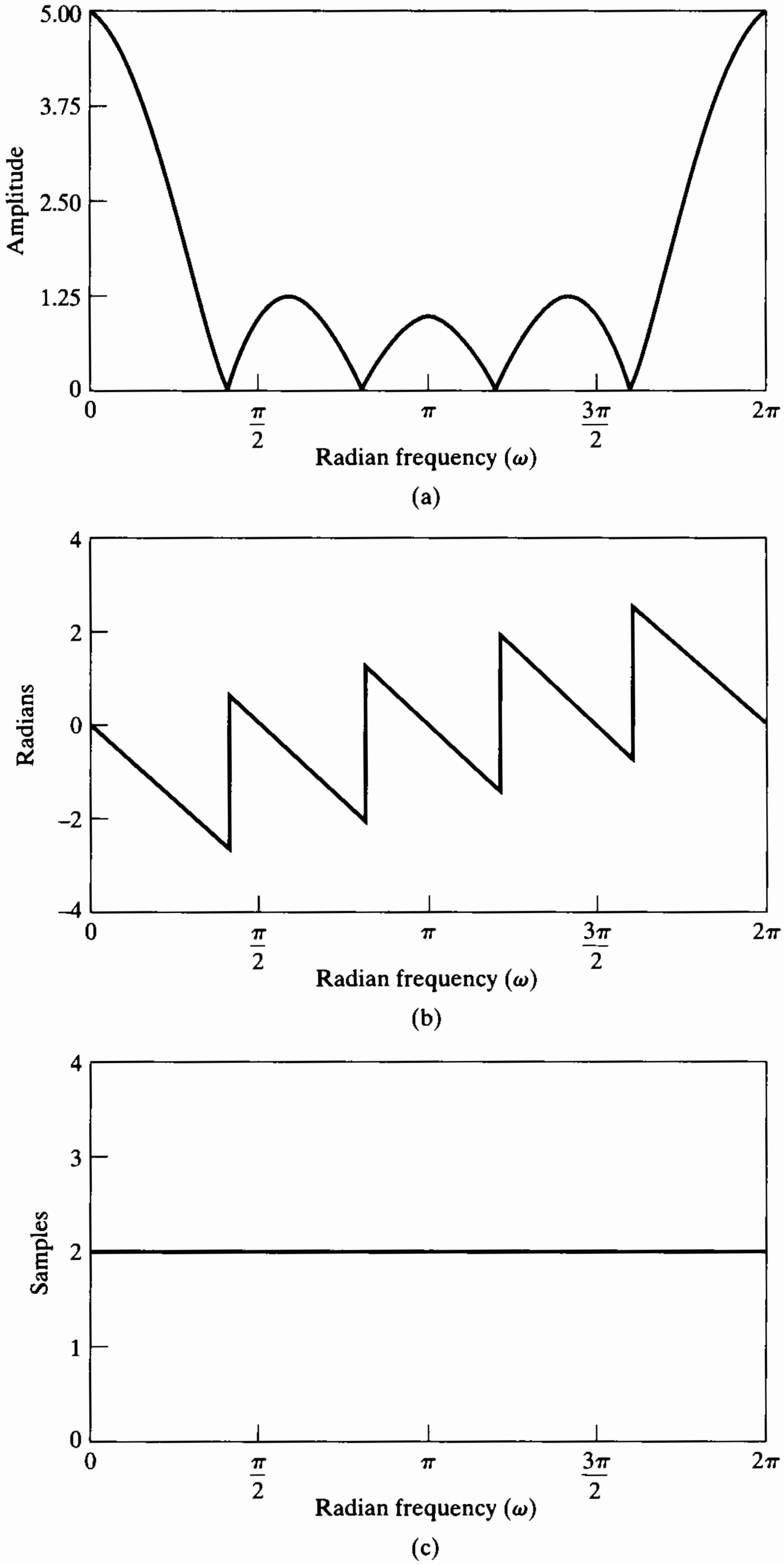


Figure 5.37 Frequency response of type I system of Example 5.17. (a) Magnitude. (b) Phase. (c) Group delay.

as shown in Figure 5.36(a), the system satisfies the condition of Eq. (5.148). The frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{4} e^{-j\omega n} = \frac{1-e^{-j\omega 5}}{1-e^{-j\omega}} \\ &= e^{-j\omega 2}\frac{\sin(5\omega/2)}{\sin(\omega/2)}. \end{aligned} \tag{5.156}$$

The magnitude, phase, and group delay of the system are shown in Figure 5.37. Since $M = 4$ is even, the group delay is an integer, i.e., $\alpha = 2$.

Example 5.18 Type II Linear-Phase System

If the length of the impulse response of the previous example is extended by one sample, we obtain the impulse response of Figure 5.36(b), which has frequency response

$$H(e^{j\omega}) = e^{-j\omega 5/2}\frac{\sin(3\omega)}{\sin(\omega/2)}. \tag{5.157}$$

The frequency-response functions for this system are shown in Figure 5.38. Note that the group delay in this case is constant with $\alpha = 5/2$.

Example 5.19 Type III Linear-Phase System

If the impulse response is

$$h[n] = \delta[n] - \delta[n-2], \tag{5.158}$$

as in Figure 5.36(c), then

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j2\omega} \\ &= j[2\sin(\omega)]e^{-j\omega}. \end{aligned} \tag{5.159}$$

The frequency-response plots for this example are given in Figure 5.39. Note that the group delay in this case is constant with $\alpha = 1$.

Example 5.20 Type IV Linear-Phase System

In this case (Figure 5.36(d)), the impulse response is

$$h[n] = \delta[n] - \delta[n-1], \tag{5.160}$$

for which the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j\omega} \\ &= j[2\sin(\omega/2)]e^{-j\omega/2}. \end{aligned} \tag{5.161}$$

The frequency response for this system is shown in Figure 5.40. Note that the group delay is equal to $\frac{1}{2}$ for all ω.

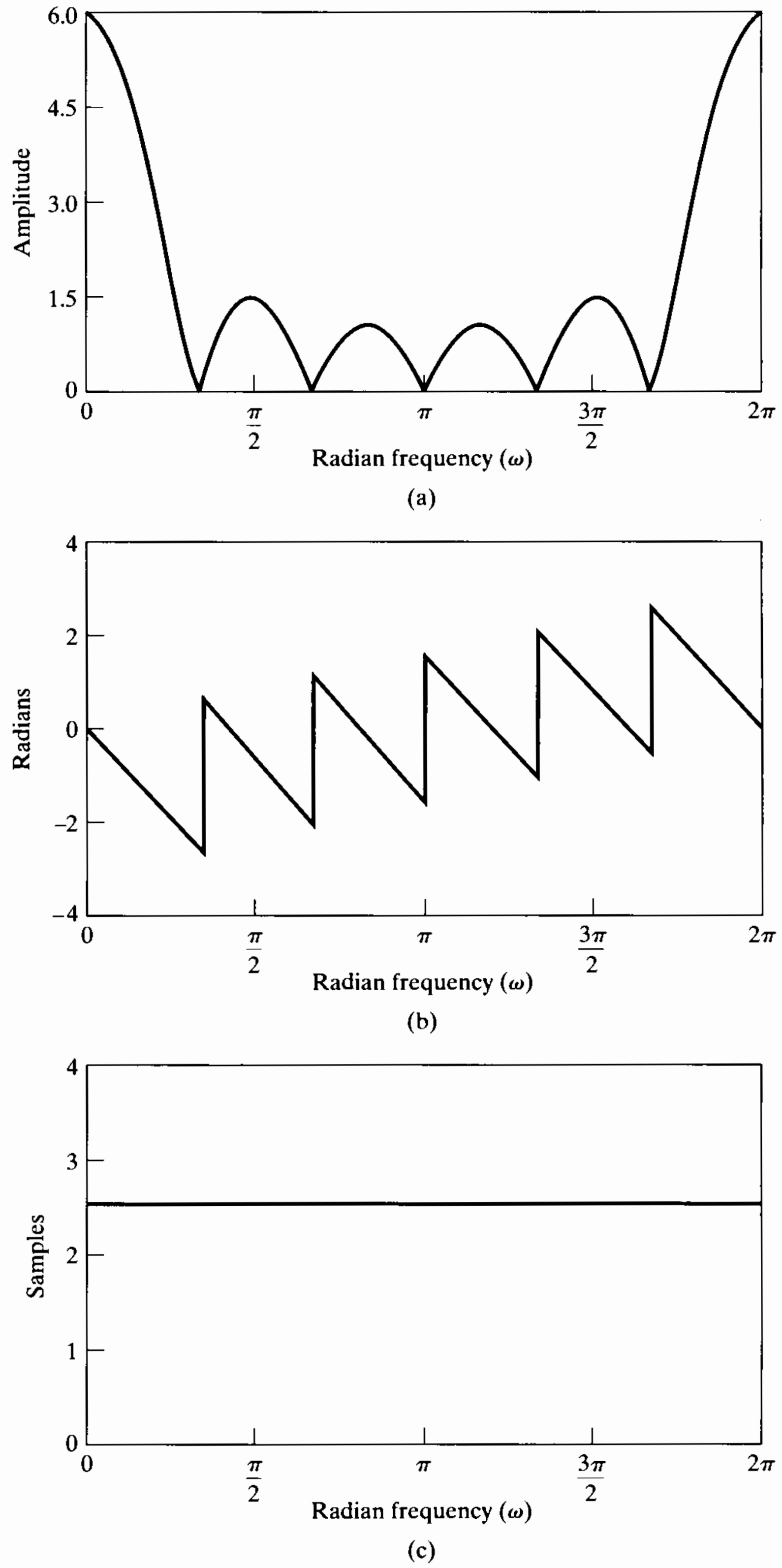


Figure 5.38 Frequency response of type II system of Example 5.18. (a) Magnitude. (b) Phase. (c) Group delay.

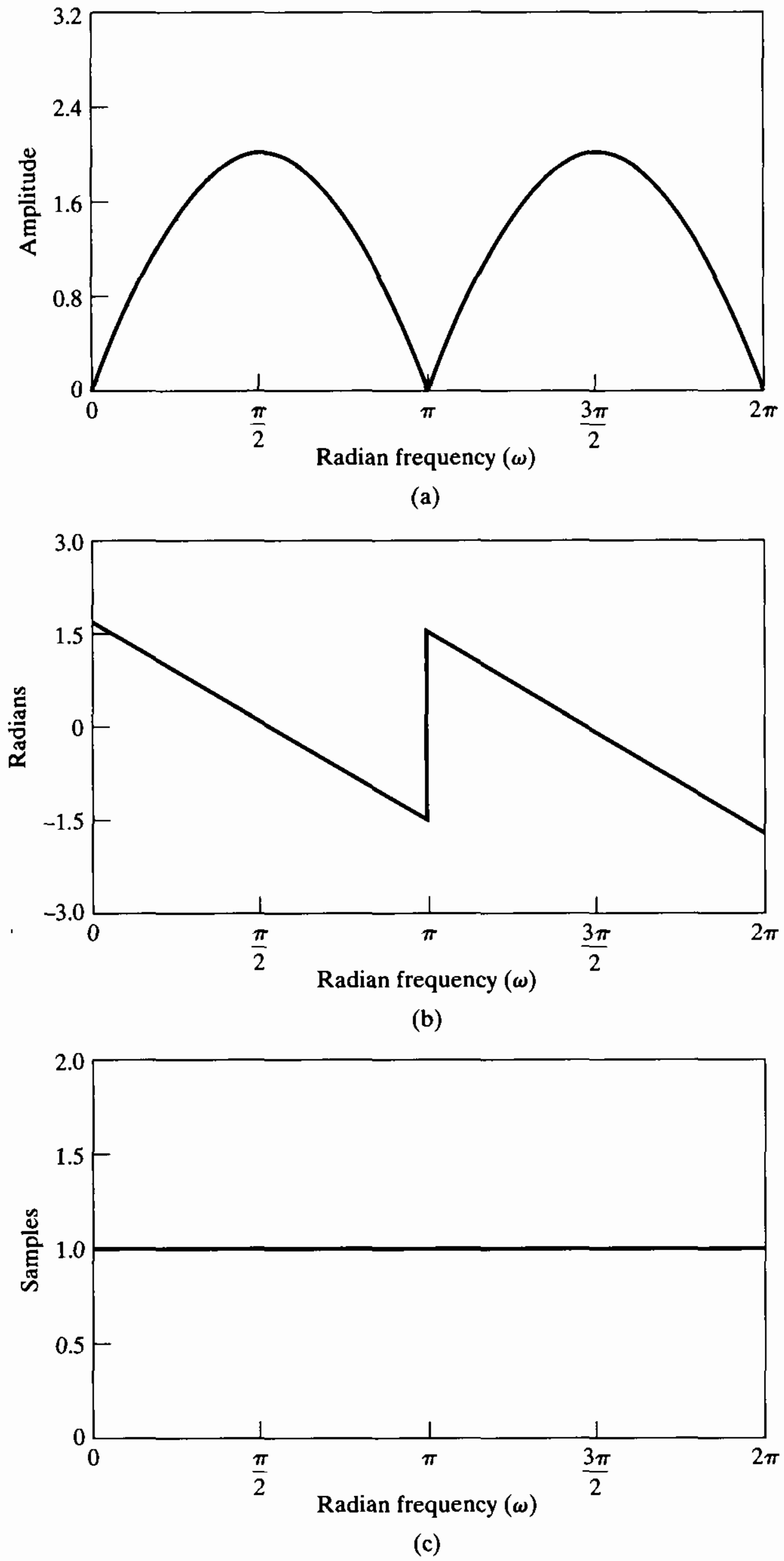


Figure 5.39 Frequency response of type III system of Example 5.19. (a) Magnitude. (b) Phase. (c) Group delay.

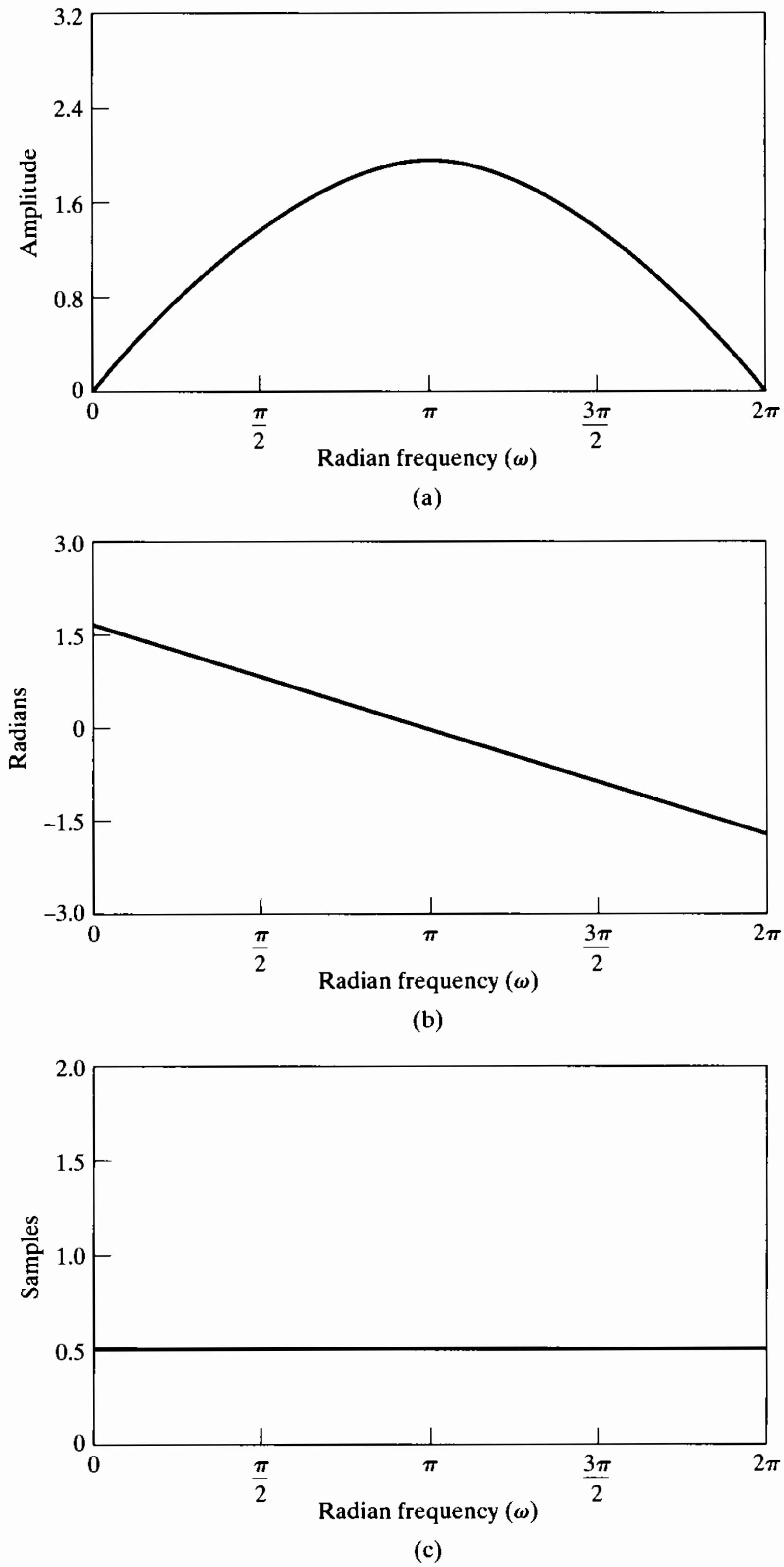


Figure 5.40 Frequency response of type IV system of Example 5.20. (a) Magnitude. (b) Phase. (c) Group delay.

Locations of Zeros for FIR Linear-Phase Systems

The preceding examples illustrate the properties of the impulse response and the frequency response for all four types of FIR linear-phase systems. It is also instructive to consider the locations of the zeros of the system function for FIR linear-phase systems. The system function is

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}. \tag{5.162}$$

In the symmetric cases (types I and II), we can use Eq. (5.148) to express $H(z)$ as

$$\begin{aligned} H(z) &= \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M} \\ &= z^{-M}H(z^{-1}). \end{aligned} \tag{5.163}$$

From Eq. (5.163), we conclude that if z_0 is a zero of $H(z)$, then

$$H(z_0) = z_0^{-M}H(z_0^{-1}) = 0. \tag{5.164}$$

This implies that if $z_0 = re^{j\theta}$ is a zero of $H(z)$, then $z_0^{-1} = r^{-1}e^{-j\theta}$ is also a zero of $H(z)$. When $h[n]$ is real and z_0 is a zero of $H(z)$, $z_0^* = re^{-j\theta}$ will also be a zero of $H(z)$, and by the preceding argument, so will $(z_0^*)^{-1} = r^{-1}e^{j\theta}$. Therefore, when $h[n]$ is real, each complex zero not on the unit circle will be part of a set of four conjugate reciprocal zeros of the form

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1}).$$

If a zero of $H(z)$ is on the unit circle, i.e., $z_0 = e^{j\theta}$, then $z_0^{-1} = e^{-j\theta} = z_0^*$, so zeros on the unit circle come in pairs of the form

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

If a zero of $H(z)$ is real and not on the unit circle, the reciprocal will also be a zero of $H(z)$, and $H(z)$ will have factors of the form

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

Finally, a zero of $H(z)$ at $z = \pm 1$ can appear by itself, since ± 1 is its own reciprocal and its own conjugate. Thus, we may also have factors of $H(z)$ of the form

$$(1 \pm z^{-1}).$$

The case of a zero at $z = -1$ is particularly important. From Eq. (5.163),

$$H(-1) = (-1)^M H(-1).$$

If M is even, we have a simple identity, but if M is odd, $H(-1) = -H(-1)$, so $H(-1)$ must be zero. Thus, for symmetric impulse responses with M odd, the system function

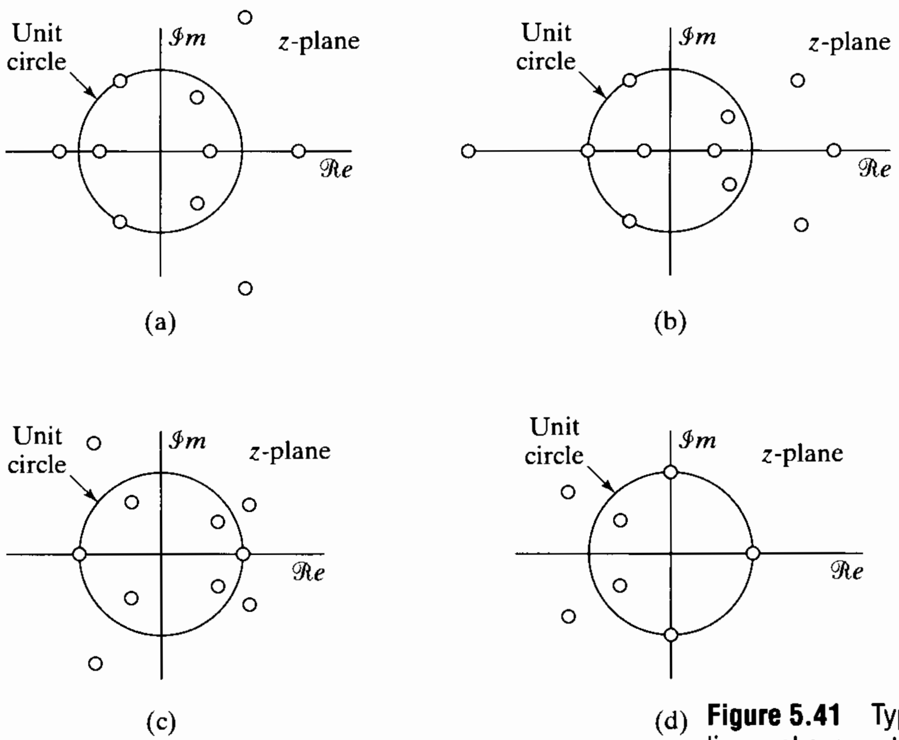


Figure 5.41 Typical plots of zeros for linear-phase systems. (a) Type I. (b) Type II. (c) Type III. (d) Type IV.

must have a zero at $z = -1$. Figures 5.41(a) and 5.41(b) show typical locations of zeros for type I (M even) and type II (M odd) systems, respectively.

If the impulse response is antisymmetric (types III and IV), then, following the approach used to obtain Eq. (5.163), we can show that

$$H(z) = -z^{-M}H(z^{-1}). \tag{5.165}$$

This equation can be used to show that the zeros of $H(z)$ for the antisymmetric case are constrained in the same way as the zeros for the symmetric case. In the antisymmetric case, however, both $z = 1$ and $z = -1$ are of special interest. If $z = 1$, Eq. (5.165) becomes

$$H(1) = -H(1). \tag{5.166}$$

Thus, $H(z)$ *must* have a zero at $z = 1$ for both M even and M odd. If $z = -1$, Eq. (5.165) gives

$$H(-1) = (-1)^{-M+1}H(-1). \tag{5.167}$$

In this case, if $(M - 1)$ is odd (i.e., if M is even), $H(-1) = -H(-1)$, so $z = -1$ *must* be a zero of $H(z)$ if M is even. Figures 5.41(c) and 5.41(d) show typical zero locations for type III and IV systems, respectively.

These constraints on the zeros are important in designing FIR linear-phase systems, since they impose limitations on the types of frequency responses that can be achieved. For example, we note that, in approximating a highpass filter using a symmetric impulse response, M should not be odd, since the frequency response is constrained to be zero at $\omega = \pi (z = -1)$.

5.7.4 Relation of FIR Linear-Phase Systems to Minimum-Phase Systems

The previous discussion shows that all FIR linear-phase systems with real impulse response have zeros either on the unit circle or at conjugate reciprocal locations. Thus, it is easily shown that the system function of any FIR linear-phase system can be factored into a minimum-phase term $H_{\text{min}}(z)$, a maximum-phase term $H_{\text{max}}(z)$, and a term $H_{\text{uc}}(z)$ containing only zeros on the unit circle; i.e.,

$$H(z) = H_{\text{min}}(z)H_{\text{uc}}(z)H_{\text{max}}(z), \tag{5.168a}$$

where

$$H_{\text{max}}(z) = H_{\text{min}}(z^{-1})z^{-M_i} \tag{5.168b}$$

and M_i is the number of zeros of $H_{\text{min}}(z)$. In Eq. (5.168a), $H_{\text{min}}(z)$ has all M_i of its zeros *inside* the unit circle, and $H_{\text{uc}}(z)$ has all M_o of its zeros *on* the unit circle. $H_{\text{max}}(z)$ has all M_i of its zeros *outside* the unit circle, and, from Eq. (5.168b), its zeros are the reciprocals of the zeros of $H_{\text{min}}(z)$. The order of the system function $H(z)$ is therefore $M = 2M_i + M_o$.

Example 5.21 Decomposition of a Linear-Phase System

As a simple example of the use of Eqs. (5.168), consider the minimum-phase system function of Eq. (5.109), for which the frequency response is plotted in Figure 5.28. The system obtained by applying Eq. (5.168b) to $H_{\text{min}}(z)$ in Eq. (5.109) is

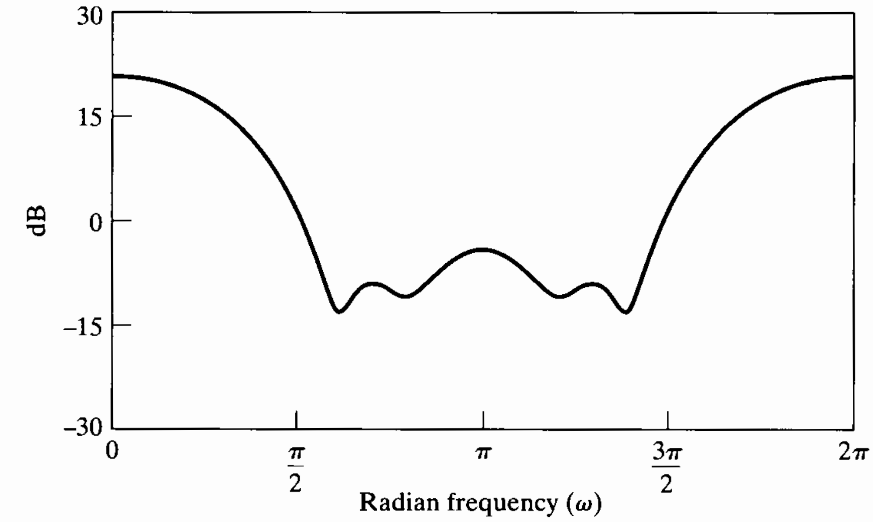
$$\begin{aligned} H_{\text{max}}(z) &= (0.9)^2(1 - 1.1111e^{j0.6\pi}z^{-1})(1 - 1.1111e^{-j0.6\pi}z^{-1}) \\ &\quad \times (1 - 1.25e^{-j0.8\pi}z^{-1})(1 - 1.25e^{j0.8\pi}z^{-1}). \end{aligned}$$

$H_{\text{max}}(z)$ has the frequency response shown in Figure 5.42. Now, if these two systems are cascaded, it follows from Eq. (5.168) that the overall system

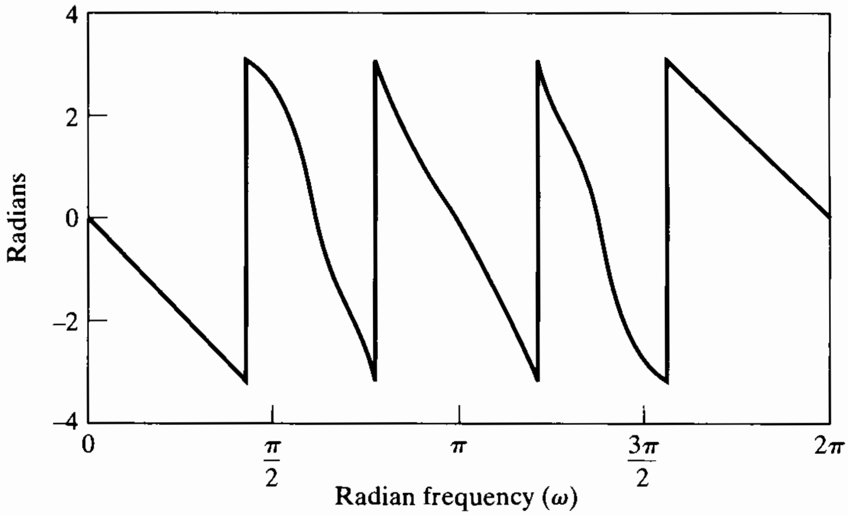
$$H(z) = H_{\text{min}}(z)H_{\text{max}}(z)$$

has linear phase. The frequency response of the composite system would be obtained by adding the respective log magnitude, phase, and group-delay functions. Therefore,

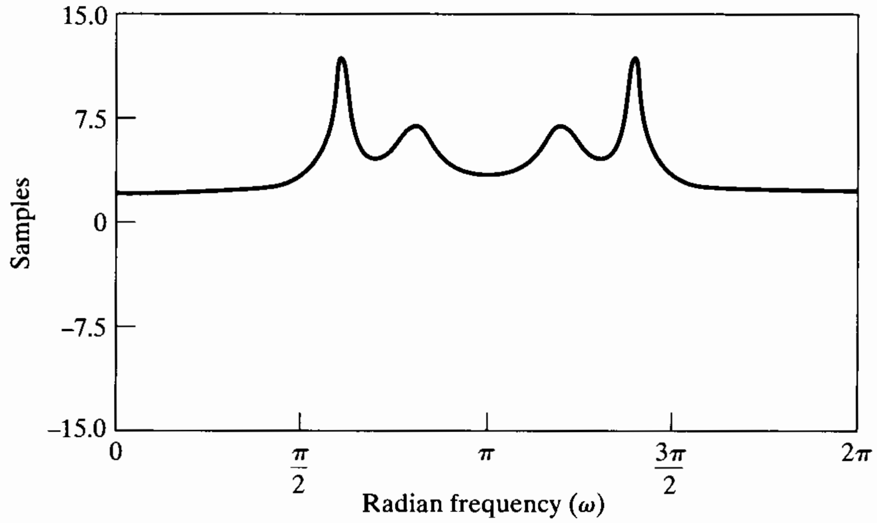
$$\begin{aligned} 20\log_{10}|H(e^{j\omega})| &= 20\log_{10}|H_{\text{min}}(e^{j\omega})| + 20\log_{10}|H_{\text{max}}(e^{j\omega})| \\ &= 40\log_{10}|H_{\text{min}}(e^{j\omega})|. \end{aligned} \tag{5.169}$$



(a)



(b)



(c)

Figure 5.42 Frequency response of maximum-phase system having the same magnitude as the system in Figure 5.28. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

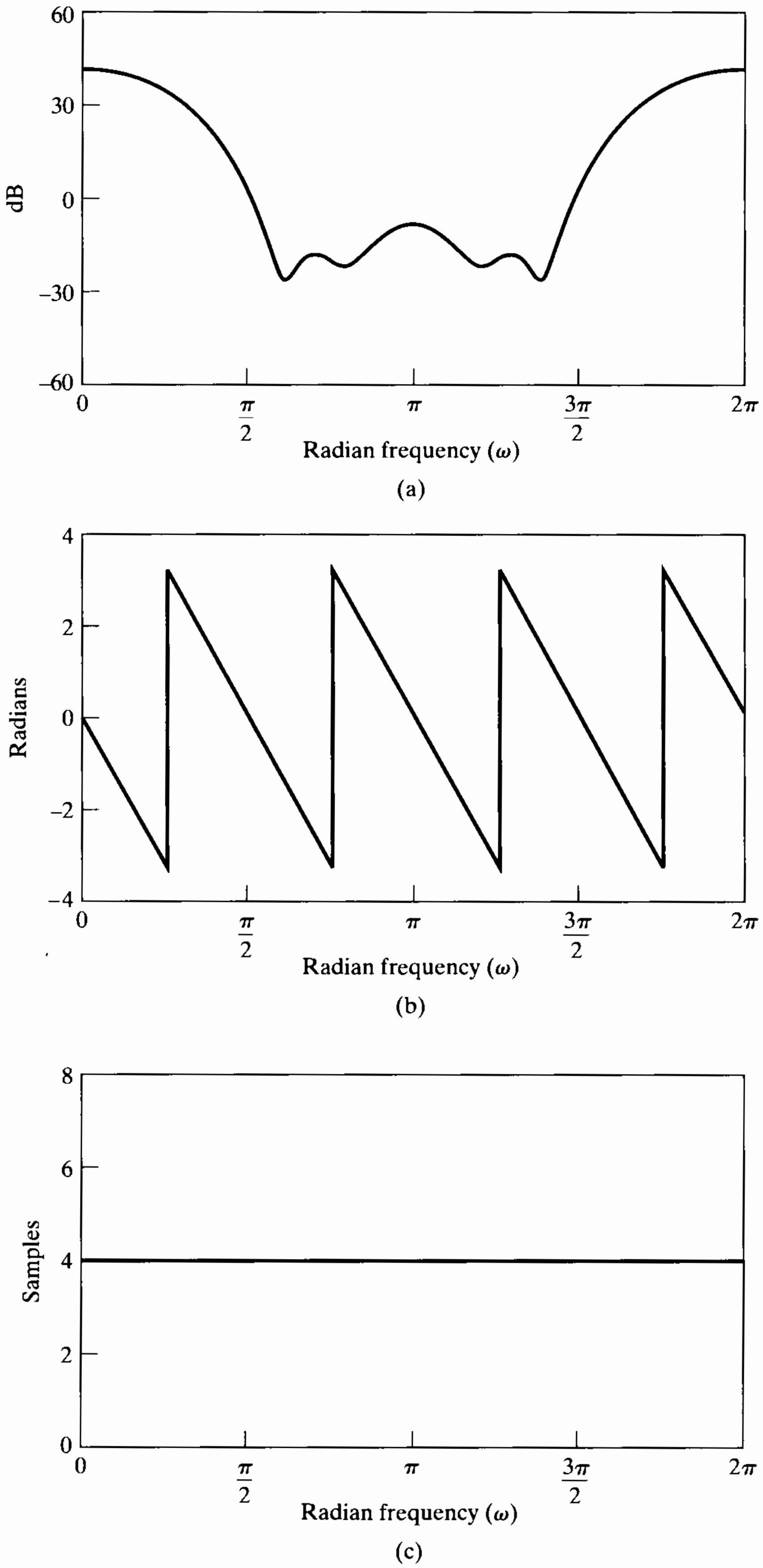


Figure 5.43 Frequency response of cascade of maximum-phase and minimum-phase systems, yielding a linear-phase system. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

Similarly,

$$\sphericalangle H(e^{j\omega}) = \sphericalangle H_{\min}(e^{j\omega}) + \sphericalangle H_{\max}(e^{j\omega}). \tag{5.170}$$

From Eq. (5.168b), it follows that

$$\sphericalangle H_{\max}(e^{j\omega}) = -\omega M_i - \sphericalangle H_{\min}(e^{j\omega}). \tag{5.171}$$

and

$$\sphericalangle H(e^{j\omega}) = -\omega M_i,$$

where $M_i = 4$ is the number of zeros of $H_{\min}(z)$. In like manner, the group-delay functions of $H_{\min}(e^{j\omega})$ and $H_{\max}(e^{j\omega})$ combine to give

$$\text{grd}[H(e^{j\omega})] = M_i = 4.$$

The frequency-response plots for the composite system are given in Figure 5.43. Note that the curves are sums of the corresponding functions in Figures 5.28 and 5.42.

5.8 SUMMARY

In this chapter, we developed and explored the representation and analysis of LTI systems using the Fourier and z-transforms. The importance of transform analysis for LTI systems stems directly from the fact that complex exponentials are eigenfunctions of such systems and the associated eigenvalues correspond to the system function or frequency response.

A particularly important class of LTI systems is that characterized by linear constant-coefficient difference equations. Transform analysis is particularly useful for analyzing these systems, since the Fourier transform or z-transform converts a difference equation to an algebraic equation. In particular, the system function is a ratio of polynomials, the coefficients of which correspond directly to the coefficients in the difference equation. The roots of these polynomials provide a useful system representation in terms of the pole–zero plot. Systems characterized by difference equations may have an impulse response that is infinite in duration (IIR) or finite in duration (FIR).

The frequency response of LTI systems is often characterized in terms of magnitude and phase or group delay, which is the negative of the derivative of the phase. Linear phase is often a desirable characteristic of a system frequency response, since it is a relatively mild form of phase distortion, corresponding to a time shift. The importance of FIR systems lies in part in the fact that such systems can be easily designed to have exactly linear phase (or generalized linear phase), while, for a given set of frequency response magnitude specifications, IIR systems are more efficient. These and other trade-offs will be discussed in detail in Chapter 7.

While, in general, for LTI systems, the frequency-response magnitude and phase are independent, for minimum-phase systems the magnitude uniquely specifies the phase and the phase uniquely specifies the magnitude to within a scale factor. Nonminimum-phase systems can be represented as the cascade combination of a minimum-phase system and an all-pass system. Relations between Fourier transform magnitude and phase will be discussed in considerably more detail in Chapter 11.

PROBLEMS

Basic Problems with Answers

5.1. In the system shown in Figure P5.1-1, $H(e^{j\omega})$ is an ideal lowpass filter. Determine whether for some choice of input $x[n]$ and cutoff frequency ω_c, the output can be the pulse

$$y[n] = \begin{cases} 1, & 0 \le n \le 10, \\ 0, & \text{otherwise,} \end{cases}$$

shown in Figure P5.1-2.

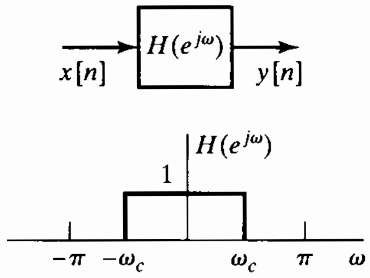


Figure P5.1-1

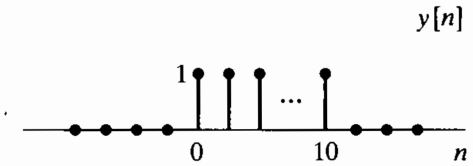


Figure P5.1-2

5.2. Consider a stable linear time-invariant system with input $x[n]$ and output $y[n]$. The input and output satisfy the difference equation

$$y[n-1] - \tfrac{10}{3}y[n] + y[n+1] = x[n].$$

(a) Plot the poles and zeros in the z-plane.
(b) Find the impulse response $h[n]$.

5.3. Consider a linear time-invariant discrete-time system for which the input $x[n]$ and output $y[n]$ are related by the second-order difference equation

$$y[n-1] + \tfrac{1}{3}y[n-2] = x[n].$$

From the following list, choose *two* possible impulse responses for the system:

(a) $\left(-\frac{1}{3}\right)^{n+1} u[n+1]$
(b) $3^{n+1}u[n+1]$
(c) $3(-3)^{n+2}u[-n-2]$
(d) $\frac{1}{3}\left(-\frac{1}{3}\right)^{n} u[-n-2]$

(e) $\left(-\frac{1}{3}\right)^{n+1} u[-n-2]$
(f) $\left(\frac{1}{3}\right)^{n+1} u[n+1]$
(g) $(-3)^{n+1}u[n]$
(h) $n^{1/3}u[n]$

5.4. When the input to a linear time-invariant system is

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + (2)^n u[-n-1],$$

the output is

$$y[n] = 6\left(\tfrac{1}{2}\right)^n u[n] - 6\left(\tfrac{3}{4}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the region of convergence.
(b) Find the impulse response $h[n]$ of the system for all values of n.
(c) Write the difference equation that characterizes the system.
(d) Is the system stable? Is it causal?

5.5. Consider a system described by a linear constant-coefficient difference equation with initial-rest conditions. The step response of the system is given by

$$y[n] = \left(\tfrac{1}{3}\right)^n u[n] + \left(\tfrac{1}{4}\right)^n u[n] + u[n].$$

(a) Determine the difference equation.
(b) Determine the impulse response of the system.
(c) Determine whether or not the system is stable.

5.6. The following information is known about a linear time-invariant system:
(a) The system is causal.
(b) When the input is

$$x[n] = -\tfrac{1}{3}\left(\tfrac{1}{2}\right)^n u[n] - \tfrac{4}{3}(2)^n u[-n-1],$$

the z-transform of the output is

$$Y(z) = \frac{1 - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}.$$

(c) Find the z-transform of $x[n]$.
(d) What are the possible choices for the region of convergence of $Y(z)$?
(e) What are the possible choices for the impulse response of the system?

5.7. When the input to a linear time-invariant system is

$$x[n] = 5u[n],$$

the output is

$$y[n] = \left[2\left(\tfrac{1}{2}\right)^n + 3\left(-\tfrac{3}{4}\right)^n\right] u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the region of convergence.
(b) Find the impulse response of the system for all values of n.
(c) Write the difference equation that characterizes the system.

5.8. A causal linear time-invariant system is described by the difference equation

$$y[n] = \tfrac{3}{2}y[n-1] + y[n-2] + x[n-1].$$

(a) Find the system function $H(z) = Y(z)/X(z)$ for this system. Plot the poles and zeros of $H(z)$, and indicate the region of convergence.
(b) Find the impulse response of the system.
(c) You should have found the system to be unstable. Find a stable (noncausal) impulse response that satisfies the difference equation.

5.9. Consider a linear time-invariant system with input $x[n]$ and output $y[n]$ for which

$$y[n-1] - \tfrac{5}{2}y[n] + y[n+1] = x[n].$$

The system may or may not be stable or causal.

By considering the pole–zero pattern associated with the preceding difference equation, determine three possible choices for the impulse response of the system. Show that each choice satisfies the difference equation. Indicate which choice corresponds to a stable system and which choice corresponds to a causal system.

5.10. If the system function $H(z)$ of a linear time-invariant system has a pole–zero diagram as shown in Figure P5.10-1 and the system is causal, can the inverse system $H_i(z)$, where $H(z)H_i(z) = 1$, be both causal and stable? Clearly justify your answer.

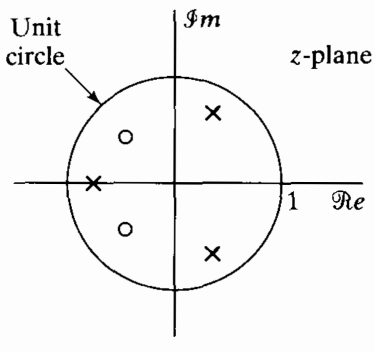


Figure P5.10-1

5.11. The system function of a linear time-invariant system has the pole–zero plot shown in Figure P5.11-1 Specify whether each of the following statements is true, is false, or cannot be determined from the information given.
(a) The system is stable.
(b) The system is causal.
(c) If the system is causal, then it must be stable.
(d) If the system is stable, then it must have a two-sided impulse response.

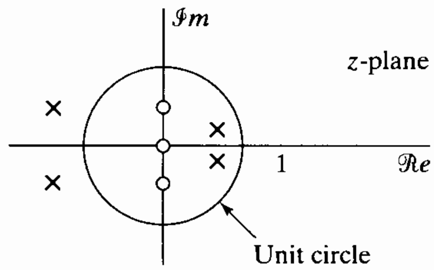


Figure P5.11-1

5.12. A discrete-time causal LTI system has the system function

$$H(z) = \frac{(1 + 0.2z^{-1})(1 - 9z^{-2})}{(1 + 0.81z^{-2})}.$$

(a) Is the system stable?

(b) Find expressions for a minimum-phase system $H_1(z)$ and an all-pass system $H_{ap}(z)$ such that

$$H(z) = H_1(z)H_{ap}(z).$$

5.13. Figure P5.13-1 shows the pole–zero plots for four different LTI systems. Based on these plots, state whether or not each system is an all-pass system.

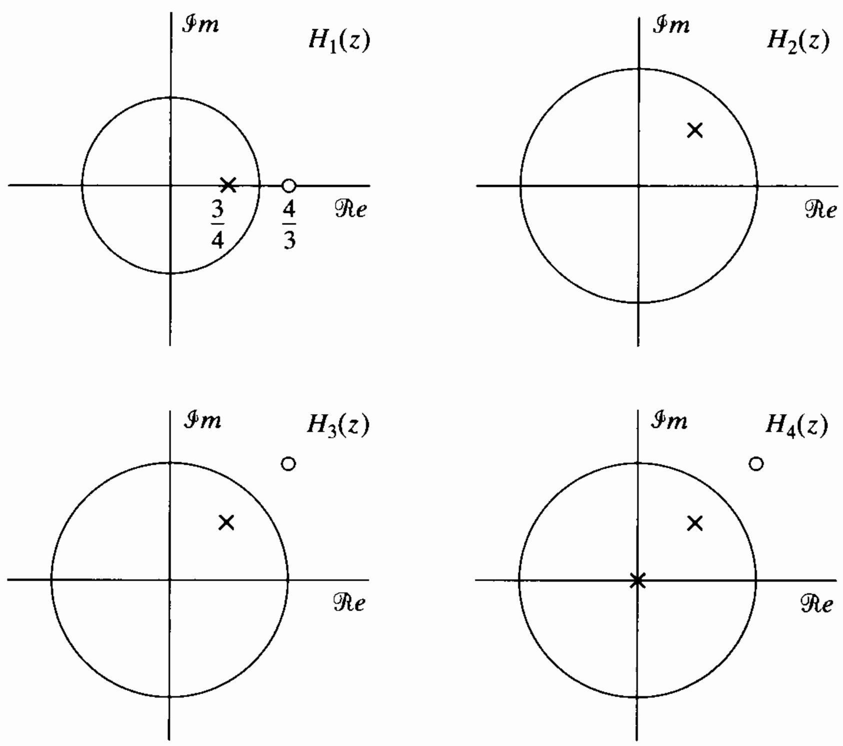


Figure P5.13-1

5.14. Determine the group delay for $0 < \omega < \pi$ for each of the following sequences:

(a)

$$x_1[n] = \begin{cases} n-1, & 1 \le n \le 5, \\ 9-n, & 5 < n \le 9, \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$x_2[n] = \left(\frac{1}{2}\right)^{|n-1|} + \left(\frac{1}{2}\right)^{|n|}.$$

5.15. Consider the class of discrete-time filters whose frequency response has the form

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\alpha\omega},$$

where $|H(e^{j\omega})|$ is a real and nonnegative function of ω and α is a real constant. As discussed in Section 5.7.1, this class of filters is referred to as *linear-phase* filters.

Consider also the class of discrete-time filters whose frequency response has the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega+j\beta},$$

where $A(e^{j\omega})$ is a real function of ω, α is a real constant, and β is a real constant. As discussed in Section 5.7.2, filters in this class are referred to as *generalized linear-phase* filters.

For each of the filters in Figure P5.15-1, determine whether it is a generalized linear-phase filter. If it is, then find $A(e^{j\omega})$, α, and β. In addition, for each filter you determine to be a generalized linear-phase filter, indicate whether it also meets the more stringent criterion for being a linear-phase filter.

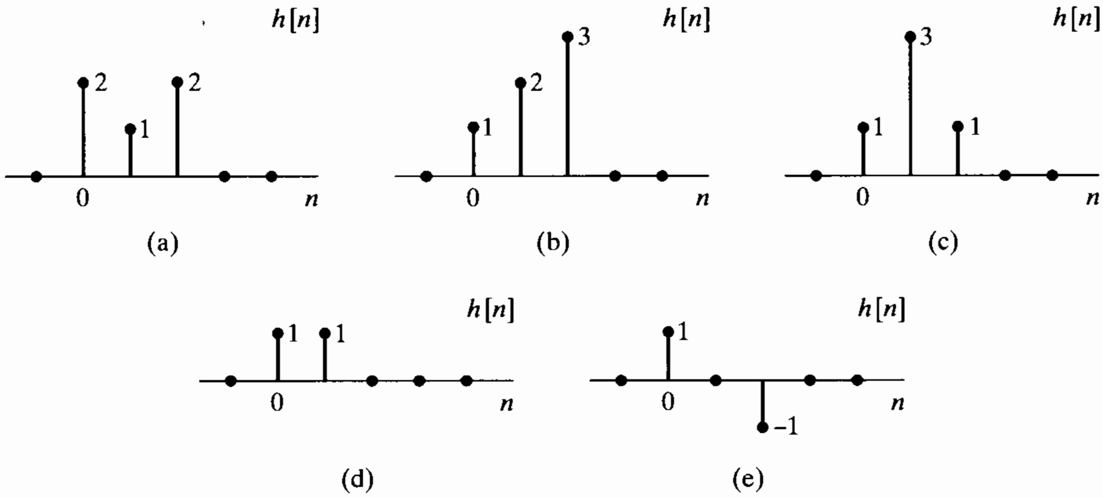


Figure P5.15-1

5.16. Figure P5.16-1 plots the continuous-phase $\arg[H(e^{j\omega})]$ for the frequency response of a specific LTI system, where

$$\arg[H(e^{j\omega})] = -\alpha\omega$$

for $|\omega| < \pi$ and α is a positive integer.

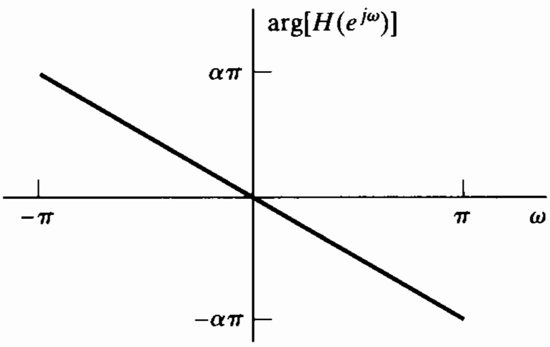


Figure P5.16-1

Is the impulse response $h[n]$ of this system a causal sequence? If the system is definitely causal, or if it is definitely not causal, give a proof. If the causality of the system cannot be determined from Figure P5.16-1, give examples of a noncausal sequence and a causal sequence that both have the foregoing phase response $\arg[H(e^{j\omega})]$.

5.17. For each of the following system functions, state whether or not it is a minimum-phase system. Justify your answers:

$$H_1(z) = \frac{(1 - 2z^{-1})\left(1 + \frac{1}{2}z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)},$$

$$H_2(z) = \frac{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 - \frac{2}{3}z^{-1}\right)\left(1 + \frac{2}{3}z^{-1}\right)},$$

$$H_3(z) = \frac{1 - \frac{1}{3}z^{-1}}{\left(1 - \frac{j}{2}z^{-1}\right)\left(1 + \frac{j}{2}z^{-1}\right)},$$

$$H_4(z) = \frac{z^{-1}\left(1 - \frac{1}{3}z^{-1}\right)}{\left(1 - \frac{j}{2}z^{-1}\right)\left(1 + \frac{j}{2}z^{-1}\right)}.$$

5.18. For each of the following system functions $H_k(z)$, specify a minimum-phase system function $H_{\min}(z)$ such that the frequency-response magnitudes of the two systems are equal, i.e., $|H_k(e^{j\omega})| = |H_{\min}(e^{j\omega})|$.

(a)

$$H_1(z) = \frac{1 - 2z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

(b)

$$H_2(z) = \frac{(1 + 3z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)}{z^{-1}\left(1 + \frac{1}{3}z^{-1}\right)}$$

(c)

$$H_3(z) = \frac{(1 - 3z^{-1})\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 - \frac{3}{4}z^{-1}\right)\left(1 - \frac{4}{3}z^{-1}\right)}$$

5.19. Figure P5.19-1 shows the impulse responses for several different LTI systems. Find the group delay associated with each system.

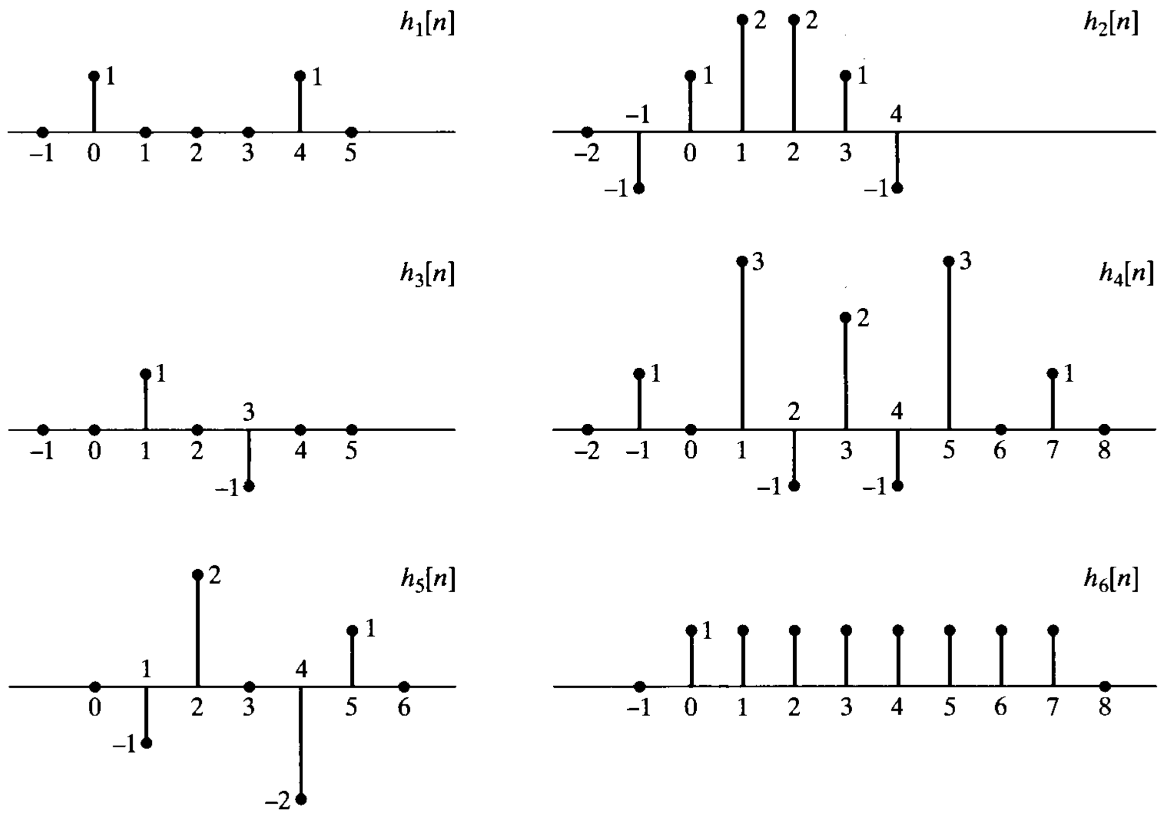


Figure P5.19-1

5.20. Figure P5.20-1 shows just the zero locations for several different system functions. For each plot, state whether the system function could be a generalized linear-phase system implemented by a linear constant-coefficient difference equation with real coefficients.

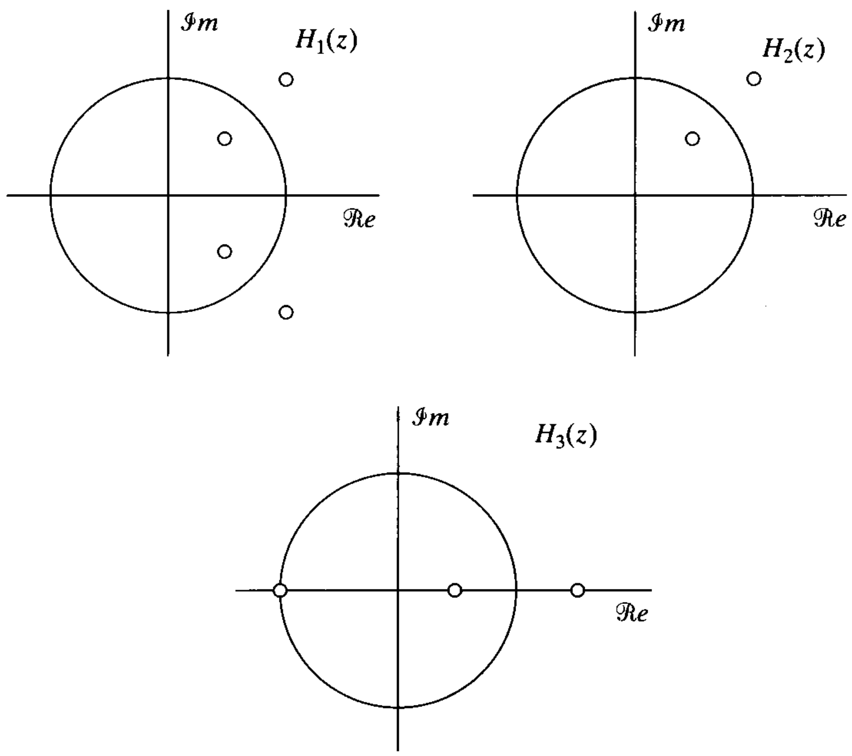


Figure P5.20-1

Basic Problems

5.21. Let $h_{lp}[n]$ denote the impulse response of an ideal lowpass filter with unity passband gain and cutoff frequency $\omega_c = \pi/4$. Figure P5.21-1 shows five systems, each of which is equivalent to an ideal LTI frequency-selective filter. For each system shown, sketch the equivalent frequency response, indicating explicitly the band-edge frequencies in terms of ω_c. In each case, specify whether the system is a lowpass, highpass, bandpass, bandstop, or multiband filter.

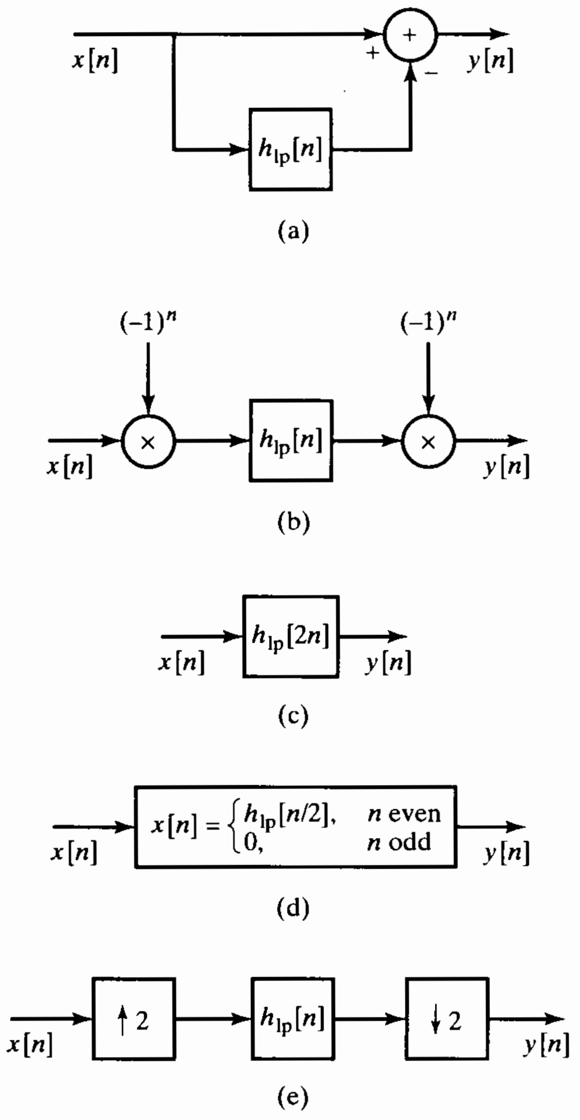


Figure P5.21-1

5.22. Consider a causal linear time-invariant system with system function

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}},$$

where a is real.

(a) Write the difference equation that relates the input and the output of this system.

(b) For what range of values of a is the system stable?

(c) For $a = \frac{1}{2}$, plot the pole–zero diagram and shade the region of convergence.

(d) Find the impulse response $h[n]$ for the system.

(e) Show that the system is an all-pass system, i.e., that the magnitude of the frequency response is a constant. Also, specify the value of the constant.

5.23. **(a)** For each of the four types of causal linear phase FIR filters discussed in Section 5.7.3, determine whether the associated symmetry imposes any constraint on the frequency response at $\omega = 0$ and/or $\omega = \pi$.

(b) For each of the following types of desired filter, indicate which of the four FIR filter types would be useful to consider in approximating the desired filter:

Lowpass
Bandpass
Highpass
Bandstop
Differentiator

5.24. Let $x[n]$ be a causal, N-point sequence that is zero outside the range $0 \le n \le N-1$. When $x[n]$ is the input to the causal LTI system represented by the difference equation

$$y[n] - \tfrac{1}{4}y[n-2] = x[n-2] - \tfrac{1}{4}x[n],$$

the output is $y[n]$, also a causal, N-point sequence.

(a) Show that the causal LTI system described by this difference equation represents an all-pass filter.

(b) Given that

$$\sum_{n=0}^{N-1} |x[n]|^2 = 5,$$

determine the value of

$$\sum_{n=0}^{N-1} |y[n]|^2.$$

5.25. Is the following statement true or false?

Statement: It is not possible for a noncausal system to have a positive constant group delay; i.e., $\text{grd}\left[H(e^{j\omega})\right] = \tau_0 > 0$.

If the statement is true, give a brief argument justifying it. If the statement is false, provide a counterexample.

5.26. Consider the z-transform

$$H(z) = \frac{rz^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}, \qquad |z| > r.$$

Assume first that $\omega_0 \ne 0$.

(a) Draw a labeled pole–zero diagram and determine $h[n]$.

(b) Repeat Part (a) when $\omega_0 = 0$. This is known as a critically damped system.

5.27. An LTI system with impulse response $h_1[n]$ is an ideal lowpass filter with cutoff frequency $\omega_c = \pi/2$. The frequency response of the system is $H_1(e^{j\omega})$. Suppose a new LTI system with impulse response $h_2[n]$ is obtained from $h_1[n]$ by

$$h_2[n] = (-1)^n h_1[n].$$

Sketch the frequency response $H_2(e^{j\omega})$.

Advanced Problems

5.28. The system function $H(z)$ of a causal linear time-invariant system has the pole-zero configuration shown in Figure P5.28-1. It is also known that $H(z) = 6$ when $z = 1$.

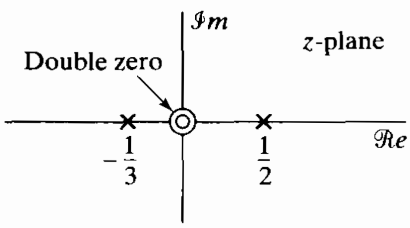


Figure P5.28-1

(a) Determine $H(z)$.
(b) Determine the impulse response $h[n]$ of the system.
(c) Determine the response of the system to the following input signals:
(i) $x[n] = u[n] - \frac{1}{2}u[n-1]$
(ii) The sequence $x[n]$ obtained from sampling the continuous-time signal

$$x(t) = 50 + 10\cos 20\pi t + 30\cos 40\pi t$$

at a sampling frequency $\Omega_s = 2\pi(40)$ rad/s

5.29. The system function of a linear time-invariant system is given by

$$H(z) = \frac{21}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})(1 - 4z^{-1})}.$$

It is known that the system is not stable and that the impulse response is two sided.
(a) Determine the impulse response $h[n]$ of the system.
(b) The impulse response found in Part (a) can be expressed as the sum of a causal impulse response $h_1[n]$ and an anticausal impulse response $h_2[n]$. Determine the corresponding system functions $H_1(z)$ and $H_2(z)$.

5.30. A signal $x[n]$ is processed by a linear time-invariant system $H(z)$ and then downsampled by a factor of 2 to yield $y[n]$, as shown in Figure P5.30-1. The pole–zero plot for $H(z)$ is shown in Figure P5.30-2.
(a) Determine and sketch $h[n]$, the impulse response of the system $H(z)$.

(b) A second system is shown in Figure P5.30-3, in which the signal $x[n]$ is first time compressed by a factor of 2 and then passed through an LTI system $G(z)$ to obtain $r[n]$.

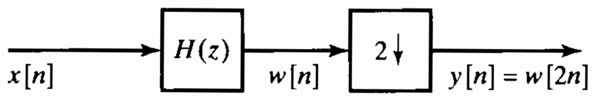


Figure P5.30-1

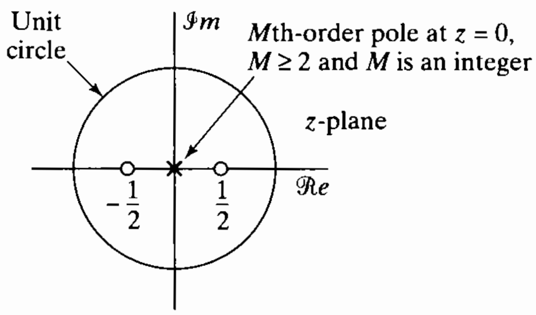


Figure P5.30-2

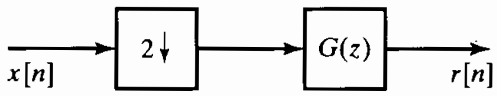


Figure P5.30-3

Determine whether $G(z)$ can be chosen so that $y[n] = r[n]$ for any input $x[n]$. If your answer is no, clearly explain. If your answer is yes, specify $G(z)$. If your answer depends on the value of M, clearly explain how. (M is constrained to be an integer greater than or equal to 2.)

5.31. Consider a linear time-invariant system whose system function is

$$H(z) = \frac{z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 3z^{-1})}.$$

(a) Suppose the system is known to be stable. Determine the output $y[n]$ when the input $x[n]$ is the unit step sequence.

(b) Suppose the region of convergence of $H(z)$ includes $z = \infty$. Determine $y[n]$ evaluated at $n = 2$ when $x[n]$ is as shown in Figure P5.31-1.

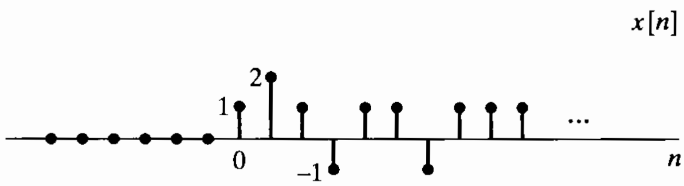


Figure P5.31-1

(c) Suppose we wish to recover $x[n]$ from $y[n]$ by processing $y[n]$ with an LTI system whose impulse response is given by $h_i[n]$. Determine $h_i[n]$. Does $h_i[n]$ depend on the region of convergence of $H(z)$?

5.32. The Fourier transform of a stable linear time-invariant system is purely real and is shown in Figure P5.32-1. Determine whether this system has a stable inverse system.

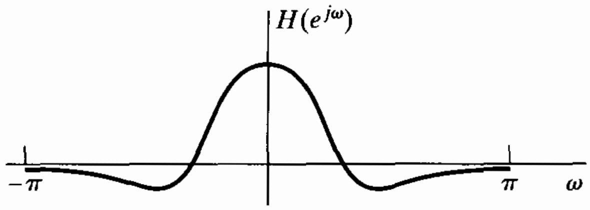


Figure P5.32-1

5.33. A sequence $x[n]$ is the output of a linear time-invariant system whose input is $s[n]$. This system is described by the difference equation

$$x[n] = s[n] - e^{-8\alpha}s[n-8], \tag{P5.33-1}$$

where $0 < \alpha$.

(a) Find the system function

$$H_1(z) = \frac{X(z)}{S(z)},$$

and plot its poles and zeros in the z-plane. Indicate the region of convergence.

(b) We wish to recover $s[n]$ from $x[n]$ with a linear time-invariant system. Find the system function

$$H_2(z) = \frac{Y(z)}{X(z)}$$

such that $y[n] = s[n]$. Find all possible regions of convergence for $H_2(z)$, and for each, tell whether or not the system is causal and/or stable.

(c) Find all possible choices for the impulse response $h_2[n]$ such that

$$y[n] = h_2[n] * x[n] = s[n]. \tag{P5.33-2}$$

(d) For all choices determined in Part (c), demonstrate, by explicitly evaluating the convolution in Eq. P5.33-2, that when $s[n] = \delta[n]$, $y[n] = \delta[n]$.

Note: As discussed in Problem 4.7, Eq. P5.33-1 represents a simple model for a multipath channel. The systems determined in Parts (b) and (c), then, correspond to compensation systems to correct for the multipath distortion.

5.34. Consider a linear time-invariant system whose impulse response is

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n].$$

The input $x[n]$ is zero for $n < 0$, but in general, may be nonzero for $0 \le n \le \infty$. We would like to compute the output $y[n]$ for $0 \le n \le 10^9$, and in particular, we want to compare the use of an FIR filter with that of an IIR filter for obtaining $y[n]$ over this interval.

(a) Determine the linear constant-coefficient difference equation for the IIR system relating $x[n]$ and $y[n]$.

(b) Determine the impulse response $h_1[n]$ of the minimum-length LTI FIR filter whose output $y_1[n]$ is identical to the output $y[n]$ for $0 \le n \le 10^9$.

(c) Specify the linear constant-coefficient difference equation associated with the FIR filter in Part (b).

(d) Compare the number of arithmetic operations (multiplications and additions) required to obtain $y[n]$ for $0 \leq n \leq 10^9$ using the linear constant-coefficient difference equations in Part (a) and in Part (c).

5.35. Consider a causal linear time-invariant system with system function $H(z)$ and real impulse response. $H(z)$ evaluated for $z = e^{j\omega}$ is shown in Figure P5.35-1.

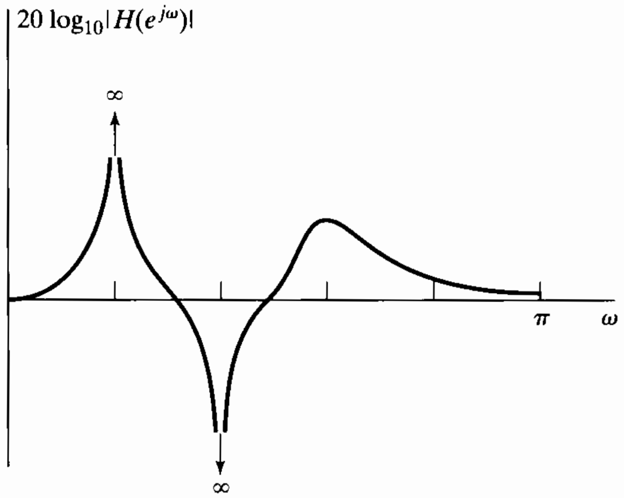


Figure P5.35-1

(a) Carefully sketch a pole–zero plot for $H(z)$ showing all information about the pole and zero locations that can be inferred from the figure.

(b) What can be said about the length of the impulse response?

(c) Specify whether $\sphericalangle H(e^{j\omega})$ is linear.

(d) Specify whether the system is stable.

5.36. A causal linear time-invariant system has the system function

$$H(z) = \frac{(1 - 1.5z^{-1} - z^{-2})(1 + 0.9z^{-1})}{(1 - z^{-1})(1 + 0.7jz^{-1})(1 - 0.7jz^{-1})}.$$

(a) Write the difference equation that is satisfied by the input and the output of the system.

(b) Plot the pole–zero diagram and indicate the region of convergence for the system function.

(c) Sketch $|H(e^{j\omega})|$.

(d) State whether the following are true or false about the system:

(i) The system is stable.

(ii) The impulse response approaches a constant for large n.

(iii) The magnitude of the frequency response has a peak at approximately $\omega = \pm\pi/4$.

(iv) The system has a stable and causal inverse.

5.37. Consider a causal sequence $x[n]$ with the z-transform

$$X(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{5}z\right)}{\left(1 - \frac{1}{6}z\right)}.$$

For what values of α is $\alpha^n x[n]$ a real, minimum-phase sequence?

5.38. Consider the linear time-invariant system whose system function is

$$H(z) = (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1})(1 - 1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1}).$$

(a) Find all causal system functions that result in the same frequency-response magnitude as $H(z)$ and for which the impulse responses are real valued and of the same length as the impulse response associated with $H(z)$. (There are four different such system functions.) Identify which system function is minimum phase and which, to within a time shift, is maximum phase.

(b) Find the impulse responses for the system functions in Part (a).

(c) For each of the sequences in Part (b), compute and plot the quantity

$$E[n] = \sum_{m=0}^{n} (h[m])^2$$

for $0 \le n \le 5$. Indicate explicitly which plot corresponds to the minimum-phase system.

5.39. Shown in Figure P5.39-1 are eight different finite-duration sequences. Each sequence is four points long. The magnitude of the Fourier transform is the same for all sequences. Which of the sequences has all the zeros of its z-transform *inside* the unit circle?

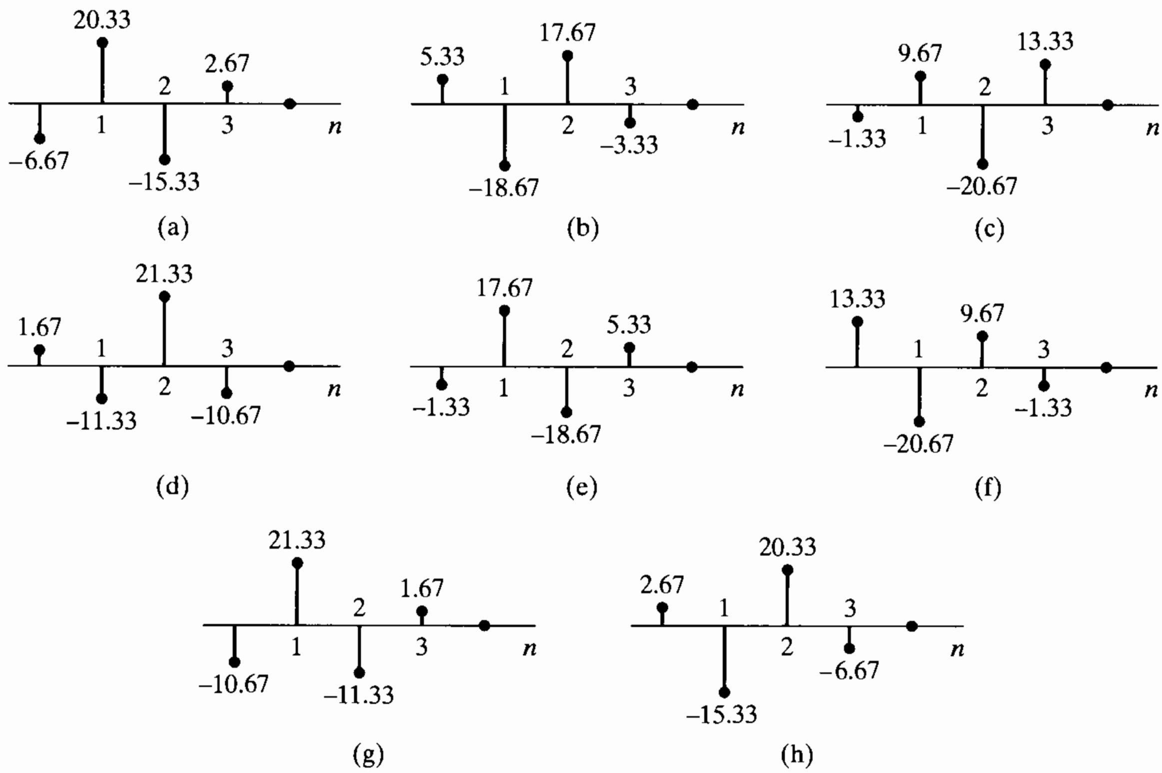


Figure P5.39-1

5.40. Each of the pole–zero plots in Figure P5.40-1, together with the specification of the region of convergence, describes a linear time-invariant system with system function $H(z)$. In each case, determine whether any of the following statements are true. Justify your answer with a brief statement or a counterexample.

(a) The system is a zero-phase or a generalized linear-phase system.

(b) The system has a stable inverse $H_i(z)$.

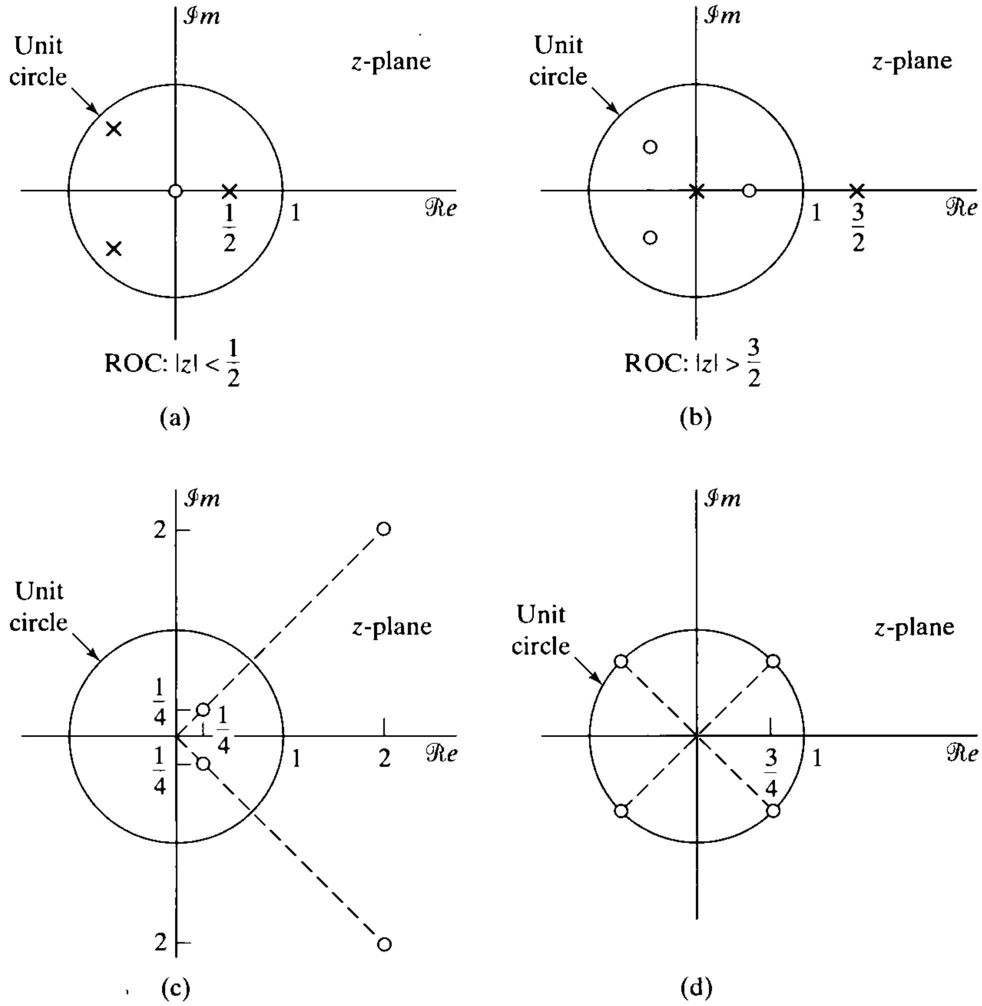


Figure P5.40-1

5.41. Figure P5.41-1 shows two different interconnections of three systems. The impulse responses $h_1[n]$, $h_2[n]$, and $h_3[n]$ are as shown in Figure P5.41-2. Determine whether system A and/or system B is a generalized linear-phase system.

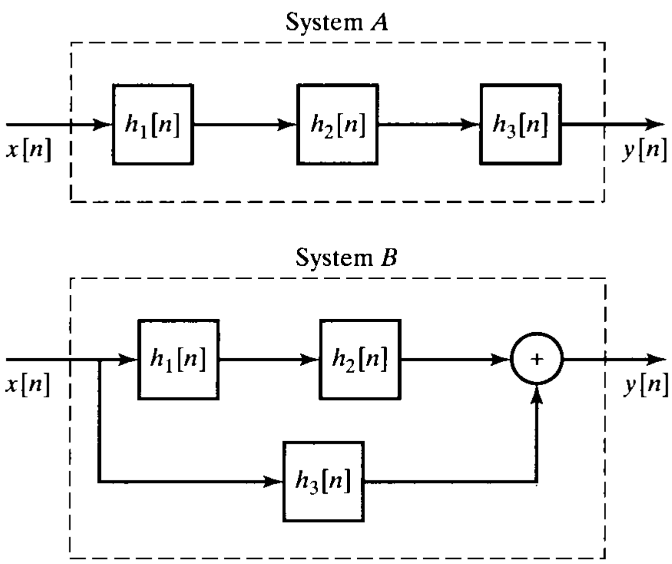


Figure P5.41-1

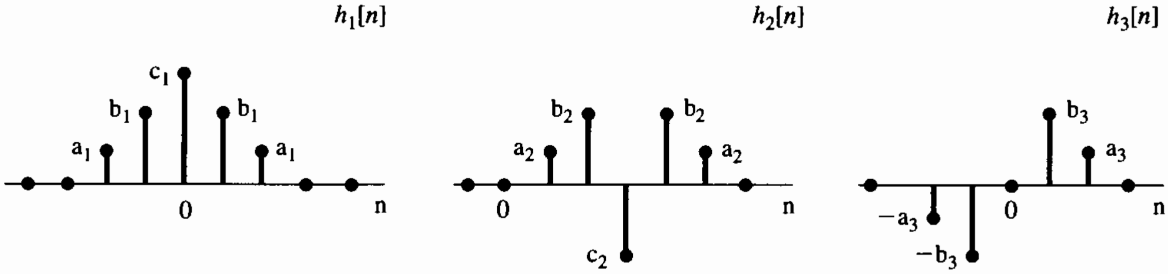


Figure P5.41-2

5.42. The overall system of Figure P5.42-1 is a discrete-time linear time-invariant system with frequency response $H(e^{j\omega})$ and impulse response $h[n]$.

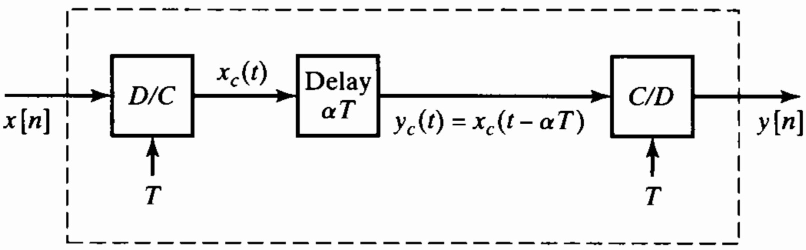


Figure P5.42-1

(a) $H(e^{j\omega})$ can be expressed in the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{j\phi(\omega)},$$

with $A(e^{j\omega})$ real. Determine and sketch $A(e^{j\omega})$ and $\phi(\omega)$ for $|\omega| < \pi$.

(b) Sketch $h[n]$ for the following:

(i) $\alpha = 3$

(ii) $\alpha = 3\frac{1}{2}$

(iii) $\alpha = 3\frac{1}{4}$

(c) Consider a discrete-time linear time-invariant system for which

$$H(e^{j\omega}) = A(e^{j\omega})e^{j\alpha\omega}, \qquad |\omega| < \pi,$$

with $A(e^{j\omega})$ real. What can be said about the symmetry of $h[n]$ for the following?

(i) α = integer

(ii) $\alpha = M/2$, where M is an odd integer

(iii) General α

5.43. Consider the class of FIR filters that have $h[n]$ real, $h[n] = 0$ for $n < 0$ and $n > M$, and one of the following symmetry properties:

$$\text{Symmetric: } h[n] = h[M - n]$$

$$\text{Antisymmetric: } h[n] = -h[M - n]$$

All filters in this class have generalized linear phase, i.e., have frequency response of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta},$$

where $A(e^{j\omega})$ is a real function of ω, α is a real constant, and β is a real constant.

For the following table, show that $A(e^{j\omega})$ has the indicated form, and find the values of α and β.

Type	Symmetry	Filter length $(M+1)$	Form of $A(e^{j\omega})$	α	β
I	Symmetric	Odd	$\sum_{n=0}^{M/2} a[n]\cos\omega n$		
II	Symmetric	Even	$\sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-1/2)$		
III	Antisymmetric	Odd	$\sum_{n=1}^{M/2} c[n]\sin\omega n$		
IV	Antisymmetric	Even	$\sum_{n=1}^{(M+1)/2} d[n]\sin\omega(n-1/2)$		

Here are several helpful suggestions.

- For type I filters, first show that $H(e^{j\omega})$ can be written in the form

$$H(e^{j\omega}) = \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{n=0}^{(M-2)/2} h[M-n]e^{-j\omega[M-n]} + h[M/2]e^{-j\omega(M/2)}.$$

- The analysis for type III filters is very similar to that for type I, with the exception of a sign change and removal of one of the preceding terms.
- For type II filters, first write $H(e^{j\omega})$ in the form

$$H(e^{j\omega}) = \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=0}^{(M-1)/2} h[M-n]e^{-j\omega[M-n]},$$

and then pull out a common factor of $e^{-j\omega(M/2)}$ from both sums.
- The analysis for type IV filters is very similar to that for type II filters.

5.44. Let $h_{\text{lp}}[n]$ denote the impulse response of an FIR generalized linear-phase lowpass filter. The impulse response $h_{\text{hp}}[n]$ of an FIR generalized linear-phase highpass filter can be obtained by the transformation

$$h_{\text{hp}}[n] = (-1)^n h_{\text{lp}}[n].$$

If we decide to design a highpass filter using this transformation and we wish the resulting highpass filter to be symmetric, which of the four types of generalized linear-phase FIR filters can we use for the design of the lowpass filter? Your answer should consider *all* the possible types.

5.45. A causal linear time-invariant discrete-time system has system function

$$H(z) = \frac{(1-0.5z^{-1})(1+4z^{-2})}{(1-0.64z^{-2})}.$$

(a) Find expressions for a minimum-phase system $H_1(z)$ and an all-pass system $H_{\text{ap}}(z)$ such that

$$H(z) = H_1(z)H_{\text{ap}}(z).$$

(b) Find expressions for a different minimum-phase system $H_2(z)$ and a generalized linear-phase FIR system $H_{\text{lin}}(z)$ such that

$$H(z) = H_2(z)H_{\text{lin}}(z).$$

5.46. **(a)** A minimum-phase system has system function $H_{\text{min}}(z)$ is such that

$$H_{\text{min}}(z)H_{\text{ap}}(z) = H_{\text{lin}}(z),$$

where $H_{\text{ap}}(z)$ is an all-pass system function and $H_{\text{lin}}(z)$ is a causal generalized linear-phase system. What does this information tell you about the poles and zeros of $H_{\text{min}}(z)$?

(b) A generalized linear-phase FIR system has an impulse response with real values and $h[n] = 0$ for $n < 0$ and for $n \geq 8$, and $h[n] = -h[7-n]$. The system function of this system has a zero at $z = 0.8e^{j\pi/4}$ and another zero at $z = -2$. What is $H(z)$?

5.47. Consider an LTI system with input $x[n]$ and output $y[n]$. When the input to the system is

$$x[n] = 5\frac{\sin(0.4\pi n)}{\pi n} + 10\cos(0.5\pi n),$$

the corresponding output is

$$y[n] = 10\frac{\sin[0.3\pi(n-10)]}{\pi(n-10)}.$$

Determine the frequency response $H(e^{j\omega})$ and the impulse response $h[n]$ for the LTI system.

5.48. Figure P5.48-1 shows the pole–zero plots for three different causal LTI systems with real impulse responses. Indicate which of the following properties apply to each of the systems pictured: stable, IIR, FIR, minimum phase, all-pass, generalized linear phase, positive group delay at all ω.

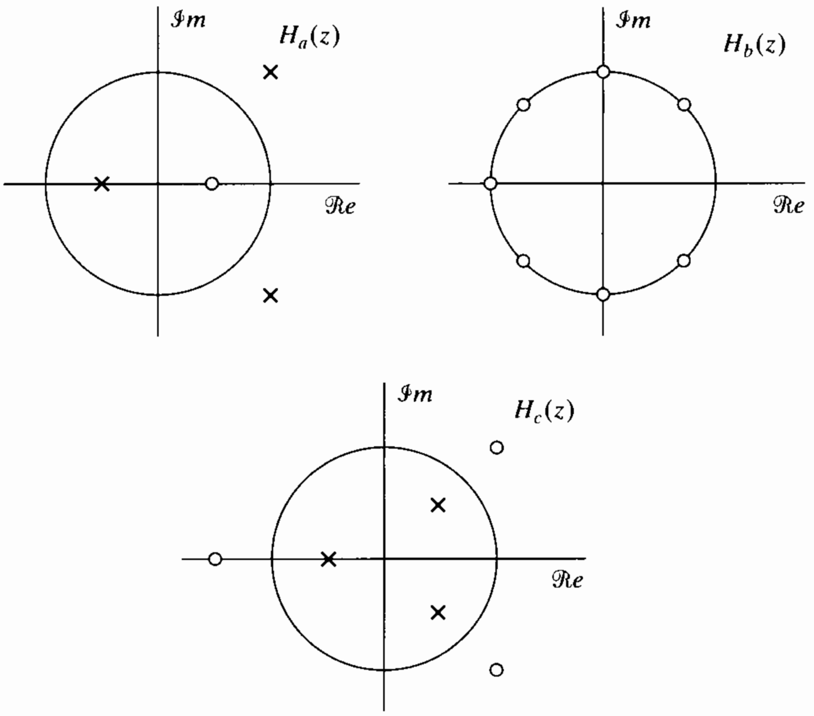


Figure P5.48-1

5.49. Let S_1 be an LTI system with system function:

$$H_1(z) = \frac{1-z^{-5}}{1-z^{-1}}, \qquad |z| > 0,$$

and impulse response $h_1[n]$.

(a) Is S_1 causal? Explain.

(b) Let $g[n] = h_1[n] * h_2[n]$. Specify an $h_2[n]$ such that $g[n]$ has at least nine nonzero samples and $g[n]$ can be considered the impulse response of a causal LTI system with strictly linear phase; i.e., $G(e^{j\omega}) = |G(e^{j\omega})|e^{-j\omega n_0}$ for some integer n_0.

(c) Let $q[n] = h_1[n] * h_3[n]$. Specify an $h_3[n]$ such that

$$q[n] = \delta[n] \qquad \text{for } 0 \le n \le 19.$$

5.50. The LTI systems $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are generalized linear-phase systems. Which, if any, of the following systems also must be generalized linear-phase systems?

(a)

$$G_1(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

(b)

$$G_2(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$$

(c)

$$G_3(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\theta})H_2(e^{j(\omega-\theta)})d\theta$$

5.51. This problem concerns a discrete-time filter with a real-valued impulse response $h[n]$. Determine whether the following statement is true or false:

> **Statement:** If the group delay of the filter is a constant for $0 < \omega < \pi$, then the impulse response must have the property that either
>
> $$h[n] = h[M - n]$$
>
> or
>
> $$h[n] = -h[M - n],$$
>
> where M is an integer.

If you believe that the statement is true, clearly show your reasoning. If you believe that it is false, provide a counterexample.

5.52. The system function $H_{II}(z)$ represents a type II FIR generalized linear-phase system with impulse response $h_{II}[n]$. This system is cascaded with an LTI system whose system function is $(1 - z^{-1})$ to produce a third system with system function $H(z)$ and impulse response $h[n]$. Prove that the overall system is a generalized linear-phase system, and determine what type of linear phase system it is.

5.53. In this problem, you will consider three different LTI systems. All three are causal and have real impulse responses. You will be given additional information about each system. Using this information, state as much as possible about the poles and zeros of each system function and about the length of the impulse response of the system.

(a) $H_1(z)$ has a pole at $z = 0.9e^{j\pi/3}$, and when $x[n] = u[n]$, $\lim_{n\to\infty} y[n] = 0$.

(b) $H_2(z)$ has a zero at $z = 0.8e^{j\pi/4}$, $H_2(e^{j\omega})$ has linear phase with $\sphericalangle H_2(e^{j\omega}) = -2.5\omega$, and $20\log_{10}|H_2(e^{j0})| = -\infty$.

(c) $H_3(z)$ has a pole at $z = 0.8e^{j\pi/4}$ and $|H_3(e^{j\omega})| = 1$ for all ω.

5.54. The following three things are known about a signal $x[n]$ with z-transform $X(z)$:

(i) $x[n]$ is real valued and minimum phase,

(ii) $x[n]$ is zero outside the interval $0 \le n \le 4$,

(iii) $X(z)$ has a zero at $z = \frac{1}{2}e^{j\pi/4}$ and a zero at $z = \frac{1}{2}e^{j3\pi/4}$.

Based on this information, answer the following questions:

(a) Is $X(z)$ rational? Justify your answer.

(b) Sketch the complete pole–zero plot for $X(z)$ and specify its ROC.

(c) If $y[n]*x[n] = \delta[n]$ and $y[n]$ is rightsided, sketch the pole–zero plot for $Y(z)$ and specify its ROC.

5.55. Consider a real sequence $x[n]$ and its DTFT $X(e^{j\omega})$. Given the following information, determine and plot the sequence $x[n]$:

1. $x[n]$ is a finite-length sequence.
2. At $z = 0$, $X(z)$ has exactly five poles and no zeros. $X(z)$ *may* have poles or zeros at other locations.
3. The unwrapped phase function is

$$\arg\left[X(e^{j\omega})\right] = \begin{cases} -\alpha\omega + \frac{\pi}{2}, & 0 < \omega < \pi, \\ -\alpha\omega - \frac{\pi}{2}, & -\pi < \omega < 0, \end{cases}$$

for some real constant α.

4. The group delay of the sequence evaluated at $\omega = \frac{\pi}{2}$ is 2; i.e.,

$$\text{grd}\left[X(e^{j\omega})\right]_{\omega=\pi/2} = 2.$$

5.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 28.$$

6. If $y[n] = x[n] * u[n]$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})d\omega = 4,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega}d\omega = 6.$$

7. $X(e^{j\omega})|_{\omega=\pi} = 0.$
8. The sequence $v[n]$ whose DTFT is $V(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$ satisfies $v[5] = -\frac{3}{2}$.

5.56. Let S_1 be a causal and stable LTI system with impulse response $h_1[n]$ and frequency response $H_1(e^{j\omega})$. The input $x[n]$ and output $y[n]$ for S_1 are related by the difference equation

$$y[n] - y[n-1] + \tfrac{1}{4}y[n-2] = x[n].$$

(a) If an LTI system S_2 has a frequency response given by $H_2(e^{j\omega}) = H_1(-e^{j\omega})$, would you characterize S_2 as being a lowpass filter, a bandpass filter, or a highpass filter? Justify your answer.

(b) Let S_3 be a causal LTI system whose frequency response $H_3(e^{j\omega})$ has the property that

$$H_3(e^{j\omega})H_1(e^{j\omega}) = 1.$$

Is S_3 a minimum-phase filter? Could S_3 be classified as one of the four types of FIR filters with generalized linear phase? Justify your answers.

(c) Let S_4 be a stable and *noncausal* LTI system whose frequency response is $H_4(e^{j\omega})$ and whose input $x[n]$ and output $y[n]$ are related by the difference equation:

$$y[n] + \alpha_1 y[n-1] + \alpha_2 y[n-2] = \beta_0 x[n],$$

where α_1, α_2, and β_0 are all real and nonzero constants. Specify a value for α_1, a value for α_2, and a value for β_0 such that $|H_4(e^{j\omega})| = |H_1(e^{j\omega})|$.

(d) Let S_5 be an FIR filter whose impulse response is $h_5[n]$ and whose frequency response, $H_5(e^{j\omega})$, has the property that $H_5(e^{j\omega}) = |A(e^{j\omega})|^2$ for some DTFT $A(e^{j\omega})$ (i.e., S_5 is a *zero-phase* filter). Find $h_5[n]$ such that $h_5[n] * h_1[n]$ is the impulse response of a noncausal FIR filter.

Extension Problems

5.57. In the system shown in Figure P5.57-1, assume that the input can be expressed in the form

$$x[n] = s[n]\cos(\omega_0 n).$$

Assume also that $s[n]$ is lowpass and relatively narrowband; i.e., $S(e^{j\omega}) = 0$ for $|\omega| > \Delta$, with Δ very small and $\Delta \ll \omega_0$, so that $X(e^{j\omega})$ is narrowband around $\omega = \pm\omega_0$.

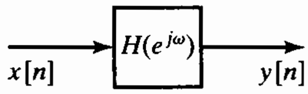


Figure P5.57-1

(a) If $|H(e^{j\omega})| = 1$ and $\sphericalangle H(e^{j\omega})$ is as illustrated in Figure P5.57-2, show that $y[n] = s[n]\cos(\omega_0 n - \phi_0)$.

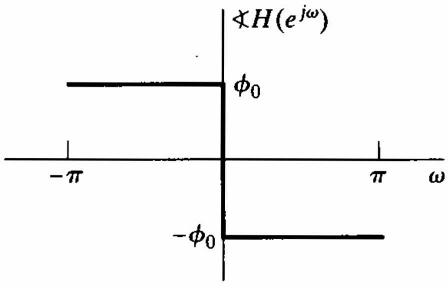


Figure P5.57-2

(b) If $|H(e^{j\omega})| = 1$ and $\sphericalangle H(e^{j\omega})$ is as illustrated in Figure P5.57-3, show that $y[n]$ can be expressed in the form

$$y[n] = s[n - n_d]\cos(\omega_0 n - \phi_0 - \omega_0 n_d).$$

Show also that $y[n]$ can be equivalently expressed as

$$y[n] = s[n - n_d]\cos(\omega_0 n - \phi_1),$$

where $-\phi_1$ is the phase of $H(e^{j\omega})$ at $\omega = \omega_0$.

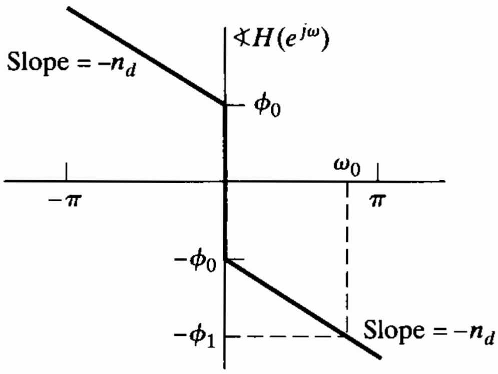


Figure P5.57-3

(c) The group delay associated with $H(e^{j\omega})$ is defined as

$$\tau_{gr}(\omega) = -\frac{d}{d\omega}\arg[H(e^{j\omega})],$$

and the phase delay is defined as $\tau_{ph}(\omega) = -(1/\omega)\sphericalangle H(e^{j\omega})$. Assume that $|H(e^{j\omega})|$ is unity over the bandwidth of $x[n]$. Based on your results in Parts (a) and (b) and on the assumption that $x[n]$ is narrowband, show that if $\tau_{gr}(\omega_0)$ and $\tau_{ph}(\omega_0)$ are both integers, then

$$y[n] = s[n - \tau_{gr}(\omega_0)]\cos\{\omega_0[n - \tau_{ph}(\omega_0)]\}.$$

This equation shows that, for a narrowband signal $x[n]$, $\sphericalangle H(e^{j\omega})$ effectively applies a delay of $\tau_{gr}(\omega_0)$ to the envelope $s[n]$ of $x[n]$ and a delay of $\tau_{ph}(\omega_0)$ to the carrier $\cos \omega_0 n$.

(d) Referring to the discussion in Section 4.5 associated with noninteger delays of a sequence, how would you interpret the effect of group delay and phase delay if $\tau_{gr}(\omega_0)$ or $\tau_{ph}(\omega_0)$ (or both) is not an integer?

5.58. The signal $y[n]$ is the output of a linear time-invariant system with input $x[n]$, which is zero-mean white noise. The system is described by the difference equation

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k], \qquad b_0 = 1.$$

(a) What is the z-transform $\Phi_{yy}(z)$ of the autocorrelation function $\phi_{yy}[n]$?

Sometimes it is of interest to process $y[n]$ with a linear filter such that the power spectrum of the linear filter's output will be flat when the input to the linear filter is $y[n]$. This procedure is known as "whitening" $y[n]$, and the linear filter that accomplishes the task is said to be the "whitening filter" for the signal $y[n]$. Suppose that we know the autocorrelation function $\phi_{yy}[n]$ and its z-transform $\Phi_{yy}(z)$, but not the a_k's and the b_k's.

(b) Discuss a procedure for finding a system function $H_w(z)$ of the whitening filter.

(c) Is the whitening filter unique?

5.59. In many practical situations, we are faced with the problem of recovering a signal that has been "blurred" by a convolution process. We can model this blurring process as a linear filtering operation, as depicted in Figure P5.59-1, where the blurring impulse response is as shown in Figure P5.59-2. This problem will consider ways to recover $x[n]$ from $y[n]$.

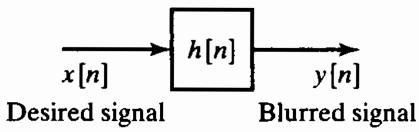


Figure P5.59-1

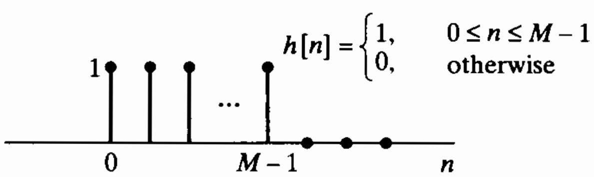


Figure P5.59-2

(a) One approach to recovering $x[n]$ from $y[n]$ is to use an inverse filter; i.e., $y[n]$ is filtered by a system whose frequency response is

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})},$$

where $H(e^{j\omega})$ is the Fourier transform of $h[n]$. For the impulse response $h[n]$ shown in Figure P5.59-2, discuss the practical problems involved in implementing the inverse filtering approach. Be complete, but also be brief and to the point.

(b) Because of the difficulties involved in inverse filtering, the following approach is suggested for recovering $x[n]$ from $y[n]$: The blurred signal $y[n]$ is processed by the system shown in Figure P5.59-3, which produces an output $w[n]$ from which we can extract an improved replica of $x[n]$. The impulse responses $h_1[n]$ and $h_2[n]$ are shown in Figure P5.59-4. Explain in detail the working of this system. In particular, state precisely the conditions under which we can recover $x[n]$ exactly from $w[n]$. *Hint:* Consider the impulse response of the overall system from $x[n]$ to $w[n]$.

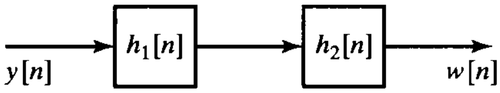


Figure P5.59-3

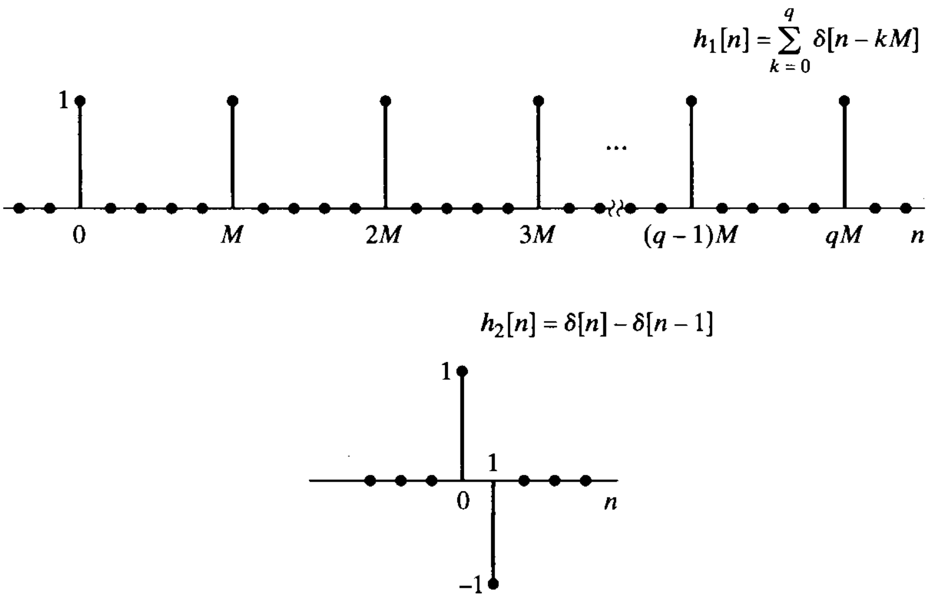


Figure P5.59-4

(c) Let us now attempt to generalize this approach to arbitrary finite-length blurring impulse responses $h[n]$; i.e., assume only that $h[n] = 0$ for $n < 0$ or $n \geq M$. Further, assume that $h_1[n]$ is the same as in Figure P5.59-4. How must $H_2(z)$ and $H(z)$ be related for the system to work as in Part (b)? What condition must $H(z)$ satisfy in order that it be possible to implement $H_2(z)$ as a causal system?

5.60. In this problem, we demonstrate that, for a rational z-transform, a factor of the form $(z - z_0)$ and a factor of the form $z/(z - z_0^*)$ contribute the same phase.

(a) Let $H(z) = z - 1/a$, where a is real and $0 < a < 1$. Sketch the poles and zeros of the system, including an indication of those at $z = \infty$. Determine $\sphericalangle H(e^{j\omega})$, the phase of the system.

(b) Let $G(z)$ be specified such that it has poles at the conjugate-reciprocal locations of zeros of $H(z)$ and zeros at the conjugate-reciprocal locations of poles of $H(z)$, including those at zero and ∞. Sketch the pole–zero diagram of $G(z)$. Determine $\sphericalangle G(e^{j\omega})$, the phase of the system, and show that it is identical to $\sphericalangle H(e^{j\omega})$.

5.61. Prove the validity of the following two statements:

(a) The convolution of two minimum-phase sequences is also a minimum-phase sequence.

(b) The sum of two minimum-phase sequences is not necessarily a minimum-phase sequence. Specifically, give an example of both a minimum-phase and a nonminimum-phase sequence that can be formed as the sum of two minimum-phase sequences.

5.62. A sequence is defined by the relationship

$$r[n] = \sum_{m=-\infty}^{\infty} h[m]h[n+m] = h[n] * h[-n],$$

where $h[n]$ is a minimum-phase sequence and

$$r[n] = \tfrac{4}{3}\left(\tfrac{1}{2}\right)^n u[n] + \tfrac{4}{3}2^n u[-n-1].$$

(a) Find $R(z)$ and sketch the pole–zero diagram.

(b) Determine the minimum-phase sequence $h[n]$ to within a scale factor of ± 1. Also, determine the z-transform $H(z)$ of $h[n]$.

5.63. A *maximum-phase* sequence is a stable sequence whose z-transform has all its poles and zeros *outside* the unit circle.

(a) Show that maximum-phase sequences are anticausal, i.e., that they are zero for $n > 0$.

FIR maximum-phase sequences can be made causal by including a finite amount of delay. A finite-duration causal maximum-phase sequence having a Fourier transform of a given magnitude can be obtained by reflecting all the zeros of the z-transform of a minimum-phase sequence to conjugate-reciprocal positions outside the unit circle. That is, we can express the z-transform of a maximum-phase causal finite-duration sequence as

$$H_{\max}(z) = H_{\min}(z) H_{\text{ap}}(z).$$

Obviously, this process ensures that $|H_{\max}(e^{j\omega})| = |H_{\min}(e^{j\omega})|$. Now, the z-transform of a finite-duration minimum-phase sequence can be expressed as

$$H_{\min}(z) = h_{\min}[0] \prod_{k=1}^{M} (1 - c_k z^{-1}), \qquad |c_k| < 1.$$

(b) Obtain an expression for the all-pass system function required to reflect all the zeros of $H_{\min}(z)$ to positions outside the unit circle.

(c) Show that $H_{\max}(z)$ can be expressed as

$$H_{\max}(z) = z^{-M} H_{\min}(z^{-1}).$$

(d) Using the result of Part (c), express the maximum-phase sequence $h_{\max}[n]$ in terms of $h_{\min}[n]$.

5.64. It is not possible to obtain a causal and stable inverse system (a perfect compensator) for a nonminimum-phase system. In this problem, we study an approach to compensating for only the magnitude of the frequency response of a nonminimum-phase system.

Suppose that a stable nonminimum-phase linear time-invariant discrete-time system with a rational system function $H(z)$ is cascaded with a compensating system $H_c(z)$ as shown in Figure P5.64-1.

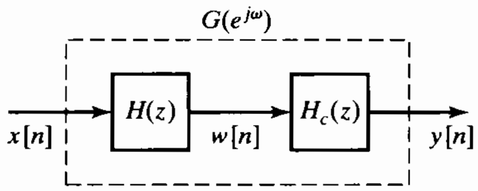


Figure P5.64-1

(a) How should $H_c(z)$ be chosen so that it is stable and causal and so that the magnitude of the overall effective frequency response is unity? (Recall that $H(z)$ can always be represented as $H(z) = H_{\text{ap}}(z) H_{\min}(z)$.)

(b) What are the corresponding system functions $H_c(z)$ and $G(z)$?

(c) Assume that

$$H(z) = (1 - 0.8e^{j0.3\pi} z^{-1})(1 - 0.8e^{-j0.3\pi} z^{-1})(1 - 1.2e^{j0.7\pi} z^{-1})(1 - 1.2e^{-j0.7\pi} z^{-1}).$$

Find $H_{\min}(z)$, $H_{\text{ap}}(z)$, $H_c(z)$, and $G(z)$ for this case, and construct the pole–zero plots for each system function.

5.65. Let $h_{\min}[n]$ denote a minimum-phase sequence with z-transform $H_{\min}(z)$. If $h[n]$ is a causal nonminimum-phase sequence whose Fourier transform magnitude is equal to $|H_{\min}(e^{j\omega})|$, show that

$$|h[0]| < |h_{\min}[0]|.$$

(Use the initial-value theorem together with Eq. (5.103).)

5.66. One of the interesting and important properties of minimum-phase sequences is the minimum-energy delay property; i.e., of all the causal sequences having the same Fourier transform magnitude function $|H(e^{j\omega})|$, the quantity

$$E[n] = \sum_{m=0}^{n} |h[m]|^2$$

is maximum for all $n \geq 0$ when $h[n]$ is the minimum-phase sequence. This result is proved as follows: Let $h_{\min}[n]$ be a minimum-phase sequence with z-transform $H_{\min}(z)$. Furthermore, let z_k be a zero of $H_{\min}(z)$ so that we can express $H_{\min}(z)$ as

$$H_{\min}(z) = Q(z)(1 - z_k z^{-1}), \qquad |z_k| < 1,$$

where $Q(z)$ is again minimum phase. Now consider another sequence $h[n]$ with z-transform $H(z)$ such that

$$|H(e^{j\omega})| = |H_{\min}(e^{j\omega})|$$

and such that $H(z)$ has a zero at $z = 1/z_k^*$ instead of at z_k.

(a) Express $H(z)$ in terms of $Q(z)$.

(b) Express $h[n]$ and $h_{\min}[n]$ in terms of the minimum-phase sequence $q[n]$ that has z-transform $Q(z)$.

(c) To compare the distribution of energy of the two sequences, show that

$$\varepsilon = \sum_{m=0}^{n} |h_{\min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 = (1 - |z_k|^2)|q[n]|^2.$$

(d) Using the result of Part (c), argue that

$$\sum_{m=0}^{n} |h[m]|^2 \leq \sum_{m=0}^{n} |h_{\min}[m]|^2 \qquad \text{for all } n.$$

5.67. A causal all-pass system $H_{ap}(z)$ has input $x[n]$ and output $y[n]$.

(a) If $x[n]$ is a real minimum-phase sequence (which also implies that $x[n] = 0$ for $n < 0$), using Eq. (5.118), show that

$$\sum_{k=0}^{n} |x[k]|^2 \geq \sum_{k=0}^{n} |y[k]|^2. \qquad \text{(P5.67-1)}$$

(b) Show that Eq. (P5.67-1) holds even if $x[n]$ is not minimum phase, but is zero for $n < 0$.

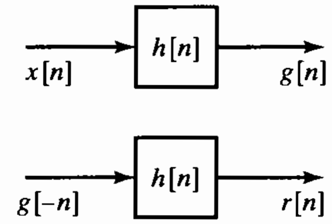
5.68. In the design of either continuous-time or discrete-time filters, we often approximate a specified magnitude characteristic without particular regard to the phase. For example, standard design techniques for lowpass and bandpass filters are derived from a consideration of the magnitude characteristics only.

In many filtering problems, we would prefer that the phase characteristics be zero or linear. For causal filters, it is impossible to have zero phase. However, for many filtering applications, it is not necessary that the impulse response of the filter be zero for $n < 0$ if the processing is not to be carried out in real time.

One technique commonly used in discrete-time filtering when the data to be filtered are of finite duration and are stored, for example, in computer memory is to process the data forward and then backward through the same filter.

Let $h[n]$ be the impulse response of a causal filter with an arbitrary phase characteristic. Assume that $h[n]$ is real, and denote its Fourier transform by $H(e^{j\omega})$. Let $x[n]$ be the data that we want to filter.

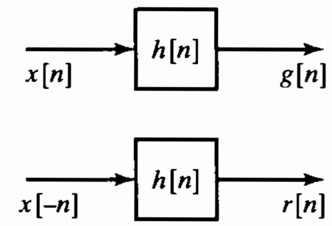
(a) *Method A:* The filtering operation is performed as shown in Figure P5.68-1.



$s[n] = r[-n]$ **Figure P5.68-1**

1. Determine the overall impulse response $h_1[n]$ that relates $x[n]$ to $s[n]$, and show that it has a zero-phase characteristic.
2. Determine $|H_1(e^{j\omega})|$, and express it in terms of $|H(e^{j\omega})|$ and $\sphericalangle H(e^{j\omega})$.

(b) *Method B:* As depicted in Figure P5.68b-2, process $x[n]$ through the filter $h[n]$ to get $g[n]$. Also, process $x[n]$ backward through $h[n]$ to get $r[n]$. The output $y[n]$ is then taken as the sum of $g[n]$ and $r[-n]$. This composite set of operations can be represented by a filter with input $x[n]$, output $y[n]$, and impulse response $h_2[n]$.



$y[n] = g[n] + r[-n]$ **Figure P5.68-2**

1. Show that the composite filter $h_2[n]$ has a zero-phase characteristic.
2. Determine $|H_2(e^{j\omega})|$, and express it in terms of $|H(e^{j\omega})|$ and $\sphericalangle H(e^{j\omega})$.

(c) Suppose that we are given a sequence of finite duration on which we would like to perform a bandpass zero-phase filtering operation. Furthermore, assume that we are given the bandpass filter $h[n]$, with frequency response as specified in Figure P5.68-3, which has the magnitude characteristic that we desire, but has linear phase. To achieve zero phase, we could use either method A or B. Determine and sketch $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$. From these results, which method would you use to achieve the desired bandpass filtering operation? Explain why. More generally, if $h[n]$ has the desired magnitude, but a nonlinear phase characteristic, which method is preferable to achieve a zero-phase characteristic?

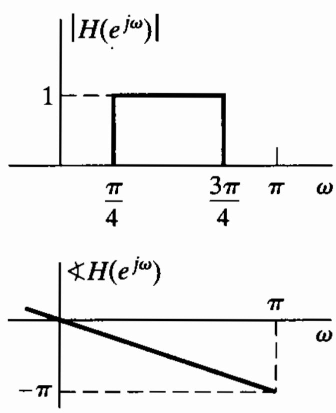


Figure P5.68-3

5.69. Determine whether the following statement is true or false. If it is true, concisely state your reasoning. If it is false, give a counterexample.

Statement: If the system function $H(z)$ has poles anywhere other than at the origin or infinity, then the system cannot be a zero-phase or a generalized linear-phase system.

5.70. Figure P5.70-1 shows the zeros of the system function $H(z)$ for a real causal linear-phase FIR filter. All of the indicated zeros represent factors of the form $(1 - az^{-1})$. The corresponding poles at $z = 0$ for these factors are not shown in the figure. The filter has approximately unity gain in its passband.

(a) One of the zeros has magnitude 0.5 and angle 153 degrees. Determine the exact location of as many other zeros as you can from this information.

(b) The system function $H(z)$ is used in the system for discrete-time processing of continuous-time signals shown in Figure 4.11, with the sampling period $T = 0.5$ msec. Assume that the continuous-time input $X_c(j\Omega)$ is bandlimited and that the sampling rate is high enough to avoid aliasing. What is the time delay (in msec) through the entire system, assuming that both C/D and D/C conversion require negligible amounts of time?

(c) For the system in Part (b), sketch the overall effective continuous-time frequency response $20 \log_{10} |H_{\text{eff}}(j\Omega)|$ for $0 \leq \Omega \leq \pi/T$ as accurately as possible using the given information. Estimate the frequencies at which $H_{\text{eff}}(j\Omega) = 0$, and mark them on your plot.

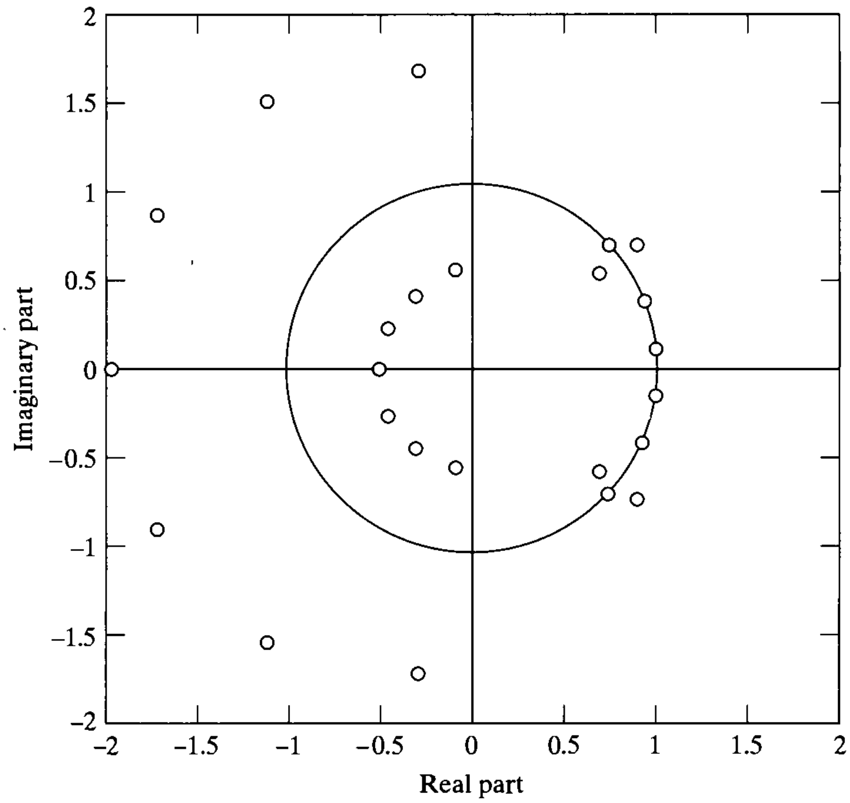


Figure P5.70-1

5.71. A signal $x[n]$ is processed through an LTI system $H(z)$ and then downsampled by a factor of 2 to yield $y[n]$ as indicated in Figure P5.71-1. Also, as shown in the same figure, $x[n]$ is first downsampled and then processed through an LTI system $G(z)$ to obtain $r[n]$.

(a) Specify a choice for $H(z)$ (other than a constant) and $G(z)$ so that $r[n] = y[n]$ for an arbitrary $x[n]$.

(b) Specify a choice for $H(z)$ so that there is no choice for $G(z)$ that will result in $r[n] = y[n]$ for an arbitrary $x[n]$.

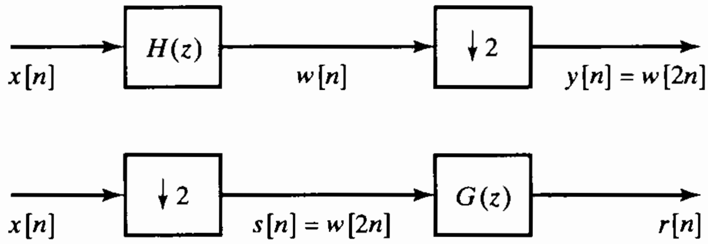


Figure P5.71-1

(c) Determine as general a set of conditions as you can on $H(z)$ such that $G(z)$ can be chosen so that $r[n] = y[n]$ for an arbitrary $x[n]$. The conditions should not depend on $x[n]$. If you first develop the conditions in terms of $h[n]$, restate them in terms of $H(z)$.

(d) For the conditions determined in Part (c), what is $g[n]$ in terms of $h[n]$ so that $r[n] = y[n]$.

5.72. Consider a discrete-time LTI system with a real-valued impulse response $h[n]$. We want to find $h[n]$, or equivalently, the system function $H(z)$ from the autocorrelation $c_{hh}[\ell]$ of the impulse response. The definition of the autocorrelation is

$$c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]h[k+\ell].$$

(a) If the system $h[n]$ is causal and stable, can you uniquely recover $h[n]$ from $c_{hh}[\ell]$? Justify your answer.

(b) Assume that $h[n]$ is causal and stable and that, in addition, you know that the system function has the form

$$H(z) = \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

for some finite a_k. Can you uniquely recover $h[n]$ from $c_{hh}[\ell]$? Clearly justify your answer.

5.73. Let $h[n]$ and $H(z)$ denote the impulse response and system function of a stable all-pass LTI system. Let $h_i[n]$ denote the impulse response of the (stable) LTI inverse system. Assume that $h[n]$ is real. Show that $h_i[n] = h[-n]$.

5.74. Consider a real-valued sequence $x[n]$ for which $X(e^{j\omega}) = 0$ for $\frac{\pi}{4} \le |\omega| \le \pi$. One sequence value of $x[n]$ may have been corrupted, and we would like to recover it approximately or exactly. With $g[n]$ denoting the corrupted signal,

$$g[n] = x[n] \qquad \text{for } n \neq n_0,$$

and $g[n_0]$ is real but not related to $x[n_0]$. In each of the following two cases, specify a practical algorithm for recovering $x[n]$ from $g[n]$ exactly or approximately.

(a) The exact value of n_0 is not known, but we know that n_0 is an odd number.

(b) Nothing about n_0 is known.

5.75. Show that if $h[n]$ is an N-point FIR filter such that $h[n] = h[N-1-n]$ and $H(z_0) = 0$, then $H(1/z_0) = 0$. This shows that even symmetric linear-phase FIR filters have zeros that are reciprocal images. (If $h[n]$ is real, the zeros also will be real or will occur in complex conjugates.)