B

CONTINUOUS-TIME FILTERS

The techniques discussed in Sections 7.1 and 7.2 for designing IIR digital filters rely on the availability of appropriate continuous-time filter designs. In this appendix,we briefly summarize the characteristics of several classes of lowpass filter approximations that we referred to in Chapter 7. More detailed discussions of these classes of filters appear in Guillemin (1957), Weinberg (1975), and Parks and Burrus (1987). Extensive design tables and formulas are found in Zverev (1967), and a variety of computer programs are available for computer-aided design of IIR digital filters based on analog filter approximations. (See, for example, Gray and Markel, 1976; DSP Committee 1979; Mersereau et al., 1984; and Mathworks, 1998.)

B.1 BUTTERWORTH LOWPASS FILTERS

Butterworth lowpass filters are defined by the property that the magnitude response is maximally flat in the passband. For an Nth-order lowpass filter, this means that the first $(2N-1)$ derivatives of the magnitude-squared function are zero at $\Omega = 0$. Another property is that the magnitude response is monotonic in the passband and the stopband. The magnitude-squared function for a continuous-time Butterworth lowpass filter is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (j\Omega/j\Omega_c)^{2N}}. \tag{B.1}$$

This function is plotted in Figure B.1.

As the parameter N in Eq. (B.1) increases, the filter characteristics become sharper: that is, they remain close to unity over more of the passband and become close to zero

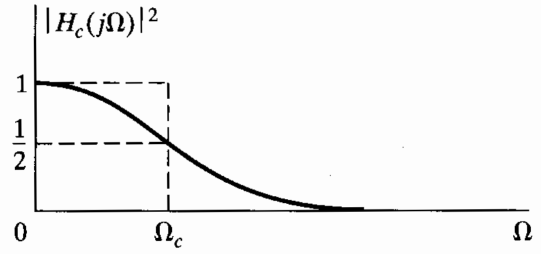

Figure B.1 Magnitude-squared function for continuous-time Butterworth filter.

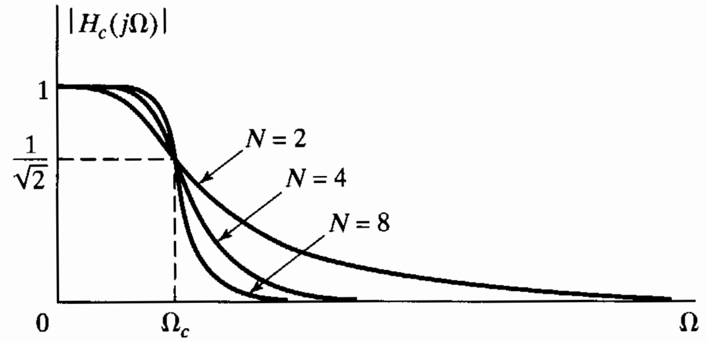

Figure B.2 Dependence of Butterworth magnitude characteristics on the order N.

more rapidly in the stopband, although the magnitude-squared function at the cutoff frequency Ω_c will always be equal to one-half because of the nature of Eq. (B.1). The dependence of the Butterworth filter characteristic on the parameter N is indicated in Figure B.2.

From the magnitude-squared function in Eq. (B.1), we observe by substituting $j\Omega = s$ that $H_c(s)H_c(-s)$ must be of the form

$$H_c(s)H_c(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}}. \tag{B.2}$$

The roots of the denominator polynomial (the poles of the magnitude-squared function) are therefore located at values of s satisfying $1 + (s/j\Omega_c)^{2N} = 0$; i.e.,

$$s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{(j\pi/2N)(2k+N-1)}, \qquad k = 0, 1, \ldots, 2N - 1. \tag{B.3}$$

Thus, there are $2N$ poles equally spaced in angle on a circle of radius Ω_c in the s-plane. The poles are symmetrically located with respect to the imaginary axis. A pole never falls on the imaginary axis, and one occurs on the real axis for N odd, but not for N even. The angular spacing between the poles on the circle is π/N radians. For example, for $N = 3$, the poles are spaced by $\pi/3$ radians, or 60 degrees, as indicated in Figure B.3. To determine the system function of the analog filter to associate with the Butterworth magnitude-squared function, we must perform the factorization $H_c(s)H_c(-s)$. The poles of the magnitude-squared function always occur in pairs; i.e., if there is a pole at $s = s_k$, then a pole also occurs at $s = -s_k$. Consequently, to construct $H_c(s)$ from the magnitude-squared function, we would choose one pole from each such pair. To obtain a stable and causal filter, we should choose the poles on the left-half-plane part of the s-plane.

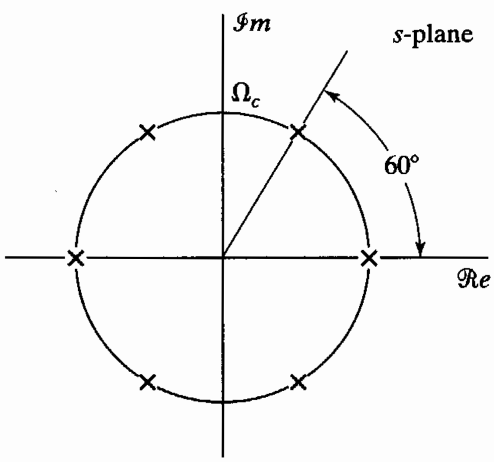

Figure B.3 s-plane pole locations for a third-order Butterworth filter.

B.2 CHEBYSHEV FILTERS

In a Butterworth filter, the magnitude response is monotonic in both the passband and the stopband. Consequently, if the filter specifications are in terms of maximum passband and stopband approximation error, the specifications are exceeded toward the low-frequency end of the passband and above the stopband cutoff frequency. A more efficient approach, which usually leads to a lower order filter, is to distribute the accuracy of the approximation uniformly over the passband or the stopband (or both). This is accomplished by choosing an approximation that has an equiripple behavior rather than a monotonic behavior. The class of Chebyshev filters has the property that the magnitude of the frequency response is either equiripple in the passband and monotonic in the stopband (referred to as a type I Chebyshev filter) or monotonic in the passband and equiripple in the stopband (a type II Chebyshev filter). The frequency response of a type I Chebyshev filter is shown in Figure B.4. The magnitude-squared function for this filter is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)}, \tag{B.4}$$

where $V_N(x)$ is the Nth-order Chebyshev polynomial defined as

$$V_N(x) = \cos(N\cos^{-1}x). \tag{B.5}$$

For example, for $N = 0$, $V_0(x) = 1$; for $N = 1$, $V_1(x) = \cos(\cos^{-1}x) = x$; for $N = 2$, $V_2(x) = \cos(2\cos^{-1}x) = 2x^2 - 1$; and so on.

From Eq. (B.5), which defines the Chebyshev polynomials, it is straightforward to obtain a recurrence formula from which $V_{N+1}(x)$ can be obtained from $V_N(x)$ and $V_{N-1}(x)$. By applying trigonometric identities to Eq. (B.5), it follows that

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x). \tag{B.6}$$

From Eq. (B.5), we note that $V_N^2(x)$ varies between zero and unity for $0 < x < 1$. For $x > 1$, $\cos^{-1}x$ is imaginary, so $V_N(x)$ behaves as a hyperbolic cosine and consequently increases monotonically. Referring to Eq. (B.4), we see that $|H_c(j\Omega)|^2$ ripples between 1 and $1/(1+\varepsilon^2)$ for $0 \le \Omega/\Omega_c \le 1$ and decreases monotonically for $\Omega/\Omega_c > 1$. Three parameters are required to specify the filter: ε, Ω_c, and N. In a typical design, ε is

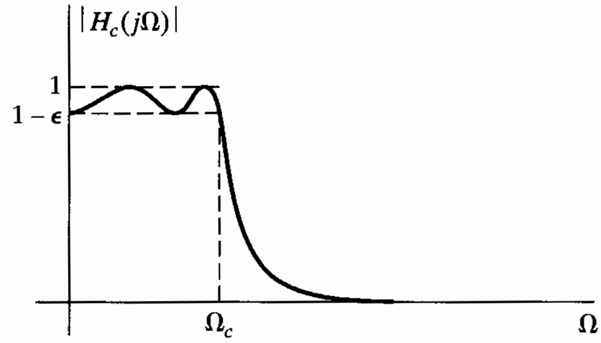

Figure B.4 Type I Chebyshev lowpass filter approximation.

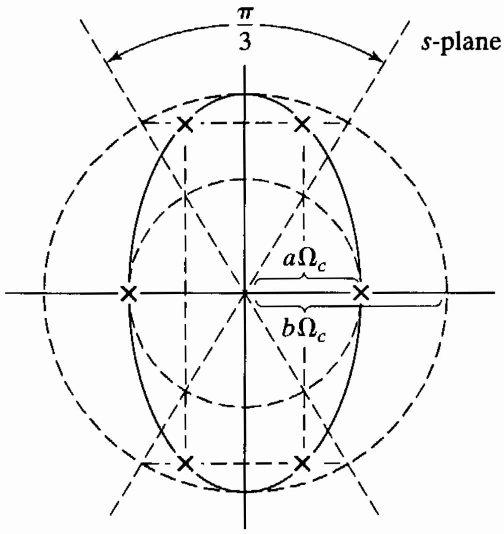

Figure B.5 Location of poles for a third-order type I lowpass Chebyshev filter.

specified by the allowable passband ripple and Ω_c is specified by the desired passband cutoff frequency. The order N is then chosen so that the stopband specifications are met.

The poles of the Chebyshev filter lie on an ellipse in the s-plane. As shown in Figure B.5, the ellipse is defined by two circles whose diameters are equal to the minor and major axes of the ellipse. The length of the minor axis is $2a\Omega_c$, where

$$a = \tfrac{1}{2}(\alpha^{1/N} - \alpha^{-1/N}) \tag{B.7}$$

with

$$\alpha = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}. \tag{B.8}$$

The length of the major axis is $2b\Omega_c$, where

$$b = \tfrac{1}{2}(\alpha^{1/N} + \alpha^{-1/N}). \tag{B.9}$$

To locate the poles of the Chebyshev filter on the ellipse, we first identify the points on the major and minor circles equally spaced in angle with a spacing of π/N in such a way that the points are symmetrically located with respect to the imaginary axis and such that a point never falls on the imaginary axis and a point occurs on the real axis for N odd but not for N even. This division of the major and minor circles corresponds exactly to the manner in which the circle is divided in locating the poles of a Butterworth filter as in Eq. (B.3). The poles of a Chebyshev filter fall on the ellipse, with the ordinate specified by the points identified on the major circle and the abscissa specified by the points identified on the minor circle. In Figure B.5, the poles are shown for $N = 3$.

A type II Chebyshev lowpass filter can be related to a type I filter through a transformation. Specifically, if in Eq. (B.4) we replace the term $\varepsilon^2 V_N^2(\Omega/\Omega_c)$ by its reciprocal and also replace the argument of V_N^2 by its reciprocal, we obtain

$$|H_c(j\Omega)|^2 = \frac{1}{1 + [\varepsilon^2 V_N^2(\Omega_c/\Omega)]^{-1}}. \tag{B.10}$$

This is the analytic form for the type II Chebyshev lowpass filter. One approach to designing a type II Chebyshev filter is to first design a type I filter and then apply the preceding transformation.

B.3 ELLIPTIC FILTERS

If we distribute the error uniformly across the entire passband or across the entire stopband, as in the Chebyshev cases, we are able to meet the design specifications with a lower order filter than if we permit a monotonically increasing error in the passband, as in the Butterworth case. We note that in the type I Chebyshev and Butterworth approximations, the stopband error decreases monotonically with frequency, raising the possibility of further improvements if we distribute the stopband error uniformly across the stopband. This suggests the lowpass filter approximation in Figure B.6. Indeed, it can be shown (Papoulis, 1957) that this type of approximation (i.e., equiripple in the passband and the stopband) is the best that can be achieved for a given filter order N, in the sense that for given values of Ω_p, δ_1, and δ_2, the transition band $(\Omega_s - \Omega_p)$ is as small as possible.

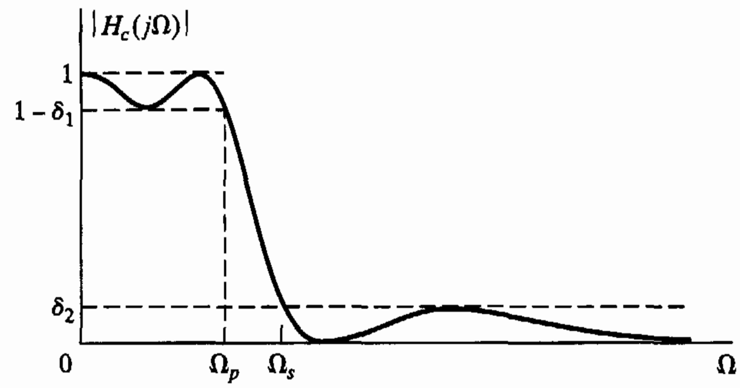

Figure B.6 Equiripple approximation in both passband and stopband.

This class of approximations, referred to as elliptic filters, has the form

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 U_N^2(\Omega)}, \tag{B.11}$$

where $U_N(\Omega)$ is a Jacobian elliptic function. To obtain equiripple error in both the passband and the stopband, elliptic filters must have both poles and zeros. As can be seen from Figure B.6, such a filter will have zeros on the $j\Omega$-axis of the s-plane. A discussion of elliptic filter design, even on a superficial level, is beyond the scope of this appendix. The reader is referred to the texts by Guillemin (1957), Storer (1957), Gold and Rader (1969), and Parks and Burrus (1987) for more detailed discussions.