From Fourier Series to Fourier integral :-

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier Series.

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

we have $a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx, \quad n=1,2,\dots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx, \quad n=1,2,\dots$$

$$\therefore f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos w_n x\int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x\int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big].$$

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos w_n x)\Delta w\int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\Delta w\int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big]. \quad \text{------} \ (1)$$

This representation is valid for any fixed L, arbitrary Large, but finite.

let $L \to \infty$ and assume that the resulting nonperiodic function.

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is absolutely integrable on the X-axis; that is, the following (finite) limits exist:

$$\lim_{a\to\infty}\int_a^0 + \lim_{b\to\infty}\int_0^b |f(x)|\,dx \quad \left(\text{written } \int_{-\infty}^{\infty} |f(x)|\,dx\right). \qquad \text{------ (2)}$$

then $1/L \to 0$, and the value of the first term on the right Side of (1) approaches Zero. Also $\Delta w = \pi/L \to 0$ and it Seems plausible that the infinite sereis in (1) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v) \sin wv\,dv\right]dw.$$

if we introduce the notations

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv,$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv. \qquad \text{---- (4)}$$

We can write this in the form

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw.$$

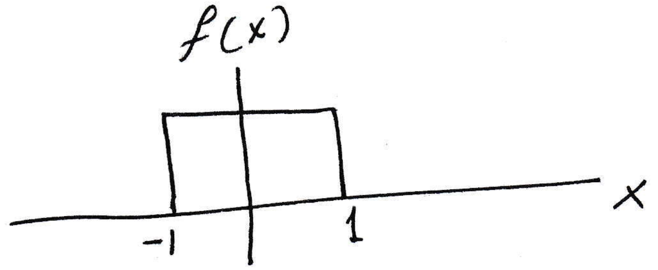
We can write this in the form

$$\boxed{f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw.} \quad \cdots\cdots (5)$$

Fourier integral

EX Find the Fourier integral representation of the function?

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$$

$$= \frac{1}{\pi}\int_{-1}^{1}\cos wv\, dv = \frac{\sin wv}{\pi w}\Big|_{-1}^{1} = \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1}\sin wv\, dv = 0$$

$$\therefore \; f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\cos wx \sin w}{w}\, dw.$$

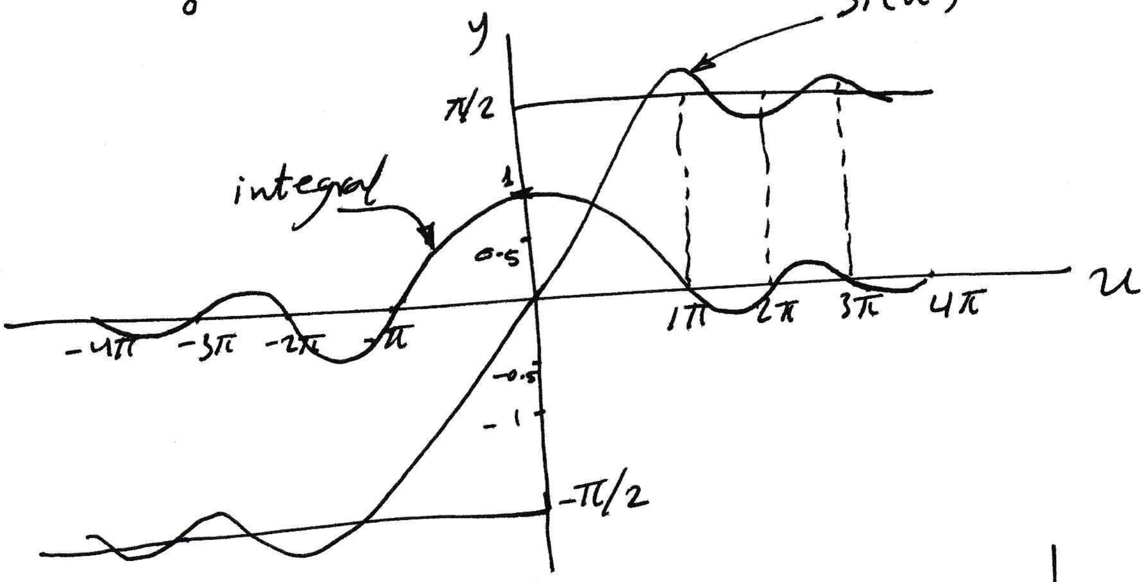
$$\int_0^\infty \frac{\cos wx \sin w}{w}\,dw = \begin{cases} \pi/2 & \text{if } 0 \le x < 1, \\ \pi/4 & \text{if } x = 1, \\ 0 & \text{if } x > 1 \end{cases}$$

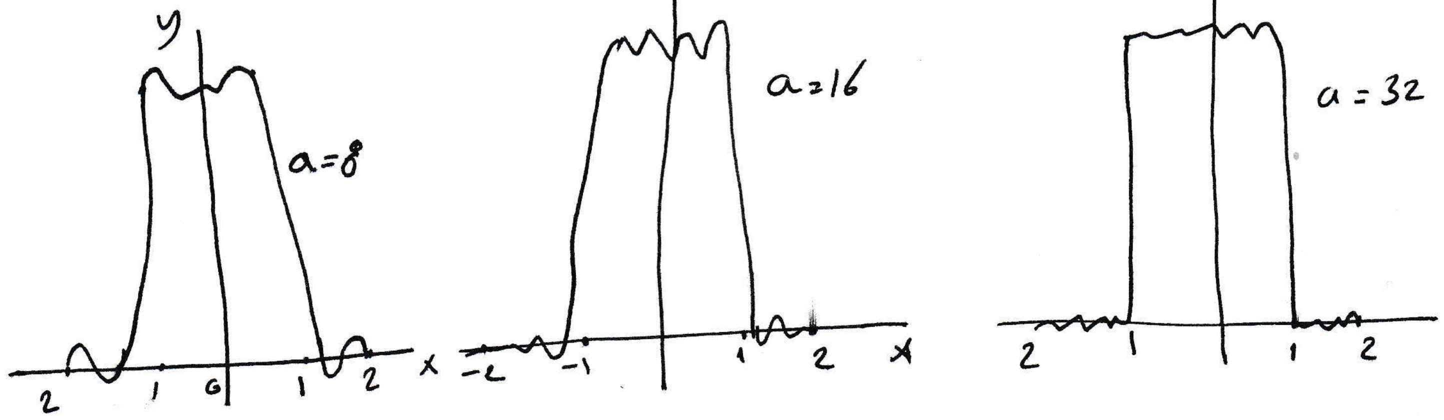
if $x = 0 \Rightarrow \int_0^\infty \frac{\sin w}{w}\,dw = \frac{\pi}{2}$

"Sine integral" $\Rightarrow Si(u) = \int_0^u \frac{\sin w}{w}\,dw$

$u \to \infty$, and by replacing ∞ by numbers a

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw$$





$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^a \frac{\sin(w+wx)}{w}\,dw + \frac{1}{\pi}\int_0^a \frac{\sin(w-wx)}{w}\,dw$$

we have $\sin(-t) = -\sin t$

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t}\,dt.$$

$$= \frac{1}{\pi} Si(a[x+1]) - \frac{1}{\pi} Si(a[x-1]).$$

Fourier Cosine Integral and Fourier Sine Integral

If f has a fourier integral representation and is even, then $B(w) = 0$. this holds because the integrand of $B(w)$ is odd., then (5) reduces to a "Fourier Cosine integral"

$$f(x) = \int_0^\infty A(w)\cos wx\,dw$$

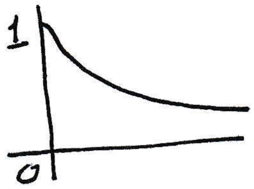
where $A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv.$

If f has a Fourier integral representation and is odd, then $A(w) = 0$. "the Fourier Sine integral"

$$f(x) = \int_0^\infty B(w)\sin wx\,dw \quad \text{where } B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv.$$

EX Find the Fourier Cosine and Fourier Sine integrals of

$f(x) = e^{-Kx}$, where $x > 0$ and $K > 0$



Sol

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

$$= \frac{2}{\pi} \int_0^\infty e^{-Kv} \cos wv \, dv$$

by integration by parts,

$$\int_0^\infty e^{-Kv} \cos wv \, dv = \frac{-K}{K^2 + w^2} e^{-Kv} \left(-\frac{w}{K} \sin wv + \cos wv \right).$$

at $v \to 0 \to$ the first term is $\frac{-K}{K^2 + w^2}$

at $v \to \infty \to$ the second term will be 0

thus $\frac{2}{\pi}$ times the integral from 0 to ∞ gives

$$A(w) = \frac{2K}{\pi (K^2 + w^2)}$$

the Fourier Cosin itegral is

We have $f(x) = \int_0^\infty A(w) \cos wx \, dw$

$$f(x) = \frac{2K}{\pi} \int_0^\infty \frac{\cos wx}{K^2 + w^2} \, dw$$

$$\boxed{\int_0^\infty \frac{\cos wx}{K^2+w^2}\,dw = \frac{\pi}{2K}\,e^{-Kx}} \qquad x>0,\ K>0$$

we have $B(w) = \frac{2}{\pi}\int_0^\infty e^{-Kv}\sin wv\,dv$

By integration by parts,

$$\int e^{-Kv}\sin wv\,dv = \frac{-w}{K^2+w^2}\,e^{-Kv}\left(\frac{K}{w}\sin wv + \cos wv\right).$$

$$= \frac{-w}{K^2+w^2}$$

$$\therefore\ B(w) = \frac{2w}{\pi(K^2+w^2)}$$

$$\therefore\ f(x) = e^{-Kx} = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{K^2+w^2}\,dw.$$

$$\therefore\ \boxed{\int_0^\infty \frac{w\sin wx}{K^2+w^2}\,dw = \frac{\pi}{2}\,e^{-Kx}} \qquad x>0,\ K>0$$

Fourier Transform :-

Fourier Series enable us to represent aperiodic function as a sum of sinusoids and to obtain the frequency spectrum from the series. the Fourier Transform (F.T) allows us to extend the concept of frequency spectrum to a non-periodic function. The transform assumes that a non-periodic function is aperiodic function with initial period.

The F.T allows a transform from the time domain to the frequency domain.

the exponential form of a fourier series as a quation

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jnw_o t} \, dt \quad \text{——} \ (1)$$

where $$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cdot e^{-jnw_o t} \, dt \quad \text{——} \ (2)$$

the fundamental frequency is

$$w_o = \frac{2\pi}{T}$$

and the spacing between adjacent harmonic is

$$\Delta w = (n+1) w_o - n w_o \Rightarrow w_o = \frac{2\pi}{T}$$

Sub (2) in (1) and obtain

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt \right] e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2T} \int_{-T/2}^{T/2} f(t) \cdot e^{-jn\omega_0 t}\, dt \right] e^{jn\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f(t) \cdot e^{-jn\omega_0 t}\, dt \right] \Delta\omega\, e^{jn\omega_0 t}$$

$$\sum_{n=-\infty}^{\infty} \Rightarrow \int_{-\infty}^{\infty} \quad \text{and} \quad \Delta\omega \Rightarrow d\omega \quad \text{and} \quad n\omega_0 \Rightarrow \omega$$

eq(3) will become:-

$$f(t) = \frac{1}{2T} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \right] e^{j\omega t}\, d\omega \quad \text{---} \; (4)$$

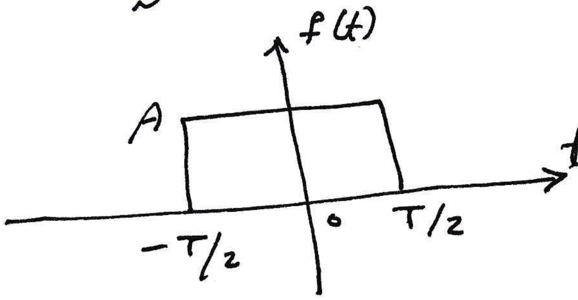
and can be represent by $F(\omega)$.

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt \quad \text{---} \; (5)$$

$F(\omega)$ is a Complex function, its magnitud Called the "amplitud Spectrum" with phase is Called "phase spectrum".

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t}\, d\omega \quad \text{---} \; (6)$$

EX Derive the Fourier transform of a single rectangular pulse of width T and high A, shown in figure below.

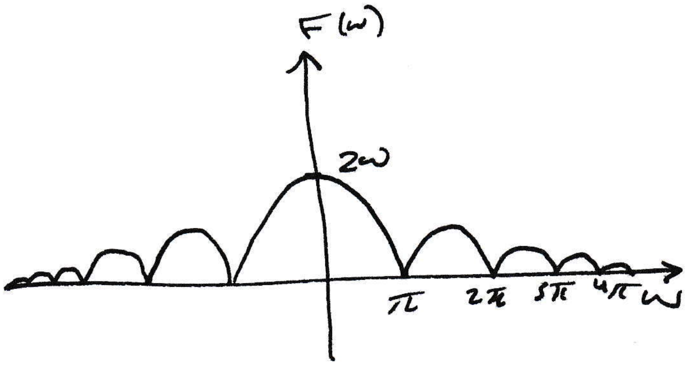


$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} A\, e^{-jwt} = -\frac{A}{jw} e^{-jwt} \Big|_{-T/2}^{T/2}$$

$$= \frac{2A}{w} \left(\frac{e^{jwT/2} - e^{-jwT/2}}{2j} \right) = AT \frac{\sin(w\frac{T}{2})}{w\frac{T}{2}}$$

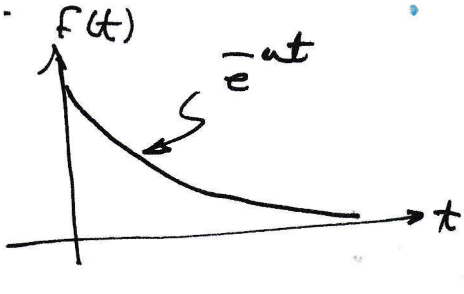
$$= AT \operatorname{sinc} \frac{wT}{2}$$



if $A = 10$, $T = 2$

$$F(w) = 20 \operatorname{Sinc}(w)$$

EX obtain the Fourier transform of the switch-on). ~~explain~~ exponential function below.



$$f(t) = e^{-at} = \begin{cases} e^{-at}, & t > 0 \\ 0 & t < 0 \end{cases}$$

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, e^{-jwt}\, dt = \int_{0}^{\infty} e^{-(a+jw)t}\, dt$$

$$= \frac{-1}{a+jw} e^{-(a+jw)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a+jw}$$

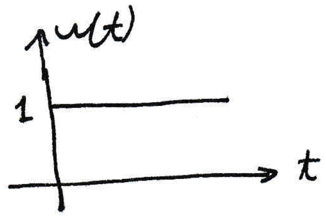
Ex Find the Fourier transform for these functions?

a) $\delta(t-t_o)$ b) $e^{j w_o t}$ c) $\cos w_o t$

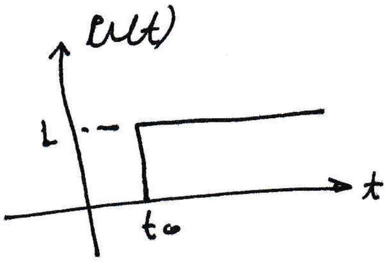
a) for the impulse function

$$F(w) = \mathcal{F}[\delta(t-t_o)] = \int_{-\infty}^{\infty} \delta(t-t_o)\, e^{-jwt}\, dt = e^{-jwt_o}$$

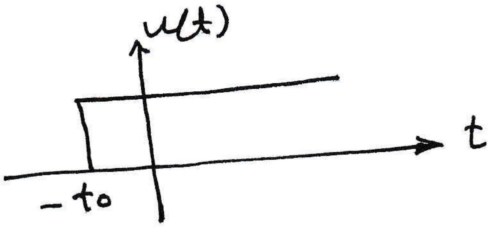
*



$$u(t) = \begin{cases} 0 , & t < 0 \\ 1 , & t > 0 \end{cases}$$



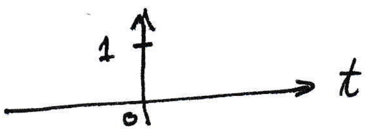
$$u(t-t_o) = \begin{cases} 0 , & t < t_o \\ 1 , & t > t_o \end{cases}$$

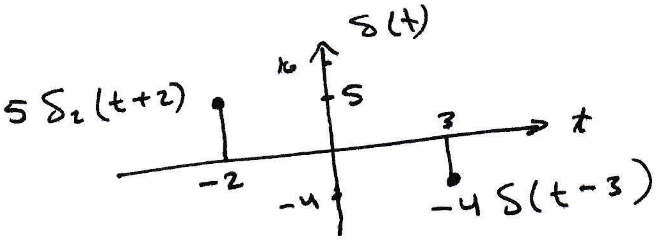


$$u(t+t_o) = \begin{cases} 0 , & t < -t_o \\ 1 , & t > t_o \end{cases}$$

$$\delta(t) = \frac{d}{dt} u(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} & t = 0 \\ 0 & t > -t_o \end{cases}$$

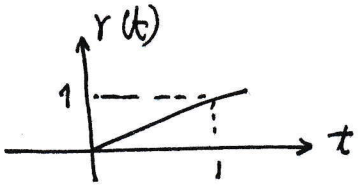
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



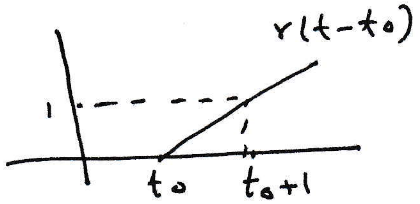


$r(t) = \int_{-\infty}^{t} u(t)\,dt = t\,u(t)$ $\qquad$ $r(t)$: ramp

$$r(t) = \begin{cases} 0 & t \leqslant 0 \\ t & t \geqslant 0 \end{cases}$$

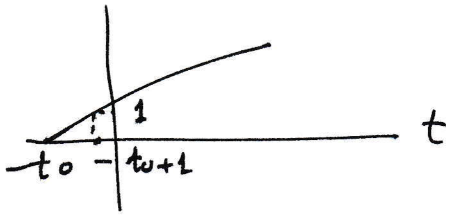


$$r(t-t_0) = \begin{cases} 0 , & t \leqslant t_0 \\ t-t_0 , & t \geqslant t_0 \end{cases}$$



$$r(t+t_0) = \begin{cases} 0 , & t \leqslant -t_0 \\ t-t_0 , & t \geqslant -t_0 \end{cases}$$

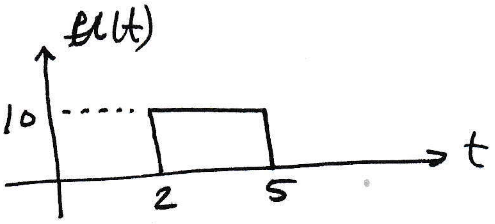
$$\delta(t) = \frac{du(t)}{dt} , \quad u(t) = \frac{dr}{dt}$$



or by integration

$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt$$

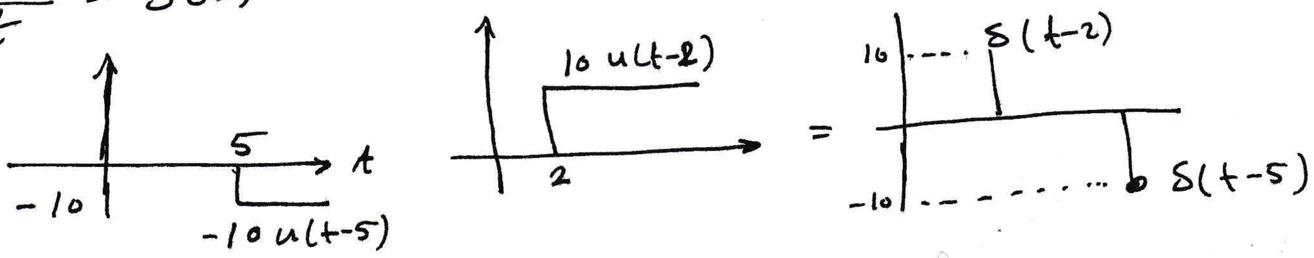
$$r(t) = \int_{-\infty}^{t} u(t)\,dt$$



$*\ u(t) = 10\,u(t-2) - 10\,u(t-5)$

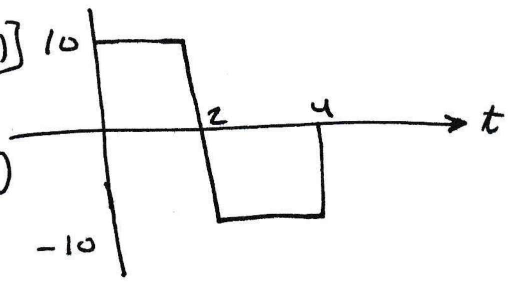
$= 10\,[u(t-2) - u(t-5)]$

$$\frac{du(t)}{dt} = \delta(t) = 10\,[\delta(t-2) - \delta(t-5)]$$



* Convert this signal to ramp:

$10u(t) - 10u(t-2) - [10u(t-2) - 10u(t-4)]$



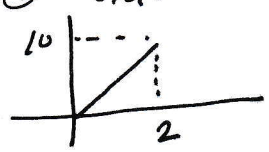
$= 10u(t) - 20u(t-2) + 10u(t-4)$

$= 10\{u(t) - 2u(t-2) + u(t-4)\}$

$r(t) = \int u(t) = 10t\,r(t) - 10t\,r(t-2) - 10t\,r(t-2) + 10t\,r(t-4)$

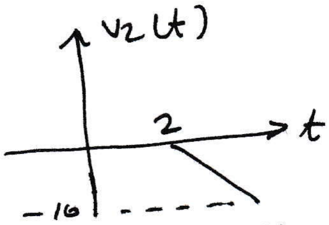
$= 10t\{r(t) - 20r(t-2) + r(t-4)\}$

<u>EX</u> ①

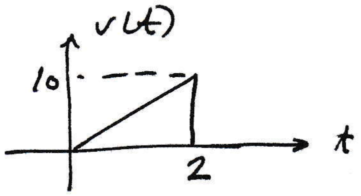


$v_1(t) = 5r(t)$

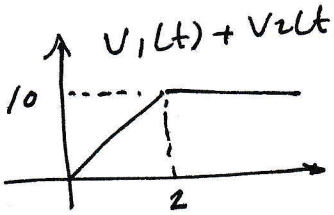
②



$v_2(t) = 5r(t-2)$

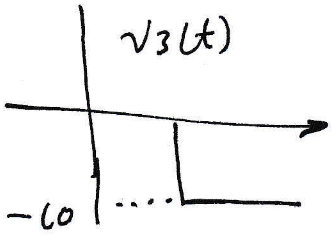


③

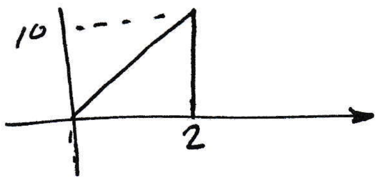


$\rightarrow y_1 + y_2 = 10$

④



③+④



$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$

<u>Second method</u> :-

$v(t) = 5t\{u(t) - u(t-2)\} = 5t\,u(t) - 5t\,u(t-2)$

$= 5r(t) - 5(t-2+2)\,u(t-2)$

$= 5r(t) - 5(t-2)\,u(t-2) - 10u(t-2)$

$= 5r(t) - 5r(t-2) - 10u(t-2).$

b) $e^{j\omega_0 t}$

$$F(\delta(t)) = 1 \Rightarrow \delta(t) = \mathcal{F}^{-1}[1]$$

Using the inverse Fourier transform formula

$$\delta(t) = \mathcal{F}^{-1}(1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{j\omega t}\, d\omega$$

or $$\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega = 2\pi\,\delta(t)$$

Inter changing variables t and ω results in

$$\int_{-\infty}^{\infty} e^{j\omega t}\, dt = 2\pi\,\delta(\omega)$$

$$\therefore \mathcal{F}[e^{j\omega_0 t}] = \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t}\, dt$$

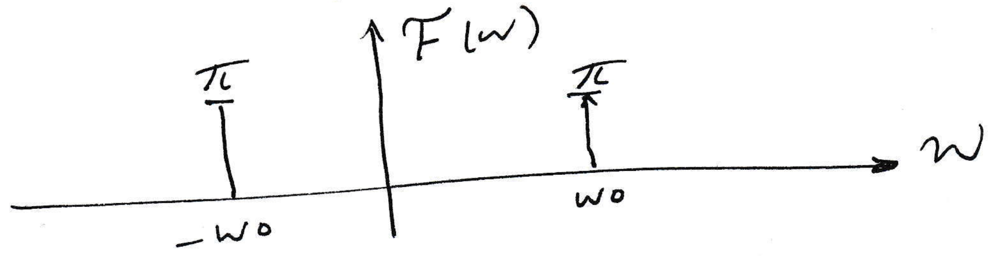
$$= 2\pi\,\delta(\omega-\omega_0)$$

$$\therefore \boxed{\begin{aligned}\mathcal{F}[e^{j\omega_0 t}] &= 2\pi\,\delta(\omega-\omega_0)\\ \mathcal{F}[e^{-j\omega_0 t}] &= 2\pi\,\delta(\omega+\omega_0)\end{aligned}}$$

c) $$\mathcal{F}[\cos\omega_0 t] = \mathcal{F}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{1}{2}\mathcal{F}[e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[e^{-j\omega_0 t}]$$

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$



Properties of the fourier transform :-

1) Linearity

If $F_1(w)$ and $F_2(w)$ are the fourier form of $f_1(t)$ & $f_2(t)$ respectively.

$$\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(w) + a_2 F_2(w)$$

where a_1 & a_2 are constant.

$$\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] = \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)] e^{-jwt} dt.$$

$$= \int_{-\infty}^{\infty} a_1 f_1(t) e^{-jwt} dt + \int_{-\infty}^{\infty} a_2 f_2(t) \cdot e^{-jwt} dt$$

$$= a_1 F_1(w) + a_2 F_2(w)$$

EX $\sin w_0 t = \frac{1}{2j}\left(e^{jw_0 t} - e^{-jw_0 t}\right) \Rightarrow \frac{1}{2j}\left[F(e^{jw_0 t}) - F(e^{-jw_0 t})\right]$

$$= \frac{\pi}{j}\left[\delta(w-w_0) - \delta(w+w_0)\right].$$

Fourier Transform of the Derivative f(x)

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x) \cdot e^{-jwx} dx$$

Integrating by parts, we obtain

$$\mathcal{F}\{f'(x)\} = \left[f(x) e^{-jwx}\Big|_{-\infty}^{\infty} - (-jw)\int_{-\infty}^{\infty} f(x) e^{-jwx} dx\right].$$

Since $f(x) \to 0$ as $|x| \to \infty$

$$\mathcal{F}\{f'(x)\} = 0 + jw\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f'(x)\} = jw\mathcal{F}\{f(x)\}$$

$\mathcal{F}(f'') = i\omega \mathcal{F}(f') = (i\omega)^2 \mathcal{F}(f).$

Since $(i\omega)^2 = -\omega^2$, we have for the transform of the second derivative of f

$$\mathcal{F}[f''(x)] = -\omega^2 \mathcal{F}\{f(x)\}.$$

<u>EX</u> find the Fourier transform of xe^{-x^2}.

$$\mathcal{F}(xe^{-x^2}) = \mathcal{F}\{-\tfrac{1}{2}(e^{-x^2})'\}$$

$$= -\tfrac{1}{2}\mathcal{F}\{(e^{-x^2})'\}$$

$$= -\tfrac{1}{2} j\omega \mathcal{F}(e^{-x^2})$$

$$= -\tfrac{1}{2} j\omega \frac{1}{\sqrt{2}} e^{-\omega^2/4}$$

$$= \frac{-i\omega}{2\sqrt{2}} e^{-\omega^2/4}.$$

2) <u>Convolution</u>

The convolution $f * g$ of function f and g is defined by

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp = \int_{-\infty}^{\infty} f(x-p)\, g(p)\, dp$$

$$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g) \quad \text{.....} \quad (7)$$

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp\, e^{-i\omega x}\, dx.$$

An iterchange of the order of integration gives.

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)\, g(x-p)\, e^{-jwx}\, dx\, dp$$

Instead of x we now take $x-p=q$ as a new variable of integration, then $x = p+q$ and

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)\, g(q)\, e^{-jw(p+q)}\, dq\, dp.$$

This double integral can be written as a product of two integrals and gives the desired result

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{-jwp}\, dp \int_{-\infty}^{\infty} g(p)\, e^{-iwq}\, dq$$

$$= \frac{1}{\sqrt{2\pi}} [\sqrt{2\pi}\, \mathcal{F}(f)]\, [\sqrt{2\pi}\, \mathcal{F}(g)] = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g).$$

By taking the inverse Fourier transform on both sides of (7), writing $\hat{f} = \mathcal{F}(f)$ and $\hat{g} = \mathcal{F}(g)$ as before, and noting that $\sqrt{2\pi}$ in (7) cancel each other, we obtain

$$(f*g)(x) = \int_{-\infty}^{\infty} \hat{f}(w)\, \hat{g}(w)\, e^{jwx}\, dw.$$

this formula will help in solving partial differential equations.

3) <u>Time Scaling</u> :- If $F(w) = \mathcal{F}[f(t)]$ then

$$\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(\frac{w}{a}\right) \text{ where "a" is Constant}$$

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at)\, e^{-jwt}\, dt$$

let $x = at \quad \therefore dx = a\,dt$

$$\therefore \mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(x)\, e^{-jw x/a} \frac{dx}{a} = \frac{1}{a} F\left(\frac{w}{a}\right)$$

<u>EX</u>

$$e^{-2t} = \frac{1}{2} * \frac{1}{j\frac{w}{2}+1}$$

4) <u>Time shifting</u> :- If $F(w) = \mathcal{F}[f(t)]$ then

$$\mathcal{F}[f(t-t_o)] = e^{-jwt_o} F(w)$$

$$\mathcal{F}[f(t-t_o)] = \int_{-\infty}^{\infty} f(t-t_o)\, e^{-jwt}\, dt$$

let $x = t - t_o \Rightarrow dx = dt \Rightarrow t = x + t_o$ then

$$\mathcal{F}[f(t-t_o)] = \int_{-\infty}^{\infty} f(x)\, e^{-jw(x+t_o)}\, dx$$

$$= e^{-jwt_o} \int_{-\infty}^{\infty} f(x)\, e^{-jwx}\, dx = e^{-jwt_o} F(w)$$

$$\therefore \boxed{\begin{aligned} &\mathcal{F}[f(t+t_o)] = e^{jwt_o} F(w) \\ \text{or } &\mathcal{F}[f(t-t_o)] = e^{-jwt_o} F(w) \end{aligned}}$$

EX $F[e^{-at}u(t)] = \frac{1}{a+j\omega}$

EX $F[e^{-(t-2)}u(t-2)] = \frac{e^{-j2\omega}}{1+j\omega}$

5) Frequency shift or (Amplitude modulation)

This property state that if $F(\omega) = \mathcal{F}[f(t)]$ then

$$\mathcal{F}[f(t)e^{j\omega_0 t}] = F(\omega-\omega_0)$$

$$\because \mathcal{F}[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_0)t}\, dt = F(\omega-\omega_0)$$

EX $\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}F[f(t)e^{j\omega_0 t}] + \frac{1}{2}F[f(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2}F(\omega-\omega_0) + \frac{1}{2}F(\omega+\omega_0)$$

6) Time differentiation

$F(\omega) = \mathcal{F}[f(t)]$, then

$F[f'(t)] = j\omega F(\omega)$

we have $f(t) = F^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{j\omega t}\, d\omega$

taking the derivative of both sides with respect to t gives:-

$$f(t) = \frac{jw}{2\pi}\int_{-\infty}^{\infty} F(w)\, e^{jwt}\, dw = jw\, F^{-1}[F(w)]$$

or $\Rightarrow$ $$\boxed{\mathcal{F}[f'(t)] = jw\, F(w) \Rightarrow F(f^{n}(t)] = (jw)^{n}\, F(w)}$$

EX Find $F(w)$ for $f(t) = e^{-at}$?

$$f'(t) = -a e^{-at} \Rightarrow f'(t) = -a\, f(t)$$

$$jw\, F(w) = \frac{-a}{jw+a}$$

$$F(w) = \frac{-a}{jw(jw+a)}$$

⑦ Quality :- the property state that if $F(w)$ is the Fourier Transform of $f(t)$, then the fourier transform of $f(t)$ is $2\pi\, f(w)$, we write

$$\mathcal{F}[f(t)] = F(w) \Rightarrow \mathcal{F}[f(t)] = 2\pi\, f(-w)$$

$$f(t) = \mathcal{F}^{-1}[F(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\, e^{jwot}\, dt$$

or $$2\pi\, f(t) = \int_{-\infty}^{\infty} F(w)\, e^{jwt}\, dt$$

Replacing t by $-t$ gives:-

$$2\pi\, f(-t) = \int_{-\infty}^{\infty} F(w)\, e^{-jwt}\, dw$$