Laplace transform :-

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1- Introduction :-

the frequency domain analysis has been limited to circuit with sinusoidal input. In other words, we have assumed sinusoidal time-varying excitations in all our non-d.c circuits.

This introduces the "Laplace Transform" a very powerful tool for analyzing circuits with sinusoidal or non-sinusoidal input. Transform the circuit from the time domain to the frequency or phasor domain.

2- Definition of the Laplace transform :-

If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform, is the integral of $f(t)$ times e^{-st} from $t=0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$, thus

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$$

furthermore, the given function $f(t)$ is called the "inverse transform" of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$. that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F)$$

Laplace Transform for Some function

1 - Unit step function :-

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\, u(t) = \int_0^{\infty} u(t)\, e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s}$$

$$\therefore \boxed{\mathcal{L}\, u(t) = \frac{1}{s}}$$

EX $\mathcal{L}\, 5\, u(t) = \dfrac{5}{s}$

2 - exponential function :-

If $f(t) = e^{at}\, u(t)$

$$\mathcal{L}\, e^{at} = \int_0^{\infty} e^{at}\, e^{-st}\, dt = \frac{1}{a-s}\, e^{-(s-a)t}\Big|_0^{\infty} ;$$

$$\mathcal{L}\, e^{at} = \frac{1}{s-a}$$

EX $\mathcal{L}\, e^{-3t}\, u(t) = \dfrac{1}{s+3}$

3- Sine and Cosine functions :-

$$\mathcal{L}\, e^{jbt} = \int_0^{\infty} e^{jbt} e^{-st}\, dt = \frac{1}{s-jb} * \frac{s+jb}{s+jb}$$

$$= \frac{s+jb}{s^2+b^2} \Rightarrow \frac{s}{s^2+b^2} + j\frac{b}{s^2+b^2}$$

$$= \mathcal{L}\,(\cos bt + j \sin bt)$$

$$\therefore \boxed{\mathcal{L}\cos bt = \frac{s}{s^2+b^2}} \quad , \quad \boxed{\mathcal{L}\sin bt = \frac{b}{s^2+b^2}}$$

EX Find the $\cos \omega t$ and $\sin \omega t$ in detail?

$$\int_0^{\infty} \cos \omega t\, e^{-st}\, dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^{\infty} - \frac{\omega}{s}\int_0^{\infty} e^{-st} \sin \omega t\, dt$$

$$= -\left(\frac{e^{-s\infty} \cos \omega\cdot\infty - e^{s*0} \cos \omega\cdot 0}{s}\right) - \frac{\omega}{s} * \mathcal{L}\sin \omega t$$

$$= -\left[\left(\frac{0*0 - 1*(+1)}{s}\right)\right] - \frac{\omega}{s} * \frac{\omega}{s^2+\omega^2}$$

$$= +\frac{1}{s} - \frac{\omega^2}{s(s^2+\omega^2)}$$

$$= \frac{+s^2 + \cancel{\omega^2} - \cancel{\omega^2}}{s(s^2+\omega^2)} = \frac{s}{s^2+\omega^2}$$

$$\therefore \boxed{\mathcal{L}\cos \omega t = \frac{s}{s^2+\omega^2}}$$

$$\mathcal{L}\sin wt = \int_0^{\infty} e^{-st} \sin wt \, dt$$

$$= \frac{e^{-st}}{-s} \sin wt \Big|_0^{\infty} + \frac{w}{s} \int_0^{\infty} e^{-st} \cos wt \, dt$$

$$= \frac{e^{-st}}{-s} \sin wt \Big|_0^{\infty} + \frac{w}{s} * \mathcal{L}\cos wt .$$

$$= -\left(\frac{e^{-s\infty} * \sin w * \infty - e^{-s*0} \sin w * 0}{s} \right) + \frac{w}{s} * \frac{s}{s^2+w^2}$$

$$= -\frac{0-0}{s} + \frac{w}{s^2+w^2}$$

$$\boxed{\mathcal{L}\sin wt = \frac{w}{s^2+w^2}}$$

EX Find the $\mathcal{L}\cos 3t$, $\mathcal{L}\sin 2t$

$$\mathcal{L}\cos 3t = \frac{s}{s^2+9} , \quad \mathcal{L}\sin 2t = \frac{2}{s^2+4}$$

4- Laplace transform of Ramp function :-

$$f(t) = \begin{cases} t\,u(t) & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L} f(t) = \int_0^\infty t \cdot e^{-st}\, dt \quad , \quad \text{let } u = t \, , \, du = dt.$$

$$\int dv = \int e^{-st} = \frac{+e^{-st}}{-s}$$

$$\therefore \mathcal{L} f(t) = \left[\frac{t \cdot e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt$$

$$\therefore \boxed{\mathcal{L} f(t) = \mathcal{L}\, t = \frac{1}{s^2}}$$

Gamma function :-

The gamma function of variable x is defined as :-

$$\boxed{\Gamma(x) = \int_0^\infty e^{-t} \cdot t^{x-1}\, dt}$$

$$u = e^{-t} \Rightarrow du = -e^{-t} dt, \quad \int dv = \int t^{x-1} \Rightarrow v = \frac{t^x}{x}$$

$$\Gamma(x) = \left[\frac{e^{-t} \cdot t^x}{x} \right]_0^\infty + \frac{1}{x} \int_0^\infty e^{-t} \cdot t^x\, dt$$

$$= 0 + \frac{1}{x} \int_0^\infty t^x \cdot e^{-t}\, dt$$

$$\boxed{\Gamma(x) = \frac{1}{x}\Gamma(x+1)} \quad \text{or} \quad \boxed{\Gamma(x+1) = x\,\Gamma(x)}$$

EX $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$, $\Gamma(2) = 1\Gamma(1) = 1$, $\Gamma(3) = 2\Gamma(2) = 2$

$\Gamma(4) = 3\Gamma(3) = 3 \times 2 = 3!$

$$\boxed{\Gamma(n) = (n-1)!} \quad \text{or} \quad \boxed{\Gamma(n+1) = n!}$$

5- $\mathcal{L}\, t^n$

$$\mathcal{L}\, t^n = \int_0^\infty t^n \cdot e^{-st}\,dt$$

let $st = x$, $dt = \frac{dx}{s}$

$$\therefore \mathcal{L}\, t^n = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\,\frac{dx}{ds} = \frac{1}{s^{n}+1}\int_0^\infty x^n e^{-x}\,dx$$

$$\boxed{\mathcal{L}\, t^n = \frac{\Gamma(n+1)}{s^{n+1}}} \quad \text{or} \quad \boxed{\mathcal{L}\, t^n = \frac{n!}{s^{n+1}}}$$

EX Evaluate $\mathcal{L}\, t^{-\frac{1}{2}}$, Known that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\mathcal{L}\, t^{-\frac{1}{2}} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

Properties Laplace transform :-

1- Linearity :-

If $\mathcal{L} f(t) = F(s)$ & $\mathcal{L} f_1(t) = F_1(s)$ & $\mathcal{L} f_2(t) = F_2(s)$

a, b are constant, then $\mathcal{L}\, a f(t) = a f(s)$

$$\boxed{\mathcal{L}[a f_1(t) \mp b f_2(t)] = a_1 F_1(s) \mp b F_2(s)}$$

EX

$$\mathcal{L} \cos wt = \mathcal{L} \frac{1}{2} [(e^{jwt} + e^{-jwt})]$$

$$= \frac{1}{2} \mathcal{L} [e^{jwt}] + \frac{1}{2} \mathcal{L} [e^{-jwt}]$$

$$\mathcal{L} \cos wt = \frac{1}{2} \left(\frac{1}{s-jw} + \frac{1}{s+jw} \right) = \frac{s}{s^2+w^2}$$

2- Scaling :-

If $F(s)$ is the Laplace transform of $f(t)$ then

$$\mathcal{L}[f(t)] = \int_0^\infty f(at)\, e^{-st}\, dt$$

where "a" is a constant and $a > 0$, If we let $x = at$,

$dx = a\, dt$, then

$$\mathcal{L}[f(at)] = \int_0^\infty f(x)\, e^{-x(s/a)} \frac{dx}{a} = \frac{1}{a} \int_0^\infty f(x)\, e^{-x(s/a)}\, dx$$

Comparing this integral with the definition of the Laplace transform in equation of Laplace transform :-

~~Ex find $\mathcal{L}$ sin 2wt using~~

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Ex Find $\mathcal{L} \sin 2wt$ using the scaling property ?-

$$\mathcal{L} \sin 2wt = \frac{1}{2} \frac{w}{\left(\frac{s}{a}\right)^2 + w^2} = \frac{2w}{s^2 + 4w^2}$$

3- Time shift :-

If $F(s)$ is the Laplace transform of $f(t)$ then

$$\mathcal{L}[f(t-a)\, u(t-a)] = \int_0^\infty f(t-a)\, u(t-a)\, e^{-st}\, dt \quad a \geq 0$$

But $u(t-a) = 0$ for $t < a$ and $u(t-a) = 1$ for $t > a$

$$\therefore \mathcal{L}[f(t-a)\, u(t-a)] = \int_0^\infty f(t-a)\, e^{-st}\, dt$$

if we let $x = t - a$, then $dx = dt$ and $t = x + a$

As $t \to a$, $x \to a$ and $t \to \infty$, $x \to \infty$ thus

$$\mathcal{L}[f(t-a)\, u(t-a)] = \int_0^\infty f(x)\, e^{-s(x+a)}\, dx$$

$$\boxed{\mathcal{L}[f(t-a)\, u(t-a)] = e^{-as} \int_0^\infty f(x)\, e^{-sx}\, dx = e^{-as} F(s)}$$

EX find ① $\mathcal{L}[\cos w(t-a)\, u(t-a)]$, ② $\mathcal{L}\ e^{-(t)}\, u(t-2)$

1) $\mathcal{L}\ \cos wt = \frac{S}{S^2+w^2}$, $\Rightarrow \mathcal{L}[\cos w(t-a)\, u(t-a)] = e^{-as}\frac{S}{S^2+w^2}$

2) $e^{-t}\, u(t-2) = \mathcal{L}\, e^{-(t-2+2)}\, u(t-2) = \mathcal{L}\, e^{-(t-2)}\, e^{-2}\, u(t-2)$

$= e^{-2} \cdot e^{-2S}\, \frac{1}{S+1}$.

4- Frequency Shift :-

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[e^{-at} f(t)] = \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(s+a)t}\, dt = F(s+a)$$

$$\boxed{\mathcal{L}[e^{-at} f(t)] = F(s+a)}$$

EX Find $\mathcal{L}[e^{-at}\cos wt]$, $\mathcal{L}[e^{-at}\sin wt]$.

$\mathcal{L}\ \cos wt = \frac{S}{S^2+w^2} \Rightarrow \mathcal{L}\, e^{-at}\cos wt = \frac{S+a}{(s+a)^2+w^2}$

$\mathcal{L}\ \sin wt = \frac{w}{S^2+w^2} \Rightarrow \mathcal{L}\, e^{-at}\sin wt = \frac{w}{(S+a)^2+w^2}$

5- Time Differentiation :-

Given that F(s) is the Laplace transform of f(t), the Laplace transform of its derivative is

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}\, e^{-st}\, dt$$

To integral this by part, we let

$$u = e^{-st} \quad , \quad du = -s\, e^{-st}$$

$$dv = \left(\frac{df}{dt}\right) dt = d\, f(t) \, , \quad v = f(t)$$

then $\mathcal{L}\left[\frac{df}{dt}\right] = f(t)\, e^{-st}\Big|_0^\infty - \int_0^\infty f(t) * [-s\, e^{-st}]\, dt$

$$= 0 - f(0) + s\int_0^\infty f(t)\cdot e^{-st}\, dt$$

$$= s\,F(s) - F(0)$$

$$\therefore \boxed{\mathcal{L}[f'(t)] = s\,F(s) - f(0)} \quad \text{—} \quad (*)$$

The Laplace transform of the second derivative of f(t) is a repeated application of equation (*) as

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s\left[\mathcal{L}[f(t)] - f(0)\right] = s\left[s\,F(s) - F(0)\right] - F'(0)$$

$$\mathcal{L}\left[\frac{d^2F}{dt^2}\right] = s^2 F(s) - s\,F(0) - f'(0)$$

or $\boxed{\mathcal{L}[f''(t)] = s^2 F(s) - s\,F(0) - f'(0)}$

$$\therefore \boxed{\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s f^{(n-1)}(0)}$$

EX Solve the following differential equation!

$$y''' + 5y = 5 \quad , y(0) = y'(0) = y''(0) = 0 .$$

$$s^3 y(s) - s^2 y(0) - s y'(0) - y''(0) + 5y(s) = \frac{5}{s}$$

$$y(s)\left[s^3 + 5\right] = \frac{5}{s} \Rightarrow y(s) = \frac{5}{s(s^3+5)}$$

EX find $\mathcal{L} \sin wt$ if $f'(t) = -w \sin wt$, $f(0) = 1$.

$$\sin wt = \frac{-1}{w} f'(t)$$

$$\mathcal{L} \sin wt = -\frac{1}{w} \mathcal{L}(f'(t)) = -\frac{1}{w}\left[s F(s) - f(0)\right]$$

$$= -\frac{1}{w}\left(s \frac{s}{s^2+w^2} - 1\right) = \frac{w}{s^2+w^2}$$

6- Time integral :-

If $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of its integral is

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \int_0^\infty \left[\int_0^t f(x)\,dx\right] e^{-st}\,dt .$$

To integrate this by part we let $u = \int_0^t f(\alpha)\,d\alpha$,

$du = f(t)\,dt$ and $dv = e^{-st}\,dt \Rightarrow v = \frac{1}{s} e^{-st}$, then

$$\mathcal{L}\left[\int_0^{\infty} f(t)\,dt\right] = \int_0^t f(\alpha)\,d\alpha \left(-\frac{1}{s} e^{-st}\right)\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{s} e^{-st} f(t)\,dt$$

For the first term on the right-hand side of the equation evaluating the term at $t = \infty$ yield zero due to $e^{-s\infty}$ and evaluating at $t = 0$ gives

$\frac{1}{s}\int_0^0 f(\alpha)\,d\alpha = 0$. Thus, the first term is zero and

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\int_0^{\infty} f(t)\, e^{-st}\,dt = \frac{1}{s} F(s)$$

OR simply $\boxed{\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} F(s)}$

Ex $\mathcal{L}\int_0^t t\,dt = \frac{1}{s}\cdot\frac{1}{s^2}$

$$= \mathcal{L}\frac{t^2}{2} = \frac{1}{s^3} \Rightarrow \mathcal{L}\, t^2 = \frac{2}{s^3}$$

Ex $\mathcal{L}\int_0^t t^2\,dt$

$$= \mathcal{L}\frac{t^3}{3} = \frac{1}{3}\cdot\frac{3!}{s^4} = \frac{1}{3}\,\frac{6}{s^4}$$

$$= \frac{2}{s^4}$$

7- Frequency Differentiation :-

If F(s) is the Laplace transform of f(t), then

$$F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$$

Taking the derivative with respect to S,

$$\frac{dF(s)}{ds} = \int_0^\infty f(t)\,(-t\,e^{-st})\,dt = \int_0^\infty (-t\,f(t))\,e^{-st}\,dt$$

$$= \mathcal{L}\left[-t\,f(t)\right]$$

and the frequency differentiation property

become
$$\boxed{\mathcal{L}[t\,f(t)] = -\frac{dF(s)}{ds}}$$

Repeated application of this equation leads t

$$\boxed{\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}}$$

<u>EX</u> $\mathcal{L}[t\,e^{-at}] = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}$

<u>EX</u> $\mathcal{L}[t^2\,e^{-3t}] = \frac{2}{(s+3)^3}$

<u>EX</u> $\mathcal{L}\,t\sin t = \frac{2s}{(s^2+1)^2}$

8- $\mathcal{L}\, \frac{\sin t}{t}$

Ex $\mathcal{L}\, \frac{\sin at}{t} = \int_s^\infty F(s)\, ds \Rightarrow \int_s^\infty \frac{a}{s^2+a^2}\, ds$

$$= \int_s^\infty \frac{a/a^2}{\frac{s^2}{a^2}+1}\, ds$$

$$= \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

Ex Determine $\mathcal{L}\left(\frac{t - \cos 2t}{t}\right)$

$$\therefore \mathcal{L}\left[\frac{t-\cos 2t}{t}\right] = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty \text{ by multiply } \frac{2}{2}$$

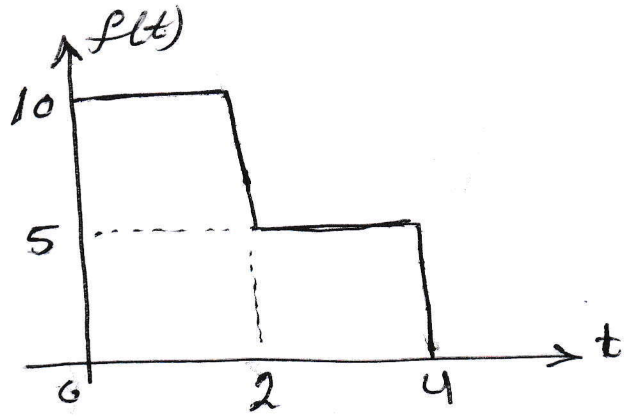
$$= \frac{1}{2}\left[2\ln s - \ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left(\ln s^2 - \ln(s^2+4)\right)_s^\infty$$

$$= \frac{1}{2}\left[\ln\left(\frac{s^2}{s^2+4}\right)\right]\Big|_s^\infty = \ln\sqrt{\frac{s^2+4}{s^2}}$$

Time non periodic

Ex Find $\mathcal{L}\, f(t)$ for figure below.



$$f(t) = 10\,[u(t) - u(t-2)] + 5\,[u(t-2) - u(t-4)].$$

$$= 10\,u(t) - 10\,u(t-2) + 5\,u(t-2) - 5\,u(t-4).$$

$$= \frac{10}{s} - \frac{10}{s} e^{-2s} + \frac{5}{s} e^{-2s} - \frac{5}{s} e^{-4s}$$

$$= \frac{10}{s} + \frac{5}{s} e^{-2s} (-2+1) - \frac{5}{s} e^{-4s}$$

$$= \frac{10}{s} - \frac{5}{s} e^{-2s} - \frac{5}{s} e^{-4s}$$

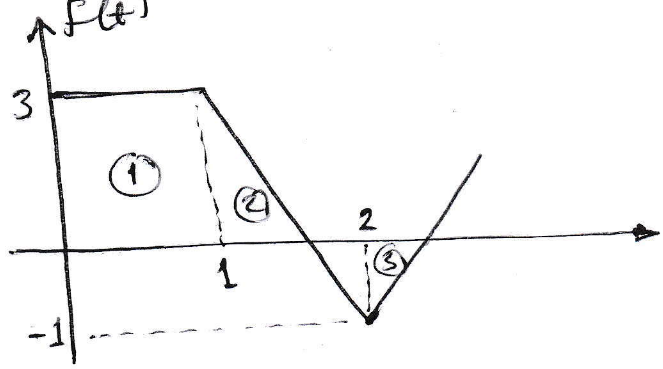
$$F(s) = \frac{5}{s}\left(2 - e^{-2s} - e^{-4s}\right)$$

Ex Find the Laplace transform for figure below?

$f_1(t) = 3(u(t) - u(t-1)\}$

$f_2(t) = -4t+7$

$f_3(t) = t-3$



$$\therefore f_1(t) = 3\{u(t) - u(t-1)]$$

$$f_2(t) = (-4t+7)[u(t-1) - u(t-2)\}$$

$$f_3(t) = (t-3)[u(t-2)\}$$

$$f(t) = 3u(t) - 3u(t-1) + (-4t)\, u(t-1) + 4t\, u(t-2) + t\, u(t-2) - 3u(t-2) + 7u(t-1) - 7u(t-2)$$

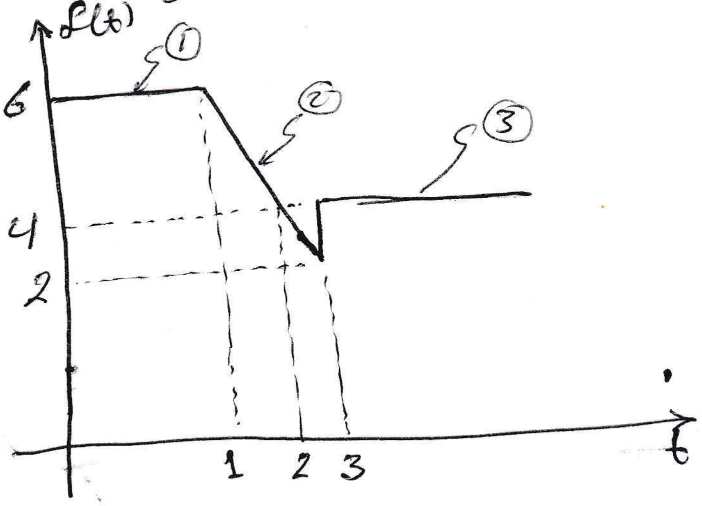
$$= 3u(t) - 3u(t-1) - 4(t-1+1)\, u(t-1) + 4(t-2+2)\, u(t-2) + (t-2+2)\, u(t-2) - 3u(t-2) + 7u(t-1) - 7u(t-2)$$

$$= 3u(t) - 3u(t-1) - 4t(t-1)u(t-1) - 4u(t-1) + 4(t-2)u(t-2) + 8u(t-2) + 7u(t-1) - 7u(t-2) + (t-2)u(t-2) + 2u(t-2) - 3u(t-2)$$

$$F(s) = \frac{3}{s} - \frac{3}{s}e^{-s} - \frac{4}{s^2}e^{-s} - \frac{4}{s}e^{-s} + \frac{4}{s^2}e^{-2s} + \frac{8}{s}e^{-2s} + \frac{7}{s}e^{-s} - \frac{7}{s}e^{-2s} + \frac{1}{s^2}e^{-2s} + \frac{2}{s}e^{-2s} - \frac{3}{s}e^{-2s}$$

$$= \frac{3}{s} - \frac{4}{s^2}e^{-s} + \frac{5}{s^2}e^{-2s}$$

Ex Find the Laplace transform for figure below?



$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

$$f_1(t) = 6\left(u(t) - u(t-1)\right) \quad 0 < t < 1$$

$$f_2(t) = -2t + 8 \qquad 1 < t < 3$$

$$f_2(t) = (-2t+8)\left(u(t-1) - u(t-3)\right)$$

$$f_2(t) = -2t\,u(t-1) + 2t\,u(t-3) + 8u(t-1) - 8u(t-3)$$

$$f_2(t) = -2(t-1+1)\,u(t-1) + 2(t-3+3)\,u(t-3) + 8u(t-1) - 8u(t-3)$$

$$f_2(t) = -2(t-1)\,u(t-1) - 2u(t-1) + 2(t-3)\,u(t-3) + 6u(t-3)$$
$$+ 8u(t-1) - 8u(t-3)$$

$$f_3(t) = 4\left[u(t-3)\right] \qquad t > 3$$

$$\therefore f(t) = 6u(t) - 6u(t-1) - 2(t-1)u(t-1) - 2u(t-1) + 2(t-3) \times$$
$$u(t-3) + 6u(t-3) + 8u(t-1) - 8u(t-3) + 4u(t-3)$$

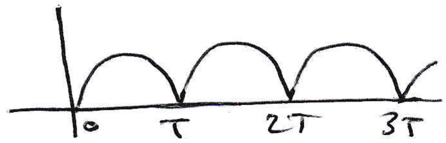
$$\therefore f(s) = \frac{6}{s} - \cancel{\frac{6}{s}e^{-s}} - \frac{2}{s^2}e^{-s} - \cancel{\frac{2}{s}e^{-s}} + \frac{2}{s^2}e^{-3s} + \frac{6}{s}e^{-3s}$$
$$+ \cancel{\frac{8}{s}e^{-s}} - \frac{8}{s}e^{-3s} + \frac{4}{s}e^{-3s}$$

$$f(s) = \frac{6}{s} - \frac{2}{s^2}e^{-s} + \frac{2}{s^2}e^{-3s} + \frac{2}{s}e^{-3s}$$

Laplace transform of a periodic function :-

If $f(t)$ is a periodic of period T then,

$$f(t) = f(t+T)$$



$$\mathcal{L} f(t) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^T f(t)\, e^{-st}\, dt + \int_T^{2T} f(t)\, e^{-st}\, dt$$

$$+ \int_{2T}^{3T} f(t)\, e^{-st}\, dt + \cdots\cdots$$

$$= F_1(s) + e^{-sT} F_1(s) + e^{-2sT} F_1(s) + \cdots$$

$$= F_1(s)\left(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right)$$

$$\therefore \mathcal{L} F(t) = F_1(s) * \frac{1}{1-e^{-sT}}$$

$F_1(s)$ is the Laplace of first periodic

EX Find Laplace transform for periodic function ?

$$f_1(t) = 3\left[u(t) - u(t-2)\right] \quad 0 < t < 2$$

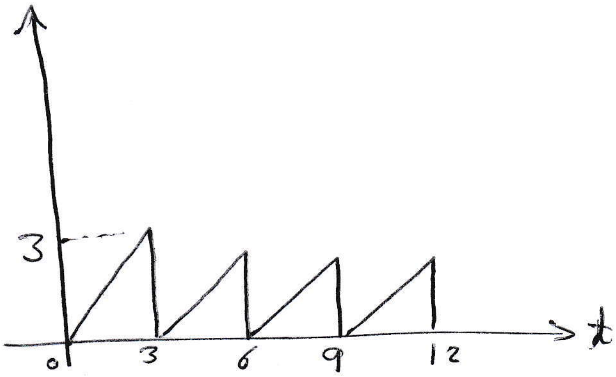
$$F_1(s) = \frac{3}{s} - \frac{3}{s} e^{-2s}$$

$$\therefore F(s) = \frac{F_1(s)}{1-e^{-sT}} = \frac{\frac{3}{s} - \frac{3}{s} e^{-2s}}{1 - e^{-4s}}$$

$$= \frac{3(1-e^{-2s})}{s(1-e^{-4s})} = \frac{3(1-e^{-2s})}{s\,(1-(e^{-s})^4)} = \frac{3\cancel{(1-e^{-2s})}}{s\,\cancel{(1-e^{-2s})}(1+e^{-2s})}$$

$$F(s) = \frac{3}{s\,(1+e^{-2s})}$$

EX Find the Laplace transform for aperiodic function?



$$f_1(t) = t\,[u(t) - u(t-3)]$$

$$= t\,u(t) - (t-3+3)\,u(t-3)$$

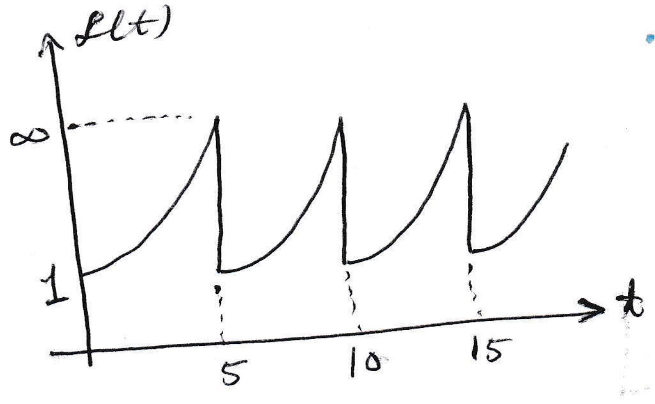
$$= t\,u(t) - (t-3)\,u(t-3) - 3\,u(t-3)$$

$$F_1(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-3s} - \frac{3}{s}e^{-3s}$$

$$F_1(s) = \frac{(1 - e^{-3s} - 3s\,e^{-3s})}{s^2} \Rightarrow F(s) = \frac{F_1(s)}{1 - e^{-sT}}$$

$$F(s) = \frac{(1 - e^{-3s} - 3s\,e^{-3s})}{s^2(1 - e^{-3s})}$$

EX Find the Laplace transform for figure below?



$$F_1(t) = e^{t}\,[u(t) - u(t-5)]$$

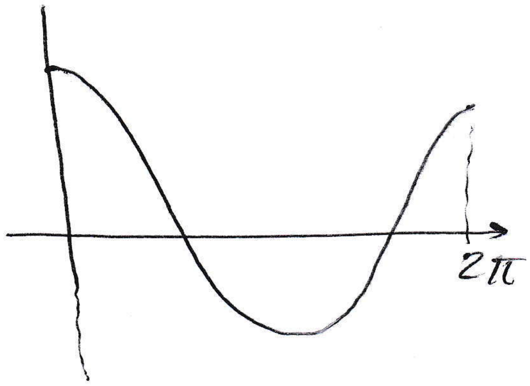
$$= e^{t}u(t) - e^{(t-5+5)}\,u(t-5)$$

$$= e^{t}u(t) - e^{5}e^{t-5}\,u(t-5)$$

$$= \frac{1}{s-1} - \frac{e^{5}\,e^{-5s}}{s-1}$$

$$\therefore\ F(s) = \frac{F_1(s)}{(1 - e^{-sT})} = \frac{1 - e^{5}e^{-5s}}{(s-1)(1 - e^{-5s})}$$

EX Find the Laplace transform for $f(t) = \cos t$ if the function is repeated at 2π sec.



$$f_1(t) = \cos t\,[u(t) - u(t-2\pi)]$$

$$f_1(t) = \cos t\,u(t) - \cos t\,u(t-2\pi)$$

$$= \cos t\,u(t) - \cos(t-2\pi)\,u(t-2\pi)$$

But $\cos t = \cos(t-2\pi)$

$$\therefore\ f_1(s) = \mathcal{L}\cos t\,u(t) - \mathcal{L}\cos(t-2\pi)\,u(t-2\pi)$$

$$f_1(s) = \frac{s}{s^2+1} - \frac{e^{-2\pi s}\,s}{s^2+1} = \frac{s(1-e^{-2\pi s})}{s^2+1}$$

$$F(s) = \frac{F(s)}{1-e^{-2\pi s}} = \frac{s\,\cancel{(1-e^{-2\pi s})}}{(s^2+1)\cancel{(1-e^{-2\pi s})}}$$

$$F(s) = \frac{s}{s^2+1}$$