EX for $u = y^3 - 3x^2 y$, check the function for harmonic and then find $f(z)$.

$$\frac{\partial u}{\partial x} = -6xy \quad , \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6y \quad , \quad \frac{\partial^2 u}{\partial y^2} = 6y$$

$\therefore$ $u$ is a harmonic fn

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -6xy \Rightarrow \partial v = -6xy\, dy$$

$$v = \int -6xy\, dy \Rightarrow v = -3xy^2 + f(x)$$

$$\frac{\partial v}{\partial x} = -3y^2 + f'(x) \equiv -\frac{\partial u}{\partial y} = -3y^2 + 3x^2$$

$$\therefore f'(x) = 3x^2$$

$$\therefore f(x) = x^3 + c$$

$$\therefore v = -3xy^2 + x^3 + c$$

$$\therefore f(z) = u + iv = y^3 - 3x^2 y + i(-3xy^2 + x^3 + c)$$

$$= y^3 - 3x^2 y + ix^3 + ic \mp i\,3xy^2$$

$$= -i^2 y^3 - 3ixy^2 + i^2 3x^2 y + ix^3 + ic$$

$$= i(x^3) + i^2 3x^2 y - 3ixy^2 - i^2 y^3 + K$$

$$= i(x^3 + 3ix^2 y - 3xy^2 + (iy)^3 + K$$

$$f(z) = iz^3 + K$$

# Standard function of complex variables :-

## (1) Exponential function :-

$$f(z) = e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y)$$

$$u = e^{x}\cos y \quad , \quad v = e^{x}\sin y$$

$$\frac{\partial u}{\partial x} = e^{x}\cos y \quad , \quad \frac{\partial v}{\partial y} = e^{x}\cos y$$

$$\frac{\partial u}{\partial y} = -e^{x}\sin y \quad , \quad \frac{\partial v}{\partial x} = e^{x}\cos y$$

C-R condition are satisfied for all $Z$.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^{x}\cos y + i\, e^{x}\sin y$$

$$= e^{x}(\cos y + i\sin y) = e^{z}$$

**EX** Find all roots of the equation $e^{z} = -i$

$$e^{z} = e^{x}\cos y + i\, e^{x}\sin y = -i$$

$$e^{x}\cos y = 0 \quad , \quad \cos y = 0$$

$$e^{x}\sin y = -1 \quad , \quad y = -\frac{\pi}{2} + 2K\pi$$

$$\therefore Z = i\left(\frac{-\pi}{2} + 2K\pi\right) = i\left(-\frac{1}{2} + 2K\right)\pi$$

$$K = 0, 1, 2, 3, \ldots$$

② Trigonometric function :-

$$\cos Z = \frac{e^{iz} + e^{-iz}}{2} \quad , \quad \sin Z = \frac{e^{iz} - e^{-iz}}{2i}$$

The trigonometric functions are analytic function :-

$$\frac{d}{dZ}[\cos Z] = \frac{i(e^{iz} - e^{-iz})}{2} = \frac{-(e^{iz} - e^{-iz})}{2i} = -\sin Z$$

$$\frac{d}{dZ}[\sin Z] = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos Z$$

$$\cos^2 Z + \sin^2 Z = 1$$

$$\cos Z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$\sin Z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

**Ex** Find the roots of the equation $\sin Z = 1$

$$\sin x \cosh y + i \cos x \cdot \sinh y = 1$$

$$\therefore \sin x \cosh y = 1 \Rightarrow \sin x = 1$$

$$\therefore x = \frac{\pi}{2} + 2K\pi$$

$$= \left(\frac{1}{2} + 2K\right)\pi \quad , K = 0, 1, 2, \dots$$

## 3) Hyperabolic function :-

The Hyperabolic functions are analytic function.

$$\frac{d}{dz}[\cosh z] = \frac{e^{z}-e^{-z}}{2} = \sinh z,$$

$$\frac{d}{dz}[\sinh z] = \frac{e^{z}+e^{-z}}{2} = \cosh z$$

$\cosh^2 z - \sinh^2 z = 1$, $\cosh z = \frac{e^{z}+e^{-z}}{2}$, $\sinh z = \frac{e^{z}-e^{-z}}{2}$

* $\cosh z = \cosh(x+iy) = \cosh x \cosh iy + \sinh x \sinh iy$
$= \cosh x \cos y + i \sinh x \sin y$

* $\sinh z = \sinh(x+iy) = \sinh x \cosh iy + i \cosh x \sinh y$
$= \sinh x \cos y + i \cosh x \sin y$

* $\tanh z = \frac{\sinh z}{\cosh z}$, $\coth = \frac{\cosh z}{\sinh z}$

$\operatorname{sech} z = \frac{1}{\cosh z}$, $\operatorname{csch} z = \frac{1}{\sinh z}$

* $\cosh iy = \cos z$, $\sinh iz = i \sin z$,

$\cos iz = \cosh z$, $\sin iz = i \sinh z$.

## ④ Logarithmic function :-

The natural logarithm of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ by the relation

$$e^{w} = z \quad , \quad 7$$

$$e^{w} = e^{u+iv}$$

$$e^{w} = r e^{i\theta}$$

$$w = \ln(r e^{i\theta})$$

$$= \ln r + i\theta$$

$$u = \ln r \quad , \quad v = \theta$$

**EX** find $\ln(1+i)$

$$\ln(1+i) = \ln r + i\theta$$

$$r = \sqrt{1+1} = \sqrt{2} \;,\; \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4} \text{ (principle value)}$$

$$\ln(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2K\pi\right) \;,\; K = 0, 1, 2, \ldots$$

**EX** Find $Z$ for $w = \ln Z = 1 - i\pi$

$$\ln Z = \ln r + i\theta = 1 - i\pi$$

$$\therefore \ln r = 1 \Rightarrow r = e \;,\; \theta = -\pi$$

$$Z = r e^{i\theta} \Rightarrow Z = e\, e^{i\theta} \Rightarrow e\, e^{-i\pi}$$

$$Z = e^{(1-i\pi)}$$

⑤ Reciprocal function :-

$f(z) = \frac{1}{z}$ , is defined & analytic

$\therefore f(z) = r^{-1} e^{-i\theta}$ , $f'(z) = -\frac{1}{z^2}$

⑥ Inverse trigonometric function :-

(A) $w = f(z) = \sin^{-1} z$ , $z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$

By multiplying $e^{iw}$ both sides

$$z e^{iw} = \frac{e^{iw} - e^{-iw}}{2i} e^{iw}$$

$$2iz e^{iw} = e^{2iw} - 1$$

$$e^{2iw} - 2iz e^{iw} - 1 = 0$$

$$e^{iw} = iz \pm \sqrt{1 - z^2}$$

$$iw = \ln(iz \pm \sqrt{1-z^2}) \Rightarrow$$

$$w = -i \ln[i(z \mp \sqrt{z^2 - 1}]$$

Ex find $\sin^{-1}(-i)$

$w = -i \ln[i(-i \pm \sqrt{(-i)^2 - 1}$

B)

$$w = f(z) = \cos^{-1}(z)$$

$$w = \cos^{-1} z \Rightarrow z = \cos w = \frac{e^{iw} + e^{-iw}}{2}$$

$$e^{2iw} - 2z e^{iw} + 1 = 0$$ multiply both sides by $e^{iw}$

$$e^{iw} = z \pm \sqrt{z^2 - 1}$$

$$w = -i \ln(z \mp \sqrt{z^2-1}$$

C) $w = \tan^{-1} z$

$$z = \tan w = \frac{e^{jw} - e^{-iw}}{i(e^{jw} + e^{-jw})}$$

$$iz = \frac{e^{2iw} - 1}{e^{2iw} + 1} \Rightarrow i e^{2iw} z + iz = e^{2iw} - 1$$

$$e^{2iw} - i e^{2iw} z = iz + 1$$

$$e^{2iw}(1 - iz) = 1 + iz$$

$$e^{2iw} = \frac{1 + iz}{1 - iz} \Rightarrow w = \frac{1}{2i} \ln \frac{1 + iz}{1 - iz}$$

z

# Complex integral :-

Complex definite integrals are called (complex) line integral

$$\int_C f(z)\,dz$$

Here the integrand $f(z)$ is integrated over a given curve "C" or a portion of it (an arc, but we will say "curve"). This curve C in the complex plane is called the "path of integration". We may represent C by a parametric representation

$$z(t) = x(t) + i\,y(t) \qquad a \leq t \leq b$$

The sense of icreasing $t$ is called the "positive sense" on "C", and we say "C" is "oriented".

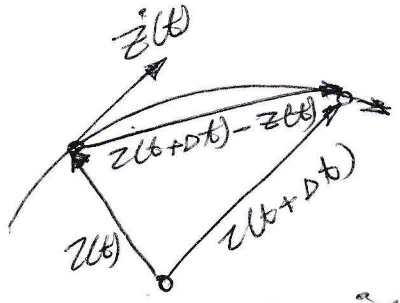
if $z(t) = t + 3it$ $\quad 0 \leq t \leq 2$ gives a portion (a segment) of line $y = 3x$.

We assume "C" to be a "smooth curve", that is, "C" has a continous and nonzero derivative

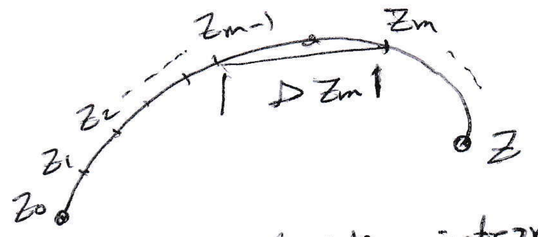
$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\,\dot{y}(t)$$

at each point. Geometrically this means that "C" has everywhere a continuously turning tangent, as follows directly from the definition

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$




tangent vector $\dot{z}(t)$ of a curve $C$ in the complex plane given by $z(t)$




Complex line integral

the line integral is denoted by

$$\int_C f(z)\,dz \,, \text{ or by } \oint_C f(z)\,dz$$

## Basic Propenties implied by the definition :-

1- linearity

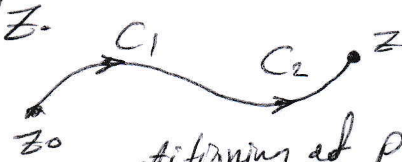
$$\int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$$

2 - Sense reversal

$$\int_{z_0}^{Z} f(z)\,dz = -\int_{Z}^{z_0} f(z)\,dz$$

3 - Pantitioning of Path

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$




pantitioning of path

for the curve, we can definite integrals by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

or $$\int_{z_0}^{z_1} f(z)\,dz = f(z_1) - F(z_0)$$

<u>EX</u> $$\int_0^{1+i} z^2\,dz = \frac{1}{3} z^3 \Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3} i$$

<u>EX</u> $$\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z \Big|_{-\pi i}^{\pi i} = 2 \sin \pi i = 2 i \sinh \pi = 23.097 i$$

<u>EX</u> $$\int_{8+\pi i}^{8-\pi i} e^{z/2}\,dz = 2 e^{z/2} \Big|_{8+\pi i}^{8-\pi i} = 2\left(e^{4-3\pi i/2} - e^{4+3\pi i/2}\right) = 0$$

since $e^z$ is periodic with period $2\pi i$.

<u>EX</u> $$\int_{-i}^{i} \frac{dz}{z} = \ln i - \ln(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) = i\pi$$

<u>EX</u> $$\int_C 2+2i$$

$$z = 2+2i \Rightarrow z = re^{i\theta} \Rightarrow dz = ri e^{i\theta} d\theta$$

$$r = \sqrt{8} = 2\sqrt{2}, \quad \theta = \tan^{-1}\frac{2}{2} = \frac{\pi}{4}$$

$$\therefore \int_C 2+2i = \int_C z\,dz = \int_0^{\pi/4} e^{i\theta} \cdot ri e^{i\theta} d\theta$$

$$= r^2 i \int_0^{\pi/4} e^{2i\theta} = \frac{r^2 i}{2}\left[e^{2i\pi/4} - 1\right]$$

$$= 4i\left[\cos\frac{\pi}{2} + i \sin\frac{\pi}{2} - 1\right] \Rightarrow \int_C 2+2i = 4i - 4$$

**EX** find $\int_C z^2 dz$ if $z = 3+3i$

$z = re^{i\theta} \Rightarrow dz = rie^{i\theta} d\theta$

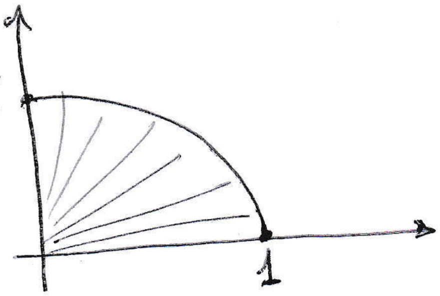
$r = 3\sqrt{2}\ ,\ \theta = \frac{\pi}{4}$

$$\therefore \int_C z^2 dz = \int_0^{\pi/4} r^2 e^{2i\theta} \cdot r i e^{i\theta} d\theta$$

$$= r^3 i \left[ \frac{e^{3i\frac{\pi}{4}} - 1}{3} \right]$$

**EX** Find integral $\int_C z^2 dz$, where $C$ is a circle between

$z = 1$ & $z = i$




$z = e^{i\theta}$

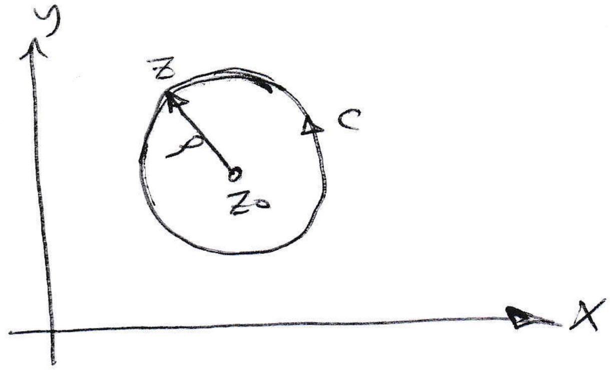
$dz = ie^{i\theta} d\theta$

$z^2 = e^{i2\theta}$

$$\int_C z^2 dz = \int_0^{\pi/2} e^{i2\theta} \cdot i \cdot e^{2\theta} d\theta$$

$\vdots$

EX Find the integral $\oint_C \frac{dz}{(z-z_0)^n}$, where C is a circle of radius r centered at $z_0$




the equation of the circle

$$(z-z_0) = r e^{i\theta}$$

$$dz = i r e^{i\theta} d\theta$$

$$\oint_C \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{i r e^{i\theta}}{r^n e^{in\theta}} d\theta = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{-i(n-1)\theta} d\theta$$

$$= \begin{cases} 2\pi i & n=1 \\ 0 & n>1 \end{cases}$$, let $z_0 = 0$, $n = 1$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{i r e^{i\theta}}{r e^{n\theta}} d\theta = 2\pi i$$

## Cauchy theorem:

If $f(z)$ is analytic within and on a boundary of ~~a boundary of a simple~~ R- a simple connected region enclosed by a closed path C, then

$$\oint_C f(z)\,dz = 0 \qquad \text{(close path)}$$

$$= \oint_C \frac{f(z)\,dz}{(z-z_0)} = 2\pi i f(z_0)$$

EX find $\oint_C \frac{dz}{z^2+4}$, where C

① is the circle $|z+i| = 2$

② is the circle $|z-i| = 2$

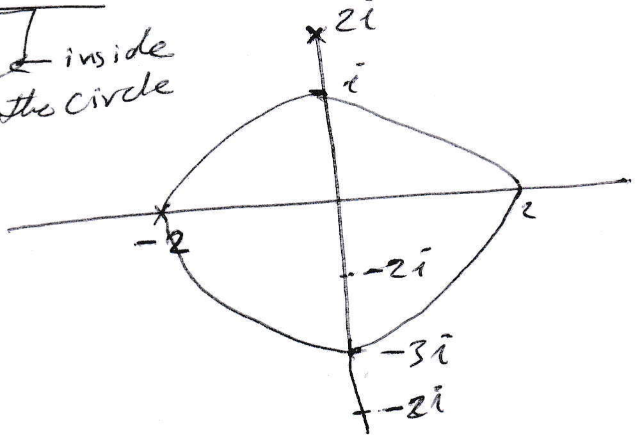
③ is the circle $|z| = 4$

① $\oint_{C_1} \frac{dz}{z^2+4} = \oint_{C_1} \frac{dz}{(z+2i)(z-2i)}$

$\oint_{C_1} \frac{dz}{z^2+4} = \oint_{C_1} \frac{\boxed{dz/z-2i}}{\boxed{z+2i}}$ ← out of the circle ← inside the circle

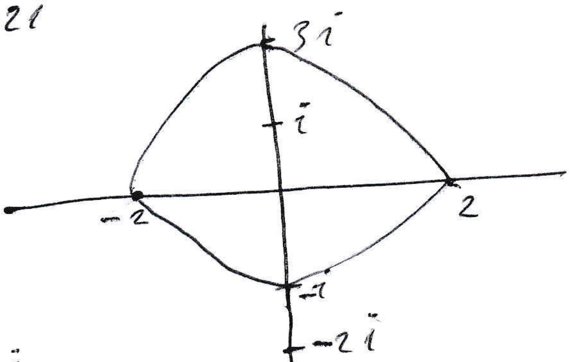
$= 2\pi i \left[\frac{1}{z-2i}\right]\Big|_{z=-2i}$

$= -\frac{\pi}{2}$




② $\oint_{C_2} \frac{dz}{z^2+4} = \oint_{C_2} \frac{dz/z+2i}{z-2i}$

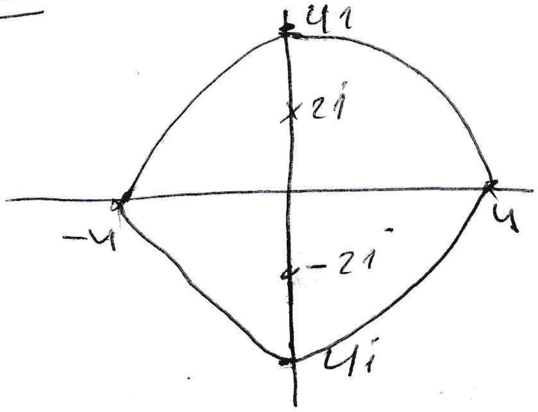
$= 2\pi i \left[\frac{1}{z+2i}\right]\Big|_{z=2i}$




③ $\oint_{C_3} \frac{dz}{z^2+4} = \oint_{C_1} \frac{dz/z-2i}{z+2i}$

$+ \oint_{C_2} \frac{dz/z+2i}{z-2i}$

$= -\frac{\pi}{2} + \frac{\pi}{2} = 0$



EX $\oint_C \frac{z^3-6}{2zi} dz$.

$$\oint_C \frac{z^3-6}{2z-i} dz = \oint_C \frac{z^3-6}{2(z-\frac{1}{2}i)} dz$$

$$= \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i} dz = 2\pi i \left[\tfrac{1}{2}z^3-3\right]\Big|_{z=i/2}$$

$$= \frac{\pi}{8} - 6\pi i \qquad (z_0 = \tfrac{1}{2}i \text{ inside } C).$$

EX Integrate $g(z) = \frac{z^2+1}{z^2-1}$

$$g(z) = \frac{z^2+1}{(z-1)(z+1)}$$

We notice that the $g(z)$ is not analytic. We consider each circle separately.

each circle separately.

Ⓐ the circle $|z-1|=1$ enclose the point $z_0 = 1$,

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1} \frac{1}{z-1};$$

$$\oint_C \frac{z^2+1}{z^2-1} dz = 2\pi i f(1)$$

$$= 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i.$$

ⓑ $\oint_C \frac{z^2+1}{z^2-1} dz = 2\pi i f(-1)$

$$= 2\pi i \left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i$$

# Derivative of an Analytic function :-

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n = 1, 2, \dots$$

**Ex** find $f'$ for any contour enclosing the point $\pi i$

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin \pi i$$

$$= 2\pi \sinh \pi .$$

**Ex** find $f''$ for any contour enclosing the point $-i$ .

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz \Rightarrow$$

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz = \pi i(z^4-3z^2+6)''\Big|_{z=-i}$$

$$= \pi i[12z^2-6]\Big|_{z=-i} = -18\pi i .$$

$\underline{\underline{EX}}$ find $\oint$ for $\oint_C \frac{e^z}{(z-1)^2(z^2+4)} dz$

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)} dz = 2\pi i \left(\frac{e^z}{z^2+4}\right)'\Big|_{z=1}$$

$$= 2\pi i \cdot \frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2}\Big|_{z=1}$$

$$= \frac{6e\pi}{25} i = 2.050 i.$$

# Taylor and Maclaurin Series :-

The "Taylor Series" of a function $f(z)$, the complex along of the real Tylor series is

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0) \quad \text{------} \; (1)$$

OR

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^* \quad \text{------------} \; (2)$$

In equation (2), we integrate around a simple closed path "C" that contains $z_0$ in ~~this~~ its interior and such that $f(z)$ is analytic in a domain containing C and every point inside C

A "Maclaurin series" is a Taylor Series with Center $z_0 = 0$.

The "remainder" of the Taylor series (1) after the term $a_n (z - z_0)^n$ is.

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1} (z^* - z)} \, dz^*$$

$$f(z) = f(z_0) + \frac{z - z_0}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots$$

$$+ \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

# Important special Taylor series :-

ⓐ Geometric Series

$$f(z) = \frac{1}{1-z} \text{, then we have } f^{n}(z) = \frac{n!}{(1-z)^{n+1}} \text{ , } f^{n}(0) = n!$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^{n} = 1 + z + z^{2} + \cdots$$

b) Exponential function.

$$e^{z} = \sum_{n=0}^{\infty} \frac{z^{n}}{n!} = 1 + z + \frac{z^{2}}{2!} + \cdots$$

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^{n}}{n!} = \sum_{k=0}^{\infty} (-1)^{k} \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^{k} \frac{y^{2k+1}}{(2k+1)!}$$

ⓒ Trigonometric and Hyperbolic functions

$$\cos z = \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!} = 1 - \frac{z^{2}}{2!} + \frac{z^{4}}{4!} - \cdots + \cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^{3}}{3!} + \frac{z^{5}}{5!} - \cdots + \cdots$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^{2}}{2!} + \frac{z^{4}}{4!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^{3}}{3!} + \frac{z^{5}}{5!} + \cdots$$

d) Logarithm

$$\ln(1+z) = z - \frac{z^{2}}{2} + \frac{z^{3}}{3} - \cdots + \cdots$$

$$\ln\left(\frac{1}{1-z}\right) = z + \frac{z^{2}}{2} + \frac{z^{3}}{3} + \cdots$$

EX Find the Maclaurin series of $f(z) = \frac{1}{1+z^2}$

we have $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$

then $\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = 1 - z^2 + z^4 - z^6 + \cdots$

EX Develope $\frac{1}{c-z}$ in powers of $z - z_0$ where $c - z_0 \neq 0$

$$\frac{1}{c-z} = \frac{1}{c-z+z_0-z_0} = \frac{1}{c-z_0-(z-z_0)}$$

$$= \frac{1}{(c-z_0)\left(1-\frac{z-z_0}{c-z_0}\right)} = \frac{1}{c-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{c-z_0}\right)^n$$

$$= \frac{1}{c-z_0}\left(1 + \frac{z-z_0}{c-z_0} + \left(\frac{z-z_0}{c-z_0}\right)^2 + \cdots\right)$$

$\left|\frac{z-z_0}{c-z_0}\right| < 1$, then $|z-z_0| < |c-z_0$

EX Find the Taylor series of the following function with center $z_0 = 1$

$$f(z) = \frac{2z^2+9z+5}{z^3+z^2-8z-12}$$

the binomial series is

$$\frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$

$$= 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

with $m=2$ and the partial fractions are

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3}$$

$$= \frac{1}{(z-1+1+2)^2} + \frac{2}{(z-1+1-3)}$$ by adding $-1$ and $+1$ for each part with $z$.

$$= \frac{1}{[3+(z-1)]^2} + \frac{2}{-2+(z-1)}$$

$$= \frac{1}{[3+(z-1)]^2} - \frac{2}{[2-(z-1)]}$$

$$= \frac{1}{9\left[1+\frac{1}{3}(z-1)\right]^2} - \frac{2}{2\left[1-\frac{1}{2}(z-1)\right]}$$

$$= \frac{1}{9}\left(\frac{1}{[1+\frac{1}{3}(z-1)]^2}\right) - \frac{1}{1-\frac{1}{2}(z-1)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n$$

$$= -\frac{8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2 - \frac{275}{1944}(z-1)^3 - \cdots$$