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ELECTROMAGNETIC FIELDS

CHAPTER TWO: COULOM'S LAW AND ELECTRIC FIELD INTENSITY

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2.1 COULOM'S LAW

Coulomb's law states that the force **F** between two point charges Q_1 and Q_2 is:

- Directly proportional to the product $Q_1 Q_2$ of the charges.
- Inversely proportional to the square of the distance R between them.
- Along the line joining them.

Expressed mathematically

$$F = k\frac{Q_1 Q_2}{R^2} \quad \text{in } N$$

Where,

- Q_1 and Q_2 are quantities of charge are measured in coulombs (C),
- R is the distance between Q_1 and Q_2 in meters (m), and
- k is a proportionality constant and equal is

$$k = \frac{1}{4\pi\epsilon_o}$$

Where, ϵ_o is called the permittivity of free space in farads per meter (F/m),

$$\epsilon_o = 8.854 \times 10^{-12}\ F/\text{m} \quad \text{or} \quad \epsilon_o = \frac{10^{-9}}{36\pi}\ F/\text{m} \tag{2.1}$$

Coulomb's law is now,

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_o R^2} \tag{2.2}$$

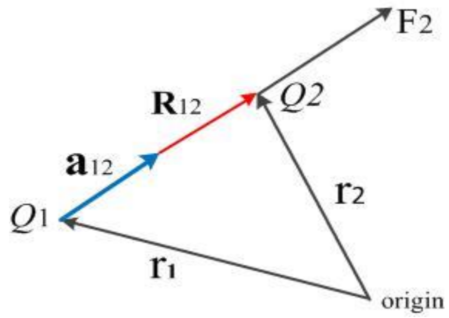


Fig. 2.1 if Q_1 and Q_2 have like signs, the vector force $\mathbf{F}_2$ on Q_2 is in the same direction of $\mathbf{R}_{12}$.

COULOM'S LAW AS A VECTOR FORM

Let,

- The position vectors $\mathbf{r_1}$ locate Q_1 while $\mathbf{r}_2$ locates Q_2 (from the origin).
- The distance vector $\mathbf{R}_{12} = \mathbf{r_2} - \mathbf{r_1}$ as shown in Fig. 2.1.

Therefore,

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_o R_{12}^2}\mathbf{a}_{12} \quad (N) \tag{2.3}$$

Where, $\mathbf{a}_{12}$ is the unit vector in the direction of vector $\mathbf{R}_{12}$, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{2.4}$$

This force $\mathbf{F}_2$ is the force on Q_2 acts along the line joining the two charges and is repulsive if the charges are alike in sign and attractive if they are of opposite sign.

Example 2.1:

If the charge of $Q_1 = 3 \times 10^{-4}$ C located at $M(1,2,3)$ and the charge of $Q_2 = -10^{-4}$ C located at $N(2,0,5)$ in a vacuum. Find the force exerted on Q_2 by Q_1.

Solution:

The vector $\mathbf{R}_{12}$ is $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z$

$$\mathbf{R}_{12} = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

$$|\mathbf{R}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

And the unit vector,

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}$$

Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi) \times 10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z \ N$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction.

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_o R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_o R_{12}^2}\mathbf{a}_{12} \tag{2.5a}$$

Example 2.2:

A charge $Q_A = -20$ μC is located at $A(-6,4,7)$, and a charge $Q_B = 50$ μC is at $B(5,8,-2)$ in free space. If distances are given in meters, find: (a) $\mathbf{R}_{AB}$; (b) R_{AB}. Determine the vector force exerted on Q_A by Q_B if (c) $\epsilon_o = \frac{1}{36\pi}10^{-9}$ F/m; (d) $\epsilon_o = 8.854 \times 10^{-12}$ F/m.

Solution:

(a) The vector $\mathbf{R}_{AB}$ is $\mathbf{R}_{AB} = (5+6)\mathbf{a}_x + (8-4)\mathbf{a}_y + (-2-7)\mathbf{a}_z$

$$\mathbf{R}_{AB} = 11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z \text{ m}$$

(b) $$R_{AB} = \sqrt{11^2 + 4^2 + 9^2} = 14.76 \text{ m}$$

(c) If $\epsilon_o = (1/36\pi) \times 10^{-9}$ F/m

$$\mathbf{F}_A = \frac{Q_1 Q_2}{4\pi\epsilon_o R_{BA}^2}\mathbf{a}_{BA} \frac{-20 \times 10^{-6}(50 \times 10^{-6})}{4\pi\left(1/_{36\pi}\right) \times 10^{-9} \times 14.76^2}\left(\frac{-11\mathbf{a}_x - 4\mathbf{a}_y + 9\mathbf{a}_z}{14.76}\right)$$

$$\mathbf{F}_A = 30.76\ \mathbf{a}_x + 11.184\ \mathbf{a}_y - 25.16\ \mathbf{a}_z\ \text{m}N$$

(d) If $\epsilon_o = 8.854 \times 10^{-12}$ F/m.

$$\mathbf{F}_A = \frac{Q_1 Q_2}{4\pi\epsilon_o R_{BA}^2}\mathbf{a}_{BA} = \frac{-20 \times 10^{-6}(50 \times 10^{-6})}{4\pi \times 8.854 \times 10^{-12} \times 14.76^2}\left(\frac{-11\mathbf{a}_x - 4\mathbf{a}_y + 9\mathbf{a}_z}{14.76}\right)$$

$$\mathbf{F}_A = 30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z\ \text{m}N$$

THE FORCE ON A CHARGE Q_1 due to n POINT CHARGES

The force on a charge Q_1 due to n other charges, Q_2, Q_3, ... , Q_n is the Vector Sum of the individual forces:

$$\mathbf{F}_1 = \frac{Q_1 Q_2}{4\pi\epsilon_o R_{21}^2}\mathbf{a}_{21} + \frac{Q_1 Q_3}{4\pi\epsilon_o R_{31}^2}\mathbf{a}_{31} + \cdots + \frac{Q_1 Q_n}{4\pi\epsilon_o R_{n1}^2}\mathbf{a}_{n1} = \frac{Q_1}{4\pi\epsilon_o}\sum_{m=2}^{n}\frac{Q_m}{R_{m1}^2}\mathbf{a}_{m1} \qquad (2.5b)$$

Example 2.3:

Three point charges, $Q_1 = Q_2 = Q_3 = 20\ \mu C$ are located at (4,0,0)m, (0,0,0)m, and (−4,0,0)m respectively and $Q_4 = 100\ \mu C$ located at (0,0,3)m.

(a) Determine the force on a charge Q_4.

Solution:

$$\mathbf{R}_{14} = (0-4)\mathbf{a}_x + (0-0)\mathbf{a}_y + (3-0)\mathbf{a}_z = -4\ \mathbf{a}_x + 3\ \mathbf{a}_z$$

$$|R_{14}| = \sqrt{(-4)^2 + (3)^2} = 5\ \text{m}$$

$$\mathbf{R}_{24} = (0-0)\mathbf{a}_x + (0-0)\mathbf{a}_y + (3-0)\mathbf{a}_z = 3\ \mathbf{a}_z$$

$$|R_{24}| = \sqrt{(3)^2} = 3\ \text{m}$$

$$\mathbf{R}_{34} = \big(0-(-4)\big)\mathbf{a}_x + (0-0)\mathbf{a}_y + (3-0)\mathbf{a}_z = 4\ \mathbf{a}_x + 3\ \mathbf{a}_z$$

$$|R_{34}| = \sqrt{(4)^2 + (3)^2} = 5\ \text{m}$$

$$\mathbf{F}_4 = \sum_{k=1}^{3}\frac{Q_4 Q_k}{4\pi\epsilon_o R_{k4}^2}\mathbf{a}_{k4} = \sum_{k=1}^{3}\frac{Q_4 Q_k}{4\pi\epsilon_o R_{k4}^3}\mathbf{R}_{k4}$$

$$\mathbf{F}_4 = \frac{20 \times 10^{-6} \times 100 \times 10^{-6}}{4\pi\frac{10^{-9}}{36\,\pi}}\left[\frac{(-4\ \mathbf{a}_x + 3\ \mathbf{a}_z)}{(5)^3} + \frac{3\ \mathbf{a}_z}{(3)^3} + \frac{(4\ \mathbf{a}_x + 3\ \mathbf{a}_z)}{(5)^3}\right]$$

$$\mathbf{F}_4 = 18 \times (0.024\mathbf{a}_z + 0.111\mathbf{a}_z + 0.024\mathbf{a}_z) = 2.862\ \mathbf{a}_z\ N$$

2.2 ELECTRIC FIELD INTENSITY

If one charge fixed in position, say Q_1, and move a second charge slowly around, we note that there exists everywhere a force on this second charge. Call this second charge a test charge, Q_t. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_o R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives,

$$\frac{\mathbf{F}_t}{Q_t} = \frac{Q_1}{4\pi\epsilon_o R_{1t}^2}\mathbf{a}_{1t} \quad (2.6)$$

Equation (2.6) is a function of Q_1 and the directed line from Q_1 to the position of the test charge, Q_t. This describes a vector field and is called the Electric Field Intensity.

The Electric Field Intensity: is define as the vector force on a unit positive test charge, $\frac{\mathbf{F}_t}{Q_t}$.

- Electric field intensity is measured by newton per coulomb (N/C) or volts per meter (V/m).
- Using a capital letter **E** for electric field intensity, we have finally

$$\mathbf{E} = \frac{\mathbf{F}_t}{Q_t} \quad (2.7)$$

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_o R_{1t}^2}\mathbf{a}_{1t} \quad (2.8)$$

Equation (2.7) is the defining expression for electric field intensity, and (2.8) is the expression for the electric field intensity due to a single point charge Q_1 in a vacuum.

Generally,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o R^2}\mathbf{a}_R \quad (2.9)$$

Where,

- R is the magnitude of the vector **R**, from the point at Q to the point at **E**,
- $\mathbf{a}_R$ is a unit vector in the direction of vector **R**.

In the Cartesian coordinates system for a charge Q at the origin, we have

$$\mathbf{R} = \mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$$

and

$$\mathbf{a}_R = \mathbf{a}_r = \frac{(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore;

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o(x^2 + y^2 + z^2)}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{a}_x + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{a}_y + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{a}_z\right) \quad (2.10)$$

If Q located at the source point $\mathbf{r}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z$, as in Figure 2.2, the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is:

By expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r_1}$, and then,

$$\mathbf{E}_{(\mathbf{r})} = \frac{Q_1}{4\pi\epsilon_o R^2}\mathbf{a}_R = \frac{Q}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r_1}|^2}\cdot\frac{\mathbf{r}-\mathbf{r_1}}{|\mathbf{r}-\mathbf{r_1}|} = \frac{Q(\mathbf{r}-\mathbf{r_1})}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r_1}|^3}$$

$$\mathbf{E}_{(\mathbf{r})} = \frac{Q[(x-x_1)\mathbf{a}_x + (y-y_1)\mathbf{a}_y + (z-z_1)\mathbf{a}_z}{4\pi\epsilon_o[(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{3/2}} \tag{2.11}$$

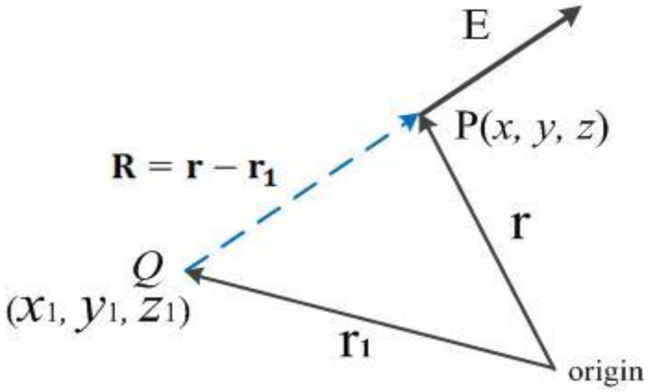


Figure 2.2 The vector $\mathbf{r_1}$ locates the point charge Q, the vector $\mathbf{r}$ identifies the general point in space $P(x, y, z)$, and the vector $\mathbf{R}$ from Q to $P(x, y, z)$ is then $\mathbf{R} = \mathbf{r} - \mathbf{r_1}$.

The electric field intensity due to two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\mathbf{E}_{(\mathbf{r})} = \frac{Q_1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r_1}|^2}\cdot\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r_2}|^2}\cdot\mathbf{a}_2$$

Where, $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of $(\mathbf{r} - \mathbf{r_1})$ and $(\mathbf{r} - \mathbf{r_2})$, as shown in Figure 2.3.

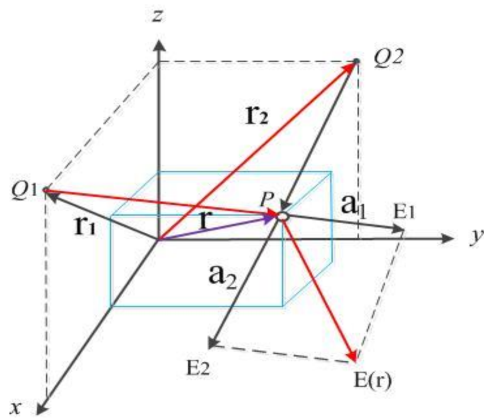


Fig 2.3 the vector addition of the total electric field intensity at point P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

THE ELECTRIC FIELD INTENSITY **E** due to n POINT CHARGES:

$$\mathbf{E}_{(\mathbf{r})} = \frac{Q_1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|^2}.\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_2|^2}.\mathbf{a}_2 + \cdots + \frac{Q_n}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_n|^2}.\mathbf{a}_n$$

Therefore;[1]

$$\mathbf{E}_{(\mathbf{r})} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m \tag{2.13}$$

Example 2.4: Find **E** at $P(0,3,4)$ m in Cartesian coordinate due to a point charge $Q = 0.5$ nC at the origin.

Solution: In this case

$$\mathbf{R}_{OP} = \mathbf{r}_P - \mathbf{r}_O = (0-0)\mathbf{a}_x + (3-0)\mathbf{a}_y + (4-0)\mathbf{a}_z = 3\mathbf{a}_y + 4\mathbf{a}_z$$

$$|\mathbf{R}_{OP}| = |\mathbf{r}_P - \mathbf{r}_O| = \sqrt{(3)^2 + (4)^2} = 5$$

And the unit vector,

$$\mathbf{a}_{OP} = \frac{\mathbf{R}_{OP}}{|\mathbf{R}_{OP}|} = \frac{\mathbf{r}_P - \mathbf{r}_O}{|\mathbf{r}_P - \mathbf{r}_O|} = \frac{3\mathbf{a}_y + 4\mathbf{a}_z}{5} = 0.6\mathbf{a}_y + 0.8\mathbf{a}_z$$

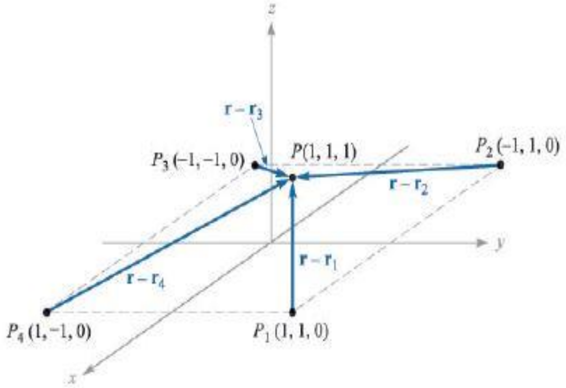
$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o R_{OP}^2}.\mathbf{a}_{OP} = \frac{0.5\times10^{-6}}{4\pi\times\left(\frac{10^{-9}}{36\pi}\right)\times(5)^2}.\left(0.6\mathbf{a}_y + 0.8\mathbf{a}_z\right)$$

$$\mathbf{E} = 180\left(0.6\mathbf{a}_y + 0.8\mathbf{a}_z\right)$$

Thus $|\mathbf{E}| = 180$ V/m in the direction of $\mathbf{a}_{OP} = 0.6\mathbf{a}_y + 0.8\mathbf{a}_z$

Example 2.5:

Find **E** at $P(1,1,1)$ caused by four identical 3 nC charges located at $P_1(1,1,0)$, $P_2(-1,1,0)$, $P_3(-1,-1,0)$, and $P_4(1,-1,0)$, as shown in Figure below.



Solution:

We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$,

$$\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y \quad : \quad \mathbf{r}_2 = -\mathbf{a}_x + \mathbf{a}_y \quad : \quad \mathbf{r}_3 = -\mathbf{a}_x - \mathbf{a}_y \quad : \quad \mathbf{r}_4 = \mathbf{a}_x - \mathbf{a}_y$$

$$\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z \quad \Rightarrow \quad |\mathbf{r} - \mathbf{r}_1| = 1$$

$$\mathbf{r}-\mathbf{r}_2 = 2\mathbf{a}_x + \mathbf{a}_z \quad \Rightarrow \quad |\mathbf{r}-\mathbf{r}_2| = \sqrt{5}$$
$$\mathbf{r}-\mathbf{r}_3 = 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z \quad \Rightarrow \quad |\mathbf{r}-\mathbf{r}_3| = 3$$
$$\mathbf{r}-\mathbf{r}_4 = 2\mathbf{a}_y + \mathbf{a}_z \quad \Rightarrow \quad |\mathbf{r}-\mathbf{r}_2| = \sqrt{5}$$

$$\mathbf{E}_{(\mathbf{r})} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$

$$\mathbf{E}_{(\mathbf{r})} = \frac{1}{4\pi\epsilon_o}\left[\frac{Q_1}{|\mathbf{r}-\mathbf{r}_1|^2}.\mathbf{a}_1 + \frac{Q_2}{|\mathbf{r}-\mathbf{r}_2|^2}.\mathbf{a}_2 + \frac{Q_3}{|\mathbf{r}-\mathbf{r}_3|^2}.\mathbf{a}_3 + \frac{Q_3}{|\mathbf{r}-\mathbf{r}_3|^2}.\mathbf{a}_3\right]$$

$$\mathbf{E} = \frac{3\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left[\frac{\mathbf{a}_z}{(1)^3} + \frac{2\mathbf{a}_x+\mathbf{a}_z}{(\sqrt{5})^3} . + \frac{2\mathbf{a}_x+2\mathbf{a}_y+\mathbf{a}_z}{(3)^3} + \frac{2\mathbf{a}_y+\mathbf{a}_z}{(\sqrt{5})^3}\right]$$

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \text{ V/m}$$

2.3 ELECTRIC FIELD DUE TO CONTINUOS CHARGE DISTRIBUTIONS:

We have only considered forces and electric fields due to point charges. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume as illustrated in Figure 2.4. It is customary to denote

- The line charge density by ρ_L (in C/m),
- Surface charge density by ρ_s (in C/m^2), and
- Volume charge density by ρ_v (in C/m^3).

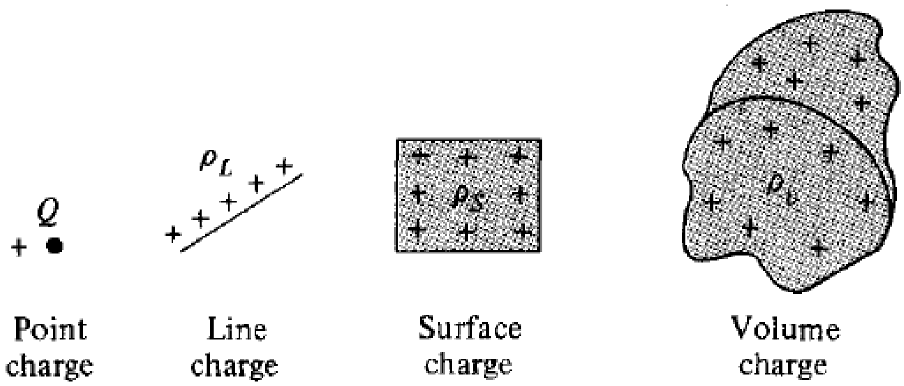


Fig. 2.4 Various charge distributions and charge elements.

The charge element dQ and the total charge Q due to these charge distributions are obtained from Figure 2.4 as:

$dQ = \rho_L dL$

$$Q = \int_L \rho_L dL \qquad \text{(Line charge)} \qquad (2.14)$$

$dQ = \rho_s ds$

$$Q = \int_s \rho_s ds \qquad \text{(Surface charge)} \qquad (2.15)$$

$dQ = \rho_v dv$

$$Q = \int_v \rho_v dv \qquad \text{(Volume charge)} \qquad (2.16)$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_s, and ρ_v may be obtained by replacing Q in equation (2.11) with charge element $dQ = \rho_L dL$, $dQ = \rho_s ds$, or $dQ = \rho_v dv$ and integrating, we get

$$\mathbf{E} = \int \frac{\rho_L dL}{4\pi\epsilon_o R^2}\mathbf{a}_R \qquad \text{(Line charge)} \qquad (2.17)$$

$$\mathbf{E} = \int \frac{\rho_s ds}{4\pi\epsilon_o R^2}\mathbf{a}_R \qquad \text{(Surface charge)} \qquad (2.18)$$

$$\mathbf{E} = \int \frac{\rho_v dv}{4\pi\epsilon_o R^2}\mathbf{a}_R \qquad \text{(Volume charge)} \qquad (2.19)$$

2.3.1 A Line Charge

Consider a line charge with uniform charge density ρ_L extending from A to B along the $z-$ axis as shown in Figure 2.5.

The charge element dQ associated with element $dl = dz$ of the line is:

$dQ = \rho_L dL = \rho_L dz$

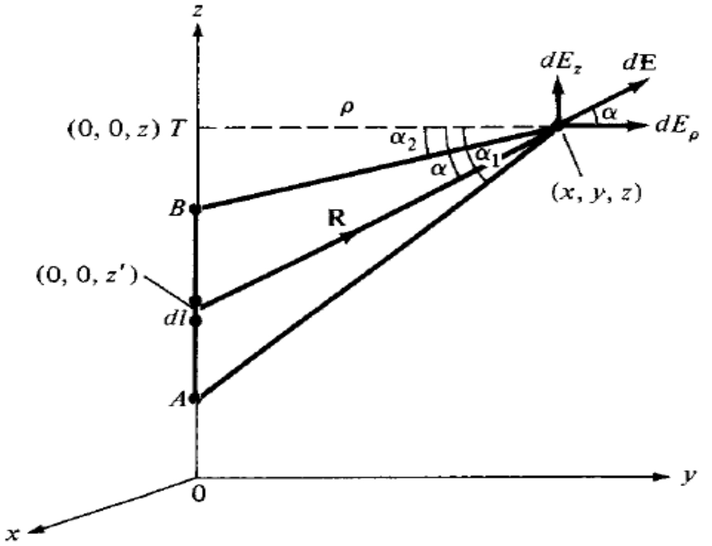


Fig. 2.5 Evaluation of the **E** field due to a line charge.

Hence, the total charge Q is:

$$Q = \int_{z_A}^{z_B} \rho_L dz \qquad (2.20)$$

The electric field intensity **E** at an arbitrary point *P(x, y, z)* can be found using equation (2.17). It is important that we learn to derive and substitute each term in equations (2.17) to (2.19) for a given charge distribution. It is customary to denote the field point by (x, y, z) and the source point by (x', y', z'). Thus from Figure 2.5.

$$dL = dz'$$

$$\mathbf{R} = (x-0)\mathbf{a}_x + (y-0)\mathbf{a}_y + (z-z')\mathbf{a}_z = x\mathbf{a}_x + y\mathbf{a}_y + (z-z')\mathbf{a}_z$$

Or in cylindrical coordinate $\mathbf{R} = \rho\mathbf{a}_\rho + (z-z')\mathbf{a}_z$

$$R^2 = |R|^2 = x^2 + y^2 + (z-z')^2 = \rho^2 + (z-z')^2$$

$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{R^3} = \frac{\rho\mathbf{a}_\rho + (z-z')\mathbf{a}_z}{[\rho^2 + (z-z')^2]^{3/2}}$$

Substituting all this into equation (2.17), we get

$$\mathbf{E} = \frac{\rho_L}{4\pi\epsilon_o}\int \frac{\rho\mathbf{a}_\rho + (z-z')\mathbf{a}_z}{[\rho^2 + (z-z')^2]^{3/2}} d_{z'} \tag{2.21}$$

To evaluate this, it is convenient that we define α, α_1 and α_2 as in Figure 2.5.

$$R = [\rho^2 + (z-z')^2]^{1/2} = \rho \sec\alpha$$

$$z' = OT - \rho\tan\alpha$$

$$dz' = -\rho\, sec^2\alpha\, d\alpha$$

Hence, equation (2.21) becomes

$$\mathrm{E} = \frac{-\rho_L}{4\pi\epsilon_o}\int_{\alpha_1}^{\alpha_2} \frac{\rho\, sec^2\alpha[\cos\alpha\,\mathbf{a}_\rho + \sin\alpha\,\mathbf{a}_z]\, d_\alpha}{\rho^2\ sec^2\alpha}$$

$$\mathbf{E} = -\frac{\rho_L}{4\pi\epsilon_o\rho}\int_{\alpha_1}^{\alpha_2} [\cos\alpha\,\mathbf{a}_\rho + \sin\alpha\,\mathbf{a}_z]\, d_\alpha \tag{2.22}$$

Thus for a finite line charge,

$$\mathbf{E} = \frac{\rho_L}{4\pi\epsilon_o\rho}[-(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_z] \tag{2.23}$$

As a special case, for an infinite line charge, point B is at $(0, 0, \infty)$ and A at $(0, 0, -\infty)$ so that $\alpha_1 = {}^{\pi}/_{2}$, $\alpha_2 = -{}^{\pi}/_{2}$; the z −component vanishes and equation (2.23) becomes:

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_o\rho}\mathbf{a}_\rho \tag{2.24}$$

Equation (2.24) is obtained for an infinite line charge along the z-axis so that ρ and $\mathbf{a}_\rho$ have their usual meaning. If the line is not along the z-axis, ρ is the perpendicular distance from the line to the point of interest and $\mathbf{a}_\rho$ is a unit vector along that distance directed from the line charge to the field point.

2.3.2 A Surface Charge

Consider an infinite sheet of charge in the $xy-$ plane with uniform charge density ρ_s. The charge associated with an elemental area dS is

$$dQ = \rho_s ds$$

Hence, the total charge is

$$Q = \int \rho_s ds$$

From equation (2.18), the contribution to the **E** field at point $P(0,0,h)$ by the elemental surface 1 shown in Figure 2.6 is:

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_o R^2}\mathbf{a}_R \qquad (2.25)$$

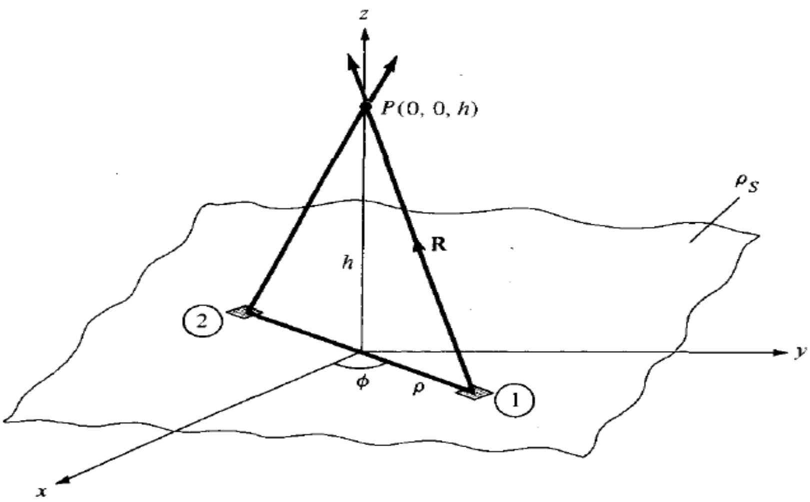


Fig. 2.6 Evaluation of the **E** field due to an infinite sheet of charge.

From Figure 2.6, $\mathbf{R} = \rho(-\mathbf{a}_\rho) + h\mathbf{a}_z = -\rho\mathbf{a}_\rho + h\mathbf{a}_z$

$$R = |\mathbf{R}| = [\rho^2 + h^2]^{1/2}$$

$$\mathbf{a}_R = \frac{\mathbf{R}}{R}$$

$$dQ = \rho_s ds = \rho_s\, \rho\, d\phi\, d\rho$$

Substitution of these terms into equation (2.25) gives:

$$d\mathbf{E} = \frac{\rho_s\, \rho\, d\Phi\, d\rho[-\rho\mathbf{a}_\rho + h\mathbf{a}_z]}{4\pi\epsilon_o[\rho^2 + h^2]^{3/2}} \qquad (2.26)$$

Due to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2 whose contribution along $\mathbf{a}_\rho$ cancels that of element 1, as illustrated in Figure 2.6. Thus, the contributions to E_ρ add up to zero so that **E** has only $z-$ component.

This can also be shown mathematically by replacing $\mathbf{a}_\rho$ with $\cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$. Integration of $\cos\phi$ or $\sin\phi$ over $0 < \phi < 2\pi$ gives zero. Therefore

$$\mathbf{E} = \int dE_z = \frac{\rho_s}{4\pi\epsilon_o} \int_{\Phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\rho\, d\rho\, d\phi}{[\rho^2 + h^2]^{3/2}} \mathbf{a}_z \tag{2.27}$$

$$\mathbf{E} = \frac{\rho_s\, h}{4\pi\epsilon_o} 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2} \frac{1}{2} d(\rho^2) \mathbf{a}_z = \frac{\rho_s\, h}{2\epsilon_o} \{-[\rho^2 + h^2]^{-1/2}\}_0^{\infty} \mathbf{a}_z$$

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_o} \mathbf{a}_z \tag{2.28}$$

That is, **E** has only $z-$ component if the charge is in the $xy-$ plane. In general, for an infinite sheet of charge

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_o} \mathbf{a}_n \tag{2.29}$$

Where, $\mathbf{a}_n$ is a unit vector normal to the sheet. From equation (2.28) or (2.29), we notice that the electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P. In a parallel plate capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_o} \mathbf{a}_n + \frac{-\rho_s}{2\epsilon_o} (-\mathbf{a}_n) = \frac{\rho_s}{\epsilon_o} \mathbf{a}_n \tag{2.30}$$

2.3.3 A Volume Charge

Let the volume charge distribution with uniform charge density ρ_v be as shown in Figure 2.7. The charge dQ associated with the elemental volume dv is

$$dQ = \rho_v dv$$

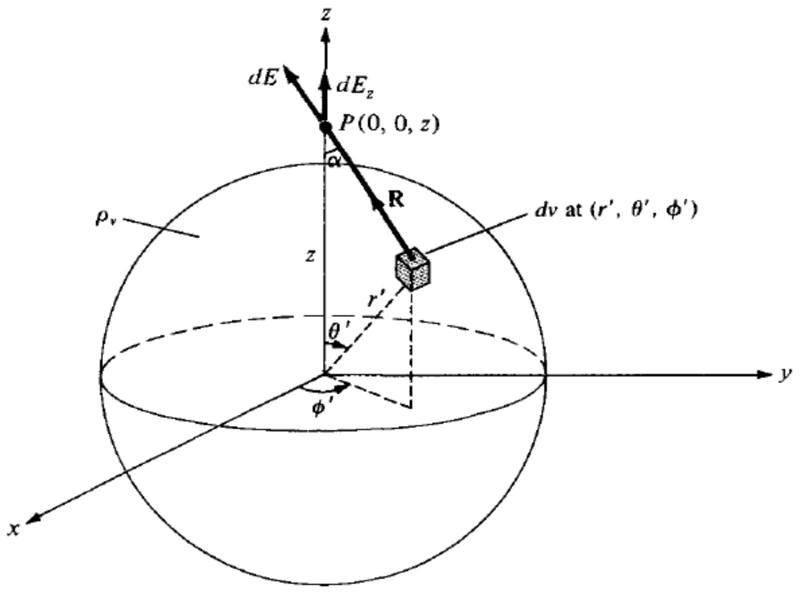


Fig. 2.7 Evaluation of the **E** field due to a volume charge distribution.

Hence, the total charge in a sphere of radius a is

$$Q = \int \rho_v dv = \rho_v \int dv = \rho_v \frac{4\pi a^3}{3} \tag{2.31}$$

The electric field $d\mathbf{E}$ at $P(0,0,z)$ due to the elementary volume charge is:

$$d\mathbf{E} = \frac{\rho_v d_v}{4\pi\epsilon_o R^2}\mathbf{a}_R$$

Where, $\mathbf{a}_R = \cos\alpha\,\mathbf{a}_z + \sin\alpha\,\mathbf{a}_\rho$. Due to the symmetry of the charge distribution, the contributions to E_x or E_y add up to zero. We are left with only E_z, given by

$$E_z = \mathbf{E}.\mathbf{a}_z = \int dE\ \cos\alpha = \frac{\rho_v}{4\pi\epsilon_o}\int\frac{d_v\cos\alpha}{R^2} \quad (2.32)$$

We need to derive expressions for dv, R^2, and $\cos\alpha$.

$$dv = r'^2\sin\theta'\,dr'\,d\,\theta'\,d\Phi' \quad (2.33)$$

Applying the cosine rule to Figure 2.7, we have

$$R^2 = z^2 + r'^2 - 2z\,r'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2z\,R\cos\alpha$$

It is convenient to evaluate the integral in equation (2.32) in terms of R and r'. Hence, we express $\cos\theta'$, $\cos\alpha$, and $\sin\theta'\,d\theta'$ in terms of R and r', that is,

$$\cos\alpha = \frac{z^2+R^2-r'^2}{2zR} \quad (2.34)$$

$$\cos\theta' = \frac{z^2+r'^2-R^2}{2zr'} \quad (2.35)$$

Differentiating equation (2.35) with respect to θ' keeping z and r' fixed, we obtain

$$\sin\theta' d\theta' = \frac{R\,dR}{z\,r'} \quad (2.36)$$

Substituting equations (2.33) to (2.36) into equation (2.32) yields

$$E_z = E.\mathbf{a}_z = \int dE\ \cos\alpha = \frac{\rho_v}{4\pi\epsilon_o}\int\frac{d_v\cos\alpha}{R^2}$$

$$E_z = \frac{\rho_v}{4\pi\epsilon_o}\int_{\Phi'=0}^{2\pi} d\Phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2\,\frac{R\,dR}{z\,r'}dr'\,\frac{z^2+R^2-r'^2}{2zR}\,\frac{1}{R^2}$$

$$E_z = \frac{2\pi\rho_v}{8\pi\epsilon_o z^2}\int_{r'=0}^{a} r'\left[1+\frac{z^2-r'^2}{R^2}\right]dRdr'$$

$$E_z = \frac{\pi\rho_v}{4\pi\epsilon_o z^2}\int_0^a r'\left[R-\frac{(z^2-r'^2)}{R}\right]_{z-r'}^{z+r'}dr'$$

$$E_z = \frac{\pi\rho_v}{4\pi\epsilon_o z^2}\int_0^a 4r'^2dr' = \frac{\pi\rho_v}{4\pi\epsilon_o z^2}\left(\frac{4}{3}a^2\pi\rho_v\right) \quad \text{or}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o z^2}\mathbf{a}_z \tag{2.37}$$

This result is obtained for **E** at $P(0,0,z)$. Due to the symmetry of the charge distribution, the electric field at $P(r,\theta,\Phi)$ is readily obtained from eq. (2.37) as:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o r^2}\mathbf{a}_r \tag{2.38}$$

Which is identical to the electric field at the same point due to a point charge Q located at the origin or the center of the spherical charge distribution.[2]

Example 2.6:

A uniform line charge, infinite in extent, with $\rho = 20$ nC/m, lies along the z axis. Find **E** at (6, 8, 3) m.

Solution:

In cylindrical coordinates $\rho = \sqrt{x^2+y^2} = \sqrt{(6)^2+(8)^2} = 10$ m. The field is constant with z. Thus

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_o\rho}\mathbf{a}_\rho = \frac{20\times 10^{-9}}{2\pi(10^{-9}/36\pi)(10)}\mathbf{a}_\rho = 36\mathbf{a}_\rho \text{ V/m}$$

Example 2.7:

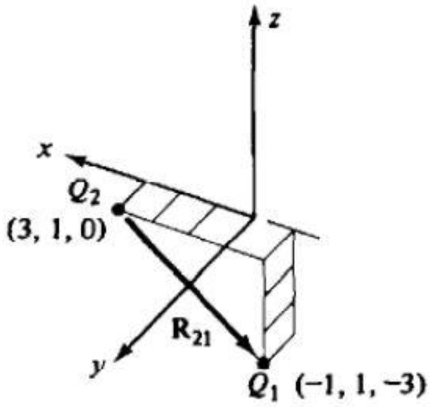
Charge is distributed uniformly over the plane $z = 10$ cm with a density $\rho_s = (1/3\pi)$ nC/m^2. Find **E**.

$$|\mathbf{E}| = \frac{\rho_s}{2\epsilon_o} = \frac{(1/3\pi)\times 10^{-9}}{2(10^{-9}/36\pi)} = 6 \text{ V/m}$$

Above the sheet ($z > 10$ cm), $\mathbf{E} = 6\mathbf{a}_z$ V/m; and for ($z < 10$ cm) $\mathbf{E} = -6\mathbf{a}_z$ V/m

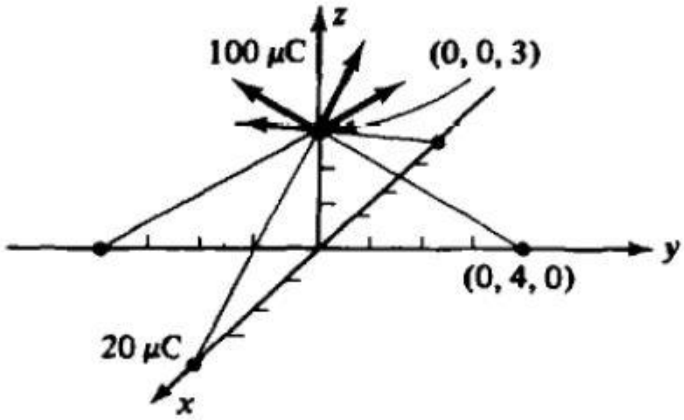
PROBLEMS

2.1.[3] Two point charges, $Q_1 = 50$ μC and $Q_2 = 10$ μC, are located at $(-1, 1, -3)$ m and $(3, 1, 0)$ m, respectively (Figure below). Find the force on Q_1.



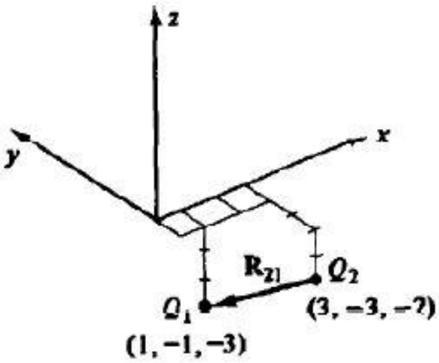
$$\left[\text{Ans}: \ \mathbf{F}_1 = 0.144\mathbf{a}_x - 0.108\mathbf{a}_z \text{ N}\right]$$

2.2.[3] Refer to Figure below. Find the force on a 100 μC charge at $P(0, 0, 3)$ m if four like charges of 20 μC are located on the x and y axes at ± 4 m.



$$\left[\text{Ans}: \mathbf{F}_P = \left(\frac{18}{25}\right) \times \left(\frac{12}{5}\mathbf{a}_z\right) = 1.73\mathbf{a}_z \text{ N}\right]$$

2.3.[3] Refer to Figure below. Point charge $Q_1 = 300$ μC, located at $(1, -1, -3)$ m, experiences a force $\mathbf{F}_1 = 8\mathbf{a}_x - 8\mathbf{a}_y + 4\mathbf{a}_z$ N, due to point charge Q_2 at $(3, -3, -2)$ m. Determine Q_2. $[\text{Ans}: Q_2 = -40 \text{ μC}]$



2.4.[1] Four 10-nC positive charges are located in the $z = 0$ plane at the comers of a square 8 cm on a side. A fifth 10-nC positive charge is located at a point $4\sqrt{2}$ cm on the z-axis. Calculate the magnitude of the total force on this fifth charge.

$$\left[\text{Ans}: \mathbf{F}_5 = (0.140 \times 10^{-3})\left[\left(\frac{16\sqrt{2}\mathbf{a}_z}{8}\right)\right] = 4 \times 10^{-4}\mathbf{a}_z \text{ N}\right]$$

The magnitude of the total force on fifth charge is: $\mathbf{F}_5 = 4 \times 10^{-4}$ N

2.5.[1] Point charges of 50 nC each are located at, $A(1,0,0)$, $B(-1,0,0)$, $C(0,1,0)$, and $D(0,-1,0)$ in free spacc. Find the total force on the charge at A.

$$\left[\text{Ans}: \mathbf{F}_A = (225 \times 10^{-7}) = \left[\frac{\mathbf{a}_x}{4} + \frac{\mathbf{a}_x}{2\sqrt{2}} + \frac{\mathbf{a}_x}{2\sqrt{2}}\right] = 21.5\mathbf{a}_x \ \mu\text{N}\right]$$

2.6.[3] Find the expression for the electric field at P due to a point charge Q at (x_1, y_1, z_1). Repeat with the charge placed at the origin.

$$\left[\text{Ans}: \mathbf{E} = \frac{Q}{4\pi\epsilon_o}\frac{(x-x_1)\mathbf{a}_x + (y-y_1)\mathbf{a}_y + (z-z_1)\mathbf{a}_z}{[(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{(3/2)}}\right]$$

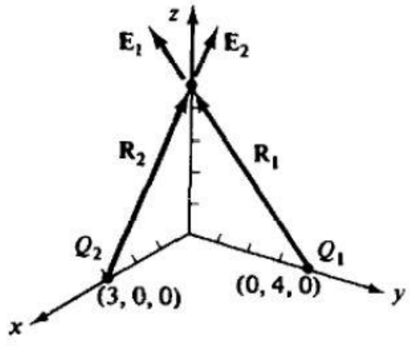
When the charge is at the origin,

$$\left[\text{Ans}: \mathbf{E} = \frac{Q}{4\pi\epsilon_o}\frac{x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{[x^2 + y^2 + z^2]^{(3/2)}}\right]$$

2.7.[3] Find $\mathbf{E}$ at the origin due to a point charge of 64.4 nC located at $(-4,3,2)$ m in Cartesian coordinates.

$$\left[\text{Ans}: \mathbf{E} = 20\left(\frac{4\mathbf{a}_x - 3\mathbf{a}_y - 2\mathbf{a}_z}{\sqrt{29}}\right)\text{V/m}\right]$$

2.8.[3] Find $\mathbf{E}$ at $(0,0,5)$ m due to $Q_1 = 0.35$ µC at $(0,4,0)$ m and $Q_2 = -0.55$ µC at $(3,0,0)$ m Figure below. $\left[\text{Ans}: \mathbf{E} = (74.9\mathbf{a}_x - 48\mathbf{a}_y - 64.9\mathbf{a}_z)\text{ V/m}\right]$



2.9.[3] Charge is distributed uniformly along an infinite straight line with constant density ρ_l. Develop the expression for $\mathbf{E}$ at the general point P.

2.10.[1] Let a point charge $Q_1 = 25$ nC be located at $P_1(4,-2,7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3,4,-2)$. (a) Find $\mathbf{E}$ at $P(1,2,3)$. $\left[\text{Ans}: \mathbf{E} = (4.58\mathbf{a}_x - 0.15\mathbf{a}_y + 5.51\mathbf{a}_z)\text{ V/m}\right]$

2.11.[1] The volume charge density $\rho_v = \rho_o e^{-|x|-|y|-|z|}$ exists over all free space. Calculate the total charge present. [Ans : $Q = 8\rho_o$]

2.12.[1] Let $\rho_v = 5e^{-0.1\rho}(\pi - |\phi|)\frac{1}{z^2+10}$ $\mu c/m^3$ in the region $0 < \rho < 10$, $-\pi < \phi < \pi$, all z, and $\rho_v = 0$ elsewhere, (a) Determine the total charge present, (b) Calculate the charge within the region $0 \le \rho \le 10$, $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$, $-10 < \phi < 10$.

[Ans : (a) $Q = 1.29$ mC (b) $Q = 0.182$ mC]

2.13.[1] The region in which $4 < r < 5$, $0 < \theta < 25^o$, and $0.9\pi < \phi < 1.1\pi$, contains the volume charge density $\rho_v = 10(r-4)(r-5)\sin\theta\cos\frac{1}{2}\phi$ $\mu c/m^2$. Outside that region $\rho_v = 0$. Find the charge within the region. [Ans : $Q = 0.57$ C]

2.14.[1] A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. (a) Find **E** at $P(1,2,3)$; (b) Find **E** at that point in the $z = 0$ plane where the direction of **E** is given by $\left(\frac{1}{3}\mathbf{a}_y - \frac{2}{3}\mathbf{a}_z\right)$. [Ans : (a) $\left(\mathbf{E}_P = 57.5\mathbf{a}_y - 28.8\mathbf{a}_z \text{ V/m}\right)$ (b)$\left(\mathbf{E}_{z=0} = 23\mathbf{a}_y - 46\mathbf{a}_z\right)$]

2.15.[1] A uniform line charge of 2 μC/m is located on the z-axis. Find **E** in Cartesian coordinates at $P(1,2,3)$ if the charge extends from: (a) $z = -\infty$ to $z = \infty$; (b) $z = -4$ to $z = 4$.

[Ans : (a) $\mathbf{E}_P = 7.2\mathbf{a}_x + 14.4\mathbf{a}_y$ kV/m (b) $\mathbf{E}_P = 4.9\mathbf{a}_x + 9.8\mathbf{a}_y + 4.9\mathbf{a}_z$ kV/m]

2.16.[1] Surface charge density is positioned in free space as follows: $20\ \text{nC/m}^2$ at $x = -3$, $-30\ \text{nC/m}^2$ at $y = 4$, and $40\ \text{nC/m}^2$ at $z = 2$. Find the magnitude of **E** at: (a) $P_A(4,3,-2)$; (b) $P_B(-2,5,-1)$; (c) $P_C(0,0,0)$.

[Ans : (a)$\mathbf{E}_A = 1130\mathbf{a}_x + 1695\mathbf{a}_y - 2260\mathbf{a}_z$ V/m and $|\mathbf{E}_A| = 3.04$ kV/m]

2.17.[1] Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC, at $P(1,0,6)$; uniform line charge density, 3 nC/m, at $x = -2$, $y = 3$; uniform surface charge density, $0.2\ \text{nC/m}^2$, at $x = 2$. [Ans : (a) $\mathbf{E} = -3.9\mathbf{a}_x - 12.4\mathbf{a}_y - 2.5\mathbf{a}_z$ V/m]

2.18.[1] A uniform line charge density of 5 nC/m is at $y = 0$, $z = 2$ m in free space, while -5 nC/m is located at $y = 0$, $z = -2$ m. A uniform surface charge density of $0.3\ \text{nC/m}^2$ is at $y = 0.2$ m, and $-0.3\ \text{nC/m}^2$ is at $y = -0.2$ m. Find $|\mathbf{E}|$ at the origin.

[Ans : (a) $\mathbf{E}_{(0,0,0)} = -33.9\mathbf{a}_y - 89.9\mathbf{a}_z$ V/m and $|\mathbf{E}| = 96.1$ V/m]

2.19.[3] Develop an expression for **E** due to charge uniformly distributed over an infinite plane with density ρ_s.

Problem 2.20.[2] Determine the total charge

(a) On line $0 < x < 5$ m if $\rho_L = 12x^2$ mC/m

(b) On the cylinder $\rho = 3$, $0 < z < 4$ m if $\rho_s = \rho z^2\ \text{nC/m}^2$

(c) Within the sphere $r = 4$ m if $\rho_v = \frac{10}{r\sin\theta}\ \text{C/m}^2$