7.1 INTRODUCTION.

In preview chapters, we limited our study and discussions to static electric fields characterized by electric field intensity **E** or electric flux density **D**. Now our study, discussions and attention on static magnetic fields, which are characterized by magnetic field intensity **H** or magnetic flux density **B**. There are similarities and dissimilarities between electric and magnetic fields. As **E** and **D** are related according to $\mathbf{D} = \varepsilon\, \mathbf{E}$ for linear material space, **H** and **B** are related according to $\mathbf{B} = \mu\mathbf{H}$. Table (7.1) shows the analogy between electric and magnetic field quantities [2].

We have noticed that, an electrostatic field is produced by static or stationary charges. If the charges are moving with constant velocity, a static magnetic (magnetostatic) field is produced. Therefore, the source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. This current flow may be due to magnetization currents as in permanent magnets, electron-beam currents as in vacuum tubes, or conduction currents as in current-carrying wires. In this chapter, will treat the magnetic fields of constant currents we consider magnetic fields of constant currents in free space [1,2,3].

Our study of static magnetic field is an indispensable necessity. The development of the motors, transformers, microphones, compasses, telephone bell ringers, television focusing controls, advertising displays, magnetically levitated high-speed vehicles, memory stores, magnetic separators, and so on, involve magnetic phenomena and play an important role in our everyday life [2].

There are two major laws governing magnetostatic fields:

(1) Biot-Savart's Law: Biot-Savart's law is the general law of static magnetic fields and it's like Coulomb's law in electric field.

(2) Ampere's Circuit Law: as Gauss's law is a special case of Coulomb's law, Ampere's law is a special case of Biot-Savart's law and is applied in problems involving symmetrical current distribution [2].

As we discussed the electric field, we shall confine our initial discussion for magnetic fields in free space conditions, and then effect of that fields in material media will also be saved for discussion [1].

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Table (7.1) Analogy between electric and magnetic field quantities.

No.	Terms	Electric field	Magnetic field
1	Basic laws	$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\mathbf{a}_R$ $\oint_S \mathbf{D}\,.\,d\mathbf{S} = Q_{enc}$	$d\mathbf{B} = \frac{\mu_o I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$ $\oint_L \mathbf{H}\,.\,d\mathbf{L} = I_{enc}$
2	Force law	$\mathbf{F} = Q\mathbf{E}$	$\mathbf{F} = Q\mathbf{u} \times \mathbf{B}$
3	Source element	dQ	$Q\mathbf{u} = Id\mathbf{L}$
4	Field intensity	$E = \frac{V}{l}$ (V/m)	$H = \frac{I}{l}$ (A/m)
5	Flux density	$\mathbf{D} = \frac{\psi}{S}$ $(\mathrm{C/m^2})$	$\mathbf{B} = \frac{\psi}{S}$ $(\mathrm{Wb/m^2})$
6	Relationship between fields	$\mathbf{D} = \varepsilon\,\mathbf{E}$	$\mathbf{B} = \mu\mathbf{H}$
7	Potentials	$\mathbf{E} = -\nabla \mathrm{V}$ $\mathrm{V} = \int \frac{\rho_L dL}{4\pi\varepsilon R}$	$\mathbf{H} = -\nabla \mathrm{V}_m$ $(\mathrm{J} = 0)$ $\mathrm{A} = \int \frac{\mu\, I\, d\mathbf{L}}{4\pi R}$
8	Flux	$\psi = \oint_S \mathbf{D}\,.\,d\mathbf{S}$ $\psi = Q = CV$ $I = C\frac{dV}{dt}$	$\psi = \oint_S \mathbf{B}\,.\,d\mathbf{S}$ $\psi = LI$ $V = L\frac{dI}{dt}$
9	Energy density	$W_E = \frac{1}{2}\mathbf{D}\,.\,\mathbf{E}$	$W_m = \frac{1}{2}\mathbf{B}\,.\,\mathbf{H}$
10	Poisson's equation	$\nabla^2 \mathrm{V} = -\frac{\rho_v}{\varepsilon}$	$\nabla^2 \mathrm{A} = -\mu \mathrm{J}$

7.2 BIOT-SAVART'S LAW.

Biot-Savart's Law: states that at any point P the magnitude of the magnetic field intensity dH produced by the differential current element Idl is proportional to the product of the current I, the magnitude of the differential length dl, and the sine of the angle α between the element and the line joining point P to the element and is inversely proportional to the square of the distance R between point P and the element as shown in Figure (7.1). That is,

$$dH \propto \frac{Idl\ \sin\alpha}{R^2} \tag{7.1}$$

or

$$dH = k\frac{Idl\sin\alpha}{R^2} \tag{7.2}$$

The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point P. where, k is the constant of proportionality. In SI units, $k = 1/4\pi$, so equation (7.2) becomes [1].

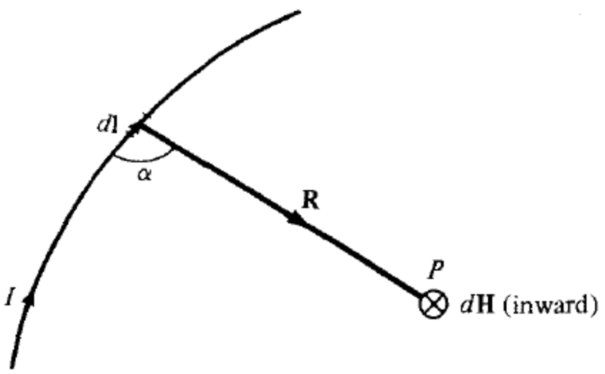


Fig. (7.1) magnetic field dH at point P due to current element Idl.

$$dH = \frac{Idl\sin\alpha}{4\pi\, R^2} \tag{7.3}$$

The Biot-Savart law, described above, may be written concisely using vector notation as:

$$d\mathbf{H} = \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2} = \frac{Id\mathbf{L}\times\mathbf{R}}{4\pi R^3} \tag{7.4}$$

Thus, the direction of $d\mathbf{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right-hand fingers encircling the wire in the direction of $d\mathbf{H}$ as shown in Figure (7.2a). Alternatively, we can use the right-handed screw rule to determine the direction of $d\mathbf{H}$: with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of $d\mathbf{H}$ as in Figure (7.2b).

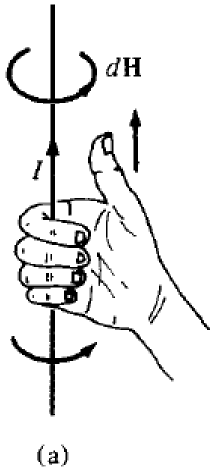


Fig. (7.2) Determining the direction of $d\mathbf{H}$ using (a) the right-hand rule (b) the right-handed screw rule.

Fig. (7.3) Conventional representation of $\mathbf{H}$ or I (a) out of the page and (b) into the page.

As in electric fields we can have different charge configurations, we can have different current distributions: line current, surface current, and volume current as shown in Figure (7.4). If we define $\mathbf{K}$ as the surface current density (in amperes/meter) and $\mathbf{J}$ as the volume current density (in amperes/meter square). The differential current element $Id\mathbf{L}$, where $d\mathbf{L}$ is in the direction of current, may be expressed in terms of surface current density $\mathbf{K}$ or volume current density J, Thus:

$$Id\mathbf{L} = \mathbf{K}dS = \vec{\mathrm{J}}dv \qquad (7.5)$$

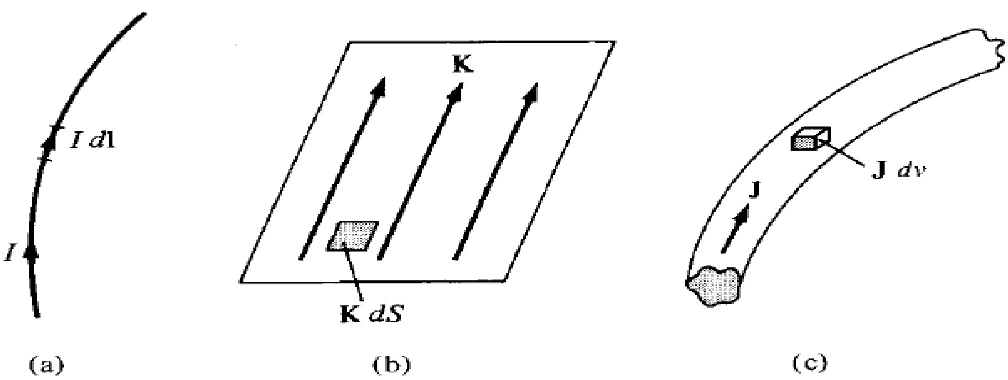


Fig. (7.4) Current distributions: (a) line current, (b) surface current, (c) volume current.

Thus, in terms of the distributed current sources, the Biot-Savart law as equation (7.4) becomes:

$$\mathbf{H} = \int_L \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi\, R^2} \qquad \text{(line current)} \tag{7.6}$$

$$\mathbf{H} = \int_S \frac{\mathbf{K}\, dS \times \mathbf{a}_R}{4\pi\, R^2} \qquad \text{(surface current)} \tag{7.7}$$

$$\mathbf{H} = \int_v \frac{\vec{\mathrm{J}}\, dv \times \mathbf{a}_R}{4\pi\, R^2} \qquad \text{(volume current)} \tag{7.8}$$

Magnetic field intensity due to a straight current carrying conductor.

Let us apply equation (7.6) to determine the field due to a straight current carrying filamentary conductor of finite length AB as in Figure (7.5). We assume that the conductor is along the z-axis with its upper and lower ends respectively subtending angles α_2 and α_1 at point P, the point at which $\mathbf{H}$ is to be determined. If we consider the contribution $d\mathbf{H}$ at point P due to an element $d\mathbf{L}$ at $(0, 0, z)$.

$$d\mathbf{H} = \frac{I d\mathbf{L} \times \mathbf{R}}{4\pi R^3} \tag{7.9}$$

But $d\mathbf{L} = dz\mathbf{a}_z$ and $\mathbf{R} = \rho\mathbf{a}_\rho - z\mathbf{a}_z$, so

$$d\mathbf{L} \times \mathbf{R} = \rho dz\, \mathbf{a}_\phi \tag{7.10}$$

Hence,

$$\mathbf{H} = \int \frac{I\rho dz}{4\pi[\rho^2 + z^2]^{3/2}} \mathbf{a}_\phi \tag{7.11}$$

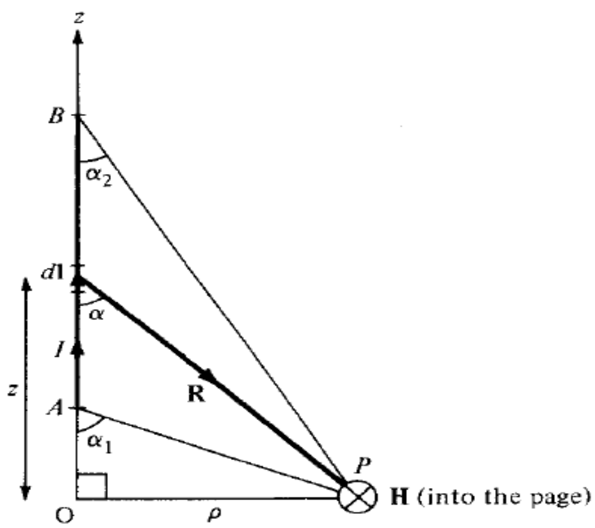


Fig. (7.5) Field at point P due to a straight filamentary conductor.

Letting $z = \rho \cot\alpha$; $dz = -\rho \operatorname{cosec}^2\alpha \, d\alpha$; and equation (7.11) becomes

$$\mathbf{H} = \int \frac{I\rho dz}{4\pi[\rho^2 + z^2]^{3/2}}\mathbf{a}_\phi = -\frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2\alpha \, d\alpha}{\rho^3 \operatorname{cosec}^3\alpha}\mathbf{a}_\phi = -\frac{I}{4\pi\rho}\mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin\alpha \, d\alpha$$

or

$$\mathbf{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi \tag{7.12}$$

This expression is generally applicable for any straight filamentary conductor of finite length. Notice from equation (7.12) that $\mathbf{H}$ is always along the unit vector $\mathbf{a}_\phi$ (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest P.

As a special case, when the conductor is semi-infinite with respect to P so that point A is now at $O(0,0,0)$ while B is at $(0,0,\infty)$; $\alpha_1 = 90^\circ$, $\alpha_2 = 0^\circ$, and equation (7.12) becomes:

$$\mathbf{H} = \frac{I}{4\pi\rho}\mathbf{a}_\phi \tag{7.13}$$

Another special case is when the conductor is infinite in length. For this case, point A is at$(0,0,-\infty)$ while B is at $(0,0,\infty)$; $\alpha_1 = 180^\circ$, $\alpha_2 = 0^\circ$, so equation (7.12) reduces to:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi \tag{7.14}$$

To find unit vector $\mathbf{a}_\phi$ in equations (7.12) to (7.14) is not always easy. A simple approach is to determine, $\mathbf{a}_\phi$, from

$$\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho \tag{7.15}$$

where,

- $\mathbf{a}_\ell$ is a unit vector along the line current and
- $\mathbf{a}_\rho$ is a unit vector along the perpendicular line from the line current to the field point [2].

Example 7.1:

The conducting triangular loop in Figure (7.6a) carries a current of 10 A. Find the magnetic field intensity $\mathbf{H}$ at $(0,0,5)$ due to side 1 of the loop [2].

Solution:

This example illustrates how equation (7.12) is applied to any straight, thin, current-carrying conductor. The key point to keep in mind in applying equation (7.12) is figuring out α_1, α_2, ρ

and $\mathbf{a}_\phi$. To find $\mathbf{H}$ at $(0,0,5)$ due to side 1 of the loop in Figure (7.6a), consider Figure (7.6b), where side 1 is treated as a straight conductor. Notice that we join the point of interest $(0,0,5)$ to the beginning and end of the line current. Observe that α_1, α_2 and ρ are assigned in the same manner as in Figure (7.5) on which equation (7.12) is based.

$$\cos\alpha_1 = \cos 90^\circ = 0 \quad : \quad \cos\alpha_2 = \frac{2}{\sqrt{29}} \quad : \quad \rho = 5$$

To determine $\mathbf{a}_\phi$

$$\mathbf{a}_\ell = \mathbf{a}_x \quad : \quad \mathbf{a}_\rho = \mathbf{a}_z \quad \text{so} \quad \mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

Hence,

$$\mathbf{H}_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi = \frac{10}{4\pi(5)}\left(\frac{2}{\sqrt{29}} - 0\right)(-\mathbf{a}_y)$$

$$\mathbf{H}_1 = -59.1\mathbf{a}_y \text{ mA/m}$$

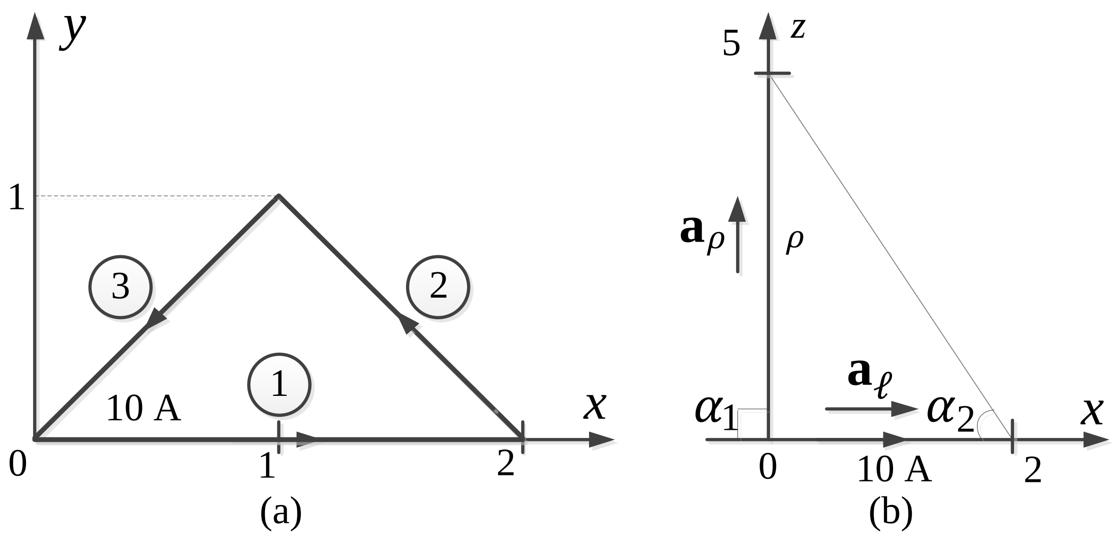


Fig. (7.6) For Example 7.1: (a) conducting triangular loop, (b) side 1 of the loop

Example 7.2:

Find the magnetic field intensity $\mathbf{H}$ at point $P(-3,4,0)$ due to the current filament shown in Figure (7.7a).

Solution: Let, $\mathbf{H} = \mathbf{H}_x + \mathbf{H}_z$, where $\mathbf{H}_x$ and $\mathbf{H}_z$ are the contributions to the magnetic field intensity at $P(-3,4,0)$ due to the portions of the filament along x and z, respectively.

$$\mathbf{H}_z = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\mathbf{a}_\phi$$

At $P(-3,4,0)$, $\rho = \sqrt{(9+16)} = 5$, $\alpha_1 = 90^\circ$, $\alpha_2 = 0^\circ$, and $\mathbf{a}_\phi$, is obtained as a unit vector along the circular path through P on plane $z = 0$ as in Figure (7.7b). The direction of $\mathbf{a}_\phi$ is determined using the right-handed screw rule or the right-hand rule. From the geometry in Figure (7.7b),

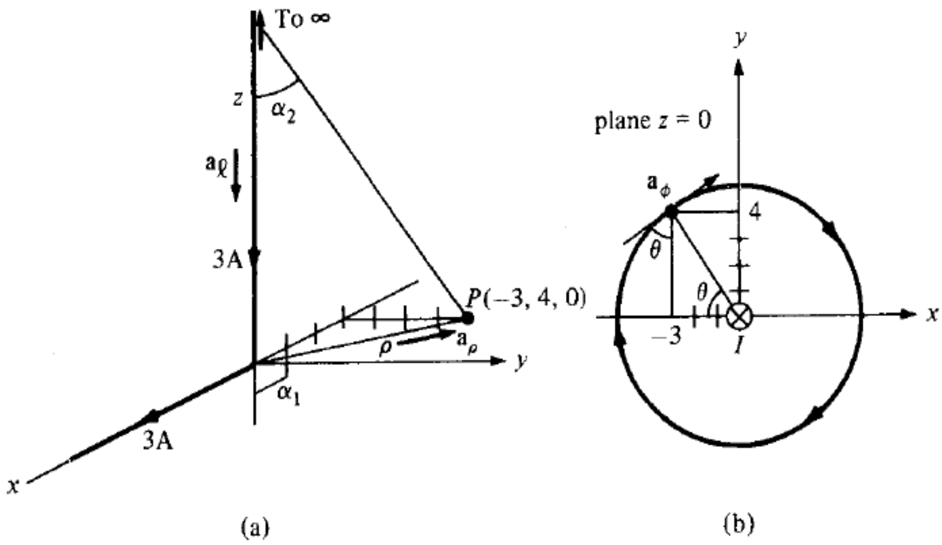


Fig. (7.7) for example 7.2: (a) current filament along semi-infinite x- and z-axes; $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ for $\mathbf{H}_z$ only; (b) determining $\mathbf{a}_\rho$ for $\mathbf{H}_z$.

At $P(-3, 4, 0)$, $\rho = \sqrt{(9+16)} = 5$, $\alpha_1 = 90^o$, $\alpha_2 = 0^o$, and $\mathbf{a}_\phi$, is obtained as a unit vector along the circular path through P on plane $z = 0$ as in Figure (7.7b). The direction of $\mathbf{a}_\phi$ is determined using the right-handed screw rule or the right-hand rule. From the geometry in Figure (7.7b),

$$\mathbf{a}_\phi = \sin\theta\, \mathbf{a}_x + \cos\theta\, \mathbf{a}_y = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

Alternatively, we can determine $\mathbf{a}_\phi$ from equation (7.15). At point P, $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ are as illustrated in Figure (7.7a) for $\mathbf{H}_z$. Hence,

$$\mathbf{a}_\phi = (-\mathbf{a}_z) \times \left(-\frac{3}{5}\mathbf{a}_x + \frac{4}{5}\mathbf{a}_y\right) = \frac{4}{5}\,\mathbf{a}_x + \frac{3}{5}\,\mathbf{a}_y$$

Hence, the magnetic field intensity $\mathbf{H}_z$ is:

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1-0)\frac{(4\mathbf{a}_x + 3\mathbf{a}_y)}{5} = 38.2\,\mathbf{a}_x + 28.65\,\mathbf{a}_y \text{ mA/m}$$

The magnetic field intensity $\mathbf{H}_z$ can be also obtained in cylindrical coordinates as

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1-0)(-\mathbf{a}_\phi) = -47.75\,\mathbf{a}_\phi \text{ mA/m}$$

Similarly, for $\mathbf{H}_x$ at P, $\rho = 4$, $\alpha_2 = 0^o$, $\cos\alpha_1 = \frac{3}{5}$ and $\mathbf{a}_\phi = \mathbf{a}_z$ or

$$\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

Hence, the magnetic field intensity $\mathbf{H}_x$ is:

$$\mathbf{H}_x = \frac{3}{4\pi(4)}\left(1 - \frac{3}{5}\right)\mathbf{a}_z = 23.88\,\mathbf{a}_z \text{ mA/m}$$

Thus, the magnetic field intensity $\mathbf{H}$ is: $\quad \mathbf{H} = \mathbf{H}_x + \mathbf{H}_z$

$\mathbf{H} = 38.2\mathbf{a}_x + 28.65\mathbf{a}_y + 23.88\mathbf{a}_z$ mA/m or $\mathbf{H} = -47.75\mathbf{a}_\phi + 23.88\mathbf{a}_z$ mA/m

Notice that although the current filaments appear semi-infinite (they occupy the positive z- and x-axes), it is only the filament along the z-axis that is semi-infinite with respect to point P. Thus $\mathbf{H}_z$ could have been found by using equation (7.13), but the equation could not have been used to find $\mathbf{H}_x$ because the filament along the x-axis is not semi-infinite with respect to P.

7.3 CURL OF A VECTOR

The curl of a vector field **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

The area ΔS is bounded by the curve L and $\mathbf{a}_n$ is the unit vector normal to the surface ΔS and is determined using the right-hand rule. Then the component of the curl of **A** in the direction $\mathbf{a}_n$ is defined as:

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint \mathbf{A} \, . \, d\mathbf{L}}{\Delta S} \right) . \, \mathbf{a}_n \tag{7.16}$$

To obtain an expression for $\nabla \times \mathbf{A}$ from the definition in equation (7.16), consider the differential area in the yz-plane as in Figure (7.8). The line integral in equation (7.16) is obtained as:

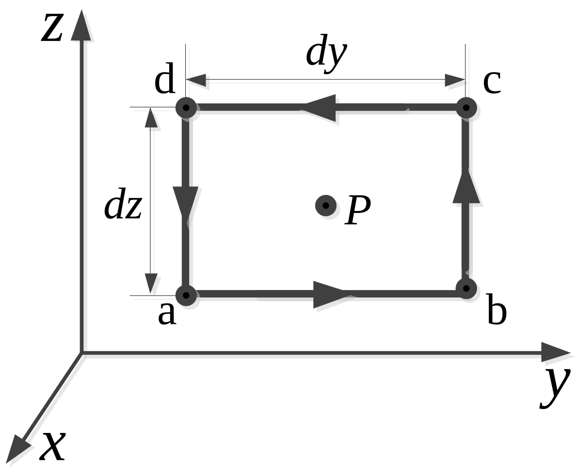


Fig. (7.8) Contour used in evaluating the x-component of $\nabla \times \mathbf{A}$ at point $P(x_o, y_o, z_o)$.

$$\oint_L \mathbf{A} \, . \, d\mathbf{L} = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \mathbf{A} \, . \, d\mathbf{L} \tag{7.17}$$

We expand the field components in a Taylor series expansion about the center point $P(x_o, y_o, z_o)$ as in equation (3.30) and evaluate equation (7.17).

On side a-b, $d\mathbf{L} = dy\mathbf{a}_y$ and $z = z_o - dz/2$, so.

$$\int_{ab} \mathbf{A}\,.\,d\mathbf{L} = dy\left[A_y(x_o, y_o, z_o) - \frac{dz}{2}\frac{\partial A_y}{\partial z}\bigg|_P\right] \tag{7.18}$$

On side b-c, $d\mathbf{L} = dz\mathbf{a}_z$ and $y = y_o + dy/2$, so.

$$\int_{bc} \mathbf{A}\,.\,d\mathbf{L} = dz\left[A_z(x_o, y_o, z_o) + \frac{dy}{2}\frac{\partial A_z}{\partial y}\bigg|_P\right] \tag{7.19}$$

On side c-d, $d\mathbf{L} = dy\mathbf{a}_y$ and $z = z_o + dz/2$, so.

$$\int_{cd} \mathbf{A}\,.\,d\mathbf{L} = -dy\left[A_y(x_o, y_o, z_o) + \frac{dz}{2}\frac{\partial A_y}{\partial z}\bigg|_P\right] \tag{7.20}$$

On side d-a, $d\mathbf{L} = dz\mathbf{a}_z$ and $y = y_o - dy/2$, so.

$$\int_{da} \mathbf{A}\,.\,d\mathbf{L} = -dz\left[A_z(x_o, y_o, z_o) - \frac{dy}{2}\frac{\partial A_z}{\partial y}\bigg|_P\right] \tag{7.21}$$

Substituting equations (7.18) to (7.21) into equation (7.17) and noting that $\Delta S = dydz$, we have

$$\lim_{\Delta S \to 0} \oint \frac{\mathbf{A}\,.\,d\mathbf{L}}{\Delta S} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

or

$$(\text{Curl } \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{7.22}$$

The y-and z-components of the curl of $\mathbf{A}$ can be found in the same maner.

$$(\text{Curl } \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{7.23}$$

$$(\text{Curl } \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{7.24}$$

The definition of $\nabla \times \mathbf{A}$ in equation (7.16) is independent of the coordinate system. In Cartesian coordinates the curl of $\mathbf{A}$ is found.

Curl **A** in Cartesian coordinate:

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{bmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix} \tag{7.25}$$

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z \tag{7.26}$$

Curl **A** in cylindrical coordinate:

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \frac{1}{\rho}\begin{bmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{bmatrix} \tag{7.27}$$

$$\nabla \times \mathbf{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\mathbf{a}_z \tag{7.28}$$

Curl **A** in spherical coordinate:

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{bmatrix} \mathbf{a}_r & r\,\mathbf{a}_\theta & r\sin\theta\,\mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{bmatrix} \tag{7.29}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\mathbf{a}_\phi \tag{7.30}$$

Note the following properties of the curl:

1. The curl of a vector field is another vector field.
2. The curl of a scalar field V, $\nabla \times V$, makes no sense.
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
4. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla.\mathbf{B}) - \mathbf{B}(\nabla.\mathbf{A}) + (\mathbf{B}.\nabla\mathbf{A} - (\mathbf{A}.\nabla\mathbf{B})$
5. $\nabla \times (V\mathbf{A}) = V\nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$

6- The divergence of the curl of a vector field vanishes, that is, $\nabla.(\nabla \times \mathbf{A}) = 0$.

7. The curl of the gradient of a scalar field vanishes, that is, $\nabla \times \nabla V = 0$.

The curl of a vector field **A** at a point *P* may be regarded as a measure of the circulation or how much the field curls around *P*. For example, Figure (7. 9a) shows that the curl of a vector field around *P* is directed out of the page. Figure (7. 9b) shows a vector field with zero curl [2].

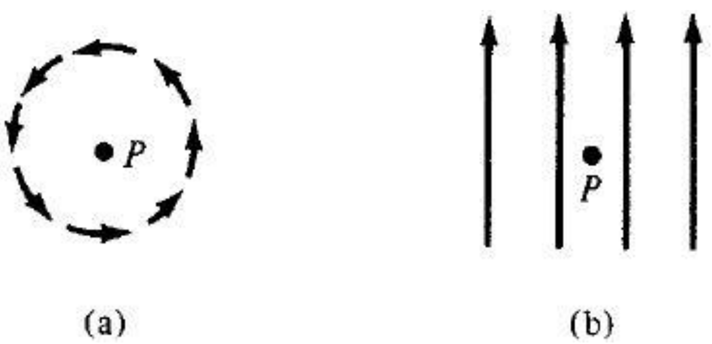


Fig. (7. 9a) Illustration of a curl: (a) curl at *P* points out of the page; (b) curl at *P* is zero.

7.4 STOKES' THEOREM

From Ampere's circuital law we derived one of Maxwell's equations, $\nabla \times \mathbf{H} = \mathrm{J}$. This equation should be considered the point form of Ampere's circuital law and applies on a "per-unit-area" basis. Also we shall devote a major share of the material to the mathematical theorem known as Stokes' theorem, and then we may obtain Ampere's circuital law from $\nabla \times \mathbf{H} = \mathrm{J}$. In other words, we can able to obtain the integral form from the point form or vies versa [1].

Consider the surface S of Figure (7.10) which is broken up into incremental surfaces of area ΔS. If we apply the definition of the curl to one of these incremental surfaces, then:

$$\frac{\oint \mathbf{H} \,.\, d\mathbf{L}_{\Delta S}}{\Delta S} = (\nabla \times \mathbf{H})_N$$

where N the subscript indicates the right-hand normal to the surface and the $d\mathbf{L}_{\Delta S}$ indicates that the closed path is the perimeter of an incremental area ΔS. This result may also be written [1].

$$\frac{\oint \mathbf{H} \,.\, d\mathbf{L}_{\Delta S}}{\Delta S} = (\nabla \times \mathbf{H}).\, \mathbf{a}_n \qquad \text{or} \qquad \oint \mathbf{H} \,.\, d\mathbf{L}_{\Delta S} = (\nabla \times \mathbf{H}).\, \mathbf{a}_n \, \Delta S = (\nabla \times \mathbf{H}) \,.\, \Delta\mathbf{S}$$

where $\mathbf{a}_n$ is a unit vector in the direction of the right-hand normal to ΔS.

Stokes's theorem states that the circulation of a vector field **A** around a closed path L is equal to the surface integral of the curl of **A** over the open surface S bounded by L. Figure (7.10) provided that **A** and $\nabla \times \mathbf{A}$ are continuous on S [2].

$$\oint \mathbf{A} \,.\, d\mathbf{L} = \int_S (\nabla \times \mathbf{A}) \,.\, d\mathbf{S} \tag{7.31}$$

If **H** is chosen to be the vector fields, Stokes' theorem gives

$$\oint \mathbf{H} \,.\, d\mathbf{L} = \int_S (\nabla \times \mathbf{H}) \,.\, d\mathbf{S} \tag{7.32}$$

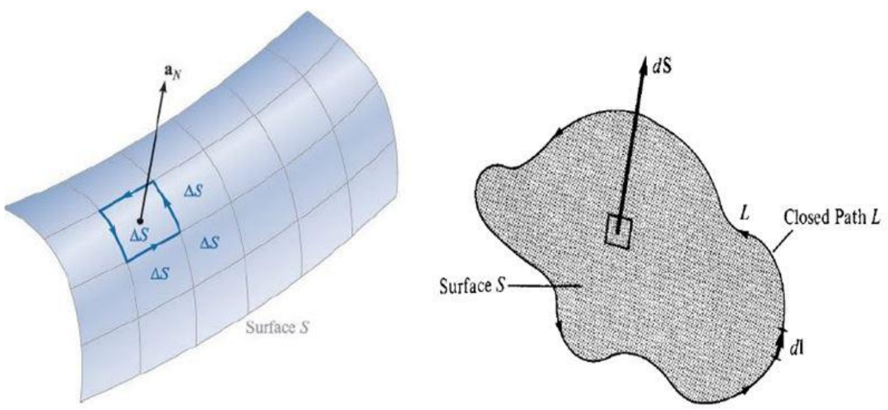


Fig. (7.10) Determining the sense of $d\mathbf{L}$ and $d\mathbf{S}$ involved in Stokes's theorem.

7.5 AMPERE'S CIRCUIT LAW

Ampere's circuit law sometimes called Ampere's work law: states that the line integral of the tangential component of the magnetic field strength (intensity) **H** around a closed path is exactly equal to the direct current enclosed by that path [1,3].

$$\oint \mathbf{H}.d\mathbf{L} = I_{enc} \tag{7.33}$$

Ampere's law is applied to determine the magnetic field intensity **H** when symmetrical current distribution exists. Ampere's law is a special case of Biot-Savart's law [2].

Two conditions must be met:

1. At each point of the closed path the magnetic field intensity **H** is either tangential or normal to the path.
2. The magnetic field intensity **H** has the same value at all points of the path where **H** is tangential [3].

By applying Stoke's theorem to the left-hand side of equation (7.33), we obtain

$$I_{enc} = \oint_L \mathbf{H}\,.\,d\mathbf{L} = \int_S (\nabla \times \mathbf{H})\,.\,d\mathbf{S} \tag{7.34}$$

But

$$I_{enc} = \int_S \vec{\mathrm{J}}\,.\,d\mathbf{S} \tag{7.35}$$

By comparing equations (7.34) and (7.35) yield.

$$\nabla \times \mathbf{H} = \vec{\mathrm{J}} \tag{7.36}$$

This is the third Maxwell's equation to be derived; it is essentially Ampere's law in differential (or point) form whereas equation (7.33) is the integral form. From equation (7.36), we should observe that $\nabla \times \mathbf{H} = \vec{\mathrm{J}} \neq 0$; that is, magnetostatic field is not conservative [2].

7.6 APPLICATIONS OF AMPERE'S LAW

7.6.1 Infinite Line Current

To find the magnetic field intensity **H** produced by an infinitely long filament carrying a current. The filament located on the z-axis in free space as in Figure (7.11), and the current flows in the direction given by $\mathbf{a}_z$. We determine which components of **H** are present by using the Biot-Savart law. Without specifically using the cross product, the direction of $d\mathbf{H}$ is perpendicular to the plane containing $d\mathbf{L}$ and **R** and therefore is in the direction of $\mathbf{a}_\phi$. Hence the only component of **H** is H_ϕ, and the magnitude of the field is a function only of ρ but not of ϕ and z. Since this path encloses the whole current I, according to Ampere's law. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as in Figure (7.11) [1].

$$I = \oint_L \mathbf{H}\,.\,d\mathbf{L} = \int (H_\phi\,\mathbf{a}_\phi)\,.\,(\rho d\phi\,\mathbf{a}_\phi) = H_\phi \int \rho d\phi = H_\phi\,.\,2\pi\rho$$

$$H_\phi = \frac{I}{2\pi\rho}$$

or

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi \tag{7.37}$$

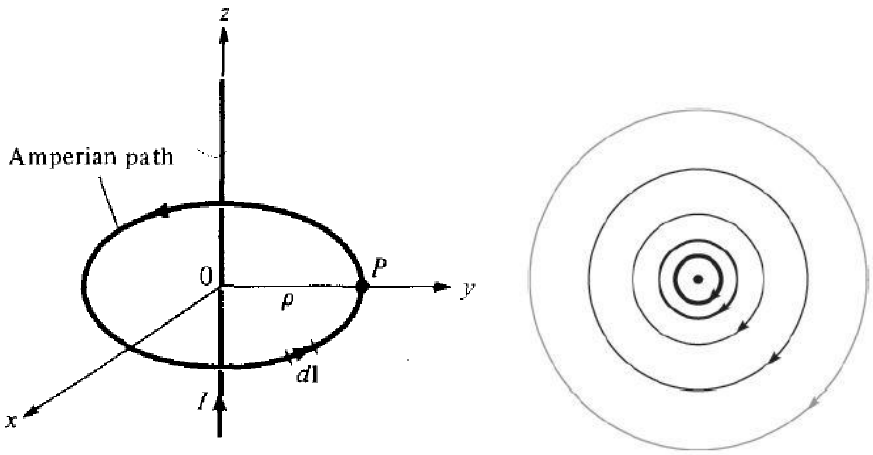


Fig. (7.11) Ampere's law applied to an infinite filamentary line current.

7.6.2 Infinite Sheet of Current

Consider an infinite sheet of current flowing in the positive y-direction and located in the plane $z = 0$. A sheet has a uniform surface current density $\mathbf{K} = K_y\mathbf{a}_y$ A/m as shown in Figure (7.12), the magnetic field intensity **H** cannot vary with x or y. If the sheet is subdivided into a number of filaments, it is evident that no produced an H_y component. The Biot-Savart law shows that the contributions to H_z produced by a symmetrically located pair of filaments cancel ($H_z = 0$).

Therefore only an H_x component is present. Applying Ampere's law to the rectangular closed path 1-2-3-4-1 (Amperian path) which are either parallel or perpendicular to H_x gives [1].

$$\oint \mathbf{H} \, . \, d\mathbf{L} = I_{enc} = K_y b \tag{7.38}$$

Due to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of **H** for a pair are the same for the infinite current sheets, that is **H** on one side of the sheet is the negative of that on the other side [2].

$$\mathbf{H} = \begin{cases} H_o \, \mathbf{a}_x & z > 0 \\ -H_o \, \mathbf{a}_x & z < 0 \end{cases} \tag{7.39}$$

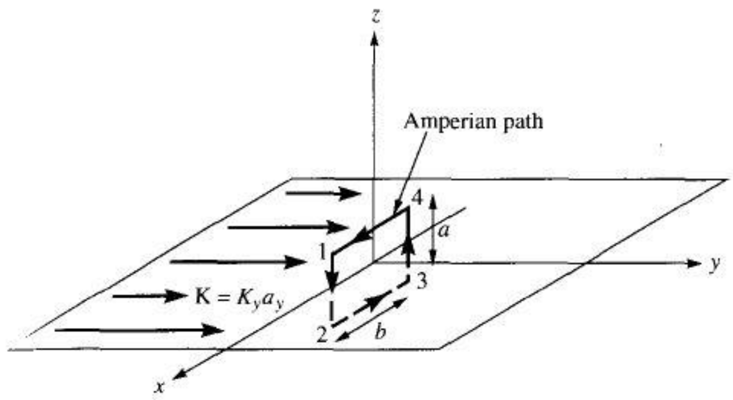


Fig. (7.12) Application of Ampere's law to an infinite sheet for closed path 1-2-3-4-1

$$\oint \mathbf{H}.d\mathbf{L} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right) \mathbf{H}.d\mathbf{L}$$

$$\oint \mathbf{H}.d\mathbf{L} = 0(-a) + (-H_o)(-b) + 0(a) + H_o(b) = 2H_o\mathrm{b} \tag{7.40}$$

From equations (7.38) and (7.40), we obtain $H_o = \frac{1}{2}K_y$. Substituting H_o in equation (7.39) gives:

$$\mathbf{H} = \begin{cases} \frac{1}{2}K_y \, \mathbf{a}_x & z > 0 \text{ (above the sheet)} \\ -\frac{1}{2}K_y \, \mathbf{a}_x & z < 0 \text{ (below the sheet)} \end{cases} \tag{7.41}$$

In general, for an infinite sheet of current density **K** A/m,

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n \tag{7.42}$$

where $\mathbf{a}_n$ is a unit vector normal to the current sheet directed from the current sheet to the point of interest [2].

7.6.3 Infinitely Long Coaxial Transmission Line

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis. The line and its cross section is shown in Figure (7.13), where the z-axis is out of the page in cross section. The inner conductor has radius a and carries current I while the outer conductor has inner radius b and thickness t and carries return current $-I$. To determine **H** everywhere assuming that current is uniformly distributed in both conductors. Since the current distribution is symmetrical [2].

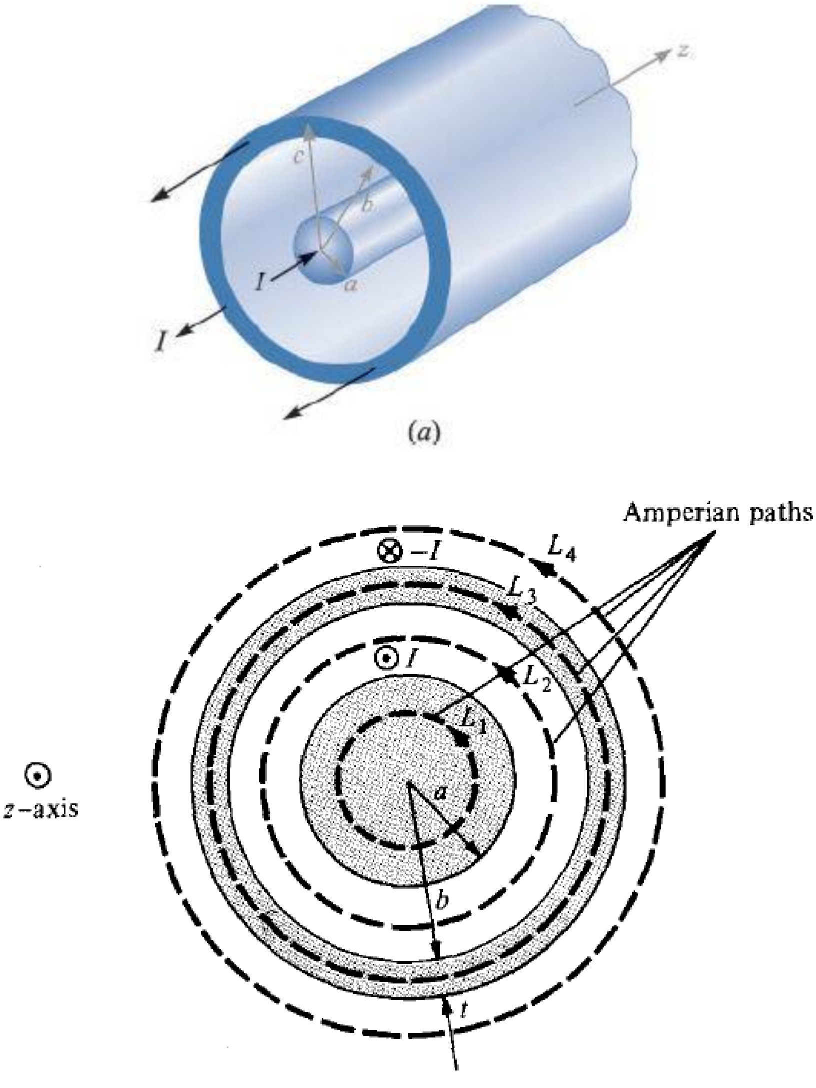


Fig. (7.13) the line and cross section of the transmission line; the positive z-direction is out of the page.

We apply Ampere's law along the Amperian path for each of the four possible regions:

$$0 \le \rho \le a; \qquad a \le \rho \le b; \qquad b \le \rho \le b+t; \qquad \text{and} \qquad \rho \ge b+t.$$

For region $0 \le \rho \le a$, we apply Ampere's law to path L_1 giving

$$\oint_{L_1} \mathbf{H} \,.\, d\mathbf{L} = I_{enc} = \int_S \vec{J} \,.\, d\mathbf{S} \tag{7.43}$$

Since the current is uniformly distributed over the cross section,

$$\vec{\mathrm{J}} = \frac{I}{\pi a^2}\mathbf{a}_z \qquad \text{and} \qquad d\mathbf{S} = \rho d\phi d\rho\, \mathbf{a}_z$$

$$I_{enc} = \int_S \vec{\mathrm{J}}\,.\, d\mathbf{S} = \int_S \left(\frac{I}{\pi a^2}\mathbf{a}_z\right).(\rho d\phi d\rho \mathbf{a}_z) = \frac{I}{\pi a^2}\int_0^{\rho}\int_0^{2\pi} \rho d\phi d\rho = \frac{I}{\pi a^2}(2\pi)\left(\frac{\rho^2}{2}\right) = \frac{I\,\rho^2}{a^2}$$

Hence equation (7.43) becomes

$$\oint_{L_1} \mathbf{H}\,.\,d\mathbf{L} = \int (H_\phi\,\mathbf{a}_\phi)\,.\,(\rho d\phi\,\mathbf{a}_\phi) = H_\phi \int dl = H_\phi \int \rho d\phi = H_\phi\,.\,2\pi\rho = I_{enc} = \frac{I\,\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\,\rho}{2\pi a^2} \tag{7.44}$$

For region $a \le \rho \le b$ we use path L_2 as the Amperian path,

$$\oint_{L_2} \mathbf{H}\,.\,d\mathbf{L} = \int (H_\phi\,\mathbf{a}_\phi)\,.\,(\rho d\phi\,\mathbf{a}_\phi) = H_\phi \int dl = H_\phi \int \rho d\phi = H_\phi\,.\,2\pi\rho = I_{enc} = I$$

$$H_\phi\, 2\pi\rho = I \qquad \text{or}$$

$$H_\phi = \frac{I}{2\pi\rho} \tag{7.45}$$

Since the whole current I is enclosed by L_2. Notice that equation (7.45) is the same as equation (7.14) and it is independent of a.

For region $b \le \rho \le b + t$, we use path L_3 getting

$$\oint_{L_3} \mathbf{H}\,.\, d\mathbf{L} = H_\phi.\,2\pi\phi = I_{enc} \tag{7.46}$$

where

$$I_{enc} = I + \int_S \vec{\mathrm{J}}\,.\, d\mathbf{S}$$

$\vec{\mathrm{J}}$ is the current density of the outer conductor and is along $-\mathbf{a}_z$ that is,

$$\vec{\mathrm{J}} = -\frac{I}{\pi[(b+t)^2 - b^2]}\,\mathbf{a}_z$$

Thus

$$I_{enc} = I - \frac{I}{\pi[(b+t)^2 - b^2]}\int_0^{2\pi}\int_{\rho=b}^{\rho} \rho d\phi d\rho$$

$$I_{enc} = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

Substituting this in equation (7.46), we have

$$H_\phi = \frac{I}{2\pi\,\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] \tag{7.47}$$

For region $\rho \geq b + t$, we use path L_4, getting

$$\oint_{L_4} \mathbf{H}\,.\,d\mathbf{L} = I_{enc} = I - I = 0 \quad \text{or}$$

$$H_\phi = 0 \tag{7.48}$$

Putting equation (7.44) to (7.48) together gives

$$\mathbf{H} = \begin{cases} \dfrac{I\,\rho}{2\pi\,a^2}\,\mathbf{a}_\phi & 0 \leq \rho \leq a \\[2ex] \dfrac{I}{2\pi\,\rho}\,\mathbf{a}_\phi & a \leq \rho \leq b \\[2ex] \dfrac{I}{2\pi\,\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]\mathbf{a}_\phi & b \leq \rho \leq b + t \\[2ex] 0 & \rho \geq b + t \end{cases} \tag{7.49}$$

The magnitude of **H** is sketched in Figure (7.14). Ampere's law can only be used to find **H** due to symmetric current distributions for which it is possible to find a closed path over which **H** is constant in magnitude [2].

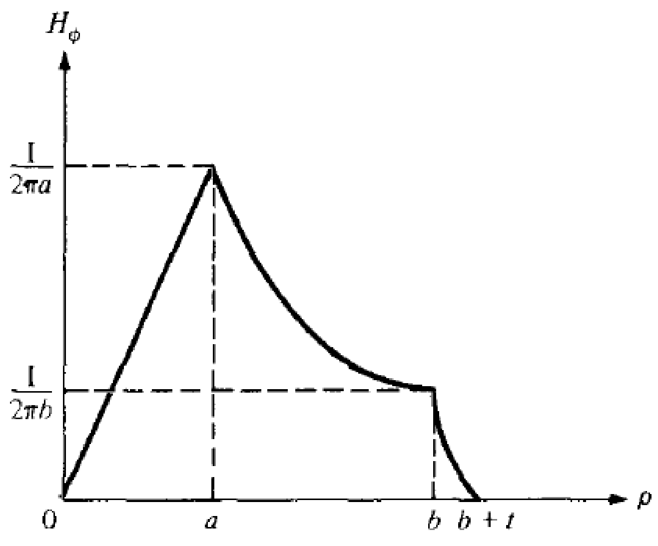


Fig. (7.14) The magnetic field intensity as a function of radius ρ in an infinitely long coaxial transmission line with the dimensions shown.

7.6.4 An N-turn solenoid carrying filamentary current.

By applying Ampere's circuital law to an infinitely long solenoid of radius a and uniform current density $K_a\mathbf{a}_\phi$, the center of the solenoid is the z-axis as shown in Figure (7.15a). The result of magnetic field intensity is [1].

$$\mathbf{H} = K_a\mathbf{a}_z \quad (\rho < a) \tag{7.50}$$

$$\mathbf{H} = 0 \qquad (\rho > a) \tag{7.51}$$

If the solenoid has a finite length d and consists of N closely wound turns of a filament that carries a current I Figure (7.15b), then the magnetic field intensity $\mathbf{H}$ at points well within the solenoid is given closely by [1].

$$\mathbf{H} = \frac{NI}{d}\mathbf{a}_z \qquad \text{(within the solenoid)} \tag{7.52}$$

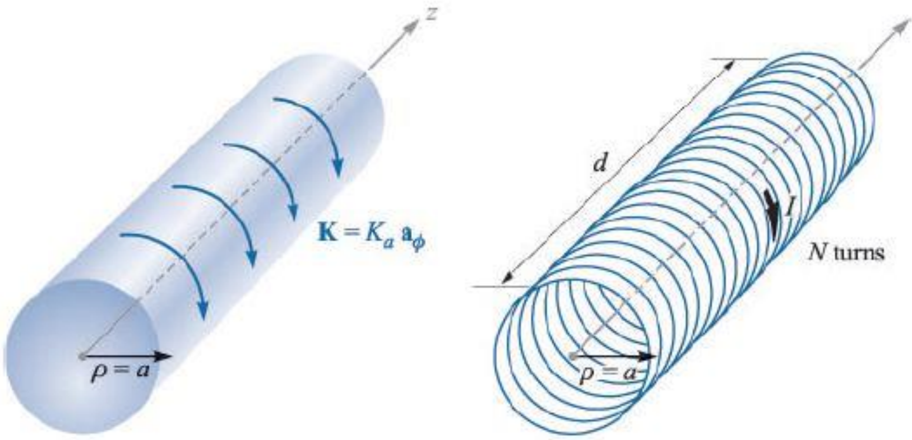


Fig. (7.15) (a) An ideal solenoid of infinite length with a circular current sheet. (b) An N-turn solenoid of finite length d.

7.6.5 An N-turn toroid carrying filamentary current.

For the toroids shown in Figure (7.16a), it can be shown that the magnetic field intensity for the ideal case is:

$$\mathbf{H} = K_a\frac{\rho_o - a}{\rho}\mathbf{a}_\phi \qquad \text{(inside toroid)} \tag{7.53}$$

$$\mathbf{H} = 0 \qquad \text{(outside toroid)} \tag{7.54}$$

A toroid whose dimensions are shown in Figure (7.16b) has N turns and carries filamentary current I. To determine the magnetic field intensity $\mathbf{H}$ inside and outside the toroid apply Ampere's circuit law to the Amperian path, which is a circle of radius ρ show in Figure (7.16b).

Since N wires cut through this path each carrying current I, the net current enclosed by the Amperian path is NI. Hence,

$$\oint_L \mathbf{H}\,.\,d\mathbf{L} = I_{enc} \quad \Rightarrow \quad H.\,2\pi\rho = NI$$

or

$$H = \frac{NI}{2\pi\rho} \qquad \text{for} \qquad \rho_o - a < \rho < \rho_o + a \tag{7.55}$$

where ρ_o is the mean radius of the toroid. An approximate value of H is

$$H_{approx.} = \frac{NI}{2\pi\rho_o} = \frac{NI}{\ell} \tag{7.56}$$

Notice that this is the same as the formula obtained for H for points well inside a very long solenoid ($\ell \gg a$). Thus a straight solenoid may be regarded as a special toroidal coil for which $\rho_o \to \infty$. Outside toroid, the current enclosed by an Amperian path is $NI - NI = 0$, hence [1,2].

$$H = 0 \tag{7.57}$$

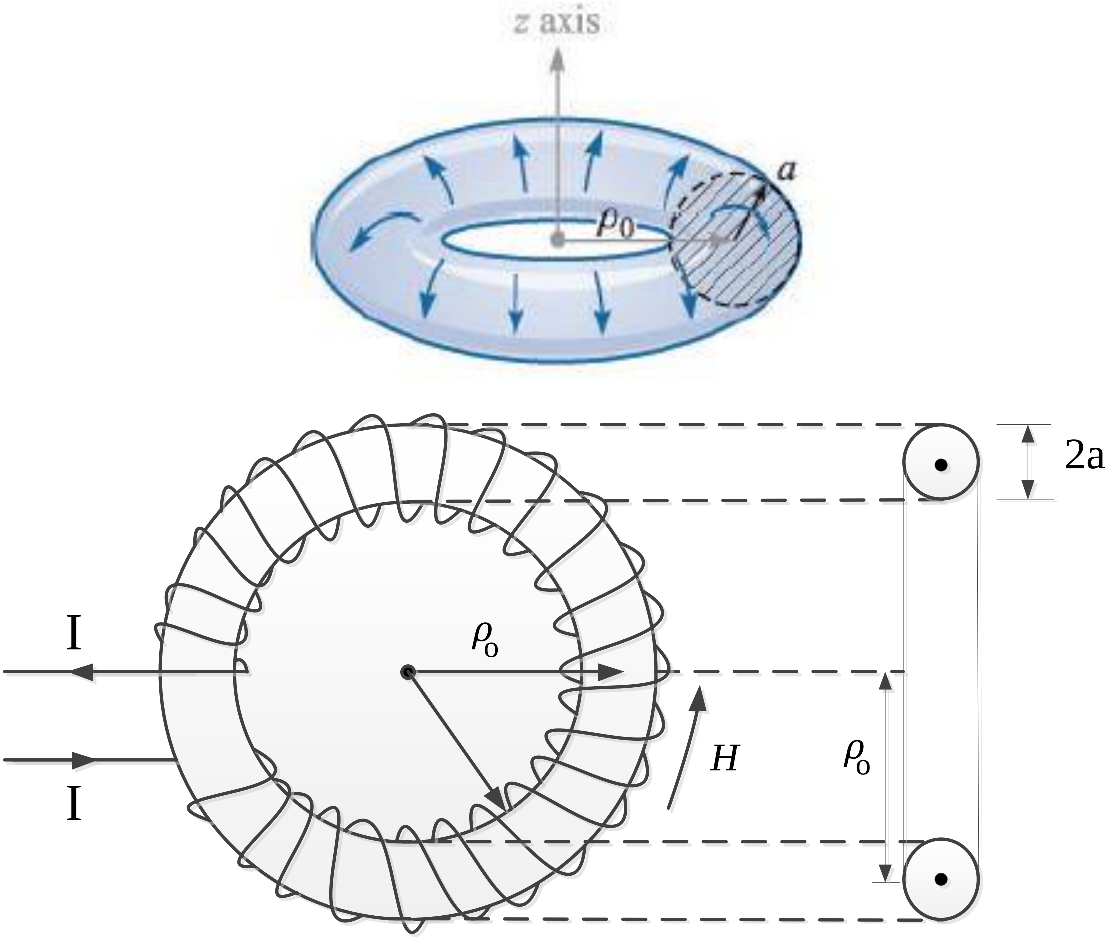


Fig. 7.16 a toroid with a circular cross section.

Example 7.3: Planes $z = 0$ and $z = 4$ carry current $\mathbf{K} = -10\mathbf{a}_x$ A/m and $\mathbf{K} = 10\mathbf{a}_x$ A/m, respectively. Determine **H** at

(a) $(1, 1, 1)$

(b) $(0, -3, 10)$

Solution: Let the parallel current sheets be as in Figure (7.17). Also let

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_4$$

where $\mathbf{H}_0$ and $\mathbf{H}_4$ are the contributions due to the current sheets $z = 0$ and $z = 4$, respectively use of equation (7.23).

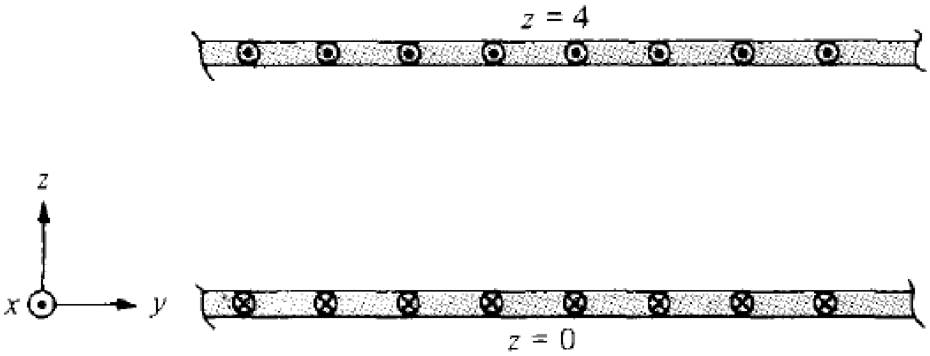


Fig. (7.17) For Example 7.3; parallel infinite current sheets

(a) At $(1, 1, 1)$, which is between the plates $(0 < z = 1 < 4)$,

$$\mathbf{H}_0 = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y$$

$$\mathbf{H}_4 = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}(10\mathbf{a}_x) \times (-\mathbf{a}_z) = 5\mathbf{a}_y$$

Hence,

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_4 = 10\ \mathbf{a}_y \quad \text{A/m}$$

(b) At $(0, -3, 10)$, which is above the two sheets $(z = 10 > 4 > 0)$,

$$\mathbf{H}_0 = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y$$

$$\mathbf{H}_4 = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}(10\mathbf{a}_x) \times (\mathbf{a}_z) = -5\mathbf{a}_y$$

Hence,

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_4 = 0 \quad \text{A/m}$$

Example 7.4:

Plane $y = 1$ carries current $= 50\ \mathbf{a}_z$ mA/m. Find $\boldsymbol{H}$ at (a) (0,0,0); (b) $(1,5,-3)$

Solution:

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n \quad \Rightarrow \quad \mathbf{H}_{(0,0,0)} = \frac{1}{2}50\ \mathbf{a}_z \times (-\mathbf{a}_y) = 25\ \mathbf{a}_x\ \text{mA/m}$$

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n \quad \Rightarrow \quad \mathbf{H}_{(1,5,-3)} = \frac{1}{2}50\ \mathbf{a}_z \times (\mathbf{a}_y) = -25\ \mathbf{a}_x\ \text{mA/m}$$

Example 7.5: A toroid of circular cross section whose center is at the origin and axis the same as the z-axis has 1000 turns with $\rho_o = 10$ cm, $a = 1$ cm. If the toroid carries a 100 mA current, find $|H|$ at (a) $(3 \text{ cm}, \ -4 \text{ cm}, 0)$ and (b) $(6 \text{ cm}, 9 \text{ cm}, 0)$

Solution:

$$|H| = \begin{cases} \dfrac{NI}{2\pi\rho} & \rho_o - a < \rho < \rho_o + a = 9 < \rho < 11 \\ 0 & \text{otherewise} \end{cases}$$

(a) at $(3, \ -4, 0)$, $\rho = \sqrt{3^2 + 4^2} = 5 \text{ cm} < 9 \text{ cm}$

$$|H| = 0$$

(b) at $(6, \ 9, 0)$, $\rho = \sqrt{6^2 + 9^2} = \sqrt{117} < 11 \text{ cm}$

$$|H| = \frac{NI}{2\pi\rho} = \frac{10^3 \times 100 \times 10^{-3}}{2\pi \times \sqrt{117} \times 10^2} = 147.1 \text{ A/m}$$

7.7 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY:

7.7.1 Magnetic Flux Density B

In free space, the magnetic flux density **B** is related to the magnetic field intensity **H** according to

$$\mathbf{B} = \mu_o \mathbf{H} \tag{7.58}$$

where

- **B** is measured in webers per square meter ($\mathrm{Wb/m^2}$) or in the (IS) International System of Units is tesla (T).
- The older unit of the magnetic flux density is the gauss (G), where 1T or 1 $\mathrm{Wb/m^2}$ is equal to 10,000G.
- μ_o is a constant is not dimensionless known as the permeability of free space in henrys per meter (H/m).[1]

$$\mu_o = 4\pi \times 10^{-7} \quad (\mathrm{H/m}) \tag{7.59}$$

7.7.2 Magnetic Flux Φ

Let, the magnetic flux represented by Φ and define as the flux passing through any designated area, Thus, the magnetic flux through a surface S is given by [1,2].

$$\psi = \Phi = \int_S \mathbf{B} \, . \, d\mathbf{S} \tag{7.60}$$

where,

The magnetic flux Φ is measured in webers (Wb). The sign on Φ may be positive or negative depending upon the choice of the surface normal in $d\mathbf{S}$ [3].

The magnetic flux lines due to a straight long wire are formed concentric circles about the filament wire as shown in Figure (7.18). Though Figure (7.18) is for a straight, current-carrying conductor, it is generally true that magnetic flux lines are closed and do not cross each other regardless of the current distribution [1,2].

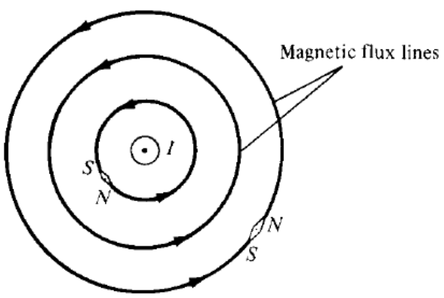


Fig. (7.18) Magnetic flux lines due to a straight wire with current coming out of the page.

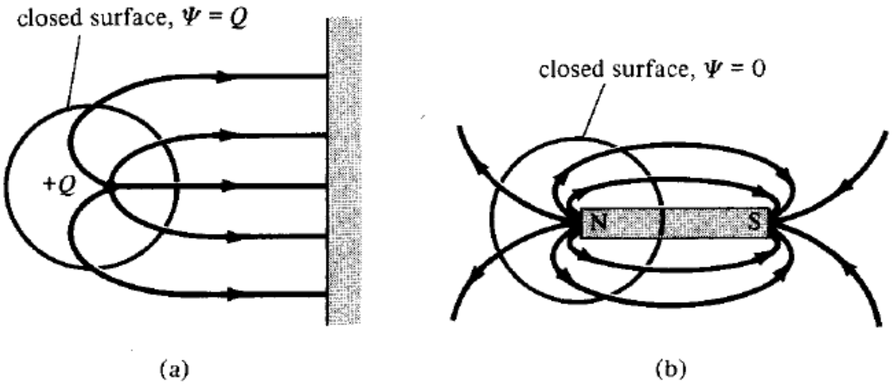


Fig. (7.19) Flux leaving a closed surface due to: (a) isolated electric charge, (b) magnetic charge

If we desire to have an isolated magnetic pole by dividing a magnetic bar into two, we end up with pieces each having north and south poles as illustrated in Figure (7.20). We find it impossible to separate the north pole from the south pole [2].

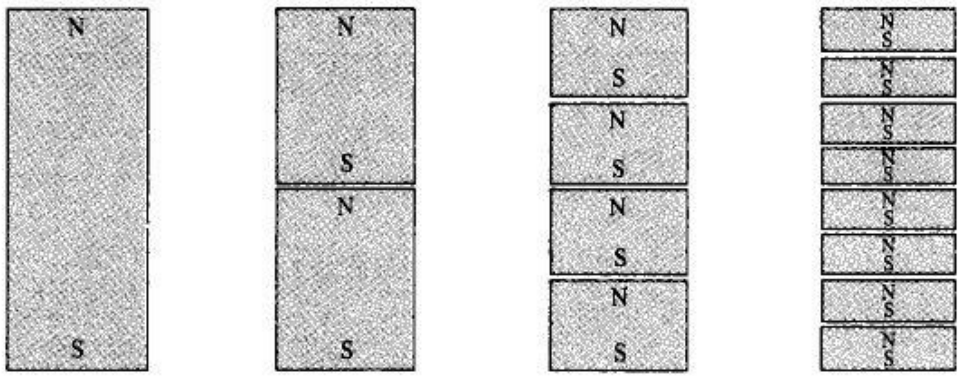


Fig. (7.20) Successive division of a bar magnet results in pieces with north and south poles.

Unlike electric flux lines, magnetic flux lines always close upon themselves as in Figure (7.17). This is due to the fact that it is not possible to have isolated magnetic poles or magnetic charges. For this reason Gauss's law for the magnetic field is [1,2].

$$\oint_S \mathbf{B} \, . \, d\mathbf{S} = 0 \tag{7.61}$$

By applying the divergence theorem to equation (7.61), we obtain

$$\oint_S \mathbf{B} \, . \, d\mathbf{S} = \int_v \nabla \, . \, \mathbf{B} \, dv = 0 \quad : \quad \text{or}$$

$$\nabla \, . \, \mathbf{B} = 0 \tag{7.62}$$

This equation is the fourth Maxwell's equation to be derived. Equation (7.61) or (7.62) shows that magnetostatic fields have no sources or sinks. Equation (7.62) suggests that magnetic field lines are always continuous [2].

7.8 MAXWELL'S EQUATIONS FOR STATIC E.M. FIELDS

Having derived Maxwell's four equations for static electromagnetic fields, we may take a moment to put them together as in Table (7.2) [2].

Table (7.2) Maxwell's Equations for Static EM Fields

No.	Differential or Point Form	Integral Form	Remarks
1	$\nabla \cdot \mathbf{D} = \rho_v$	$\psi = \oint_s \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \; dv = Q$	Gauss's law
2	$\nabla \cdot \mathbf{B} = 0$	$\oint_s \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
3	$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{L} = 0$	Conservativeness of electrostatic field
4	$\nabla \times \mathbf{H} = \vec{J}$	$\oint_L \mathbf{H} \cdot \, d\mathbf{L} = I_{enc} = \int_s \vec{J} \cdot \, d\mathbf{S}$	Ampere's law

7.9 MAGNETIC SCALAR AND VECTOR MAGNETIC POTENTIALS

We can define a potential associated with static magnetic field **B**. The magnetic potential could be scalar V_m or vector **A**. To define V_m and **A** involves recalling two important identities [2].

$$\nabla \times (\nabla V) = 0 \tag{7.63}$$

$$\nabla . (\nabla \times \mathbf{A}) = 0 \tag{7.64}$$

7.9.1 MAGNETIC SCALAR

As in electric field ($\mathbf{E} = -\nabla V$) the magnetic scalar potential V_m in amperes is related to magnetic field intensity **H**. The existence of a scalar magnetic potential, whose negative gradient gives the magnetic field intensity. In other words the magnetic field intensity **H** is to be defined as the gradient of a scalar magnetic potential, then current density must be zero throughout the region in which the scalar magnetic potential is defined [1,2].

$$\mathbf{H} = -\nabla V_m \qquad \text{If} \quad (\vec{J} = 0) \tag{7.65}$$

where $\vec{J} = \nabla \times \mathbf{H}$ equation (7.36). Substitution equation (7.65) in equation (7.36) we get.

$$\vec{J} = \nabla \times \mathbf{H} = \nabla \times (-\nabla V_m) = 0 \tag{7.66}$$

The magnetic scalar potential V_m is only defined in a region where $(\vec{J} = 0)$. We should also note that V_m satisfies Laplace's equation just as V does for electrostatic fields. Thus [2].

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mu_o \nabla \cdot \mathbf{H} = 0 \quad \Rightarrow \quad \mu_o \nabla \cdot (-\nabla V_m) = 0 \quad \text{or}$$

$$\nabla^2 V_m = 0 \qquad \text{If} \quad (\vec{J} = 0) \tag{7.67}$$

We shall see later that V_m continues to satisfy Laplace's equation in homogeneous magnetic materials; it is not defined in any region in which current density is present [1].

7.9.2 MAGNETIC VECTOR POTENTIAL

Our choice of a vector magnetic potential is indicated by noting that.

$$\nabla \cdot \mathbf{B} = 0$$

We can define the vector magnetic potential **A** in (Wb/m) such that.

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{7.68}$$

We defined

$$V = \int \frac{dQ}{4\pi\, \varepsilon_o\, r} \tag{7.69}$$

We can define

$$\mathrm{A} = \int_L \frac{\mu_0\, \mathrm{IdL}}{4\pi \mathrm{R}} \qquad \text{for line current} \tag{7.70}$$

$$\mathbf{A} = \int_S \frac{\mu_0\, \mathbf{K} d\mathbf{S}}{4\pi R} \qquad \text{for surface current} \tag{7.71}$$

$$\mathbf{A} = \int_v \frac{\mu_0\, \vec{J} dv}{4\pi R} \qquad \text{for volume current} \tag{7.72}$$

Also we can obtain equations (7.70) to (7.72) from equations (7.6) to (7.8). Now, we can derive equation (7.70) from equation (7.6) in conjunction with equation (7.68). To do this, we write equation (7.6) as

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_L \frac{\boldsymbol{IdL} \times \mathbf{R}}{\mathrm{R}^3} \tag{7.73}$$

where **R** is the distance vector from the line element $\boldsymbol{dL}$ at the source point (x_1, y_1, z_1) to the field point (x, y, z) as shown in Figure (7.22) and $R = |\mathbf{R}|$, that is,

$$R = |\mathbf{r}_1 - \mathbf{r}_2| = [(x - x_1) + (y - y_1) + (z - z_1)]^{1/2}$$

Hence,

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x-x_1)\mathbf{a}_x + (y-y_1)\mathbf{a}_y + (z-z_1)\mathbf{a}_z}{[(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{\frac{3}{2}}} = -\frac{\mathbf{R}}{\mathrm{R}^3}$$

or

$$\frac{\mathbf{R}}{\mathrm{R}^3} = -\nabla\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{\mathrm{R}^2} \tag{7.74}$$

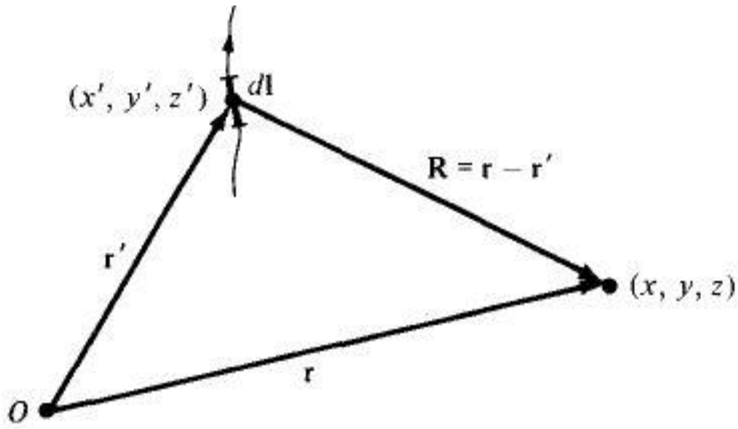


Fig. (7.22) Illustration of the source point (x_1, y_1, z_1) and the field point (x, y, z).

where the differentiation is with respect to x, y, and z. Substituting this into equation (7.54), we obtain.

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\int_L \boldsymbol{IdL} \times \nabla\left(\frac{1}{R}\right) \tag{7.75}$$

By applying the vector identity

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{E} \tag{7.76}$$

Where f is a scalar field and $\mathbf{F}$ is a vector field. Taking $f = 1/R$ and $\mathbf{F} = d\mathbf{L}$, we have

$$d\mathbf{L} \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\nabla \times d\mathbf{L} - \nabla \times \left(\frac{d\mathbf{L}}{R}\right)$$

Since ∇ operates with respect to (x, y, z) while $d\mathbf{L}$ is a function of (x_1, y_1, z_1), $\nabla \times d\mathbf{L} = 0$. Hence,

$$d\mathbf{L} \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \left(\frac{d\mathbf{L}}{R}\right) \tag{7.77}$$

With this equation, equation (7.75) reduces to

$$\mathbf{B} = \nabla \times \int_L \frac{\mu_0\, \boldsymbol{IdL}}{4\pi R} \tag{7.78}$$

Comparing equation (7.78) with equation (7.68) shows that

$$A = \int_L \frac{\mu_0 \, IdL}{4\pi R}$$

By substituting equation (7.68) into equation (7.65) and applying Stokes's theorem, we obtain

$$\psi = \int_S \mathbf{B} \,.\, d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \,.\, d\mathbf{S} = \oint_L \mathbf{A} \,.\, d\mathbf{L}$$

or

$$\psi = \oint_L \mathbf{A} \,.\, d\mathbf{L} \tag{7.79}$$

Thus the magnetic flux through a given area can be found using either equation (7.60) or (7.79). Also, the magnetic field can be determined using either V_m or $\mathbf{A}$. The V_m can only be used in a source-free region. The use of the magnetic vector potential provides a powerful, elegant approach to solving EM problems [2].

Example 7.6: Given the magnetic vector potential $\mathbf{A} = -\frac{\rho^2}{4}\mathbf{a}_z \; Wb/m$, calculate the total magnetic flux crossing the surface $\phi = \pi/2$, $1 \le \rho \le 2 \; m$, $0 \le z \le 5$.

Method 1:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \frac{\rho}{2}\mathbf{a}_\phi \qquad \text{and} \qquad d\mathbf{S} = d\rho dz \, \mathbf{a}_\phi$$

$$\psi = \int_S \mathbf{B} \,.\, d\mathbf{S} = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho \, d\rho \, dz = \frac{15}{4} = 3.75 \; Wb$$

Method 2: We use

$$\psi = \oint_L \mathbf{A} \,.\, d\mathbf{L} = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

where L is the path bounding surface S; ψ_1, ψ_2, ψ_3, and ψ_4 are, respectively, the evaluations of $\int \mathbf{A} \,.\, d\mathbf{L}$ along the segments of L labeled 1 to 4 in Figure 7.20. Since $\mathbf{A}$ has only a Z-component,

$$\psi_1 = \psi_3 = 0$$

$$\psi = \psi_2 + \psi_4 = -\frac{1}{4}\left[(1)^2\int_0^5 dz + (2)^2\int_5^0 dz\right] = -\frac{1}{4}[(1-4)(5)] = 3.75 \; Wb$$

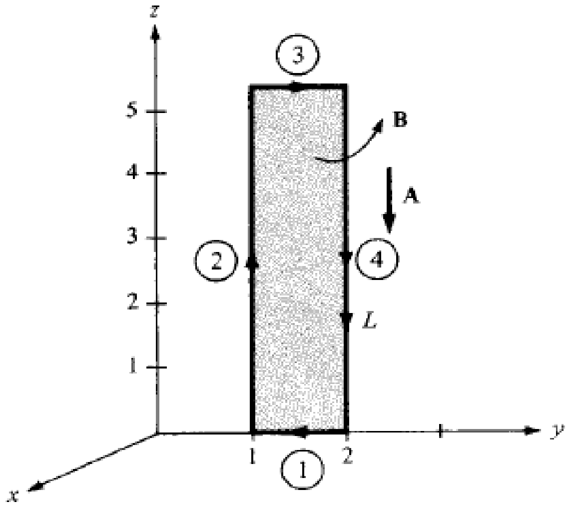


Figure 7.18 For Example 7.4.

Example 7.7:

Find the flux between the conductors of the coaxial line of Figure (7.21).

Solution: The magnetic field intensity was found to be

$$\frac{I}{2\pi\rho}\,\mathbf{a}_\phi \qquad a \le \rho \le b$$

$$\mathbf{B} = \mu_o\mathbf{H} = \frac{\mu_o I}{2\pi\rho}\,\mathbf{a}_\phi$$

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from $\rho = a$ to $\rho = b$ and from, $z = 0$ to $z = d$.

$$\psi = \Phi = \int_s \mathbf{B}\,.\,d\mathbf{S} = \int_{z=0}^{d}\int_{\rho=a}^{b} \frac{\mu_o I}{2\pi\rho}\,\mathbf{a}_\phi\,.\,d\rho dz\,\mathbf{a}_\phi = \frac{\mu_o I}{2\pi}\int_{z=0}^{d}\int_{\rho=a}^{b}\frac{d\rho}{\rho}\,dz = \frac{\mu_o I d}{2\pi}\ln\frac{b}{a}$$

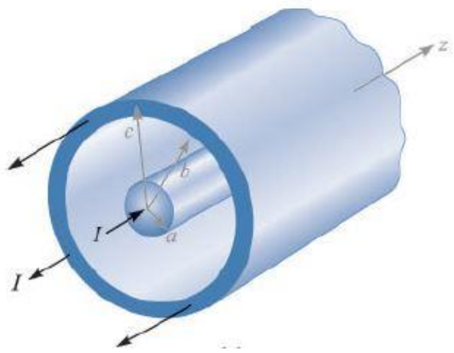


Fig. (7.21) for example 7.7

Example 7.8: A current distribution gives rise to the vector magnetic potential

$\mathbf{A} = x^2y\,\mathbf{a}_x + y^2x\,\mathbf{a}_y - 4xyz\,\mathbf{a}_z$ Wb/m. Calculate

(a) $\mathbf{B}$ at $(-1, 2, 5)$ (b) The flux through the surface defined by $z = 1$, $0 \le x \le 1$, $-1 \le y \le 4$

Solution: $\mathbf{B} = \nabla \times \mathbf{A}$

$$\nabla\times\mathbf{A}=\begin{bmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{bmatrix}=\left[\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right]\mathbf{a}_x+\left[\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right]\mathbf{a}_y+\left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

$$\nabla\times\mathbf{A}=\begin{bmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & y^2x & -4xyz\end{bmatrix}=[-4xz-0]\mathbf{a}_x+[0+4yz]\mathbf{a}_y+[y^2-x^2]\mathbf{a}_z$$

$$\mathbf{B}_{(-1,2,5)}=20\,\mathbf{a}_x+40\,\mathbf{a}_y+3\,\mathbf{a}_z\quad \mathrm{wb/m^2}$$

$$\psi=\int_S \mathbf{B}\,.\,d\mathbf{S}$$

$$\psi=\int_{y=-1}^{4}\int_{x=0}^{1}\left[\left([-4xz-0]\mathbf{a}_x+[0+4yz]\mathbf{a}_y+[y^2-x^2]\mathbf{a}_z\right).\,(dxdy\,\mathbf{a}_z)\right]$$

$$\psi=\int_{-1}^{4}\int_{0}^{1}\left[[y^2-x^2].\,(dxdy)\right]=\int_{-1}^{4}y^2dy-5\int_{0}^{1}x^2dx=\left(\frac{64}{3}+\frac{1}{3}\right)-\frac{5}{3}=20\ \mathrm{Wb}$$

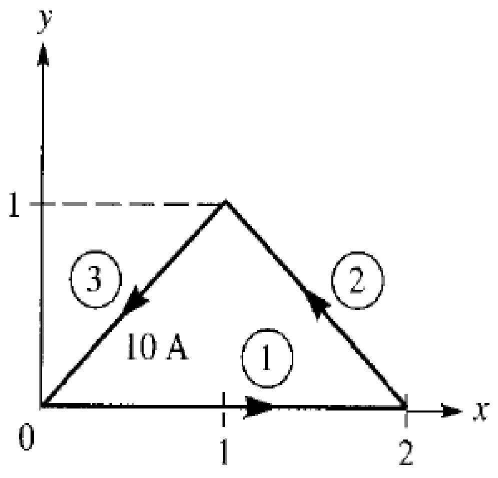
Second method:

$$\psi=\int_L \mathbf{A}\,.\,d\mathbf{L}=\int_0^1 x^2(-1)dx+\int_{-1}^{4}y^2(1)dy+\int_1^0 x^2(4)dx+\int_4^{-1}y^2(0)dy$$

$$\psi=-\frac{5}{3}+\frac{65}{3}=20\ \mathrm{Wb}$$

Problems

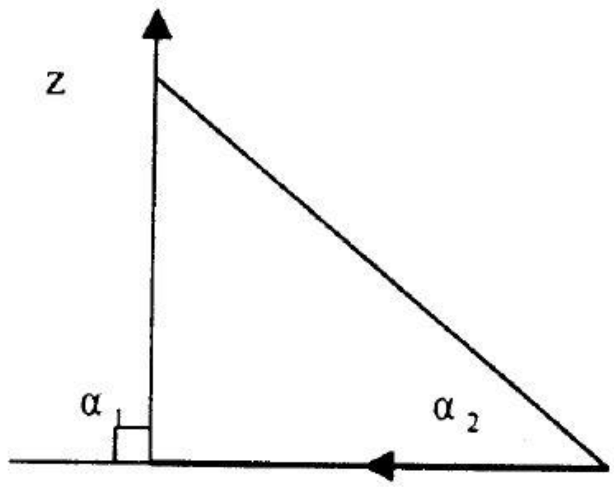
7.1.[2] Find the magnetic field intensity **H** at point $P(0,0,5)$ due to side 3 of the triangular loop in Figure below. [Answer : $\mathbf{H} = -30.63\mathbf{a}_x + 30.63\mathbf{a}_z$ mA/m]



(a)

Solution

P.E. 7.1



$$\rho = 5, \cos\alpha_1 = 0, \cos\alpha_2 = \sqrt{\frac{2}{27}}$$

$$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \left(\frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}\right) \times \mathbf{a}_z = \frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$$

$$H_3 = \frac{10}{4\pi(5)}\left(\sqrt{\frac{2}{27}} - 0\right)\left(\frac{-a_x + a_y}{2}\right) = \underline{\underline{-30.03a_x + 30.6a_y}} \quad \text{mA/m}$$

7.2.[2] The positive y-axis (semi-infinite line with respect to the origin) carries a filamentary current of 2A in the $-\mathbf{a}_y$ direction. Assume it is part of a large circuit. Find the magnetic field intensity $\mathbf{H}$ (a) at point $A(2,3,0)$ and (b) at point $B(3,12,-4)$.

[Answer : (a) $145.8\mathbf{a}_z$ mA/m : (b) $48.97\mathbf{a}_x + 36.73\mathbf{a}_z$ mA/m]

Solution

P.E. 7.2

(a) $$\mathbf{H} = \frac{2}{4\pi(2)}\left(1+\frac{3}{\sqrt{13}}\right)\mathbf{a}_z = \underline{\underline{0.1458}} \text{ A/m}$$

(b) $$\rho = \sqrt{3^2+4^2} = 5, \alpha_2 = 0, \cos\alpha_1 = -\frac{12}{13},$$

$$\mathbf{a}_\phi = \mathbf{a}_y \times \left(\frac{3\mathbf{a}_x - 4\mathbf{a}_z}{5}\right) = \frac{4\mathbf{a}_x + 3\mathbf{a}_z}{5}$$

$$\mathbf{H} = \frac{2}{4\pi(5)}\left(1+\frac{12}{13}\right)\left(\frac{4\mathbf{a}_x + 3\mathbf{a}_z}{5}\right) = \frac{1}{26\pi}(4\mathbf{a}_x + 3\mathbf{a}_z)$$

$$= \underline{\underline{48.97\mathbf{a}_x + 36.73\mathbf{a}_z}} \text{ mA/m}$$

7.3.[2] A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at $(0,0,4)$ and $(0,0,-4)$.

[Answer : (a) $0.63\mathbf{a}_z$ A/m : (b) $0.63\mathbf{a}_z$ A/m]

Solution

Consider the circular loop shown in Figure 7.8(a). The magnetic field intensity dH at point $P(0,0,h)$ contributed by current element $Id\mathbf{L}$ is given by Biot-Savart's law:

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho\mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}}(\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero because the radial components produced by pairs of current element 180° apart cancel. This may also be shown mathematically by writing $\mathbf{a}_\rho$ in rectangular coordinate systems (i.e., $\mathbf{a}_\rho = \cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$)

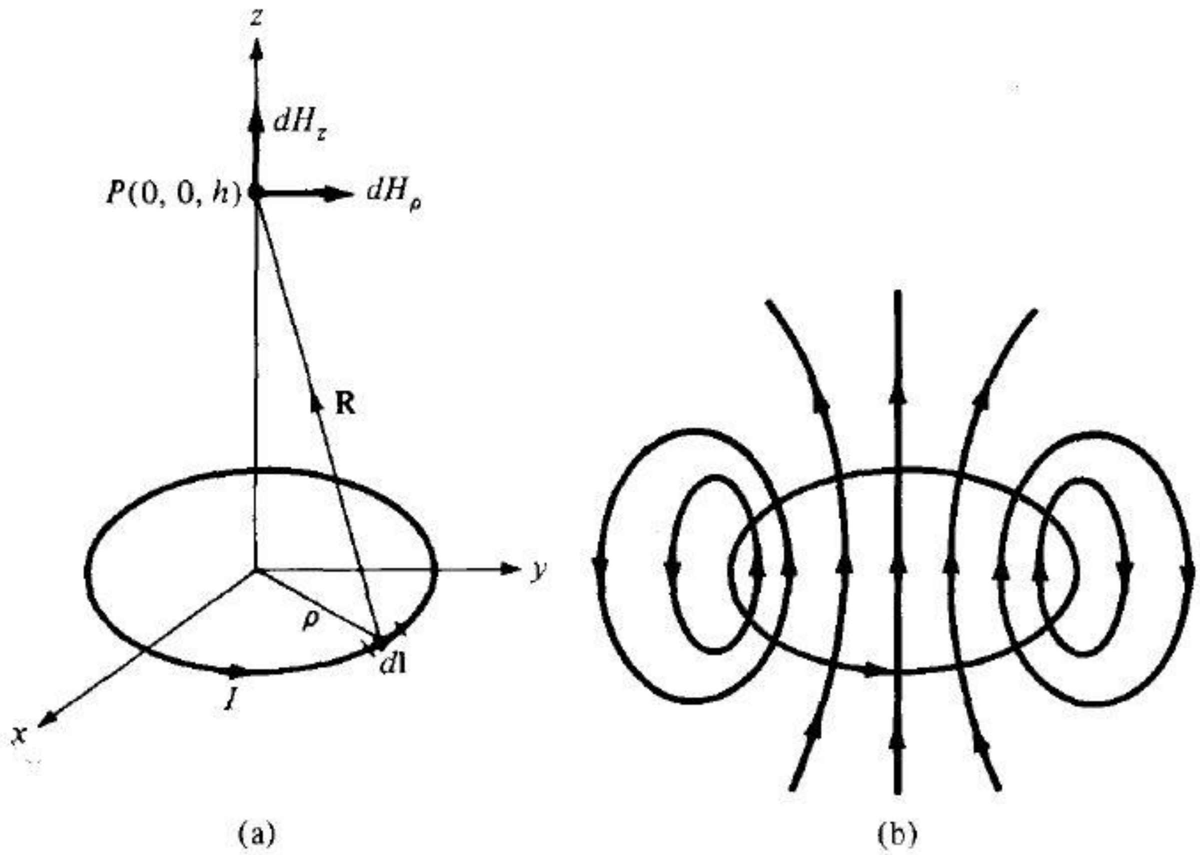


Figure 7.8 For Example 7.3: (a) circular current loop, (b) flux lines due to the current loop.

Integrating $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$.

Thus

$$\mathbf{H} = \int dH_z \, \mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2 \, d\phi \, \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

Or

$$\mathbf{H} = \frac{I\rho^2 \mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\text{A}$, $\rho = 3$, $h = 4$ gives

$$\mathbf{H}_{(0,0,4)} = \frac{10(3)^2 \mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

(b) Notice from $d\mathbf{L} \times \mathbf{R}$ above that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence:

$$\mathbf{H}_{(0,0,-4)} = \mathbf{H}_{(0,0,4)} = 0.36\mathbf{a}_z \text{ A/m}$$

The flux lines due to the circular current loop are sketched in Figure (7.8b).

7.4.[2] A thin ring of radius 5 cm is placed on plane $z = 1$ cm so that its center is at $(0, 0, 1)$ cm. If the ring carries 50mA along $\mathbf{a}_\phi$, find magnetic field $\mathbf{H}$ at (a) $(0\,, 0, -1)$ cm (b) $(0, 0, 10)$ cm.

[Answer : (a) $400\mathbf{a}_z$ mA/m : (b) $57.3\mathbf{a}_z$ mA/m]

Solution

P.E. 7.3

(a) From Example 7.3,

$$H = \frac{Ia^2}{2(a^2 + z^2)^{3/2}} a_z$$

At (0,0,1), z = 2cm,

$$H = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 2^2)^{3/2} \times 10^{-6}} a_z \quad \text{A/m}$$

$$= 400.2 a_z \quad \text{A/m}$$

(b) At (0,0,10cm), z = 9cm,

$$H = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 9^2)^{3/2} \times 10^{-6}} a_z$$

$$= \underline{57.3 a_z \quad \text{mA/m}}$$

7.5.[2] A solenoid of length ℓ and radius a consists of N turns of wire carrying current I. Show that at point P along its axis,

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$

where $n = N/\ell$, θ_1 and θ_2 are the angles subtended at P by the end turns as illustrated in Figure 7.9. Also show that if $\ell \gg a$, at the center of the solenoid,

$$\mathbf{H} = nI\mathbf{a}_z$$

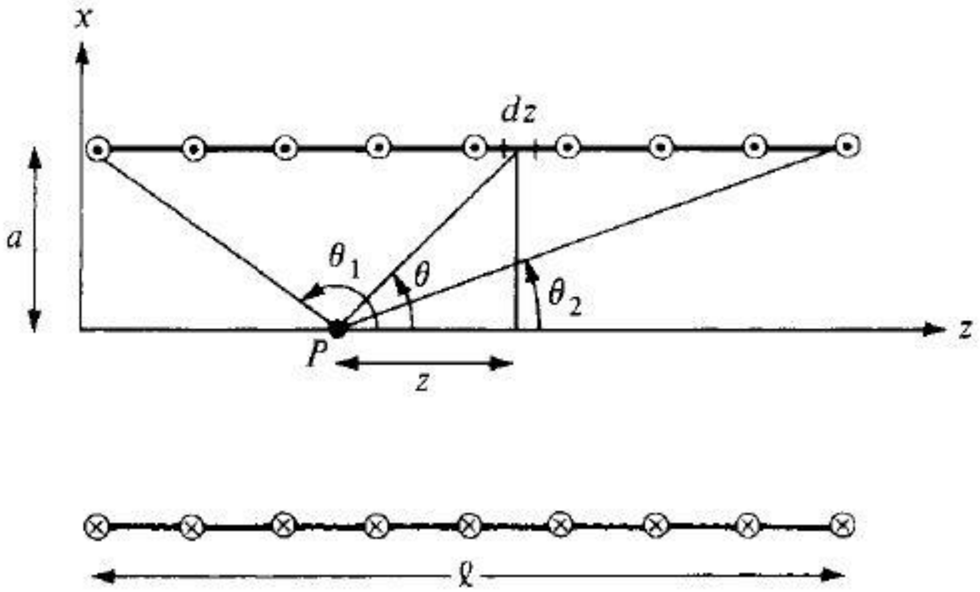


Figure 7.9 For Example 7.4; cross section of a solenoid.

Solution:

Consider the cross section of the solenoid as shown in Figure 7.9. Since the solenoid consists of circular loops, we apply the result of Example 7.3. The contribution to the magnetic field **H** at P by an element of the solenoid of length dz is

$$dH_z = \frac{I\,dL\,a^2}{2[a^2 + z^2]^{3/2}} = \frac{I\,a^2\,n\,dz}{2[a^2 + z^2]^{3/2}}$$

where $dL = ndz = (NI\ell)dz$. From Figure 7.9, $\tan\theta = \frac{a}{z}$ that is,

$$dz = -a\,\text{cosec}^2\theta\,d\theta = -\frac{[z^2 + a^2]^{3/2}}{a^2}\sin\theta\,d\theta$$

Hence,

$$dH_z = -\frac{nI}{2}\sin\theta\,d\theta$$

Or

$$H_z = -\frac{nI}{2}\int_{\theta_1}^{\theta_2} \sin\theta \, d\theta$$

Thus

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$

As required. Substituting $n = N/\ell$, gives

$$\mathbf{H} = \frac{nI}{2\ell}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$

At the center of the solenoid,

$$\cos\theta_2 = \frac{\ell/2}{\left[a^2 + \left(\frac{\ell}{2}\right)^2\right]^{1/2}} = -\cos\theta_1$$

And

$$\mathbf{H} = \frac{In\ell}{2\left[a^2 + \left(\frac{\ell}{2}\right)^2\right]^{1/2}}\mathbf{a}_z$$

If $\ell \gg a$ or $\theta_2 \approx 0^{\mathrm{o}}, \theta_1 \approx 180^{\mathrm{o}}$

$$\mathbf{H} = nI\mathbf{a}_z = \frac{NI}{\ell}\mathbf{a}_z$$

7.6.[2] If the solenoid of Figure 7.9 has 2,000 turns, a length of 75 cm, a radius of 5 cm, and carries a current of 50 mA along $\mathbf{a}_\phi$, find $\mathbf{H}$ at (a) $(0,0,0)$ cm (b) $(0,0,75)$ cm (c) $(0,0,50)$ cm.

[Answer : (a) $66.52\mathbf{a}_z$ A/m : (b) $66.52\mathbf{a}_z$ A/m : (c) $131.7\mathbf{a}_z$ A/m]

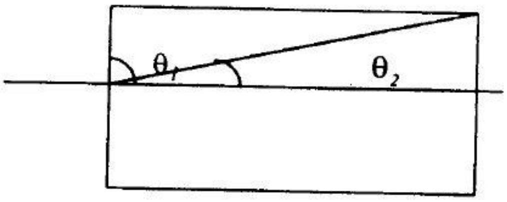
Solution:

P.E. 7.4

$$H = \frac{NI}{2L}(\cos\theta_2 - \cos\theta_1)a_z = \frac{2\times 10^3 \times 50\times 10^{-3}(\cos\theta_2 - \cos\theta_1)a_z}{2\times 0.75}$$

$$= \frac{100}{1.5}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$

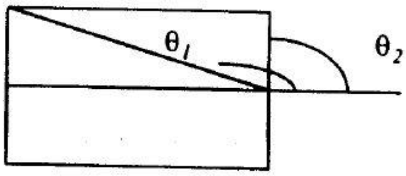
(a) At (0,0,0), $\theta = 90^o$, $\cos\theta_2 = \dfrac{0.75}{\sqrt{0.75^2 + 0.05^2}}$
$= 0.9978$



$$\boldsymbol{H} = \frac{100}{1.5}(0.9978 - 1)\boldsymbol{a}_z$$

$$= 66.52\,\boldsymbol{a}_z \text{ A/m}$$

(b) At (0,0,0.75), $\theta_2 = 90^o, \cos\theta_1 = -0.9978$

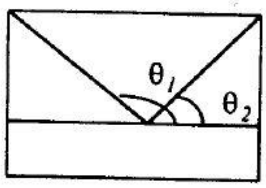


$$\boldsymbol{H} = \frac{100}{1.5}(0 + 0.9978)\boldsymbol{a}_z$$

$$= 66.52\,\boldsymbol{a}_z \text{ A/m}$$

(c) At (0,0,0.5), $\cos\theta_1 = \dfrac{-0.5}{\sqrt{0.5^2 + 0.05^2}} = -0.995$

$$\cos\theta_1 = \frac{0.25}{\sqrt{0.25^2 + 0.05^2}} = 0.9806$$



$$\boldsymbol{H} = \frac{100}{1.5}(0.9806 + 0.995)\boldsymbol{a}_z$$

$$= 131.7\,\boldsymbol{a}_z \text{ A/m}$$