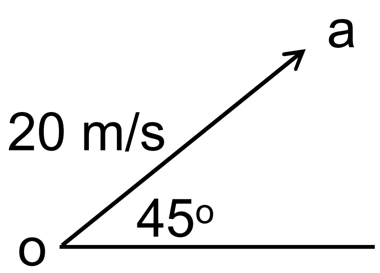
Vectors

DEFINITION

A vector may be represented by a straight line, the length of line being directly proportional to the magnitude of the quantity and the direction of the line being in the same direction as the line of action of the quantity. An arrow is used to denote the sense of the vector, that is, for a horizontal vector, say, whether it acts from left to right or vice-versa. The arrow is positioned at the end of the vector. Figure shows a velocity of 20 m/s at an angle of 45° to the horizontal and may be depicted by oa = 20 m/s at 45° to the horizontal.



VECTORS AND SCALAR QUANTITIES

Some of the things we measure are completely determined by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. But we need more than that to describe a force, displacement, or velocity, for these quantities have direction as well as magnitude. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moved as well as how far it moved. To describe a body's velocity at any given time, we have to know where the body is headed as well as how fast it is going.

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

Vector in Space

Quantities that have direction as well as magnitude are usually represented by arrows that point in the direction of the action and whose lengths give the magnitude of the action in terms of a suitably chosen unit. When we describe these arrows abstractly, as directed line segments in the plane or in space, we call them *vectors*. Typical examples of vectors are displacement, velocity, and force as shown in figure below:-

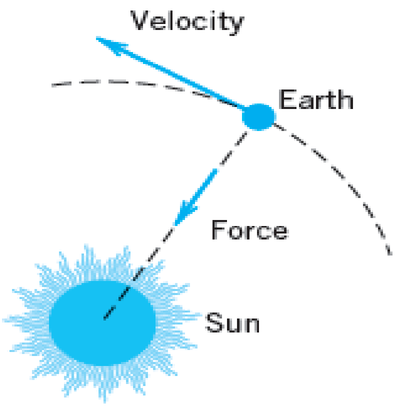


Figure: Force and velocity

Equality of Vectors

Two vectors **a** & **b** are equal, written **a** = **b**, if they have the same length and the same direction as explained in Figure.

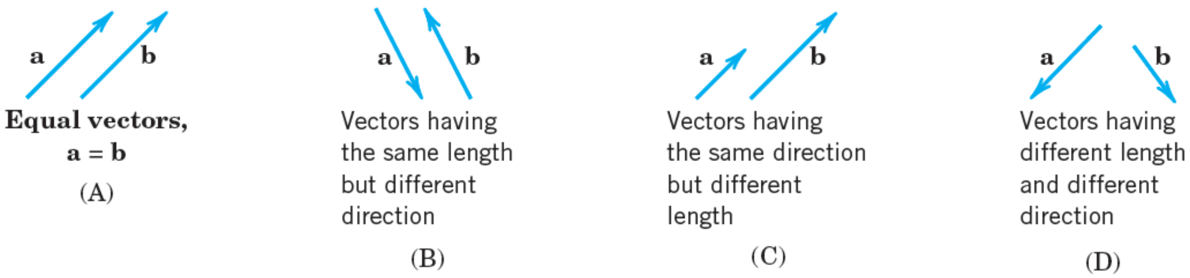


Figure: (A) Equal vectors. (B)– (D) Different vectors

Zero Vectors

Vector which has length equal to (0) and no direction

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

Vector Representation

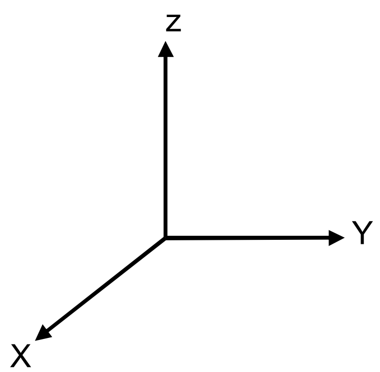
In print, vectors are usually represented by single letter, as in **A** ("vector A"). The vector defined by the directed line segment from point A to point B, however, is written as $\overrightarrow{AB}$ ("vector AB "). In handwritten work it is customary to draw small arrows above all letters representing vectors.

VECTOR IN SPACE

In order to describe a vector accurately, some specific lengths, directions, angles, projections, or components must be given. There are three simple methods of doing this, and about eight or ten other methods which are useful in very special cases.

We are going to use only the three simple methods, and the simplest of these is the *Cartesian,* or *rectangular, coordinate system.*

In the Cartesian coordinate system we set up three coordinate axes mutually at right angles to each other, and call them the x, *y,* and z axes. It is customary to choose a *right-handed* coordinate system, in which a rotation (through the smaller angle) of the x axis into the y axis would cause a right-handed screw to progress in the direction of the z axis. If the right hand is used, then the thumb, forefinger, and middle finger

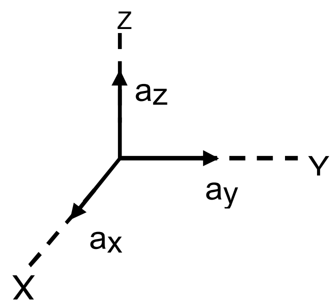


May then be identified, respectively, as the x, y, and z axes. Figure below shows a right-handed Cartesian coordinate system.

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

To describe a vector in the Cartesian coordinate system, let us first consider a vector r extending outward from the origin. A logical way to identify this vector is by giving the three **component vectors**, lying along the three coordinate axes, whose vector sum must be the given vector. If the component vectors of the vector r are x, y, and z, then r = x + y + z. The component vectors are shown in Figure below. Instead of one vector, we now have three, but this is a step forward, because the three vectors are of a very simple nature; each is always directed along one of the coordinate axes.

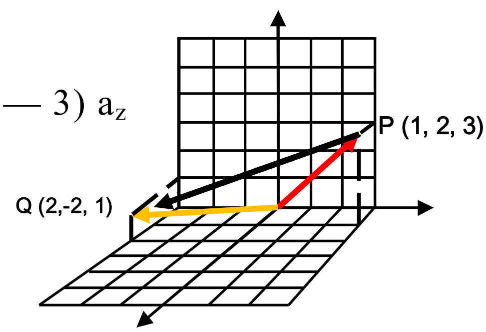
In other words, the component vectors have magnitudes which depend on the given vector (such as r above), but they each have a known and constant direction. This suggests the use of ***unit vectors*** having unit magnitude, by definition, and directed along the coordinate axes in the direction of the increasing coordinate values. We shall reserve the symbol **A** for a unit vector and identify the direction of the unit vector by an appropriate subscript. Thus $\mathbf{a_x}$, $\mathbf{a_y}$, and $\mathbf{a_z}$ are the unit vectors in the cartesian coordinate system. They are directed along the x, y, and z axes, respectively, as shown in Figure.



A vector $\mathbf{r_P}$ pointing from the origin to point P(1, 2, 3) and another one r_Q pointing from the origin to the point Q(2, —2, 1). Then the vector from P to Q may be obtained by applying the rule of vector addition. This rule shows that the vector from the origin to **P** plus the vector from **P** to **Q** is equal to the vector from the origin to **Q**. The desired vector from P (1, 2, 3) to Q (2, —2, 1) is therefore,

$$R_{PQ} = r_Q - r_p = (2 - 1)\, a_x + (-2 - 2)\, a_y + (1 - 3)\, a_z$$

$$R_{PQ} = = 1\, a_x - 4\, a_y - 1\, a_z$$



Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

Vector magnitude

Any vector $\vec{A}$ then may be described by $\vec{A} = Ax\ ax + Ay\ ay + Az\ az$. The magnitude of $\vec{A}$ written |A| or simply A, is given by:

$$|A| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$$

Example: - The vector $\vec{A}$ with initial point *P*: (4, 0, 2) and terminal point *Q*: (6, -1, 2) has the components

$A_x = 6 - 4 = 2$

$A_y = -1 - 0 = -1$

$A_z = 2 - 2 = 0$.

Hence, $\vec{A} = 2\ ax - 1\ ay$ and $|A| = \sqrt{5}$.

Vector Addition

Two vectors may be added algebraically by adding their corresponding scalar components, as in below:-

If $\vec{A} = A_x\, a_x + A_y\, a_y + A_z\, a_z$ and $\vec{B} = B_x\, a_x + B_y\, a_y + B_z\, a_z$

$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\, a_x + (A_y + B_y)\, a_y + (A_z + B_z)\, a_z$$

Basic Properties of Vector Addition

- $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
- $0 + \vec{B} = \vec{B} + 0 = \vec{B}$
- $\vec{B} + (-\vec{B}) = 0$

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

Vector Subtraction

To subtract vector $\overrightarrow{V_2}$ from a vector $\overrightarrow{V_1}$ we add $-\overrightarrow{V_2}$ to $\overrightarrow{V_1}$, as in below:

If $\overrightarrow{A} = A_x\, a_x + A_y\, a_y + A_z\, a_z$ and $\overrightarrow{B} = B_x\, a_x + B_y\, a_y + B_z\, a_z$ then:

$\overrightarrow{C} = \overrightarrow{A} - \overrightarrow{B} = A_x\, a_x + A_y\, a_y + A_z\, a_z - (\overrightarrow{B} = B_x\, a_x + B_y\, a_y + B_z\, a_z)$

$\overrightarrow{C} = \overrightarrow{A} - \overrightarrow{B} = (A_x - B_x)\, a_x + (A_y - B_y)\, a_y + (A_z - B_z)\, a_z$

Unit Vector

For a vector $\overrightarrow{A}$ = Ax ax + Ay ay + Az az, then U_A is a unit vector in the direction of A and can be expressed by:-

$$U_A = \frac{\overrightarrow{A}}{|A|}$$

Dot Product (Inner product or Scalar product)

Dot product is called scalar product because resulting the products are number and not a vector, the **inner product** or **dot product** (read "*A* **dot***B*") of two vectors $\vec{A}$ and $\vec{B}$ is the product of their lengths times the cosine of their angle

$$\vec{A} \cdot \vec{B} = |A| \cdot |B| \cos\theta$$

Where θ is the smaller angle between vectors $\vec{A}$ and $\vec{B}$

$$\theta = \frac{\vec{A} \cdot \vec{B}}{|A| \cdot |B|}$$

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

If $\overrightarrow{A} = A_x\, a_x + A_y\, a_y + A_z\, a_z$ and $\overrightarrow{B} = B_x\, a_x + B_y\, a_y + B_z\, a_z$

Then,

$\vec{A} \cdot \vec{B} = (A_x * B_x)\, a_x + (A_y * By)\, a_y + (A_z * B_z)\, a_z$

PROPERTIES

- $\vec{A} \cdot \vec{A} = |A|^2$.
- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.
- If $\vec{A} \perp \vec{B}$ then $\vec{A} \cdot \vec{B} = 0$.

Example: Find the inner product and the lengths of $\overrightarrow{A} = 1\, a_x + 2\, a_y$ and $\overrightarrow{B} = 3a_x - 2\, a_y + 1\, a_z$ as well as the angle between these vectors.

Ans.

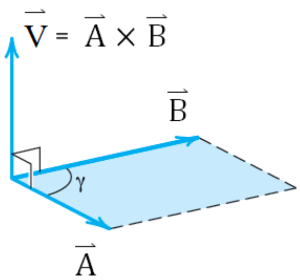
$\vec{A} \cdot \vec{B} = -1$, angle between these vectors = 1.69061= 96.865°.

The Cross Product

The **vector product** or **cross product** (read "**a** cross **b**") of two vectors $\vec{A}$ and $\vec{B}$ is the vector $\vec{V}$ denoted by:

$$\vec{V} = \overrightarrow{A} \times \overrightarrow{B}$$

Cross product is called vector product because resulting the products are vector and not a number.



If we have two vectors:

$\overrightarrow{A} = A_{1\,i} + A_{2\,j} + A_{3\,k}$

$\overrightarrow{B} = B_{1\,i} + B_{2\,j} + B_{3\,k}$

Then,

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

$$|\vec{A} \times \vec{B}| = |A| \cdot |B| \sin\theta$$

Where:

θ = angle between vectors $\vec{A}$ and $\vec{B}$.

$\overrightarrow{U_N}$ = Normal unit vector to $\vec{A}$ and $\vec{B}$.

$$\vec{V} = \vec{A} \times \vec{B} = \begin{bmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix}$$

Vector Products of the Standard Basis Vectors

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Example: Find the cross product of $\vec{A} = 1\,a_x + 1\,a_y$ and $\vec{B} = 3a_x$ as well as the angle between these vectors.

$$\vec{V} = \vec{A} \times \vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}\mathbf{k} = -3\mathbf{k}$$

PROPERTIES

- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- If $\vec{A} \parallel \vec{B}$ then $\vec{A} \times \vec{B} = 0$
- If $\vec{A} \parallel \vec{B}$ then $\vec{A} = t\vec{B}$ where t = Constant
- $i \times i = j \times j = k \times k = 0$. (☹ Why?!)

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

Triple Product

Suppose we have the following three vectors:

$\vec{A} = A_{1\,i} + A_{2\,j} + A_{3\,k}$

$\vec{B} = B_{1\,i} + B_{2\,j} + B_{3\,k}$

$\vec{C} = C_{1\,i} + C_{2\,j} + C_{3\,k}$ Then...

SCALAR TRIPLE PRODUCT

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

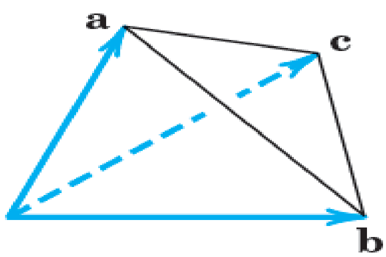
As an application we can found the volume of the tetrahedron determined by three edge vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as:-

Volume of the tetrahedron = $|\vec{A} \cdot (\vec{B} \times \vec{C})|$

Example: A tetrahedron is determined by three edge vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$), as indicated in Figure. Find the volume of the tetrahedron?

$$= \begin{vmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ 6 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= -12 - 60 = -72.$$



Tetrahedron

We take the absolute value to find that:-

V = 72 Cubic unit

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

Vector Projection

Scalar projection of $\vec{B}$ on to $\vec{A}$

$$Proj_A B = \left(\frac{\vec{A}\cdot\vec{B}}{\vec{A}\cdot\vec{A}}\right) * \vec{A}$$

Scalar component of $\vec{B}$ on the direction of $\vec{A}$ $(|\vec{B}| * cos\theta)$

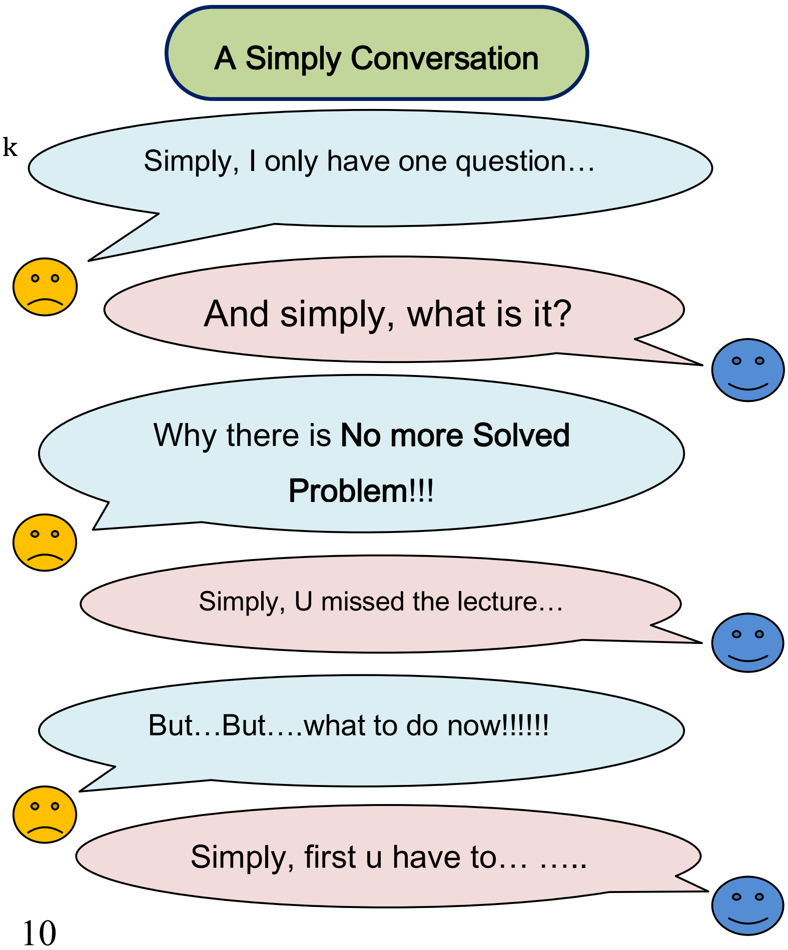
$$Bcos\theta = \vec{B}\cdot\frac{\vec{A}}{|\vec{A}|}$$

Example: - Find the vector projection of $\vec{B} = 6_{\,i} + 3_{\,j} + 2_{\,k}$ in to $\vec{N} = 1_{\,i} - 2_{\,j} - 2_{\,k}$ and the scalar component of $\vec{B}$ in the direction of $\vec{A}$?

Solution:-

$$Proj_A B = \left(\frac{\vec{A}\cdot\vec{B}}{\vec{A}\cdot\vec{A}}\right) * \vec{A} = -\frac{4}{9}_{\,i} + \frac{8}{9}_{\,j} + \frac{8}{9}_{\,k}$$

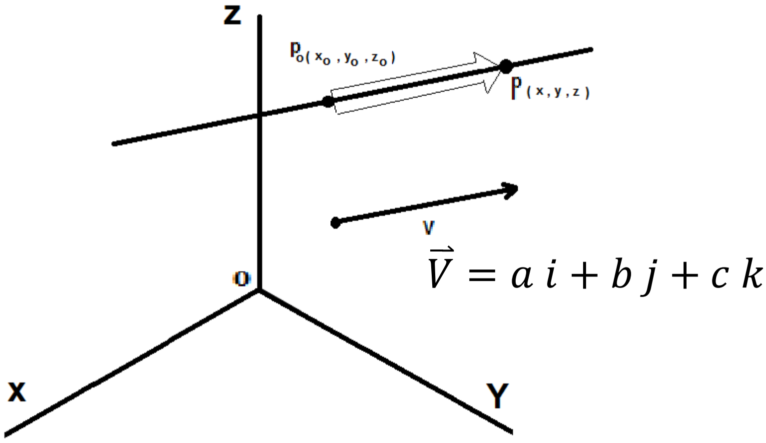
$$|\vec{B}|cos\theta = \vec{B}\cdot\frac{\vec{A}}{|\vec{A}|} = -\frac{4}{3}$$



Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

APPLICATION OF VECTORS IN SPACE

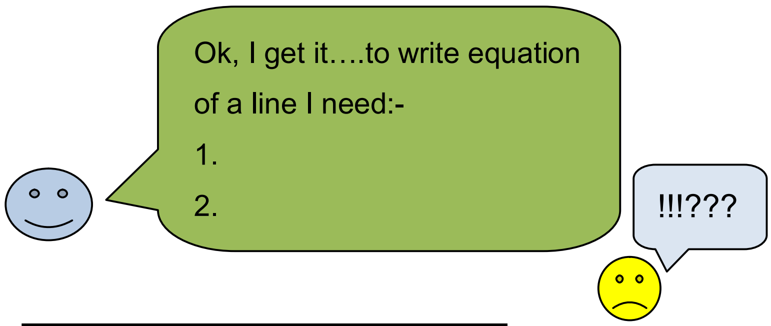
Equation of line



$\overrightarrow{P_o\ P} = (x - x_o)_i + (y - y_o)_j + (z - z_o)_z$

$\overrightarrow{P_o\ P} \parallel \vec{V} \Longrightarrow \overrightarrow{P_o\ P} = t\vec{V}$

$\vec{V} = a\,i + b\,j + c\,k$



$x - x_o = at \Longrightarrow x = x_o + at$

$y - y_o = bt \Longrightarrow y = y_o + bt$

$z - z_o = ct \Longrightarrow z = z_o + ct$

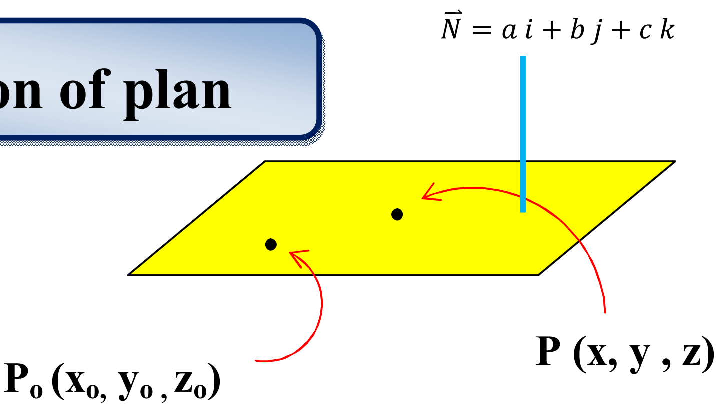
Parametric Equation

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z + z_o}{c}$$

Standard Equation

Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".

Equation of plan



$\overrightarrow{P_o\ P} = (x - x_o)\ _i + (y - y_o)\ _j + (z - z_o)\ _z$

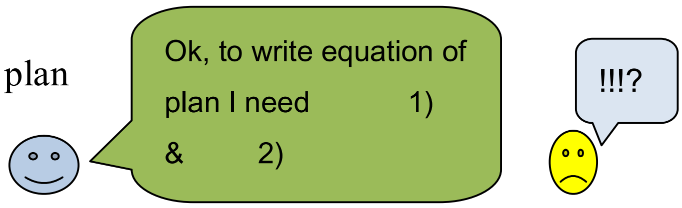
$\overrightarrow{P_o\ P} \perp \vec{N}$

$\overrightarrow{P_o\ P} \cdot \vec{N} = 0$

$a\,(x - x_o) + b\,.\,(y - y_o) + c\,.\,(z - z_o) = 0$

$a\,x + b\,y + c\,z + \underbrace{(- a\,x_o - b\,y_o - c\,z_o)}_{D} = 0$

$a\,x + b\,y + c\,z + D = 0$ …….Equation of plan



Main Ref. 1) E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS "
2) L. Finny & B. Thomas, "CALCULAS".