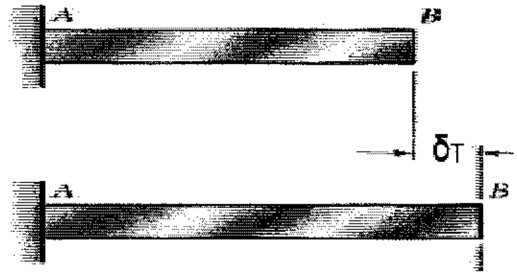
Thermal Stresses

Temperature changes cause the body to expand or contract. Mechanical stress induced in a body when some or all of its parts are not free to expand or contract in response to changes in temperature. In most continuous bodies, thermal expansion or contraction cannot occur freely in all directions because of geometry, external constraints, or the existence of temperature gradients, and so stresses are produced. Such stresses caused by a temperature change are known as thermal stresses.

The amount δ_T which gives the total thermal deflection (deformation due to temperature changes) is given by:



$$\delta_T = \alpha L\,(Tf - Ti) = \alpha L\,\Delta T$$

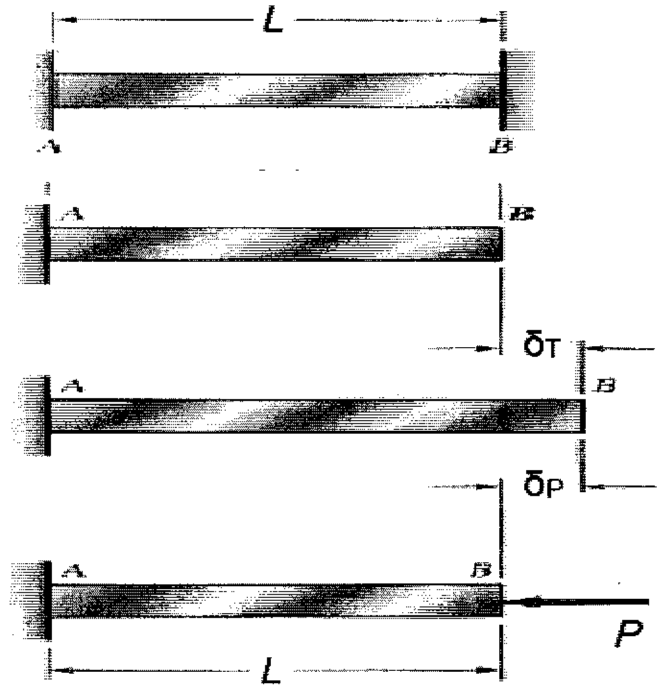
Where: δ_T: Deformation due to temperature changes

α: The coefficient of thermal expansion in m/m C°,

L: The length in meter,

T_i and T_f : are the initial and final temperatures, respectively in °C.

Deformation due to equivalent axial stress;



$$\delta_T = \delta_P$$

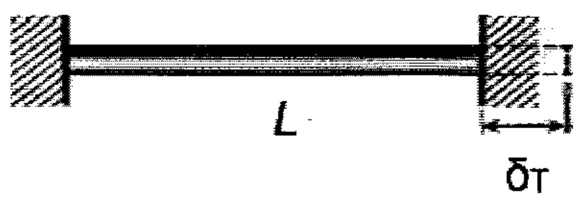
$$\delta_P = \frac{P\,.\,L}{A\;E} = \frac{\sigma\,.\,L}{E}$$

$$\alpha\,.\,L\;\Delta T = \frac{\sigma\,.\,L}{E}$$

Which means : $\sigma = \alpha\,\Delta T\,E$

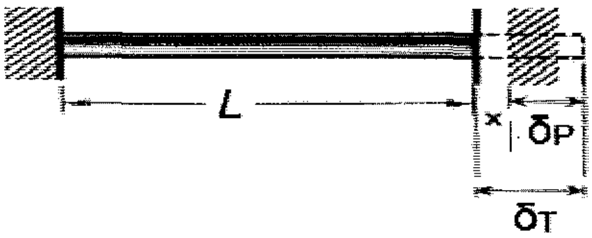
Where σ: is the thermal stress in MPa, E is the modulus of elasticity of the rod in MPa.

δ_P : Compression deformation



If the wall yields a distance of (x) as shown, the following calculations will be made:

That means: $\delta_T = x + \delta_P$



Or that: $\alpha \,.\, L \,\Delta T = x + \sigma \, L/E$

Where σ represents the thermal stress

Keep in mind that as the temperature rises above the normal, the rod will be in compression, and if the temperature drops below the normal, the rod is in tension.

Example

A 5 m aluminum flagpole is installed at 20°C. Overnight, the temperature drops to -5°C. How much does the height change, in millimeters? What is the final height of the flagpole, in meters?

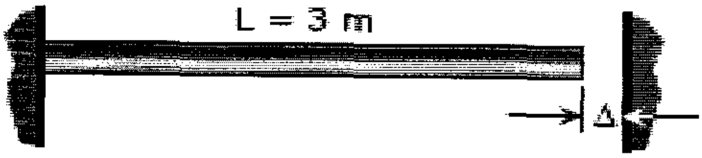
Solution First, calculate the change in length using $\delta = \alpha L(\Delta T)$. From the Appendix, the thermal expansion coefficient for aluminum is $\alpha_{Aluminum} = 23\times10^{-6}\,°C^{-1}$. Next, calculate the final length by adding the change in length to the original length.

Change in length $\delta = \alpha L(\Delta T) = \dfrac{23\times10^{-6}\,5\,\text{m}(-5\,°C - 20°C)}{°C}\left|\dfrac{10^3\,\text{mm}}{\text{m}}\right. = -2.88\,\text{mm}$. The negative sign indicates the flagpole is getting shorter.

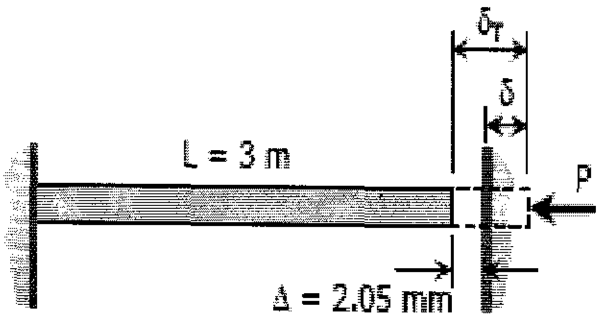
Final length $L_f = L + \delta = 5\,\text{m} - \dfrac{2.88\,\text{mm}}{}\left|\dfrac{\text{m}}{10^3\,\text{mm}}\right. = 4.997\,\text{m}$

Example:

A bronze bar 3 m long with a cross sectional area of 320 mm^2 is placed between two rigid walls as shown in the figure at a temperature of -20°C, the gap Δ = 25 mm. Find the temperature at which the compressive stress in the bar will be 35 MPa. Use α = 18.0 × 10^{-6} m/(m·°C) and E = 80 GPa.



Solution:



$$\delta_T = \delta + \Delta$$

$$\alpha L(\Delta T) = \frac{\sigma L}{E} + 2.5$$

$$(18 \times 10^{-6})(3000)(\Delta T) = \frac{35(3000)}{80000} + 2.5$$

$$\Delta T = 70.6°C$$

$$T = 70.6 - 20$$

$$T = 50.6°C$$

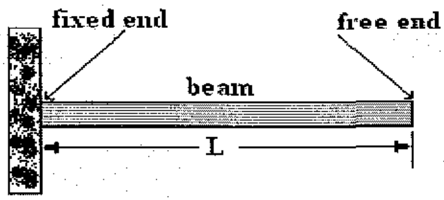
Forces on Beams & Reactions

Definition of a Beam:

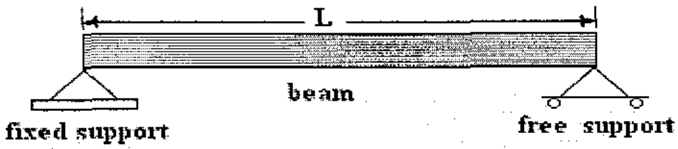
A beam is a structural member which carries loads (forces and/or couples) that lie in a plane containing the longitudinal of the bar and most often perpendicular to its longitudinal axis, but they can be of any geometry.

Types of Beams

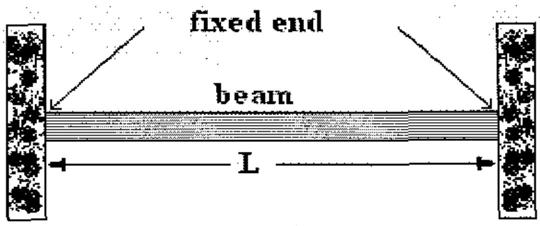
1- *Cantilever beam*: fixed or built-in at one end while it's other end is free.



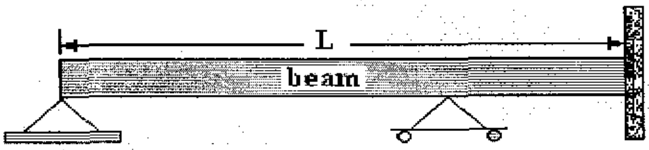
2- *Freely or simply supported beam*: the ends of a beam are made to freely rest on supports.



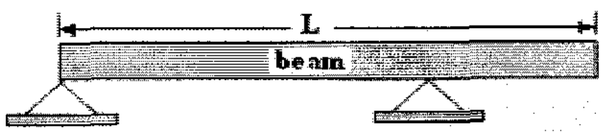
3- *Built-in or fixed beam*: the beam is fixed at both ends.



4- *Continuous beam*: a beam which is provided with more than two supports.

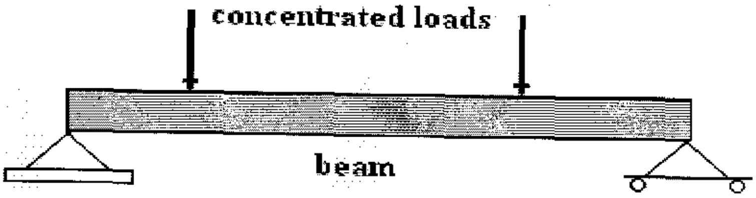


5- *Overhanging beam*: a beam which has part of the loaded beam extends outside the supports.



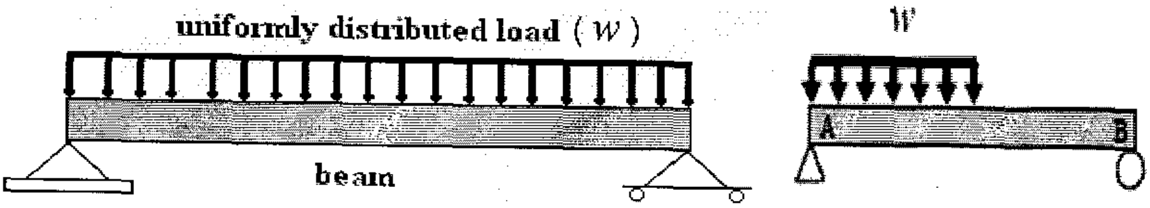
Types of Loads:

1- *Concentrated load* assumed to act at a point and immediately introduce an oversimplification since all practical loading system must be applied over a finite area.

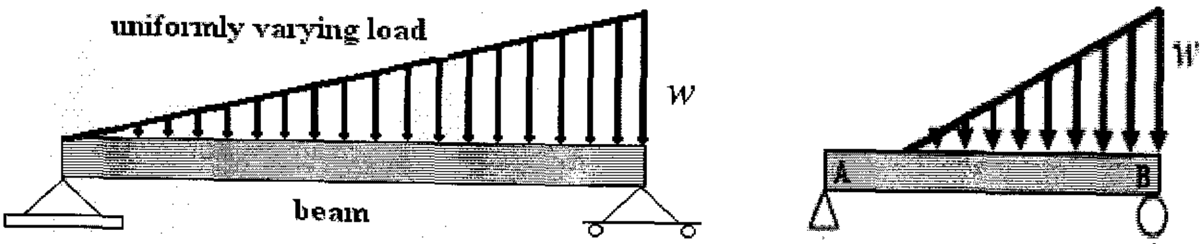


2- Distributed load are assumed to act over part, or all, of the beam and in most cases are assumed to be equally or uniformly distributed.

a- *Uniformly distributed.*

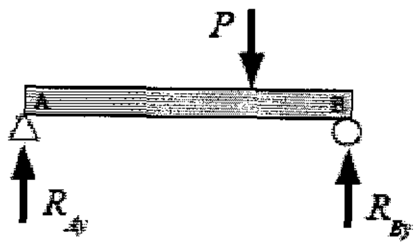


b- *Uniformly varying load.*

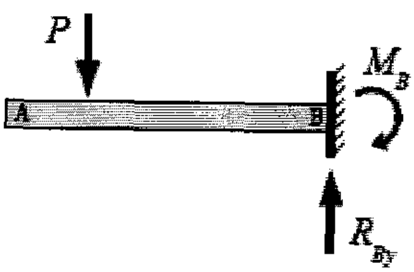


Reactions at supports of beams:

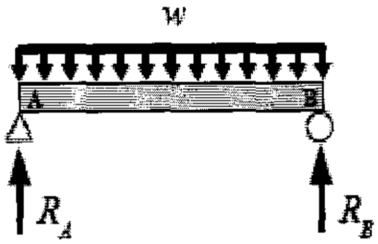
As the beam carry load, there will be reactions of forces at the beam supports. The beam shown in the figure, there are no applied horizontal forces, therefore the horizontal reaction force is zero.



A cantilever beam is embedded in a wall; therefore the beam has reaction force as well as a reaction moment.



The weight of a beam is an example of a uniform distributed load. The weight per unit length, *w*, typically has units of lb./ft., kips/ft., or kN/m.

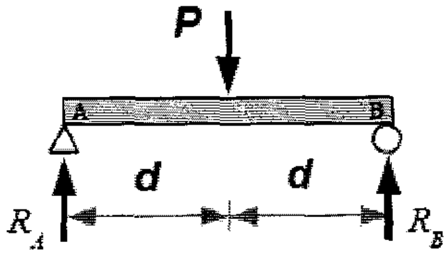


So, if the beam is 10 feet long, then the total weight, *W*, of the beam is:

$$W = wL = \frac{162\,\text{lb.}}{\text{ft.}}\frac{10\,\text{ft.}}{} = 1{,}620\,\text{lb.}$$

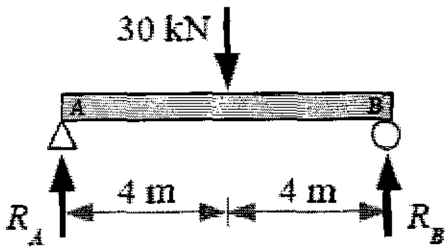
<u>Reactions for Simply-Supported Simple Beams</u>:

You can calculate the reaction forces for a symmetrically-loaded, simply-supported beam by dividing the total load by 2 if the load at the midpoint of the beam, because each end of the beam carries half the load. The reactions for the beam with a point load are *RA=RB=P/2*.



<u>Example:</u>

Calculate the reaction forces *RA* and *RB* for a beam with a 30 kN load at the mid-span.



Solution:

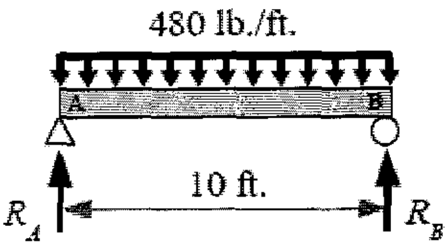
Divide the total load by 2 to obtain the reaction forces,

$$R_A = R_B = \frac{P}{2} = \frac{30\,\text{kN}}{2} = 15\,\text{kN}$$

A simply-supported beam with a uniform distributed load also has a symmetrical loading pattern. Divide the total load on the beam by 2 to find the reaction forces.

<u>Example:</u>

Calculate the reaction forces *RA* and *RB* for a 10-ft. beam with a 480 lb./ft. uniformly distributed load. Report the answer in kips.



Solution:

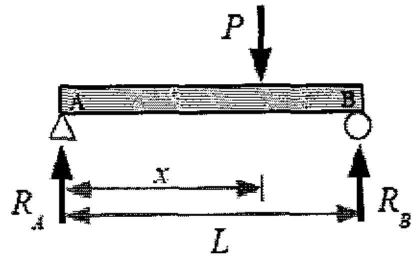
Multiply the uniform distributed load by the length to find the total load on the beam: *W = w L*

Divide the total load by 2 to obtain the reaction forces;

$$R_A = R_B = \frac{W}{2} = \frac{wL}{2} = \frac{480\,\text{lb.}}{\text{ft.}}\frac{10\,\text{ft.}}{}\frac{1}{2}\left|\frac{\text{kip}}{1000\,\text{lb.}}\right. = 2.4\,\text{kips}$$

Note: For beams with nonsymmetrical loading, we need two equations from *Statics*: the sum of the vertical forces equals zero, and the sum of the moments about a point equals zero.

You can pick a pivot point at either end of a simply supported beam.



$$\circlearrowleft\!+ \quad \sum M_A = 0 = -Px + R_B L$$

Now solve for the reaction force: $R_B = \dfrac{Px}{L}$

Use the sum of the forces in the vertical direction to calculate the other reaction force. Forces have magnitude and direction; pick upwards as positive, so:

$$\uparrow\!+ \quad \sum F_y = 0 = R_A - P + R_B$$

Now solve for the reaction force $R_A = P - R_B$

<u>Example</u>:

Calculate the reaction forces R_A and R_B for this simply-supported beam.

Solution Redraw the diagram, marking the distances to all loads and reactions from point A.

The moment about point A is $\circlearrowleft\!+ \ \sum M_A = 0 = -40\,\text{kN} \cdot 3\,\text{m} + R_B \cdot 10\,\text{m}$.

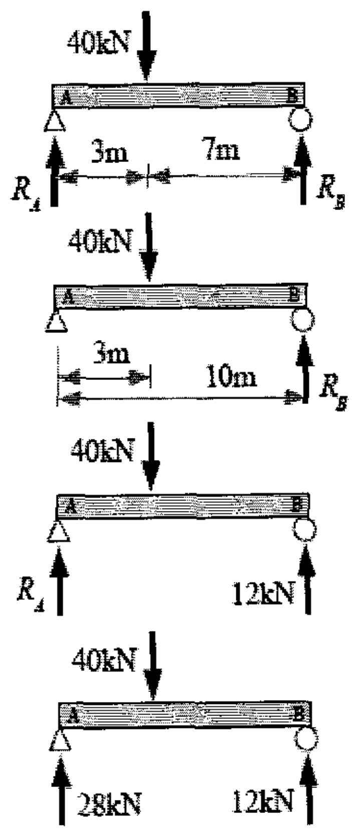
Rewrite the equation to find the reaction force $R_B = \dfrac{40\,\text{kN} \cdot 3\,\text{m}}{10\,\text{m}} = 12\,\text{kN}$.

Use the sum of the forces in the vertical direction to calculate the other reaction force:

$\uparrow\!+ \ \sum F_y = 0 = R_A - 40\,\text{kN} + 12\,\text{kN}$.

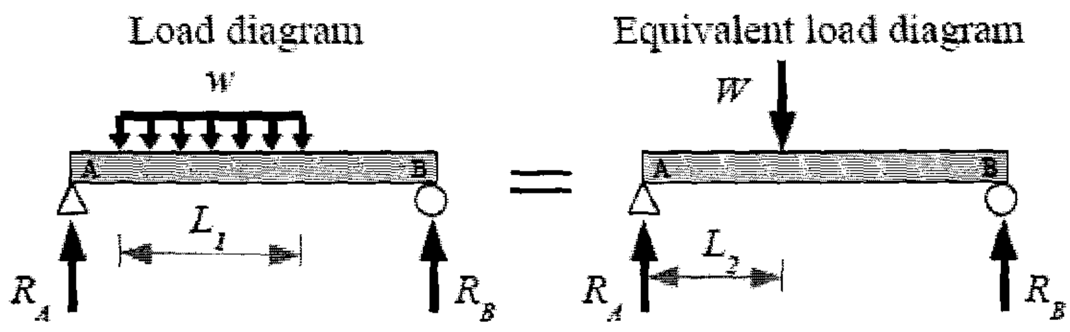
Rewrite the equation to find the reaction force $R_A = 40\,\text{kN} - 12\,\text{kN} = 28\,\text{kN}$.

You can check the answer by solving the sum of the moments about point B.

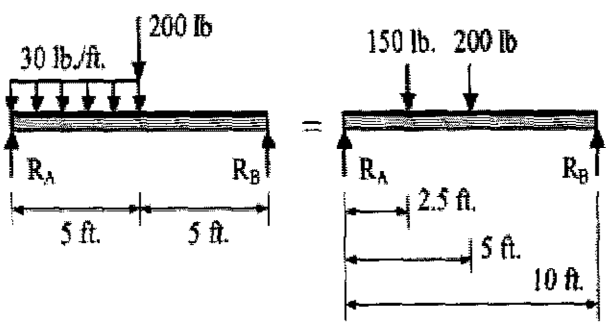


Note: Use the same technique for a simply-supported beam with multiple point loads.

If a uniformly distributed load is not symmetrical, then we need to convert the distributed load into a point load equivalent to the total load ($W = w\,L1$) where $L1$ is the length of the distributed load. The equivalent point load is located at the centroid of the distributed load...the center of the rectangle. Use the *equivalent load diagram* for calculating the reaction forces.



If the beam has a point load and a distributed load, draw an equivalent load diagram with the applied point load and the equivalent point load. The moment about point A is



$$\Sigma M_A = 0 = -150lb.\,2.5ft. - 200lb.\,5ft. + R_B\,10ft.$$

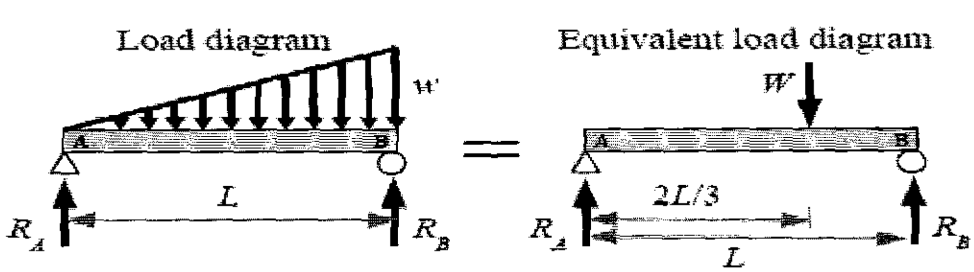
Solve for the reaction force

$$R_B = \frac{150lb.\,2.5ft. + 200lb.\,5ft.}{10ft.} = 137.5lb.$$

Sum of the forces $\Sigma F_y = 0 = R_A - 150lb. - 200lb. + 137.5lb.$

Solve for $R_A = 150lb. + 200lb. - 137.5lb. = 212.5lb.$

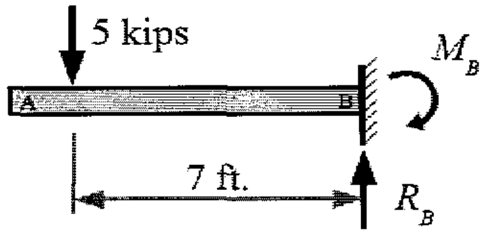
Use the same approach for a non-uniformly distributed load. Again, the location of the equivalent load is at the centroid of the distributed load. The centroid of a triangle is one third of the distance from the wide end of the triangle, so the location of the equivalent load is one third of the distance from the right end of this beam, or two thirds of the distance from the left end.



The load varies from 0 at the left end to w at the right end; therefore, the total load is the average of these loads times the beam length:

$$W = \left(\frac{0+w}{2}\right)L = \frac{wL}{2}$$

A cantilever beam with a single support has a reaction force and a reaction moment.



The reaction force *RB* equals the sum of the applied forces on the beam, so for the beam in the figure:

$R_B = P = 5$ kips.

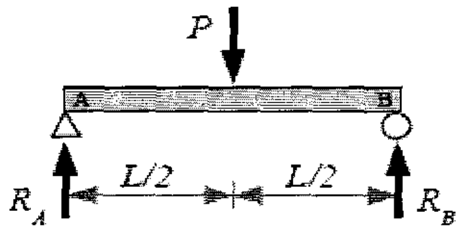
The moment reaction equals the sum of the moments about point B – the applied load times its distance from the wall – so

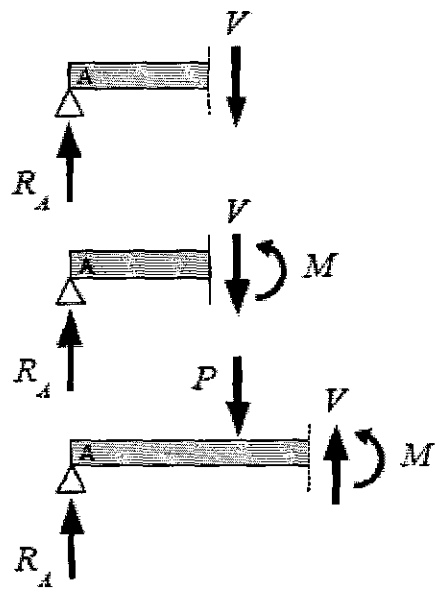
$MB = P \cdot x = 5\text{kips} \cdot 7\text{ft.} = 35$ kip ft.

Shear Diagrams

When there are forces on beams, the material resists these external loads by developing internal loads.

Imagine a simply-supported beam with a point load at the mid-span. Cut the beam to the left of the point load, and draw a free-body diagram of the beam segment. In a free body diagram, forces must balance. Therefore, a downward force at the cut edge balances the support reaction *RA*. We call this shear force *V*. It is a shear force because the force acts parallel to a surface (the cut edge of the beam).

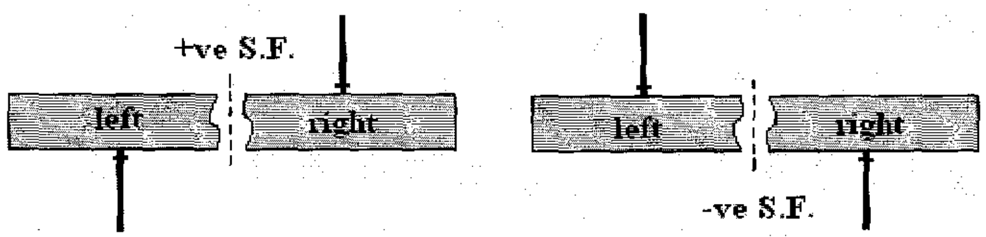




The forces *RA* and *V* are in balance (equal in value; opposite in sign), but our segment wants to spin clockwise about point A. To counteract this tendency to spin, a moment *M* develops within the beam to prevent this rotation. The moment equals the shear force times its distance from point A.

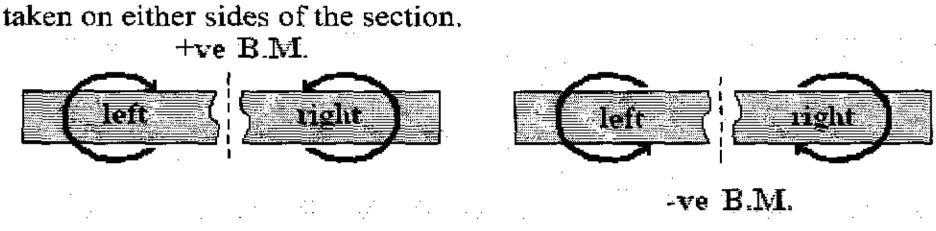
Cut the beam to the right of the point load, and draw the free-body diagram. Since *P* is larger than *RA*, force *V* points upwards.

Shearing Force (S.F.)
Shearing force at the section is defined as the algebraic sum of the forces taken on one side of the section.



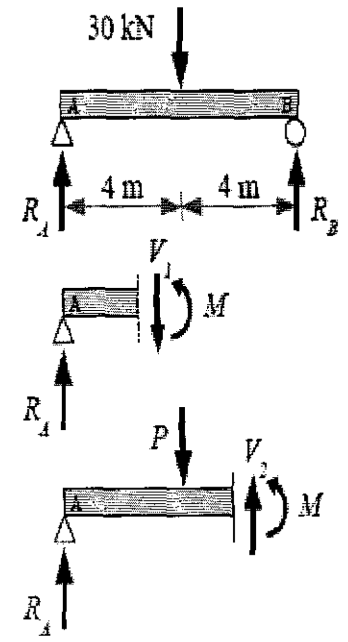
i.e. If the tendency of the section to the left of the cut is to move upward, the shear is positive; if it has a tendency to move down, it is negative.

Bending Moment (B.M)
Bending moment is defined as the algebraic sum of the moments of the forces about the section, taken on either sides of the section.



Example:

Calculate the shear forces in this beam to the left and to the right of the 30 kN point load.



Solution The loading is symmetrical, so $R_A = R_B = \frac{P}{2} = \frac{30\,\text{kN}}{2} = 15\,\text{kN}$.

Use the sum of the forces to find V.

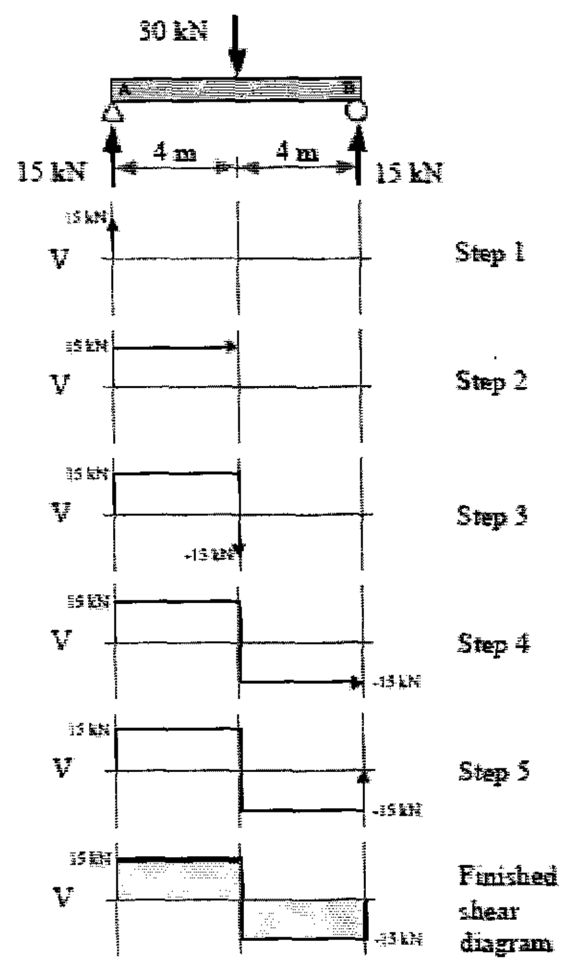
Between support A and point load P, $\uparrow +\ \sum F_y = 0 = R_A - V_1$.

Solving for shear load, $V_1 = -R_A = -15\,\text{kN}$.

Between point load P and support B, $\uparrow +\ \sum F_y = 0 = R_A - P + V_2$.

Solving for shear load, $V_2 = -R_A + P = -15\,\text{kN} + 30\,\text{kN} = 15\,\text{kN}$.

We can sketch V as a function of location along the beam using a *Shear Diagram*. Draw vertical construction lines below the load diagram wherever the applied loads and reactions occur. Draw a horizontal construction line, indicating zero shear load. Next, draw the value of V along the length of the beam, as follows:



Step 1 Starting at the left side of the shear diagram, go up 15 kN, because *RA* is 15 kN upwards.

Step 2 There are no additional loads on the beam until you get to the midspan, so the shear value remains at 15 kN.

Step 3 The applied load at the midspan is 30 kN downwards, therefore the shear load is:

$15\text{ kN} - 30\text{ kN} = -15\text{ kN}$.

Step 4 There are no additional loads on the beam until you get to point B, so the shear value remains at -15 kN.

Step 5 At point B, the reaction force *RB* = 15 kN upwards, therefore the shear load is

$-15\text{ kN} + 15\text{ kN} = 0$.

If you don't get to 0, you know you made a mistake someplace.

Finish the shear diagram by shading the areas between your line and the horizontal zero shear line. Mark all significant points (anywhere the shear line changes direction). In the next chapter, we will use the maximum absolute value of shear load, $|V|max$, *to calculate the maximum* shear stress in the beam.

A point load at the midspan of a simply-supported beam produces identical reaction forces and a symmetric shear diagram with two rectangles. If the point load is not at the midspan, use sum of the moments an sum of the forces to calculate the reaction forces. Draw vertical construction lines below the applied loads and reaction forces, draw a horizontal line at zero shear, then draw the shear value along the length of the beam.

With a shear diagram, we can identify the location and size of the largest shear load in a beam. Therefore, we know the location of the largest shear stress, and we can calculate the value of this stress. Once we know the actual stress in the material, we can compare this values with the shear strength of the material, and we can know whether the beam will fail in shear. Shear diagrams are necessary for drawing bending moment diagrams ("moment diagrams", for short), which we can use to identify the location and size of bending stresses that develop within beams. We can compare the actual bending stresses with the yield strength of the material, and we can know whether the beam will fail in bending.

A uniformly distributed load is like an infinite number of small point loads along the length of the beam, so the shear diagram is like a stepped multiple point load shear diagram with infinitely small steps.

Bending Moment Diagrams

The amount of a moment of force acting perpendicular to a beam about a point along the beam is the multiplication of this force by the distance from that point to the force, so the units are force × distance: lb.·ft. or N·m, We can graph the value of the bending moment along a beam by drawing a moment diagram.

To draw a moment diagram, sketch the value of the moment produced by the shear force V times the distance from the left end of the beam. At the first meter,

$$\text{Moment} = \text{Force} \times \text{distance},$$
$$= V \times x$$

$V = 15$ kN, so moment M_1 is:

$M_1 = 15\ kN \times 1m = 15\ kN{\cdot}m$

At 2 m, $M_2 = 15\ KN \times 2\ m = 30\ KN$

At 3 m, $M_3 = 15\ KN \times 3\ m = 45\ KN$

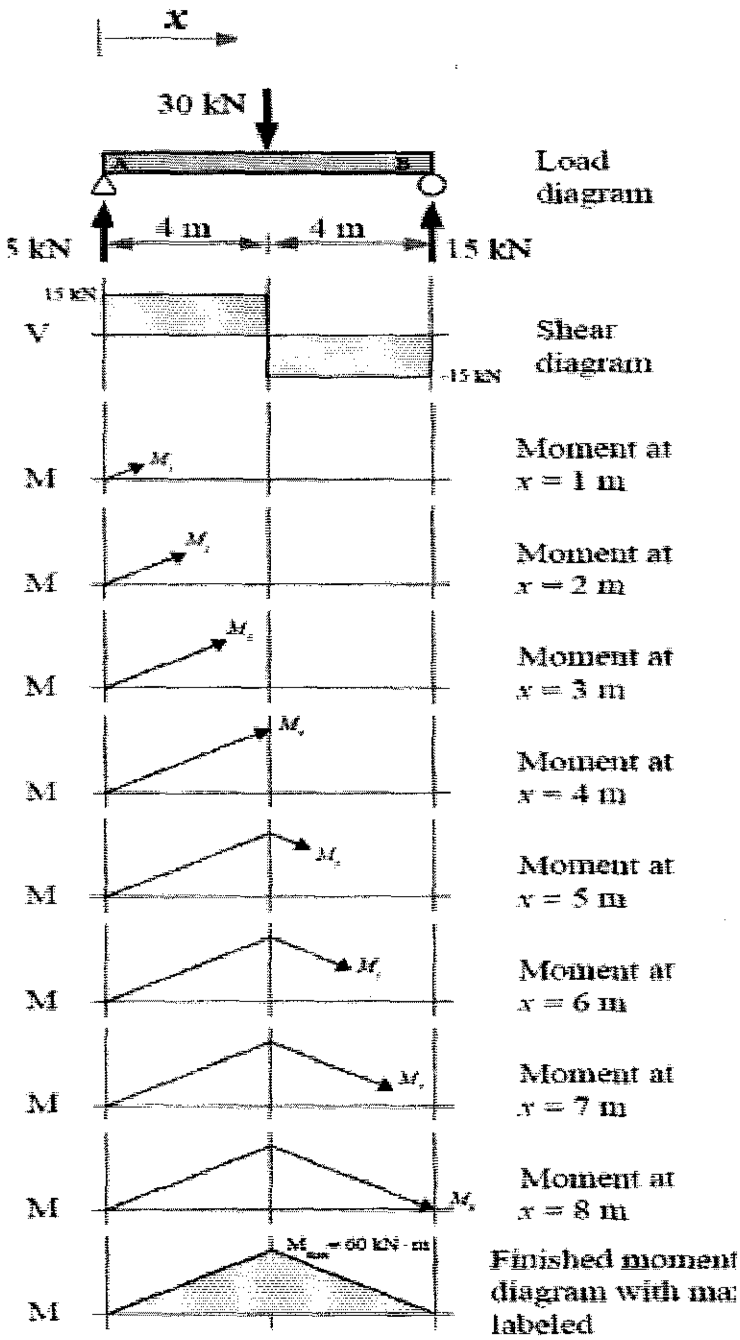
At 4 m, $M_4 = 15\ KN \times 4\ m = 60\ KN$

At 5 m, $M_5 = 15\ KN \times 5\ m - 30 \times 1m = 45\ KN$

At 6 m, $M_6 = 15\ KN \times 6\ m - 30 \times 2m = 30\ KN$

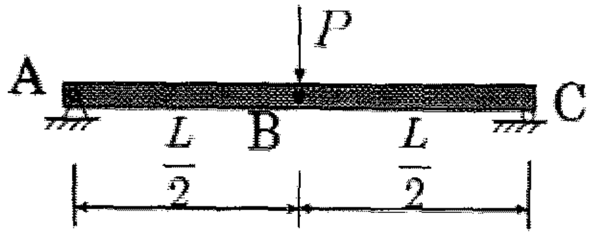
At 7 m, $M_7 = 15\ KN \times 7\ m - 30 \times 3m = 15\ KN$

$M_{max} = 15$ kN × 4m = 60kN·m

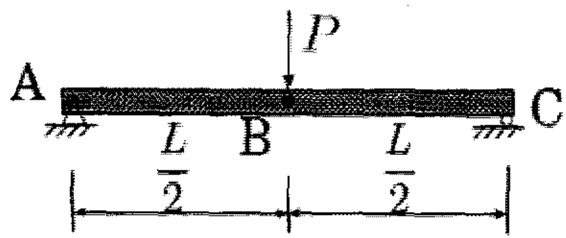


Example:,

Find the reactions at *A* & *C*, draw the shear force and bending moment diagram for the beam for simply supported beam + concentrated force as shown in the following figure:

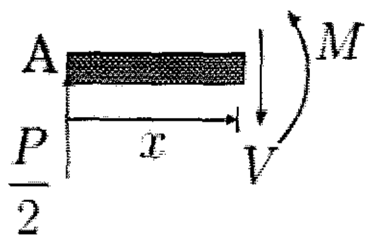


Solution:



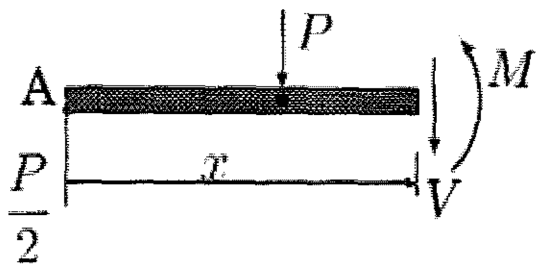
1. **support reactions:** $R_A = R_C = \frac{P}{2}$
2. **FBDs for regions AB and BC**
3. **Equilibrium:**

for AB segment:



$$\sum F_y = 0 :\longrightarrow \boxed{V = \frac{P}{2}}$$

$$\sum M = 0 :\longrightarrow \boxed{M = \frac{P}{2}x}$$



for BC segment:

$$\sum F_y = 0 : \frac{P}{2} - P - V = 0$$

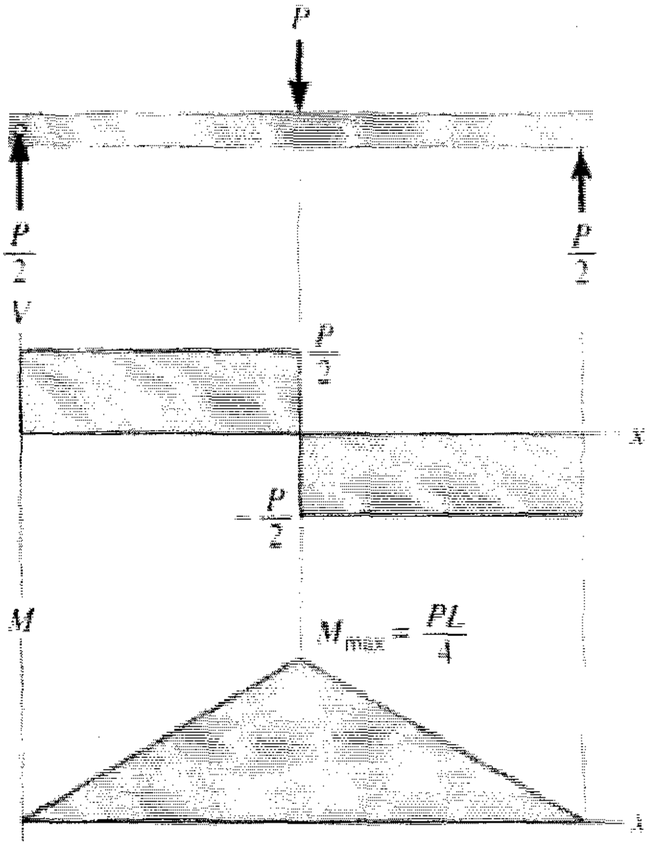
$$\longrightarrow \boxed{V = -\frac{P}{2}}$$

$$\sum M = 0 : M + P(x - \frac{L}{2}) - \frac{P}{2}x = 0$$

$$\longrightarrow \boxed{M = \frac{P}{2}(L - x)}$$

$$At: \frac{L}{2}, \quad M = \frac{P}{2}\left(L - \frac{L}{2}\right)$$

$$M = \frac{PL}{4}$$



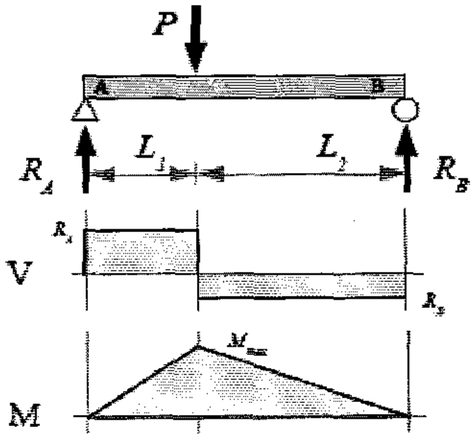
shear force diagram
(V-diagram)

bending-moment diagram
(M-diagram)

shear and moment diagrams

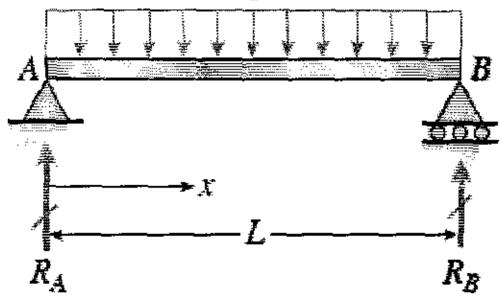
If the point load is not at the midspan, then the maximum moment will also be offset. In this example, the maximum moment is the area of the shear diagram up to the point load:

$$M_{max} = RA \cdot L1$$



The reactions, shear force and bending moment diagram for beam for simply supported beam + the distributed load

q (q: is the weight / unit volume)



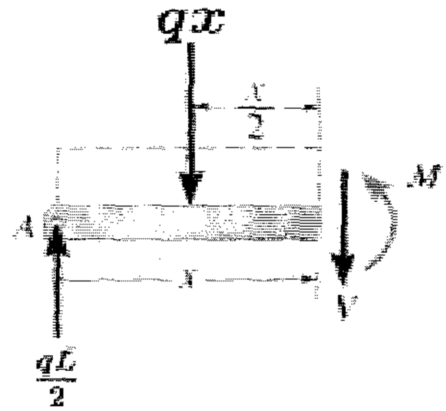
$$R_A = R_B = \frac{qL}{2}$$

$$\sum F_y = 0 : \frac{qL}{2} - qx - V = 0$$

$$\longrightarrow \boxed{V = q(\frac{L}{2} - x)}$$

$$\sum M = 0 : -\left(\frac{qL}{2}\right)x + (qx)\left(\frac{x}{2}\right) + M = 0$$

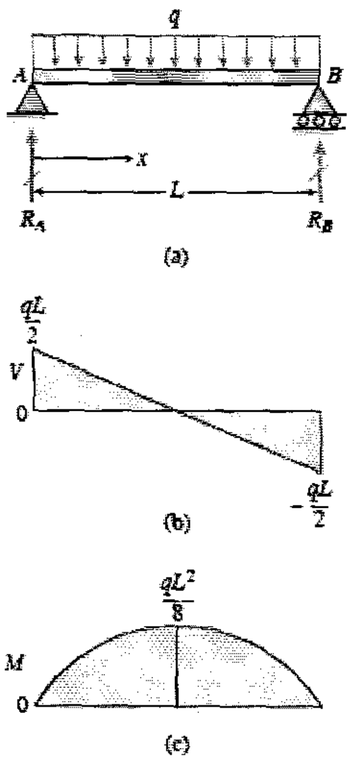
$$\longrightarrow \boxed{M = \frac{q}{2}(Lx - x^2)}$$



Observation:

$$\frac{dM}{dx} = 0 \longrightarrow x_m = \frac{L}{2},\ M_{\max} = M|_{x_m} = \frac{wL^2}{8}$$

Check $V|_{x_m} = 0$. **This is not a coincidence!**



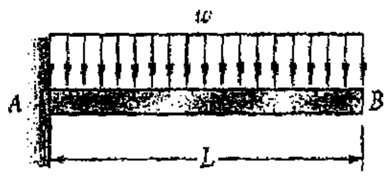
$q(x)$: **const**

$V(x)$: **linear**

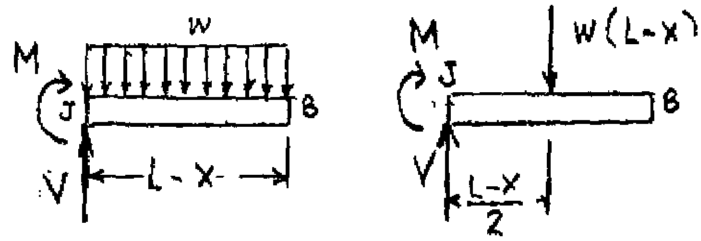
$M(x)$: **parabolic**

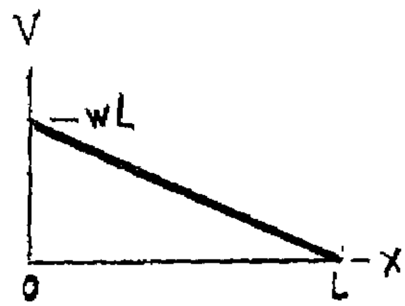
shear and moment diagrams

Example: The reactions, shear force and bending moment diagram for cantilever beam



Use portion to the right of the section as the free body.





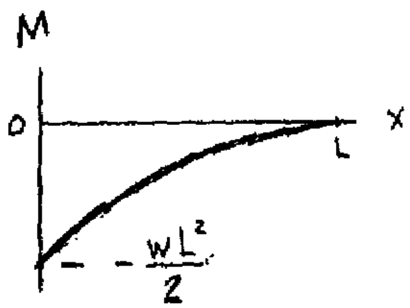
Replace distributed load by equivalent concentred load.

$+\uparrow \Sigma F_y = 0 \qquad V - w(L-x) = 0$

$$V = w(L-x)$$

$\circlearrowleft + \Sigma M_J = 0 \qquad -M - w(L-x)\left(\frac{L-x}{2}\right) = 0$

$$M = -\frac{w}{2}(L-x)^2$$



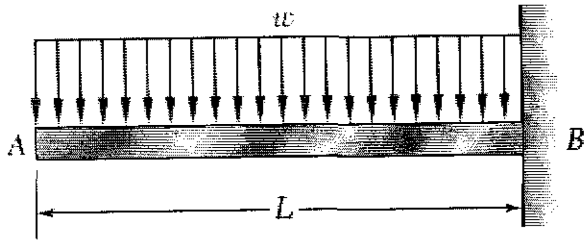
Largest negative bending moment occurs at $x = 0$.

$$M_{min} = -\frac{wL^2}{2}$$

Thus, $|M|_{max} = \frac{wL^2}{2}$

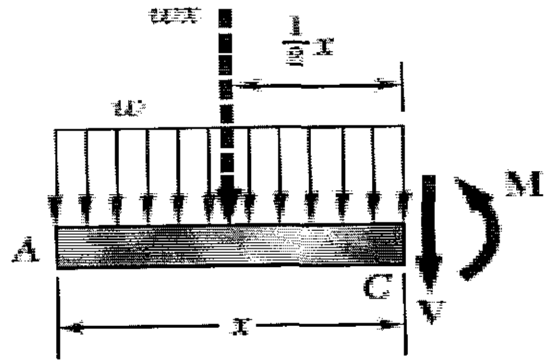
cantilever beam supporting a uniformly distributed load:

Draw the shear and bending-moment diagrams for a cantilever beam *AB* of span *L* supporting a uniformly distributed load.



We cut the beam at a point *C* between *A* and *B* and draw the free-body diagram of *AC* (Fig. *a*), directing **V** and **M** as indicated in Fig. *a*.

Denoting by *x* the distance from *A* to *C* and replacing the distributed load over *AC* by its resultant **wx** applied at the midpoint of *AC*, we write:

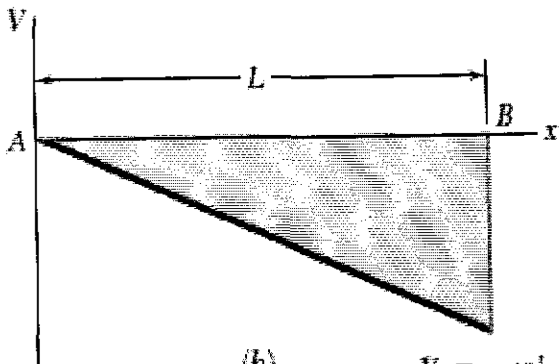


(*a*)

$$+\uparrow\Sigma F_y = 0: \qquad -wx - V = 0 \qquad V = -wx$$

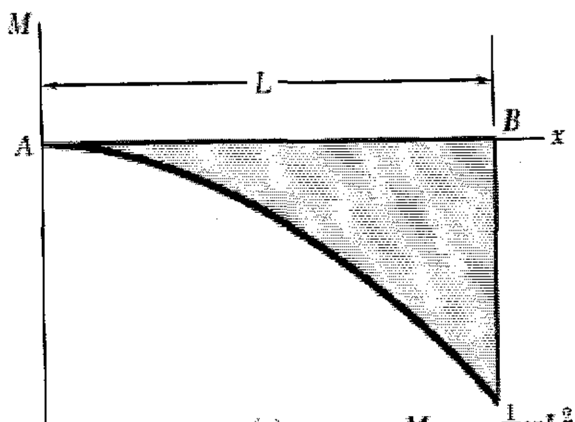
$$+\circlearrowleft\Sigma M_C = 0: \qquad wx\left(\frac{x}{2}\right) + M = 0 \qquad M = -\frac{1}{2}wx^2$$

We note that the shear diagram is represented by an oblique straight line (Fig. *b*) and the bending-moment diagram by a parabola (Fig. *c*). The maximum values of *V* and *M* both occur at *B*, where we have:



(*b*) $V_B = -wL$

$$V_B = -wL \qquad M_B = -\tfrac{1}{2}wL^2$$



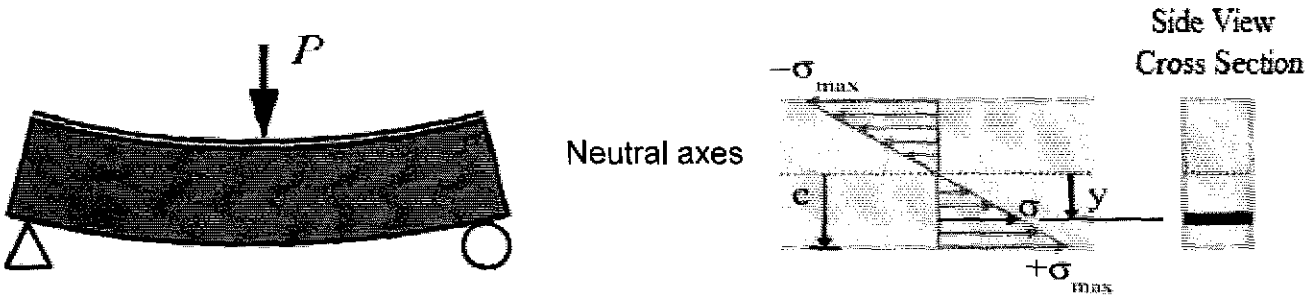
(*c*) $M_B = -\frac{1}{2}wL^2$

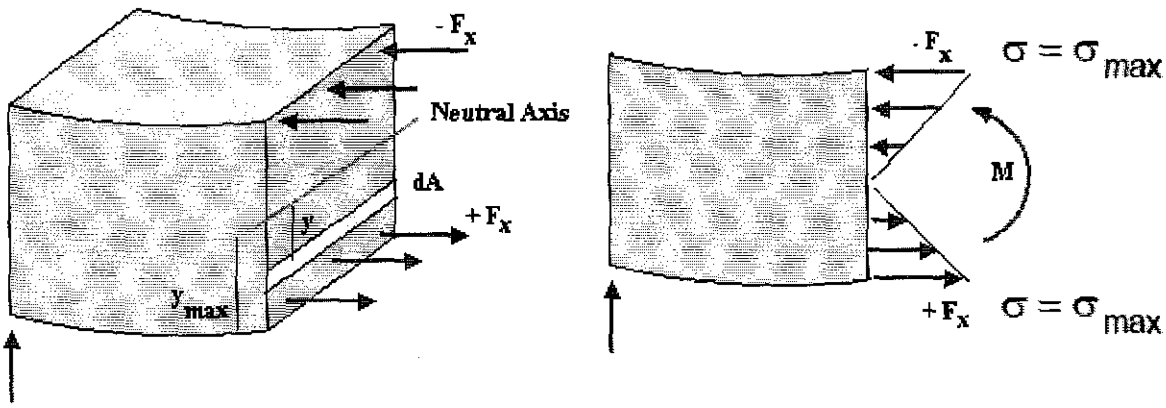
Bending & Shear Stresses on Beams

Beams are almost always designed on the basis of bending stress and, to a lesser degree, shear stress. Each of these stresses will be discussed in detail as follows.

A) Bending Stresses:

Bending stress is distributed through a beam as seen in the diagrams below:





So, in reality, bending stresses are tensile or compressive stresses in the beam. A simply-supported beam always has tensile stresses at the bottom of the beam and compressive stresses at the top of the beam.

A bending stress is NOT considered to be a simple stress. In other words, it is not load divided by area.

The formula for bending stress, σ_b, is: $$\sigma_b = \frac{M_b \cdot y}{I}$$

The maximum bending stress is at the maximum bending distance ($y_{max.}$), which is as the following:

$$\sigma_{b\,max.} = \frac{M_b \cdot y_{max.}}{I}$$

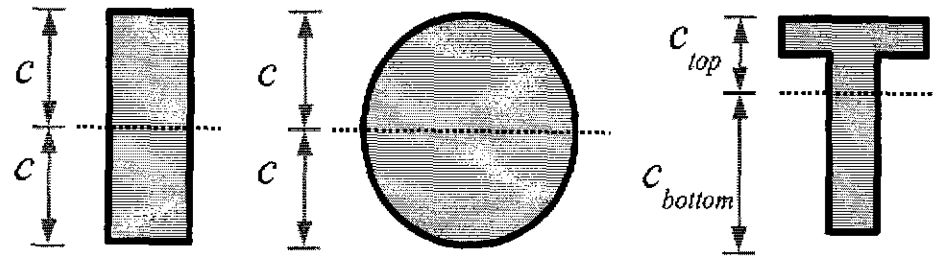
Where: M_b : moment acting on beam from moment diagram (kip-in or lb-in)

$\sigma_{b\,max.}$: Maximum bending stress

$y_{max.}$: Distance from neutral axis to extreme edge of member to the outer edge (in)

I : Moment of inertia about the axis (in4), For the standard cross section areas the maximum bending stress will be:

$$\sigma_{b\ max.} = \frac{M_b \cdot C}{I}$$



Where: C (which is equal to $y_{max.}$) the distance from neutral axis to extreme edge of member (in).

Recalling that; $S = \dfrac{I}{y}$

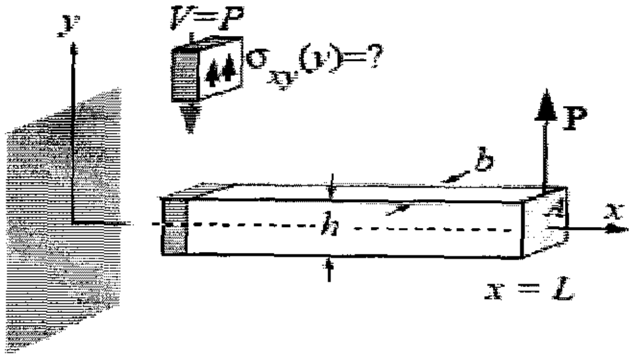
So, the bending stress formula could be re-written as: $\sigma_b = \dfrac{M_b}{S}$, and for designing: $S = \dfrac{M_b}{\sigma_{b\ all.}}$

Where S: is the section modulus about the axis (in^3). Its describing the cross section of the beam, which is used in listing beams in handbooks for beam design and selecting according to the bending moments.

For rectangle cross section beam:

$$\sigma_{b\ max.} = \frac{PL(h/2)}{I}$$

Where: $I = \dfrac{bh^3}{12}$

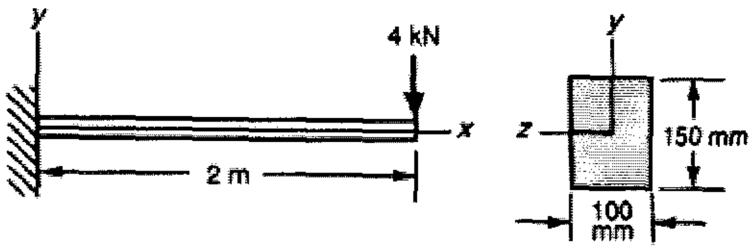


So, the maximum bending stress due to bending for rectangle cross section is then:

$$\sigma_{b\ max.} = 6\frac{PL}{(bh^2)}$$

Example:

A 100 mm × 150 mm wooden cantilever beam is 2 m long. It is loaded at its tip with a 4-kN load. Find the maximum bending stress in the beam shown in the figure The maximum bending moment occurs at the wall and is $M_{max} = 8$ kN • m.



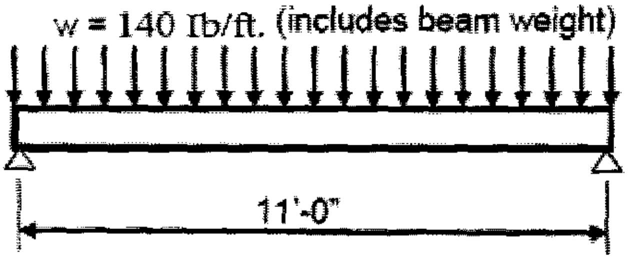
Solution

$$I = \frac{bh^3}{12} = \frac{100(150)^3}{12} = 28.1 \times 10^6 \text{ mm}^4$$

$$\sigma_{b\ max} = \frac{|M|_{max} c}{I} = \frac{(8 \text{ kN} \bullet \text{m})(75 \text{ mm})}{28.1 \times 10^6 \text{ mm}^4} = 21.3 \text{ MPa}$$

Example:

Given a beam with section modulus 21.39 in^3 simply-supported beam with loading as shown in the figure, the allowable bending stress is 1200 psi.



REQUIRED:

a) Determine the maximum moment on the beam.

b) Determine the maximum bending stress on the beam

c) Determine if the beam is acceptable based upon allowable bending stress.

Solution:

The maximum bending moment $M_{max.}$ on a simply-supported, uniformly loaded beam is:

$$M_{max} = \frac{wL^2}{8}$$

$$M_{max} = \frac{(140 \text{ Ib/ft.})(11 \text{ ft.})^2}{8}$$

$$\underline{\mathbf{M_{max} = 2117.5 \ lb\text{-}ft}}$$

The bending stress is:

$$\sigma_b = \frac{M}{S}$$

$$\sigma_b = \frac{2117.5 \text{ lb-ft}(12''/\text{ft})}{21.39 in^3}$$

$\underline{\mathbf{\sigma_b = 1187.9 \ PSI}}$

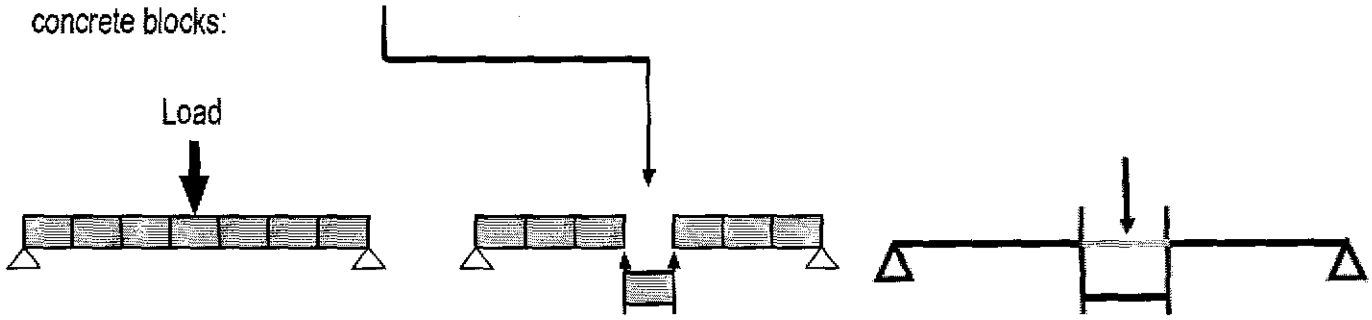
Since the actual bending stress of 1187.9 PSI is less than the allowable bending stress of 1200 PSI, **THE BEAM IS ACCEPTABLE**.

B) Shear Stress:

When a beam is subjected to non-uniform bending, both bending moments, M, and shear forces, V, act on the cross section.

There are two types of stresses **vertical** and **horizontal** shear stresses. Vertical stress causes a simple beam to shear (break).

It is easy to imagine the **vertical shear** on a beam like that was made up of concrete blocks:

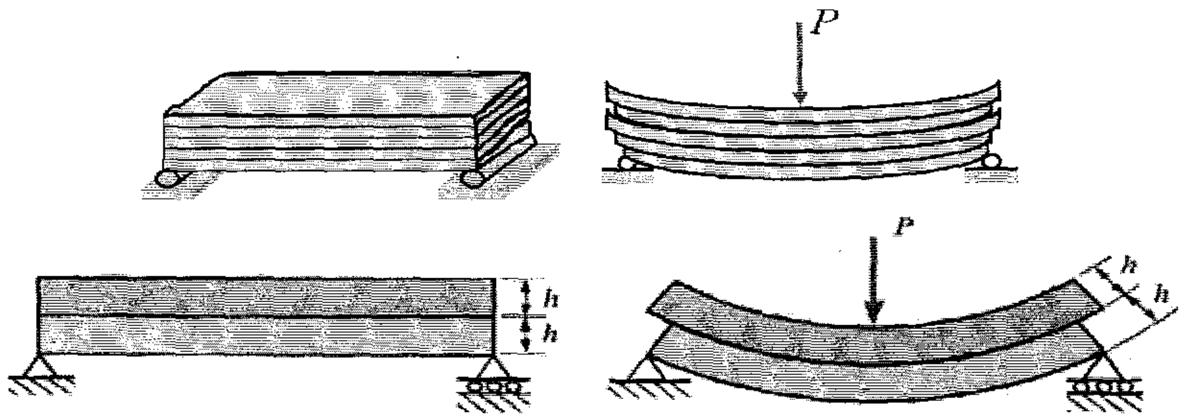


vertical shear stresses

This type of shear occurs if there are **no bending stresses present, which is a seldom case.** To calculate the vertical shear stresses (***Tv***) we divide the shear force, *V* (Kip) by the Area of cross section; A (in^2).

$$\tau_v = \frac{V}{A}$$

However, almost all real beams have bending stresses present. In this case, beams are more like a deck of cards and bending produces sliding along the horizontal planes at the interfaces of the cards as shown below:



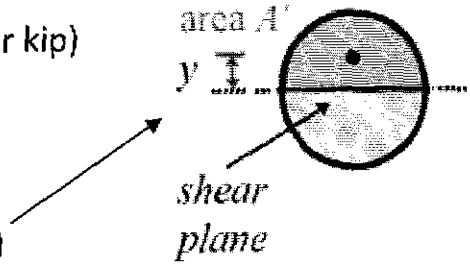
horizontal shear stresses

This type of shear is called "longitudinal" or **horizontal shear**. If the shear stress *t* is assumed to be *uniform* over the thickness *b* then the formula used for determining the maximum horizontal shear stress ($\tau_{H\,max.}$) is:

$$\tau_{H_{max}} = \frac{VQ}{Ib}$$

Where:

V = vertical shear force, usually from shear diagram (lb. or kip)
Q = first moment of the section area = $A \cdot y$
A = area of shape above or below the neutral axis (in^2)



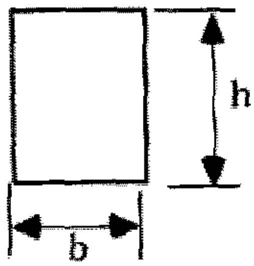
y = distance from neutral axis to centroid of area "A" (in)
I = moment of inertia of shape (in^4)
b = width of section, i.e. width of area "A" (in)

For the rectangular section the maximum shear stress is obtained as follows:

$$Q = \left(\frac{bh}{2}\right)\left(\frac{h}{4}\right) = \frac{bh^2}{8}$$

$$I = \frac{bh^3}{12}$$

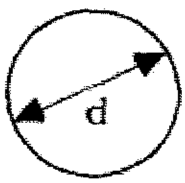
$$\tau_{H_{max}} = \frac{3V}{2A}$$



$$\tau_{H_{max}} = \frac{3V}{2A}$$

A=bh

In the same way we can find the maximum horizontal shear stress in beams $\tau_{H\,max.}$ for circular cross section area, and the formula is:

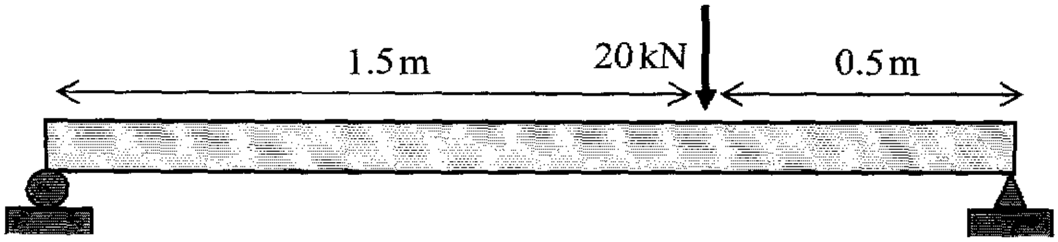


$$\tau_{H_{max}} = \frac{16V}{3\pi d^2}$$

Example:

Consider the simply supported beam loaded by a concentrated force shown in the Fig. The cross-section is rectangular with height $100\,\text{mm}$ and width $50\,\text{mm}$. The reactions at the supports are $5\,\text{kN}$ and $15\,\text{kN}$. To the left of the load, one has $V = 5\,\text{kN}$.
To the right of the load, one has $V = -15\,\text{kN}$

The maximum shear stress will occur along the neutral axis and will clearly occur where V is largest, so anywhere to the right of the load:



$$\tau_{max} = \frac{3V_{max}}{2A} = \frac{3 \times 15000}{2 \times \frac{100}{1000} \times \frac{50}{1000}} = 4.5\,\text{MPa}$$

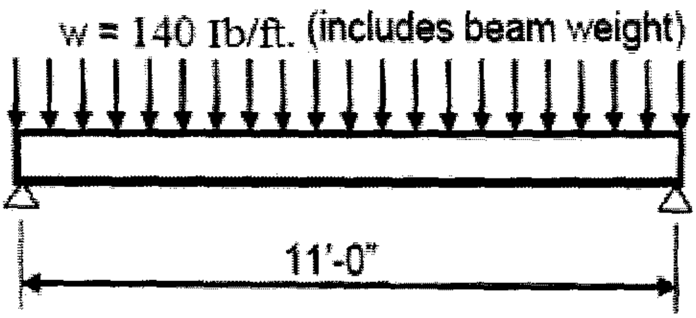
Example:

Given a 1.5 x 9.25 cross sectional wood beam with w = 140 lb/ft simply distributed beam as shown below. Given the allowable horizontal shear stress is 95 psi. Assume Moment of inertia, I = 98.93 in^4.

1) What is the maximum horizontal shear stress on the beam?

(A) 80.4 lb/in2
(B) 83.2 lb/in2
(C) 88.5 lb/in2
(D) 78.6 lb/in2

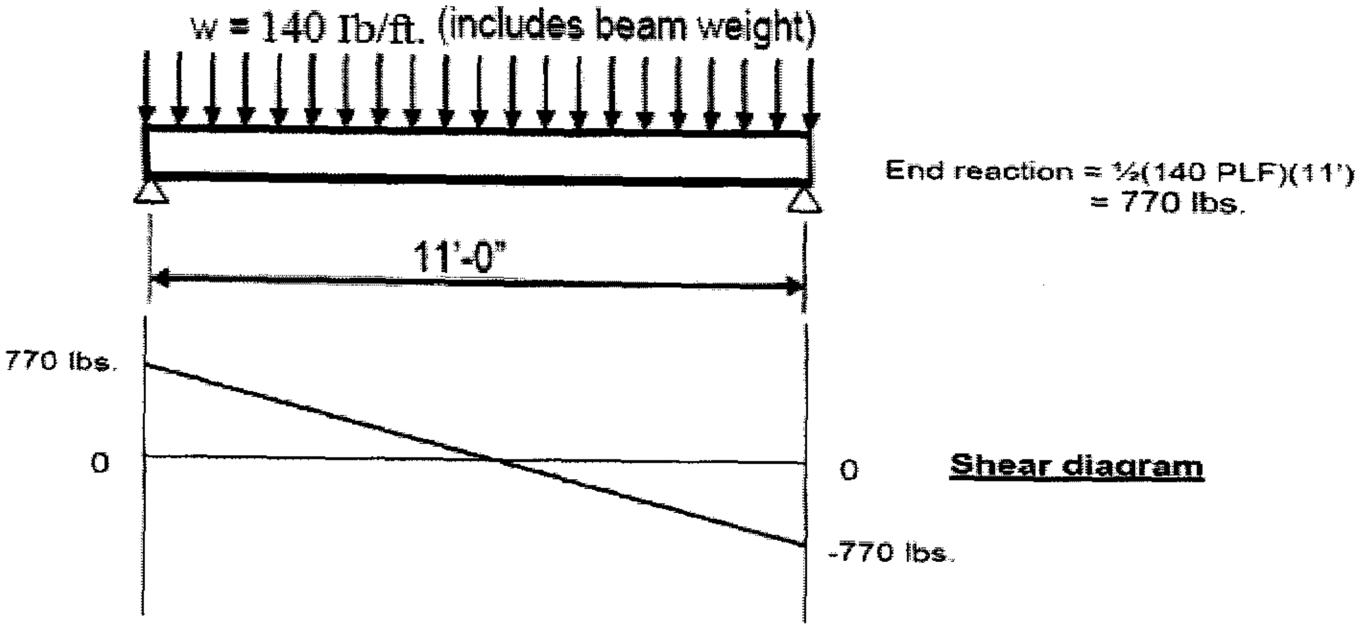
2) Determine if the beam is acceptable based upon allowable horizontal shear stress.



Solution:

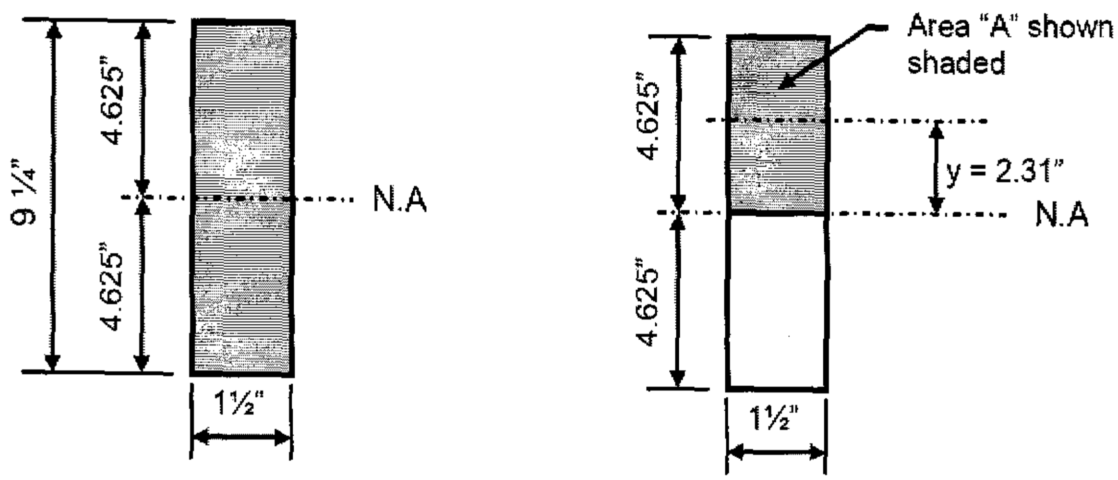
(1)

Step 1: Draw the shear diagram and calculate the Maximum shear.



F = 0.5 (140 LB/FT) (11FT) = 770 lbs

Step 2:



Beam X-Section

$$\tau_{H_{max}} = \frac{VQ}{Ib}$$

And substituting in the values as flowing:

V = 770 lbs. (from shear diagram)
Q = Ay

where: A = (4.625")(1.5")
= 6.94 in^2

y = ½(4.625")
= 2.31"

Q = (6.94 in^2)(2.31")
= 16.03 in^3

I = 98.93 in^4 (from textbook appendix, or calculate $I = \frac{bh^3}{12}$)

$$\tau_{H\,max.} = \frac{(770 lbs)(16.03 in^3)}{(98.93 in^4)(1.5")} = 83.2\ PSI$$

Answer is B

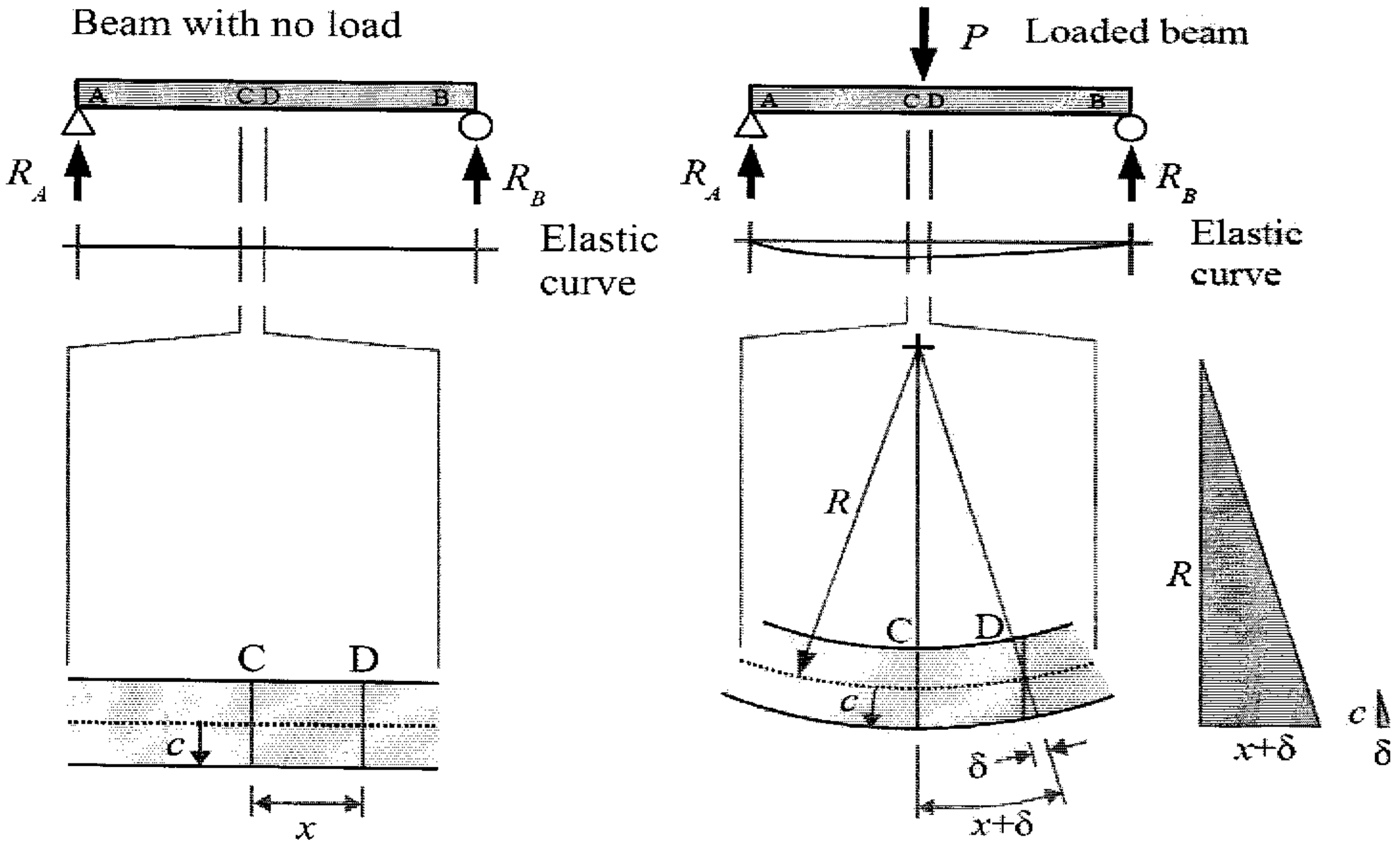
(2)

Since shear stress $\tau_{H\ max}$ = 83 psi is less than τ_H allowable = 95 psi then the beam is **acceptable**.

Deformation in Beams

Radius of Curvature Method

A weightless beam with no loads has a horizontal shape. If we add a load, the beam deflects, and the elastic curve bows downward with a *radius of curvature*, labeled *R* on the diagram.



The relationship between bending moment and curvature for pure bending remains valid for general transverse loadings, which is:

$$R = \frac{E\,I}{M} \quad \dots (7\text{-}1)$$

This equation helps us calculate the radius of curvature at a given point along the length of a beam.

Where: R: is the radius of the curvature

C: The maximum distance from the natural axes to the beam edge of the cross section area

M: Bending moment

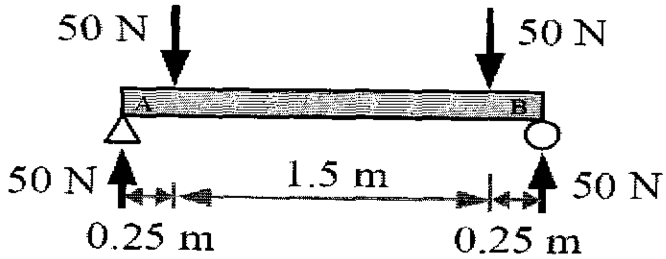
E: Young Modulus

I: Moment of inertia about the natural axis

δ: Deformation

Example:

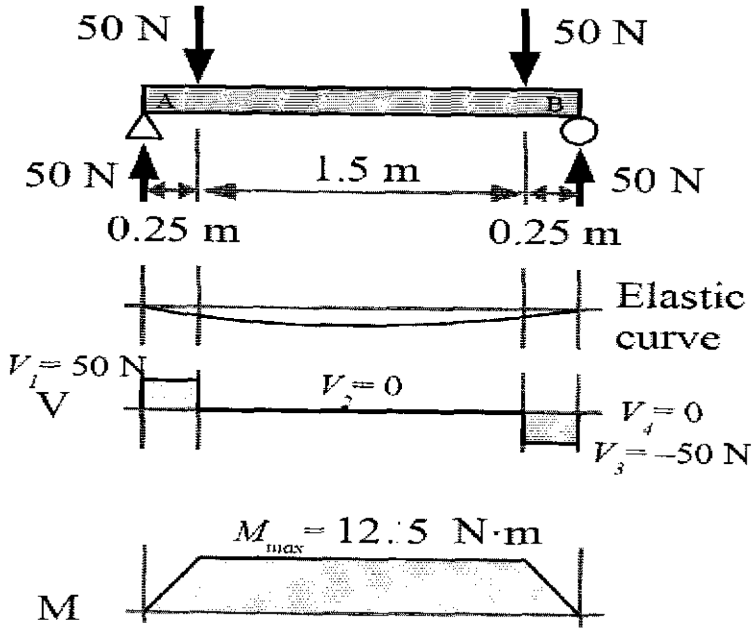
A 2 m long titanium beam is made of 1 cm square bar. Two 50 N point loads lie 0.25 meters from each end. What is the radius of curvature of the beam between the two point loads? Report the answer in meters.



Solution:

Draw the elastic curve, shear diagram, and moment diagram.

The loading is symmetrical, so the reaction forces *RA* = *RB* = 50 N.



The value of the moment diagram between the point loads equals the area of the left-hand rectangle in the shear diagram:

$M_{max.}$ = 50 * 0.25 = 12.5 N.m

The moment of inertia of a square: $I = \dfrac{b\,h^3}{12} = \dfrac{b^4}{12}$

The radius of curvature of the beam between the two point loads is:

$$R = \frac{E\,I}{M} \qquad = \frac{E\,b^4}{12\,M}$$

$$= \frac{114\ \text{GPa}\,(1\ \text{cm})^4}{12 \cdot 12.5\ \text{N}\cdot\text{m}} \left| \frac{10^9\ \text{N}}{\text{GPa m}^2} \right| \frac{\text{m}^4}{(100\,\text{cm})^4} = 7.6\ \text{m}$$

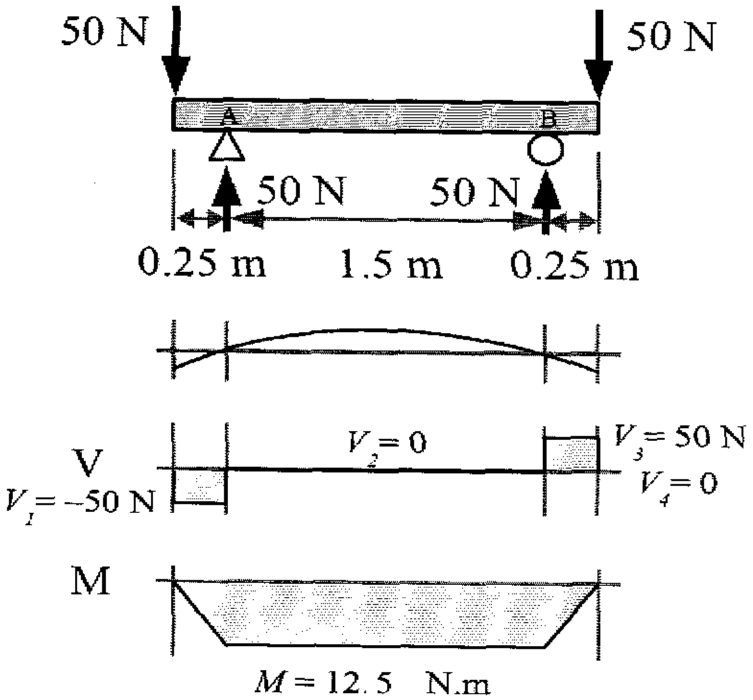
Example:

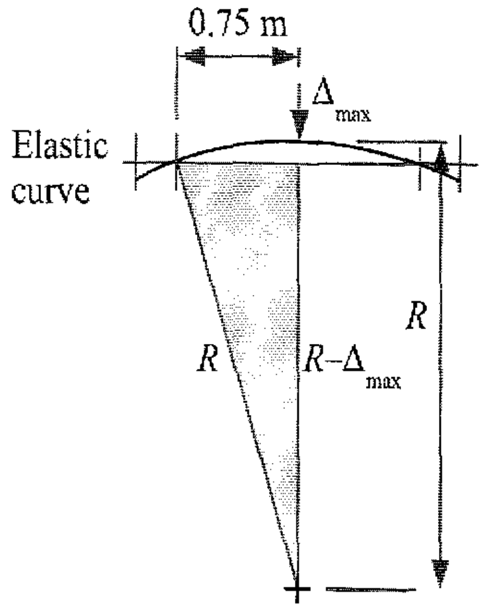
A 2 m long titanium beam is made of 1 cm square bar. Two 50 N point loads lie at the ends, and the supports lie 0.25 meters from each end. What is the maximum deflection at the mid-span? Report the answer in cm.

Solution

This problem is like the previous example, but flipped upside down.

Draw the elastic curve with its radius of curvature and the center of curvature (cross at the bottom of the figure).





The distance from the center of curvature to the beam is R;

The beam deflects a distance (**Δ_{max}**) at the mid-span;

And the distance from the center of curvature to the undeflected beam is (**$R - \Delta_{max}$**).

The base of the shaded triangle is half the distance between the supports, 0.75 m. Use the Pythagorean Theorem to find the height of the triangle,

$$R-\Delta_{max}=\sqrt{R^2-(0.75\,\text{m})^2}=\sqrt{(7.6\text{ m})^2-(0.75\,\text{m})^2}=7.562\,\text{m}$$

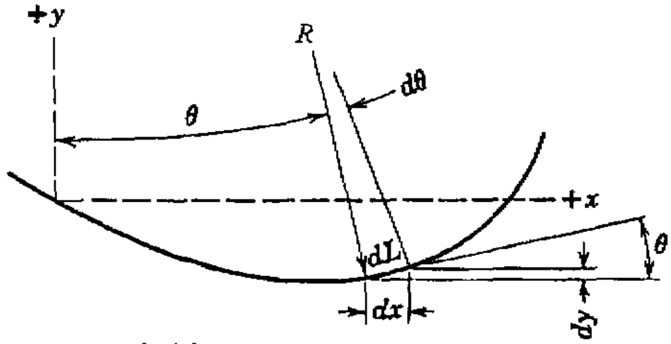
Deflection at the mid-span:

$$\Delta_{max}=R-(R-\Delta_{max})=\frac{7.6\text{m} - 7.562\text{ m}}{}\left|\frac{100\,\text{cm}}{\text{m}}\right.=3.8\text{ cm}$$

Integration Method:

For any type of material, provided the displacement is small, it can be shown that the radius of curvature R is related to the second derivative d^2y / dx^2;

$$\frac{1}{R}=\frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{3/2}}$$



For actual beams can be simplified because the slope dy/dx (which can be considered the first derivate) is small and its square is even smaller and can be neglected as a higher order term. Thus, with these simplifications, it will be:

$$\frac{1}{R}=\frac{d^2y}{dx^2}$$

Substituting this in eq. (7-1), we get: $\frac{d^2y}{dx^2}=\frac{M}{EI}$ or: $EI\frac{d^2y}{dx^2}=M$

With the moment known, this differential equation can be integrated twice to obtain the deflection. Boundary conditions must be supplied to obtain constants of integration.

By the first integration we can get the slop (dy / dx):

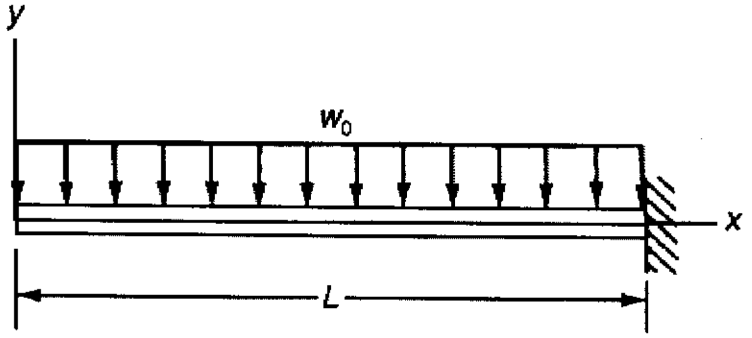
$$\frac{dy}{dx}=\int\frac{M}{EI}dx+C_1$$

Integrating twice with respect to x gives:

$$y=\iint\frac{M}{EI}dx+C_1x+C_2$$

Example:

The uniform cantilever beam shown in the figure has a constant, uniform, downward load W_0 along its length (L). Determine the deflection and slope equations of this beam using the integration method.

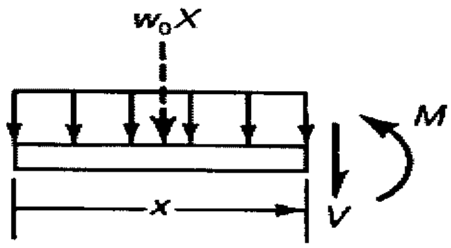


(a)

Solution:

The moment is found by drawing the free-body diagram shown in (b). The uniform load is replaced with the statically equivalent load *w0x* at the position *(x/2)*. Moments are then summed about the cut giving:

$$M = -w_0 \frac{x^2}{2}$$



(b)

Integrating twice with respect to *x*,

$$\frac{dy}{dx} = \int \frac{M}{EI} dx + C_1 = \frac{1}{EI} \int \left(-w_0 \frac{x^2}{2} \right) dx + C_1 = -\frac{1}{6} \frac{w_0 x^3}{EI} + C_1$$

$$y = \int \left(-\frac{1}{6} \frac{w_0 x^3}{EI} \right) dx + C_1 x + C_2 = -\frac{1}{24} \frac{w_0 x^4}{EI} + C_1 x + C_2$$

At $x = L$ the displacement and slope must be zero so that:

$$y(L) = 0 = -\frac{1}{24} \frac{w_0 L^4}{EI} + C_1 L + C_2$$

$$\frac{dy}{dx}(L) = 0 = -\frac{1}{6} \frac{w_0 L^3}{EI} + C_1$$

Therefore;

$$C_1 = \frac{1}{6}\frac{w_0 L^3}{EI}; \qquad C_2 = -\frac{1}{8}\frac{w_0 L^4}{EI}$$

Inserting *C*1 and *C*2 into the previous expressions gives:

$$y = -\frac{w_0}{24EI}(x^4 - 4xL^3 + 3L^4) \quad ...(7\text{-}2)$$

And the slope: $$\frac{dy}{dx} = \frac{w_0}{6EI}(L^3 - x^3) \quad ...(7\text{-}3)$$

The maximum deflection ($y_{max.}$) is at x = 0, and by substituting it in the equation (7-2) it will be:

$$Y_{max.} = -\frac{W_o\ L^4}{8\ EI}$$

The maximum slop $(dy/dx)_{max.}$ is at x = 0, and by substituting it in the equation (7- 3) it will be:

$$\left[\frac{dy}{dx}\right]_{max.} = \frac{w_o L^3}{6\ EI}$$