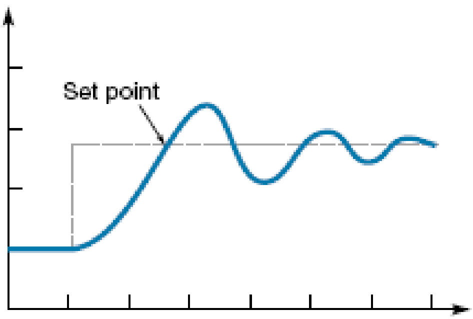
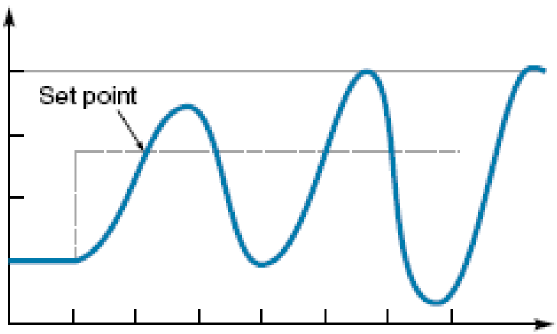
Stability Analysis

A stable system is one where the controlled variable will always settle near the set point. An unstable system is one where, under some conditions, the controlled variable drifts away from the set point or breaks into oscillations that get larger and larger until the system saturates on each side.



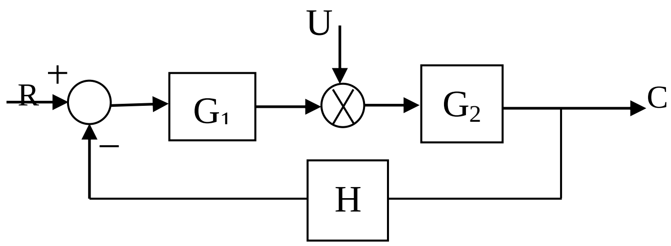
Stable system



unstable system

Methods of Stability Test

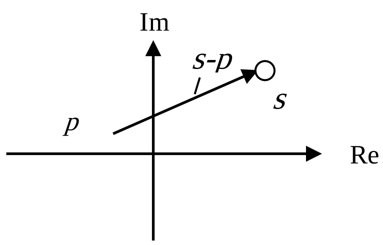
1-Determination the roots of equation



$$C = \frac{G_1G_2}{1+G_1G_2H}R(s) + \frac{G_2}{1+G_1G_2H}U(s)$$

$1+G_1G_2H=0$ (Characterstic Equation)

$(s-r_1)(s-r_2)(s-r_3)........=0$



A linear control system is ***unstable*** if any roots of its characterstic equation are to the right of imaginary axis.

If this Eqn has some roots with positive real parts, then the system is unstable, or some roots equal to zero, the system is marginally stable (oscillatory), therefore it is unstable.

Then for stability the roots of characteristic Eqn must have negative real parts.

Ex: if

$$G_1 = 10\frac{0.5s+1}{s} \quad \text{PI control}$$

$$G_2 = \frac{1}{2s+1} \quad \text{Stirred tank}$$

$$H = 1 \quad \text{Mesuring element without lag}$$

$$1+G = 1+G_1G_2H = 0$$

$$1+\frac{10(0.5s+1)}{s(2s+1)} = 0$$

$$s(2s+1)+5s+10 = 0$$

$$2s^2+6s+10 = 0$$

$$s^2+3s+5 = 0$$

$$s = \frac{-3}{2} \mp \frac{\sqrt{9-20}}{2}$$

$$\therefore s_1 = \frac{-3}{2} + j\frac{\sqrt{11}}{2} \quad \text{and} \quad s_2 = \frac{-3}{2} - j\frac{\sqrt{11}}{2}$$

Since the real part in s_1 and s_2 in -ve $(-\frac{3}{2})$ $\therefore$ The system is stable

2-Routh's Method

a-Write the characterstic eqn. on the form of a polynomial shape:

$$a_o s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots . a_n = 0 \qquad (*)$$

Where a_o is positive

It is necessary that a_o, a_1, a_2,…. a_{n-1}, a_n be positive. If any coeff. is negative, the system is ***unstable***.
If all of the coeff. are positive, the system may be stable or unstable. Then apply the next step.

b. Routh array:

Arrange the coeff. of eqn. (*) into the first two rows of the Routh array shown below.

Row				
1	a_o	a_2	a_4	a_6
2	a_1	a_3	a_5	a_7
3	A_1	A_2	A_3	
4	B_1	B_2	B_3	
n+1	C_1	C_2	C_3	

$$A_1 = \frac{a_1 a_2 - a_o a_3}{a_1} \quad , \quad A_2 = \frac{a_1 a_4 - a_o a_5}{a_1} \quad , \quad A_3 = \frac{a_1 a_6 - a_o a_7}{a_1}$$

$$B_1 = \frac{A_1 a_3 - a_1 A_2}{A_1} \quad , \quad B_2 = \frac{A_1 a_5 - a_1 A_3}{A_1}$$

$$C_1 = \frac{B_1 A_2 - A_1 B_2}{B_1} \quad , \quad C_2 = \frac{B_1 A_3 - A_1 B_3}{B_1}$$

Examine the elements of the first column of the array a_o, a_1, A_1, B_1,C_1………W_1

a) If any of these elements is negative, we have at least one root on the right of the imaginary axis and the system is unstable.
b) The number of sign changes in the elemnts of the first column is equal to the number of root to the right of the imaginary axis.

$\therefore$The system is ***stable*** if all the elements in the first column of the array are positive

<u>Ex:</u> Given the characteristic eqn.

$s^4 + 3s^3 + 5s^2 + 4s + 2 = 0$

<u>Solution:</u>

Row			
1	1	5	2
2	3	4	0
3	11/3	2	0
4	2.36	0	
5	2		

$$A_1 = \frac{3\times 5 - 4\times 11}{3} = \frac{11}{3}$$

$$A_2 = \frac{3\times 2 - 0}{3} = 2$$

$$B_1 = \frac{11/3 \times 4 - 6}{11/3} = 2.36$$

$$C_1 = \frac{2.36\times 2}{2.36} = 2$$

$\therefore$The system is **stable**

<u>Ex:</u> Apply the Routh's stability criterion to the equation:

$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

<u>Solution:</u>

s^4	1	3	5	0
s^3	2	4	0	
s^2	1	5	0	
s	-6	0		
s^0	5	0		

The system is unstable.

Ex: A system has a characteristic equation $s^3+9s^2+26s+24=0$. Using the Routh criterion, show that the system is stable.

Solution

q(s) = $s^3+9s^2+26s+24$

Using the Routh-Hurwitz criterion,

s^3	1	26
s^2	9	24
s^1	26	0
s^0	24	0

No sign change in 1^{st} column then the system is stable.

Ex: Consider the feedback control system with the characteristic eqn.

$$s^3+2s^2+(2+K_c)s+\frac{K_c}{\tau_I}=0$$

Solution:

The corresponding Routh array can now be formed

Row			
1	1	$2+K_c$	0
2	2	$\frac{K_c}{\tau_I}$	0
3	$\frac{2(2+K_c)-K_c/\tau_I}{2}$	0	0
4	K_c/τ_I	0	0

The elemnets of the first-column are positive except the third, which can be positive or negative depending on K_c and τ_I.

So state the stability

Put $\frac{2(2+K_c)-K_c/\tau_I}{2}>0 \quad \Rightarrow 2(2+K_c)>\frac{K_c}{\tau_I}$

If K_c and τ_I satisfy the condition, then the system is atable

Critical stability

Put the third element=0

i.e $2(2+K_c)=\frac{K_c}{\tau_I}$

For τ_I=0.1

$2(2+K_c)=10K_c \quad \Longrightarrow \quad 4=8K_c$

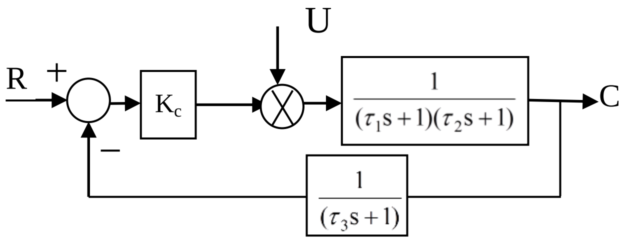
$K_c=0.5$

1) if $K_c < 0.5$, the system is stable (all of the elements in the 1st column is +ve)

2) if $K_c > 0.5$, the third element of the 1st column is negative. We have two sign change in the elements of the first column.

$\therefore$ we have two roots to the right of imaginary axis.

Ex:



If $\tau_1 = 1, \tau_2 = \frac{1}{2}, \tau_3 = \frac{1}{3}$

Determine K_c for a stable system

Solution:

The char. Eqn.

$$1 + K_c \frac{1}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)} = 0$$

$$(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1) + K_c = 0$$

$$(\frac{1}{2}s^2 + \frac{3}{2}s + 1)(\frac{1}{3}s+1) + K_c = 0$$

$$\frac{s^3}{6} + \frac{s^2}{2} + \frac{s}{3} + \frac{s^2}{2} + \frac{3s}{2} + 1 + K_c = 0$$

$$\frac{1}{6}s^3 + s^2 + \frac{11}{6}s + 1 + K_c = 0$$

Row		
1	1/6	11/6
2	1	1+K_c
3	$\frac{10-K_c}{6}$	0
4	1+K_c	

Since K_c>0

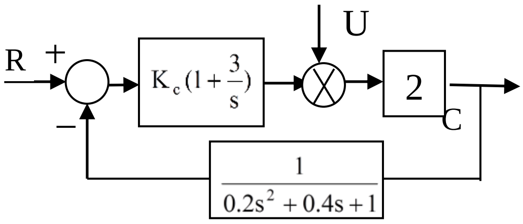
$\therefore$ The system will be stable

If 10-K_c >0

$K_c < 10$

Therfore K_c must within the range $0 < K_c < 10$

Ex:



Study the stability for K_c=2

Solution:

$$1+K_c(1+\frac{3}{s})\times 2\times\frac{1}{0.2s^2+0.4s+1}=0$$

$$1+K_c(\frac{s+3}{s})\times\frac{2}{0.2s^2+0.4s+1}=0$$

$$1+(\frac{sK_c+3K_c}{s})\times\frac{2}{0.2s^2+0.4s+1}=0$$

$$0.2s^3+0.4s^2+s+2sK_c+6K_c=0$$

$$0.2s^3+0.4s^2+(1+2K_c)s+6K_c=0$$

Row		
1	0.2	$1+2K_c$
2	0.4	$6K_c$
3	A_1	0
4	B_1	0

Row	For K_c=2	
1	0.2	5
2	0.4	12
3	$\frac{2-2.4}{0.4}$	0
4	1.2	0

$$A_1=\frac{0.4(1+2K_c)-(1.2K_c)}{0.4}=\frac{0.4+0.8K_c-1.2K_c}{0.4}=\frac{0.4-0.4K_c}{0.4}$$

$$0.4-0.4K_c>0$$

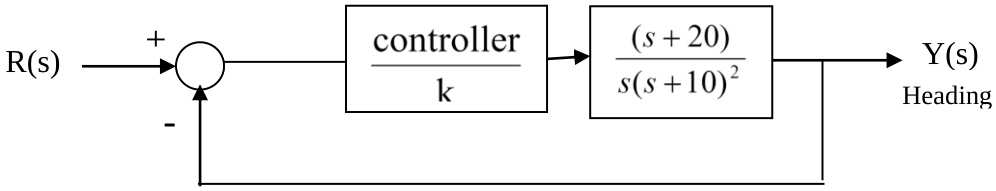
The system is stable for $K_c<1$

$B_1=6K_c \qquad \Rightarrow 6K_c>0$

And $K_c>0$

Therfore K_c must within the range $0 < K_c < 1$

Ex: Designers have developed small, fast, vertical-take off fighter aircraft that are invisible to radar. This aircraft concept uses quickly turning jet nozzles to steer the airplane. The control system for the heading or direction control is shown in figure. Determine the maximum gain of the system for stable operation.



Solution

$$G(s) = \frac{k(s+20)}{s(s+10)^2} = \frac{ks+20k}{s(s^2+20s+100)} = \frac{ks+20k}{s^3+20s^2+100s}$$

Characteristic equation,

1+GH = 0

$$1+\frac{ks+20k}{s^3+20s^2+100s}*1=0$$

$$s^3+20s^2+100s+ks+20k=0$$

$$s^3+20s^2+(100+k)s+20k=0$$

The corresponding Routh array can now be formed

Row		
1	1	100+k
2	20	20k
3	a	0
4	b	0

$$a = \frac{20(100+k)-20k}{20} = \frac{20*100+20k-20k}{20} = 100$$

$$b = \frac{a*20k-0}{a} = 20k$$

The system is stable, no sign change in 1^{st} column,

b>0

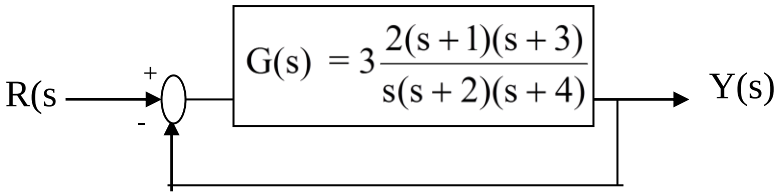
20k >0

k>0

$\therefore$ Range of k is must be k>0

Homework

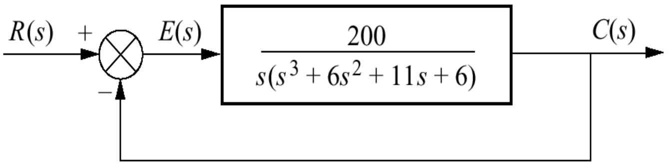
Q1)A single – loop control system is shown in figure.

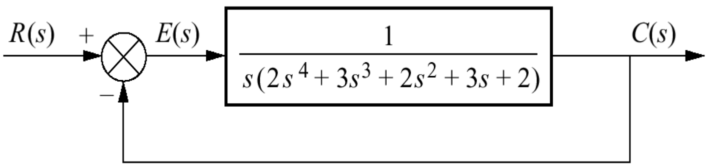


(a)Determine closed-loop transfer function $T(s) = \frac{Y(s)}{R(s)}$

(b) Determine whether or not the systems given below are stable.

Q2) **Determine whether or not the systems given below are stable.**





Q3) **Determine the range of values of K for which the following system s are stable.**