1.5 Inverse of Laplace Transform

Inversion by partial fraction

The inverse of the laplace could be obtained from tables for simple functions but for more complicated function partial fraction is needed

$$L^{-1} f(s) = f(t)$$

1- Partial fraction

Example:

$$\frac{A}{(s+a)(s+b)} = \frac{\alpha_o}{(s+a)} + \frac{\alpha_1}{(s+b)}$$

$$A = \alpha_o (s+b) + \alpha_1 (s+a)$$

$$\alpha_o = ?$$

$$\alpha_1 = ?$$

Example:

$$\frac{A}{(s+a)(s+b)^2} = \frac{\alpha_o}{(s+a)} + \frac{\alpha_1}{(s+b)} + \frac{\alpha_2}{(s+b)^2}$$

Example:

$$\frac{A}{(s^2+w^2)(s+a)} = \frac{\alpha_o}{(s+a)} + \frac{\alpha_1 s + \alpha_2}{(s^2+w^2)}$$

Example: Solve the IVP $y'' - y = t,\ y(0) = 1, y'(0) = 1.$

Solution:

Step 1: Take Laplace Transform: $(s^2 Y(s) - s - 1) - Y(s) = \frac{1}{s^2}$ (subsidiary equation)

Step 2: Solve for $Y(s) \Rightarrow Y(s) = \frac{s+1+\frac{1}{s^2}}{s^2-1} = \frac{s^3+s^2+1}{s^2(s-1)(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-1} + \frac{D}{s+1}$

Then: $A = \left.\frac{s^3+s^2+1}{(s-1)(s+1)}\right|_{s=0} = -1,\qquad C = \left.\frac{s^3+s^2+1}{s^2(s+1)}\right|_{s=1} = \frac{3}{2},$

$D = \left.\frac{s^3+s^2+1}{s^2(s-1)}\right|_{s=-1} = -\frac{1}{2}$ and lastly, to find B, substitute $s = 2,$ we get

$\frac{13}{12} = -\frac{1}{4} + \frac{B}{2} + \frac{3}{2} - \frac{1}{6}$ and obtain $B = 0.$

Step 3: Solve for $y(t)$: $Y(s) = -\frac{1}{s^2} + \frac{\frac{3}{2}}{s-1} - \frac{\frac{1}{2}}{s+1} \Longrightarrow$

$y(t) = L^{-1}[Y(s)] = -t + \frac{3}{2}e^{t} - \frac{1}{2}e^{-t}$

Example:

Find $L^{-1}[\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 4 + e^{2t}]$

where

$x(0) = 1 \quad , \quad x'(0) = 0 \quad , \quad x''(0) = -1$

Solution:

$$[s^3x(s) - s^2x(0) - sx'(0) - x''(0)] + 2[s^2x(s) - sx(0) - x'(0)] - [sx(s) - x(0)] - 2x(s)$$

$$= \frac{4}{s} + \frac{1}{(s-2)}$$

$$[s^3x(s) + 2s^2x(s) - sx(s) - 2x(s)] - s^2 + 1 - 2s + 1 = \frac{4}{s} + \frac{1}{(s-2)}$$

$$[s^3 + 2s^2 - s - 2]x(s) = \frac{4}{s} + \frac{1}{(s-2)} + s^2 + 2s - 2$$

$$x(s) = \frac{1}{(s^3 + 2s^2 - s - 2)} \times \frac{4(s-2) + s + (s^2 + 2s - 2)(s)(s-2)}{s(s-2)}$$

$$x(s) = \frac{s^4 - 6s^2 + 9s - 8}{s(s-2)(s+1)(s+2)(s-1)}$$

$$x(s) = \frac{\alpha_o}{s} \times \frac{\alpha_1}{(s-2)} \times \frac{\alpha_2}{(s+1)} \times \frac{\alpha_3}{(s+2)} \times \frac{\alpha_4}{(s-1)}$$

$$s^4 - 6s^2 + 9s - 8 = \alpha_o(s-2)(s+1)(s+2)(s-1) + \alpha_1 s(s+1)(s+2)(s-1)$$
$$+ \alpha_2 s(s-2)(s+2)(s-1) + \alpha_3 s(s-2)(s+1)(s-1) + \alpha_4 s(s-2)(s+1)(s+2)$$

assume s=0

$-8 = \alpha_o(-2)(1)(2)(-1)$

$\alpha_o = \frac{-8}{+4} = -2$

assume $s = 2 \Rightarrow \alpha_1 = 1/12$

assume $s = -1 \Rightarrow \alpha_2 = 11/3$

assume $s = -2 \Rightarrow \alpha_3 = -17/12$

assume $s = 1 \Rightarrow \alpha_4 = 2/3$

$$x(s) = \frac{-2}{s} + \frac{1/12}{s-2} + \frac{11/3}{s+1} - \frac{17/12}{s+2} + \frac{2/3}{s-1}$$

$$x(t) = -2 + \frac{1}{12}e^{2t} + \frac{11}{3}e^{-t} - \frac{17}{12}e^{-2t} + \frac{2}{3}e^{t}$$

1.6 Final Value Theorm

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s)$$

1.7 Initial Value Theorm

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sf(s)$$

Example:

Find the final value of the function x(t) for which the laplace inverse is:-

$$x(s) = \frac{1}{s(s^3+3s^2+3s+1)}$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sx(s) = \lim_{s\to 0}\frac{s\times 1}{s(s^3+3s^2+3s+1)}$$

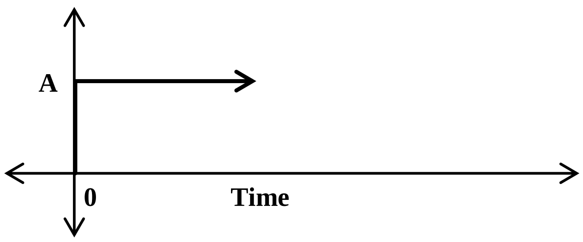
$$= \lim_{s\to 0}\frac{1}{(s^3+3s^2+3s+1)} = 1$$

1.8 Special Functions

1- Step function

$$f(t) = \begin{cases} 0 & t<0 \\ A & t\geq 0 \end{cases}$$

$$f(s) = \frac{A}{s}$$



If A=1 the change is called unit step change

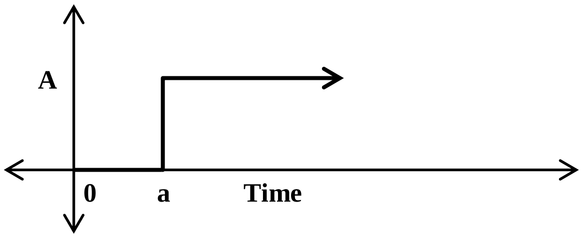
$$f(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$$

$$f(s) = \frac{1}{s}$$

Step function with Time Delay

$$f(t)=\begin{cases}0 & t<a\\ A & t\ge a\end{cases}$$

$$f(s)=\frac{A}{s}e^{-as}$$

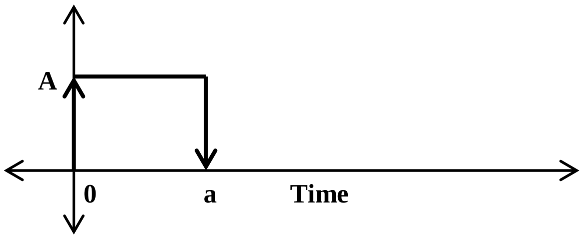


2. Pulse function

$$f(t)=\begin{cases}0 & t<0\\ A & 0\le t\le a\\ 0 & t\ge a\end{cases}$$

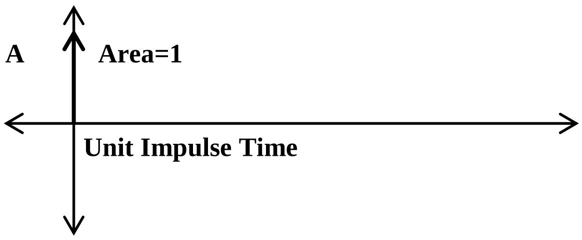
$$f(s)=\frac{A}{s}-\frac{A}{s}e^{-as}$$

$$=\frac{A}{s}(1-e^{-as})$$



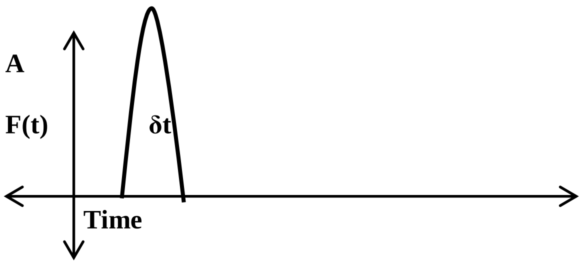
if A=1➔unit Pulse

3. Impulse function



$$f(t)=\begin{cases}0 & t<0\\ A & 0\le t\le \delta t\\ 0 & t\ge \delta t\end{cases}$$

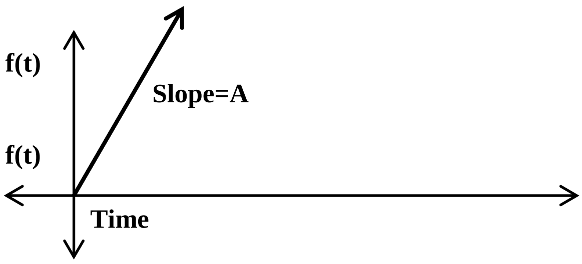
$$f(s)=area=A\times\delta t$$



4. Ramp function

$$f(t)=\begin{cases}0 & t<0\\ At & t\ge 0\end{cases}$$

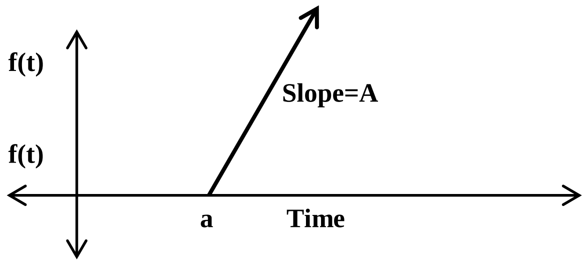
$$f(s)=\frac{A}{s^2}$$



Ramp function with time delay

$$f(t)=\begin{cases}0 & t<a\\ At & t\ge a\end{cases}$$

$$f(s)=\frac{A}{s^2}e^{-as}$$



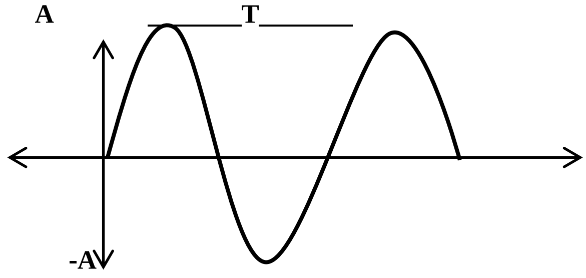
5. Sine function

$$f(t)=\begin{cases}0 & t<a\\ A\sin wt & t\ge a\end{cases}$$

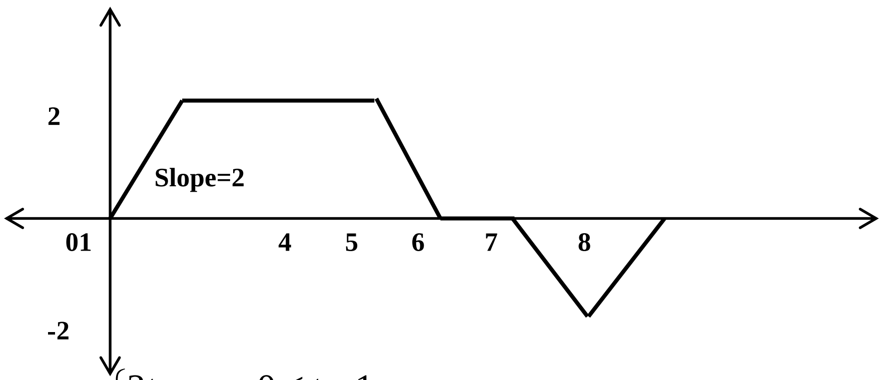
$$f(s)=\frac{A\omega}{s^2+\omega^2}$$

$$\omega=2\pi f$$

$$T=\frac{1}{f}$$



1.9 Disturbance function (complex function)

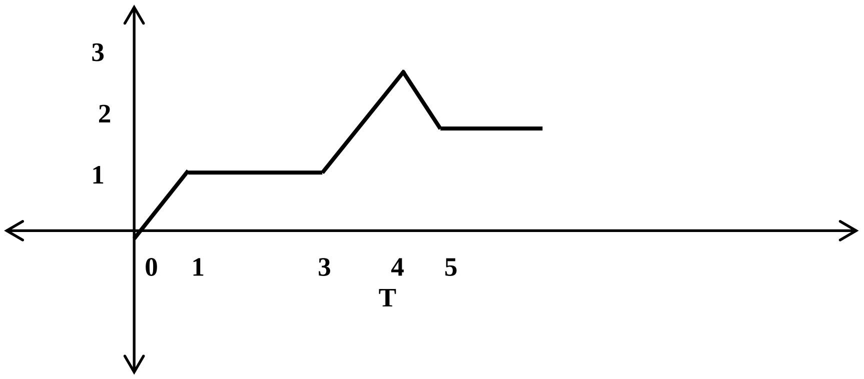


$$f(t)=\begin{cases}2t & 0\le t<1\\ 2 & 1\le t<4\\ -2t & 4\le t<5\\ 0 & 5\le t<6\\ -2t & 6\le t<7\\ 2t & 7\le t<8\\ 0 & t>8\end{cases}$$

$$f(t)=2t-2(t-1)-2(t-4)+2(t-5)-2(t-6)+2\times 2(t-7)-2(t-8)$$

$$f(s)=\frac{2}{s^2}-\frac{2}{s^2}e^{-s}-\frac{2}{s^2}e^{-4s}+\frac{2}{s^2}e^{-5s}-\frac{2}{s^2}e^{-6s}+\frac{4}{s^2}e^{-7s}-\frac{2}{s^2}e^{-8s}$$

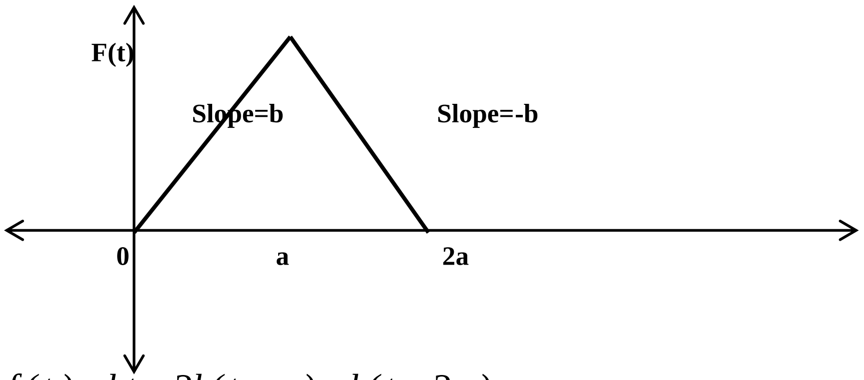
Example:



$$f(t)=t-(t-1)+2(t-3)-2(t-4)+(t-5)$$

$$f(s)=\frac{1}{s^2}-\frac{1}{s^2}e^{-s}+\frac{2}{s^2}e^{-3s}-\frac{2}{s^2}e^{-4s}+\frac{1}{s^2}e^{-5s}$$

Example:



$$f(t)=bt-2b(t-a)+b(t-2a)$$

$$f(s)=\frac{b}{s^2}-\frac{2b}{s^2}e^{-as}+\frac{b}{s^2}e^{-2as}$$

Example:

$$f(t)=-0.5+1\delta(t-0.5)-0.5\delta(t-1.5)+0.5\delta(t-2)-1\delta(t-3)+0.5\delta(t-3.5)$$

$$f(s)=-\frac{0.5}{s}+\frac{1}{s}e^{-0.5s}-\frac{0.5}{s}e^{-1.5s}+\frac{0.5}{s}e^{-2s}-\frac{1}{s}e^{-3s}+\frac{0.5}{s}e^{-3.5s}$$

Solved problems

1. Determine the inverse Laplace transform

(a) $L^{-1}\left[\dfrac{3}{(2s+5)^3}\right]=\dfrac{3}{8}L^{-1}\left[\dfrac{1}{(s+5/2)^3}\right]=\dfrac{3}{16}t^2e^{-5t/2}$

(b) $L^{-1}\left[\dfrac{s-1}{2s^2+s+3}\right]=\dfrac{1}{2}e^{-t/4}\cos(\dfrac{\sqrt{47}t}{4})-\dfrac{5}{2\sqrt{47}}e^{-t/4}\sin(\dfrac{\sqrt{47}t}{4})$

2. Determine the partial fraction expansion

(a) $\dfrac{-8s^2-5s+9}{(s+1)(s^2-3s+2)}=\dfrac{1}{s+1}+\dfrac{2}{s-1}-\dfrac{11}{s-2}$

(b) $\dfrac{-5s-36}{(s+2)(s^2+9)}=-2\dfrac{1}{s+2}+2\dfrac{s}{s^2+9}-3\dfrac{3}{s^2+9}$

(c) $\dfrac{3s^2+5s+3}{s^4+s^3}=\dfrac{3s^2+5s+3}{s^3(s+1)}=\dfrac{3}{s^3}+\dfrac{2}{s^2}+\dfrac{1}{s}-\dfrac{1}{(s+1)}$

(d) $\dfrac{s}{(s-1)(s^2-1)}=\dfrac{1}{2}\dfrac{1}{(s-1)^2}+\dfrac{1}{4}\dfrac{1}{(s-1)}-\dfrac{1}{4}\dfrac{1}{(s+1)}$

2. Determine $L^{-1}\{F(s)\}$

(a)

$$F(s)=\frac{s+11}{(s-1)(s+3)}=3\frac{1}{s-1}-2\frac{1}{s+3}$$

$$\Rightarrow L^{-1}[F(s)]=3e^t-2e^{-3t}$$

(b)

$$F(s)=\frac{7s^3-2s^2-3s+6}{s^3(s-2)}=-\frac{3}{s^3}+\frac{1}{s}+\frac{6}{s-2}$$

$$\Rightarrow L^{-1}[F(s)]=-\frac{3}{2}t^2+1+6e^{2t}$$