Part II Linear Open-Loop Systems
Response of first order systems

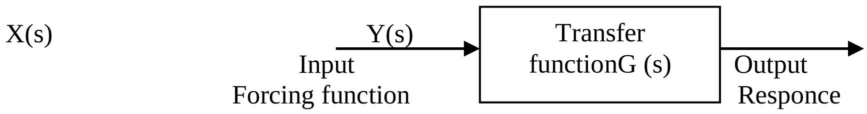
2. Dynamic behavior of first order system

Before studying the control system it is necessary to become familior with the response os some of simple basic systems (i.e study the dynamic behaviour of the first and second order systems).

2.1 The transfer function:

The dynamic behaviour of the system is described by transfer function(T.F)

$$T.F = \frac{Laplace\ transform\ of\ the\ output\ (responce)}{Laplace\ transform\ of\ the\ input\ (forcing\ function\ distubance)}$$



Block diagram

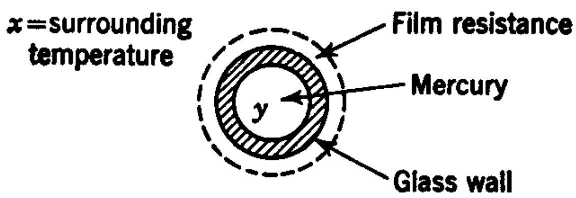
$$T.F = G(s) = \frac{y(s)}{x(s)}$$

This definition is applied to linear systems

2.2 Development of T.F for first order system:
Mercury Thermometer:

It is a measuring device use to measure the temperature of a stream.

Consider a mercury in glass thermometer to be located in a flowing stream of fluid for which the temperature x varies with time.



The opject is to calculate the time variation of the thermometer reading y for a particular change of x

The following assumptions will be used in this analysis:-

1. All the resistance to heat transfer resides in the film surrounding the bulb (i.e., the resistance offered by the glass and mercury is neglected).

2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temperature throughout.

3. The glass wall containing the mercury does not expand or contract during the transient response.

It is assumed that the thermometer is initially at steady state. This means that, before time zero, there is no change in temperature with time. At time zero the thermometer will be subjected to some change in the surrounding temperature x(t). (i.e at t<0 x(t)= y(t) =constant there is no change in temperature with time).

At t=0 there is a change in the surounding temperature x(t)

<u>Unsteady state energy balance:</u>

$m\ cp\ \frac{dy(t)}{dt} = h\ A\ (x - y) - 0 = hax - hay$ ……………… (1)

1st order differential equation

<u>Where</u>

A: area of the bulb

Cp: Heat capacity of mercury

m: Mass of mercury in the bulb

t: time

h: film heat trnfer coefficient

h depend on the flowrate and properties of the surounding fluid and the dimension of the bulb.

The dynamic behavious must be defined by a deviatopn variables.

At steady state (s.s.) t<0 x(t)=constant=x_s y(t)=constant=y_s $\frac{dy}{dt} = 0$

x(t)=constant=x_s

$m\ cp\ \frac{dy_s}{dt} = h\ A\ (x_s - y_s s) = h\ Ax_s - h\ Ay_s$ ……………… (2)

Substract eq. (2) from eq. (1)

$$m\ cp\ \frac{d(y - y_s)}{dt} = h\ A\ (x - x_s) - h\ A\ (y - y_s s)$$

$y - y_s = Y$ $\quad$ $also \quad x - x_s = X$

at t=0 Y(0)=0 and X(0)=0

$$m\ cp\ \frac{dY}{dt} = h\ A\ X - h\ A\ Y$$

$$\frac{m\ cp}{h\ A}\frac{dY}{dt} = X - Y$$

Let $\tau = \frac{m\ cp}{h\ A}$=time constant and has time units

$\tau\ \frac{dY}{dt} + Y = X$ taken laplace for the equation

$$\tau[sY(s) - Y(0)] + Y(s) = X(s)$$

$(\tau s + 1)Y(s) = X(s)$

$$\frac{Y(s)}{X(s)} = G(s) = \frac{1}{\tau s + 1} \qquad \dots\dots\dots\dots\dots\dots (2.2)$$

$$T.F = \frac{Y(s)}{X(s)} = G(s) = \frac{L.T\ of\ the\ deviation\ in\ thermometer\ reading}{L.T.of\ the\ deviation\ in\ suroundig\ Temperature}$$

Any system has a T.F of the form of equation (2.2) it is called *<u>first order</u>* system which is a first order differntial equation (Linear).

1.3 Properties of transfer functions

A T.F relates two variables in a physical process. One of these is (Forcing or Input variable) and the other is the effect (Reponce or Output).

$$T.F = \frac{Y(s)}{X(s)} = G(s)$$

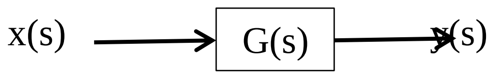
If we select a particular input variation x(t) for which the L.T is x(s) then the reponce.

$$Y(s) = G(s).X(s) \dots\dots\dots\dots\dots\dots. (2.3)$$

$$L^{-1}Y(s) = Y(t) = L^{-1}G(s).X(s)$$

If G(s) is 1st order of a thermometer

$$Y(s) = G(s).X(s) = \frac{1}{\tau s + 1}.X(s) \qquad \dots\dots\dots\dots\dots\dots\dots.. (2.4)$$

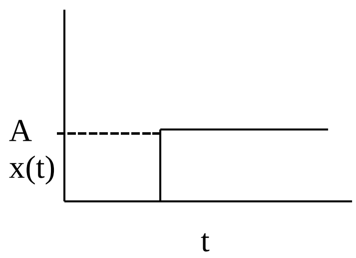


1.4 Transient response for different changes

$$Y(s) = \frac{1}{\tau s + 1}.X(s)$$

Y(t)=? For different types of x(t)

1-<u>Step Change</u>



$$X(s) = \frac{A}{s}$$

$$Y(s) = \frac{1}{\tau s + 1}.\frac{A}{s} = \frac{\alpha_o}{s} + \frac{\alpha_1}{\tau s + 1}$$

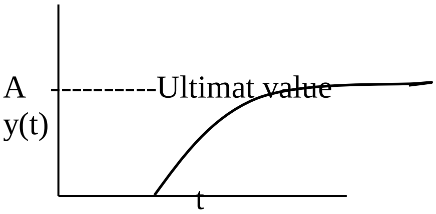
$$A = \alpha_o(\tau s + 1) + \alpha_1 s$$

S=0 $\quad \alpha_o = A$

$s = -1/\tau \quad \Longrightarrow \quad A = \alpha_o(-\tau/\tau + 1) - \alpha_1 \frac{1}{\tau}$ then $\alpha_o = -A\tau$

$$Y(s)=\frac{A}{s}-\frac{A\tau}{\tau s+1}=\frac{A}{s}-\frac{A\tau}{\tau s+1}\frac{1/\tau}{1/\tau}=\frac{A}{s}-\frac{A}{\tau+1/\tau}$$

$$Y(t)=A-Ae^{-t/\tau}=A(1-e^{-t/\tau}) \quad \text{……………….. (2.5)}$$



Several features of this response, worth remembering, are

- The value of y(t) reaches 63.2 % of its ultimate value when the time elapsed is equal to one time constant τ.
- When the time elapsed is 2τ, $3\ \tau$, and $4\ \tau$, the percent response is 86.5%, 95%, and 98%, respectively.

Where the ultimate value is find steady state value

$$U.V=\lim_{t\to\infty} y(t)=\lim_{s\to 0} sy(s) \ldots..(2.6)$$

Example (5.1): A thermometer having a time constant of 0.1 min is at a steady state temperature of 90 F^o. At time t = 0, the thermometer is placed in a temperature bath maintained at 100°F. Determine the time needed for the thermometer to read 98 F^o.

Solution:

At s.s. $x_s=y_s=90\ F^o$

Step Change $X(s)=\frac{A}{s}$

A=100-90=10

$$X(s)=\frac{10}{s}$$

$$Y(s)=\frac{1}{\tau s+1}\frac{A}{s}=\frac{1}{0.1s+1}\frac{10}{s}=\frac{10}{s(0.1s+1)}=\frac{10}{0.1s(s+10)}=\frac{A}{0.1s}+\frac{B}{s+10}$$

$$A(s+10)+B(0.1s)=10$$

$$s=0\Rightarrow A=\frac{10}{10}=1$$

$$s=-10\Rightarrow B=-10$$

$$Y(s)=\frac{1}{0.1s}-\frac{10}{s+10}=\frac{10}{s}-\frac{10}{s+10}$$

By taken lablace inverse for the equation

$$Y(t)=10-10e^{-10t}=10(1-e^{-10t})$$

Substitute $Y(t)=y(t)-y_s=98-90$

Y(t)=8

$$8=10(1-e^{-10t})$$

$$0.8=1-e^{-10t}$$

$$ln(e^{-10t})=ln(0.2)$$

$$-10t=ln(0.2)$$

$$t=-ln(0.2)\times 0.1$$

t=0.161 min

2-Impulse Input

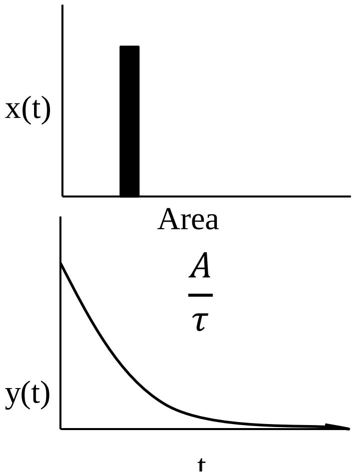
$$X(s) = A$$

$$Y(s) = \frac{1}{\tau s + 1} \times A = \frac{A}{\tau s + 1}$$

$$Y(s) = \frac{A/\tau}{s + 1/\tau}$$

$$Y(t) = y(t) - y_s = \frac{A}{\tau} e^{-t/\tau}$$

$$Y(t) = \frac{A}{\tau} e^{-t/\tau} + y_s$$



2-Sinsoidal input

$$x(t) = x_s + A \sin wt \qquad t > 0$$

$$x(t) - x_s = A \sin wt$$

$$X(t) = x(t) - x_s = A \sin wt$$

………………(2.9)

$$X(s) = \frac{Aw}{s^2 + w^2}$$

$$Y(s) = \frac{Aw}{s^2 + w^2} \times \frac{1}{(\tau s + 1)} = Aw\left[\frac{1}{(s^2 + w^2)(\tau s + 1)}\right]$$

This equation can be solved for y(t) by means of a partial fraction expansion as described in previous lectures.

$$Y(s) = Aw\left[\frac{1}{(s^2 + w^2)(\tau s + 1)}\right] = Aw\left[\frac{\alpha_o s + \alpha_1}{(s^2 + w^2)} + \frac{\alpha_2}{(\tau s + 1)}\right]$$

$$(\alpha_o s + \alpha_1)(\tau s + 1) + \alpha_2(s^2 + w^2) = 1$$

$$\alpha_o \tau s^2 + \alpha_o s + \alpha_1 \tau s + \alpha_1 + \alpha_2 s^2 + \alpha_2 w^2 = 1$$

s^0 $\qquad \alpha_1 + \alpha_2 w^2 = 1$ (1)

s^1 $\qquad \alpha_o + \alpha_1 \tau = 0 \Rightarrow \alpha_o = -\alpha_1 \tau$ (2)

s^2 $\qquad \alpha_o \tau + \alpha_2 = 0 \Rightarrow \alpha_2 = -\alpha_o \tau$ (3)

By substitution eq.(2) in eq.(3)

$$\alpha_2 = \alpha_1 \tau^2 \qquad (4)$$

By substitution eq.(4) in eq.(1)

$$\alpha_1 + \alpha_1 \tau^2 w^2 = 1$$

$$\alpha_1 = \frac{1}{1+\tau^2 w^2}$$

$$\alpha_2 = \frac{\tau^2}{1+\tau^2 w^2}$$

$$\alpha_0 = \frac{-\tau}{1+\tau^2 w^2}$$

$$Y(s) = Aw[\frac{\frac{-\tau}{1+\tau^2 w^2}s + \frac{1}{1+\tau^2 w^2}}{(s^2+w^2)} + \frac{\frac{\tau^2}{1+\tau^2 w^2}}{(\tau s+1)}]$$

$$Y(s) = Aw\frac{1}{1+\tau^2 w^2}[\frac{-\tau s+1}{(s^2+w^2)} + \frac{\tau^2}{(\tau s+1)}]$$

$$Y(s) = \frac{Aw}{1+\tau^2 w^2}[\frac{-\tau s}{(s^2+w^2)} + \frac{1}{(s^2+w^2)}\frac{w}{w} + \frac{\tau^2}{(\tau s+1)}\frac{\tau}{\tau}]$$

$$Y(s) = \frac{Aw}{1+\tau^2 w^2}[\frac{-\tau s}{(s^2+w^2)} + \frac{1}{w}\frac{w}{(s^2+w^2)} + \frac{\tau}{(s+1/\tau)}]$$

$$Y(t) = \frac{Aw}{1+\tau^2 w^2}[-\tau \cos wt + \frac{1}{w}\sin wt + \tau e^{t/\tau}]$$

Using the definition

$$p\cos\theta + q\sin\theta = r\sin(\theta+\phi)$$

$$r = \sqrt{p^2+q^2} \qquad \tan\phi = \frac{p}{q}$$

$$\phi = \tan^{-1}\frac{p}{q}$$

$$q = \frac{1}{w} \qquad p = -\tau$$

$$\phi = \tan^{-1}(-w\tau)$$

$$r = \sqrt{(\frac{1}{w})^2 + (-\tau)^2} = \sqrt{\frac{1}{w^2}+\tau^2} = \frac{\sqrt{1+w^2\tau^2}}{w}$$

$$-\tau\cos wt + \frac{1}{w}\sin wt = r\sin(wt+\phi)$$

$$\boxed{Y(t) = \frac{Aw}{1+\tau^2 w^2}[\tau e^{t/\tau} + \frac{\sqrt{1+w^2\tau^2}}{w}\sin(wt+\phi)]}$$

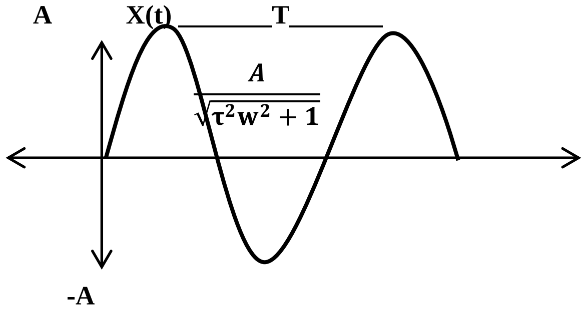
$$Y(t)=\frac{Aw\tau}{1+\tau^2 w^2}e^{t/\tau}+\frac{A}{\sqrt{1+w^2\tau^2}}sin(wt+\phi)]$$

where

$$\phi=tan^{-1}(-w\tau)$$

As $t\to\infty$ *then* $e^{-t/\tau}=0$, the first term on the right side of main equation vanishes and leavesonly the ultimate periodic solution, which is sometimes called the steady-statesolution

$$Y(t)=\frac{A}{\sqrt{1+w^2\tau^2}}sin(wt+\phi)] \qquad \text{.....(2.10)}$$



By comparing Eq. (2.9) for the input forcing function with Eq. (2.10) forthe ultimate periodic response, we see that

1. The output is a sine wave with a frequency w equal to that of the input signal.
2. The ratio of output amplitude to input amplitude is $\frac{1}{\sqrt{1+w^2\tau^2}}<1$.
3. The output lags behind the input by an angle ϕ. It is clear that lag occurs,for the sign of ϕ is always negative.

$\phi<0$ *phase lag*

$\phi>0$ *phase load*

Example 5.2. A mercury thermometer having a time constant of 0.1 min is placedin a temperature bath at 100°F and allowed to come to equilibrium with the bath. Attime t= 0, the temperature of the bath begins to vary sinusoidally about its averagetemperature of l00°F with an amplitude of 2°F If the frequency of oscillation is $10/\pi$ cycles/min, plot the ultimate response of the thermometer reading as a function oftime. What is the phase lag?

In terms of the symbols used in this chapter

$\tau = 0.1$

$t < 0 \qquad x_s = y_s = 100$

$t \geq 0 \qquad x(t) = 100 + 2\,sin(wt)$

$f = \frac{10}{\pi}$

Solution

$w = 2\pi f = 2\pi \times \frac{10}{\pi} = 10 \;\; rad/\min$

$T = \frac{1}{f} = \frac{10}{\pi} \;\; \min/cycle$

$X(t) = x(t) - x_s = 100 + 2\,sin\,20t - 100$

$X(t) = 2\,sin\,20t$

$X(s) = \frac{2 \times 20}{s^2 + 20^2}$

Ultimate responce $t \to \infty \qquad then \qquad e^{-t/\tau} = 0$

$Y(t) = \frac{A}{\sqrt{1 + w^2\tau^2}}\,sin(wt + \phi)]$

$\phi = tan^{-1}(-w\tau) = tan^{-1}(-20 \times 0.1) = tan^{-1}(-2)$

$\phi = -63.5^o$

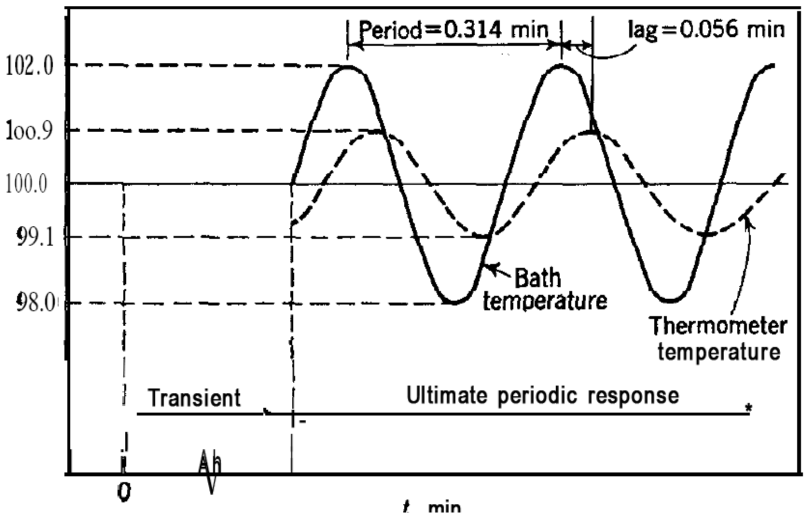
Ultimate responseat the above angle

$Y(t) = \frac{2}{\sqrt{1 + (0.1 \times 20)^2}}\,sin(20t - 63.5)]$

$Y(t) = \frac{2}{\sqrt{5}}\,sin(20t - 63.5)]$

$Y(t) = 0.896\,sin(20t - 63.5)]$

Ultimate response



In general, the lag in units of time is given by:-

$$phase\;\;lag = \frac{|\phi|}{360}\frac{1}{f}$$

$phase\;\;lag = \frac{63.5\;\;cycle}{360}\frac{\pi}{10}\frac{min}{cycle}$

$= 0.0555\,min$

A frequency of $\frac{10}{\pi}\frac{cycle}{min}$ means that a complete cycle occurs in $(\frac{10}{\pi})^{-1}\,min.$ since cycle is equivalent to 360° and lag is 63.5°

How to calculate the time constant (τ) for 1st order system

1) **Mathematical method**

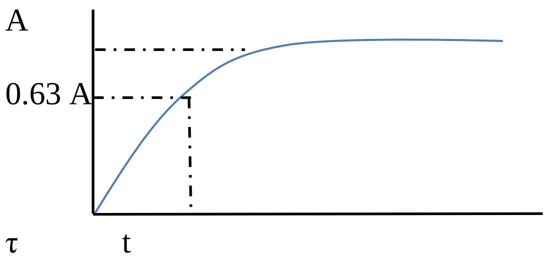
Using the definitions

$$\tau = \frac{m\ cp}{hA} \qquad \textit{Thermometer}$$

$$\tau = AR \qquad \textit{Liquid level tank}$$

$$\tau = \frac{V}{q} \qquad \textit{Mixing tank}$$

2) **Exponential method(Step change in the input variable)**



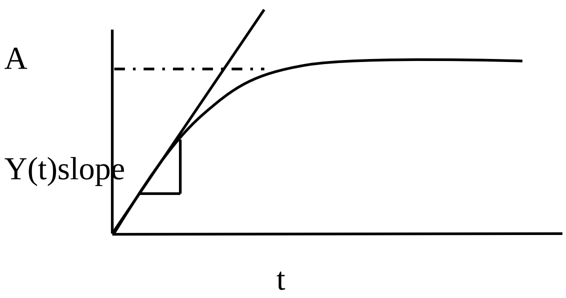
$$Y(t) = A(1 - e^{-t/\tau})$$

$$as\ t \to \infty \qquad Y(\infty) = A(1 - e^{-\infty}) = A$$

$$as\ t \to \tau \qquad Y(\tau) = A(1 - e^{-1}) = A(1 - 0.3678) \approx 0.67A$$

Time constant (τ)is the time required for the response to reach 63% of the its utimate value.

3. Third method



$$Y(t) = A(1 - e^{-t/\tau})$$

$$\frac{dy}{dt} = -Ae^{-t/\tau}(-1/\tau) = \frac{A}{\tau}e^{-t/\tau}$$

$$\lim_{t \to 0}\frac{dy}{dt} = \frac{A}{\tau}e^{0} = \frac{A}{\tau}$$

Slope of the tangent at t=0 is $\frac{A}{\tau}$

Therfore $\tau = \frac{A}{slope}$

$$Y(t) = A(1 - e^{-t/\tau})$$

$$Y(t) = A - Ae^{-t/\tau}$$

$$Ae^{-t/\tau} = A - Y(t)$$

$$e^{-t/\tau} = \frac{A - Y(t)}{A}$$

$$-t/\tau = ln\frac{A-Y(t)}{A}$$

Let $B = \frac{A-Y(t)}{A}$

$$ln B = \frac{1}{\tau}t$$

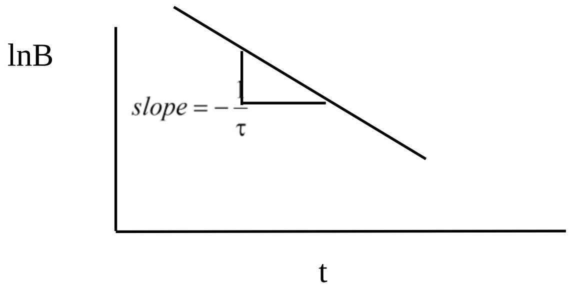
let

$$y = ln B$$

$$x = t$$

$$slope = -\frac{1}{\tau}$$

Y(t)	$B = \frac{A-Y(t)}{A}$	$ln B$	t



$$slope = -\frac{1}{\tau}$$

$$\tau = -\frac{1}{slope}$$