Second order system

A linear second order system under dynamic condition is given by the differential equation:-

$$\frac{1}{\omega_n^2}\frac{d^2Y}{dt^2}+\frac{2\psi}{\omega_n}\frac{dY}{dt}+Y=kX$$

$$\frac{1}{\omega_n}=\tau$$

$$\therefore\ \tau^2\frac{d^2Y}{dt^2}+2\psi\tau\frac{dY}{dt}+Y=kX$$

Where:-

k : Steady state gain

Y : Response value

X : Input disturbing variable

ω_n: Natural frequency of oscillation of the system.

$Y(0)=\overline{Y}(0)=0$

ψ : Damping factor (damping coefficient)

By taking laplace for the above second order differential equation

$$\tau^2s^2Y(s)+2\psi\tau sY(s)+Y(s)=kX(s)$$

$$(\tau^2s^2+2\psi\tau s+1)Y(s)=kX(s)$$

$$\boxed{G(s)=\frac{Y(s)}{X(s)}=\frac{k}{\tau^2s^2+2\psi\tau s+1}}$$ T.F. of second order system

If x is sudden force, such as,step change, inputs Y will oscillate depending on the value of damping coefficient ψ .

$\psi<1$ Response will oscillate (Under damped)

$\psi>1$ Response will oscillate (Over damped)

$\psi=1$ Response critical oscillation (critical damped)

Response of second order system

1) Step response

$$X(s)=\frac{A}{s} \quad \Rightarrow \quad Y(s)=\frac{k}{\tau^2 s^2+2\psi\tau s+1}\times\frac{A}{s}=\frac{k/\tau^2}{s^2+\frac{2\psi}{\tau}s+\frac{1}{\tau^2}}\times\frac{A}{s} \qquad \ldots\ldots(1)$$

The quadratic term in this equation may be factored into two linear terms that contain the roots

$$s_{1,2}=\frac{-\frac{2\psi}{\tau}\pm\sqrt{(\frac{2\psi}{\tau})^2-\frac{4}{\tau^2}}}{2}=\frac{-\psi}{\tau}\pm\frac{\sqrt{\frac{4\psi^2}{\tau^2}-\frac{4}{\tau^2}}}{2}=\frac{-\psi}{\tau}\pm\frac{2\sqrt{\frac{\psi^2-1}{\tau^2}}}{2}$$

$=\frac{-\psi}{\tau}\pm\frac{\sqrt{\psi^2-1}}{\tau}$ =Two real roots

$$\boxed{s_1=\frac{-\psi}{\tau}-\frac{\sqrt{\psi^2-1}}{\tau} \quad and \quad s_2=\frac{-\psi}{\tau}+\frac{\sqrt{\psi^2-1}}{\tau}} \qquad \ldots\ldots(2)$$

Eq. (1) can now be re-written as

$$Y(s)=\frac{kA/\tau^2}{s(s-s_1)(s-s_2)}$$

ψ > 1	**Overdamped**	**Two distinct real roots**
ψ = 1	**Critically Damped**	**Two equal real roots**
0 < ψ < 1	**Underdamped**	**Two complex roots**

$$Y(s)=\frac{k}{\tau^2 s^2+2\psi\tau s+1}\times\frac{A}{s}=\frac{\alpha_0}{s}+\frac{\alpha_1 s+\alpha_2}{\tau^2 s^2+2\psi\tau s+1}$$

$$\alpha_o(\tau^2 s^2+2\psi\tau s+1)+\alpha_1 s^2+\alpha_2 s=kA$$

$s^0 \qquad \alpha_0=kA$

$s^1 \qquad 2\alpha_0\psi\tau+\alpha_2=0 \quad \Rightarrow \quad \alpha_2=-2kA\psi\tau$

$s^2 \qquad \alpha_0\tau^2+\alpha_1=0 \quad \Rightarrow \quad \alpha_1=-kA\tau^2$

$$\therefore Y(s)=kA[\frac{1}{s}-\frac{\tau^2 s+2\psi\tau}{\tau^2 s^2+2\psi\tau s+1}]$$

$$Y(s)=kA[\frac{1}{s}-\frac{s+2\frac{\psi}{\tau}}{(s^2+2\frac{\psi}{\tau}s+\frac{1}{\tau^2})+\frac{\psi^2}{\tau^2}-\frac{\psi^2}{\tau^2}}]=kA[\frac{1}{s}-\frac{s+2\frac{\psi}{\tau}}{(s^2+2\frac{\psi}{\tau}s+\frac{\psi^2}{\tau^2})+\frac{1}{\tau^2}-\frac{\psi^2}{\tau^2}}]$$

$$Y(s)=kA[\frac{1}{s}-\frac{s+2\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2+\frac{1-\psi^2}{\tau^2}}]$$

1) For $\psi<1$ ==> under damped system

$$Y(s)=kA[\frac{1}{s}-\frac{s+2\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2+(\frac{\sqrt{1-\psi^2}}{\tau})^2}]=kA[\frac{1}{s}-\frac{s+\frac{\psi}{\tau}+\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2+(\frac{\sqrt{1-\psi^2}}{\tau})^2}]$$

$$=kA[\frac{1}{s}-\frac{s+\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2+(\frac{\sqrt{1-\psi^2}}{\tau})^2}-\frac{\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2+(\frac{\sqrt{1-\psi^2}}{\tau})^2}]$$

$$=kA[\frac{1}{s}-\frac{s+\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2+(\frac{\sqrt{1-\psi^2}}{\tau})^2}-\frac{\frac{\psi}{\tau}\times\frac{\tau}{\sqrt{1-\psi^2}}\times\frac{\sqrt{1-\psi^2}}{\tau}}{(s+\frac{\psi}{\tau})^2+(\frac{\sqrt{1-\psi^2}}{\tau})^2}]$$

$$Y(t)=kA[1-e^{(-\psi/\tau)t}\cos\frac{\sqrt{1-\psi^2}}{\tau}t-\frac{\psi}{\sqrt{1-\psi^2}}e^{(-\psi/\tau)t}\sin\frac{\sqrt{1-\psi^2}}{\tau}t]$$

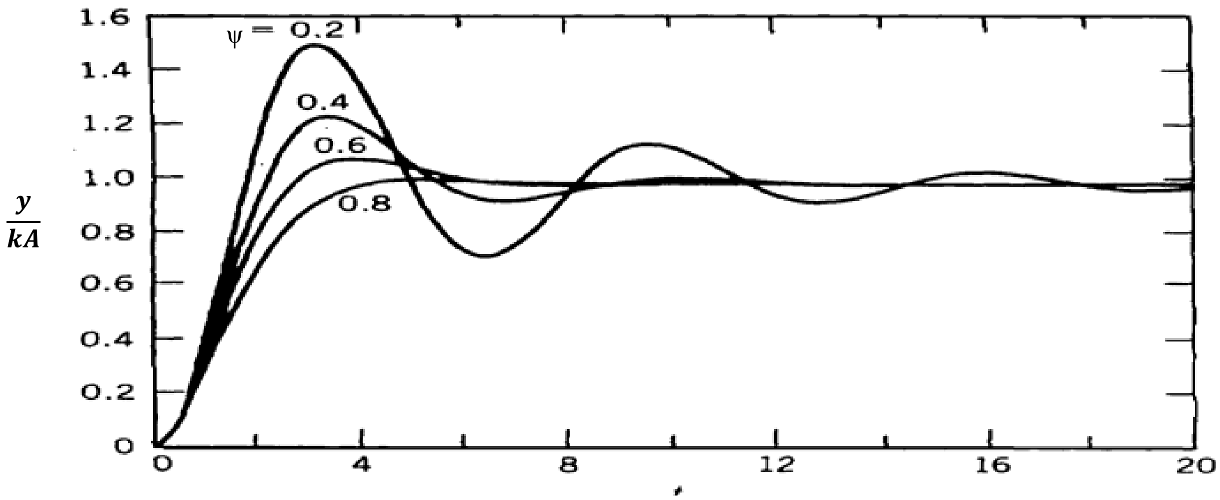
$$w=\frac{\sqrt{1-\psi^2}}{\tau}$$

$$Y(t)=kA[1-e^{(-\psi/\tau)t}(\cos wt+\frac{\psi}{\sqrt{1-\psi^2}}\sin wt]$$

$$r=\sqrt{p^2+q^2}=\sqrt{1+(\frac{\psi}{\sqrt{1-\psi^2}})^2}=\sqrt{\frac{1}{1-\psi^2}}$$

$$\phi=\tan^{-1}\frac{p}{q}=\tan^{-1}\frac{1}{\frac{\psi}{\sqrt{1-\psi^2}}}=\tan^{-1}\frac{\sqrt{1-\psi^2}}{\psi}$$

$$\boxed{Y(t)=kA[1-e^{(-\psi/\tau)t}(r\sin(wt+\phi))]}$$



2) For ψ>1 ➔Overdamped system

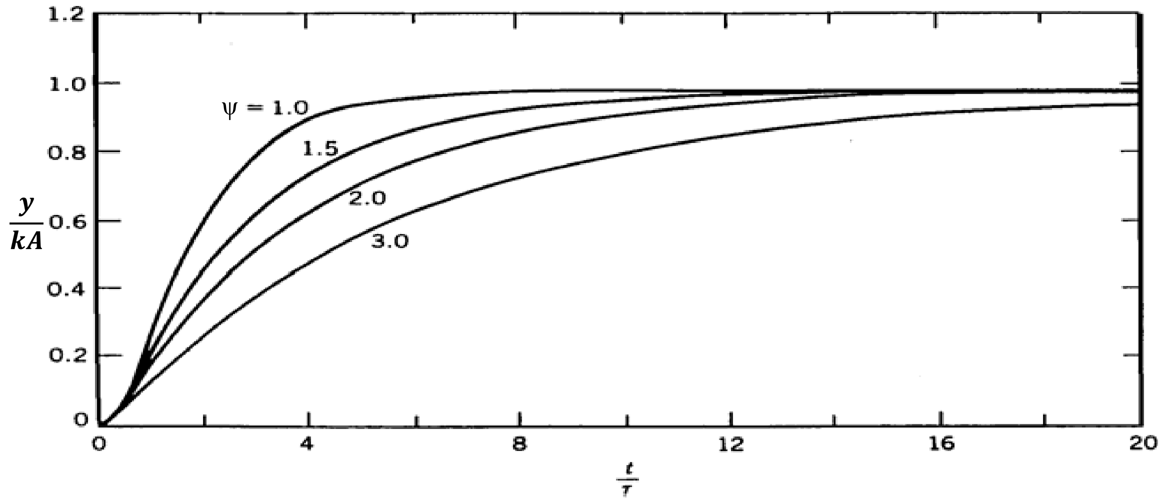
$$Y(s) = kA\left[\frac{1}{s} - \frac{s + 2\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2 + \frac{1-\psi^2}{\tau^2}}\right] = kA\left[\frac{1}{s} - \frac{s + \frac{\psi}{\tau} + \frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2 - \frac{\psi^2-1}{\tau^2}}\right]$$

$$= kA\left[\frac{1}{s} - \frac{s+\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2 - (\frac{\sqrt{\psi^2-1}}{\tau})^2} - \frac{\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2 - (\frac{\sqrt{\psi^2-1}}{\tau})^2}\right]$$

$$= kA\left[\frac{1}{s} - \frac{s+\frac{\psi}{\tau}}{(s+\frac{\psi}{\tau})^2 - (\frac{\sqrt{\psi^2-1}}{\tau})^2} - \frac{\frac{\psi}{\tau} \times \frac{\tau}{\sqrt{\psi^2-1}} \times \frac{\sqrt{\psi^2-1}}{\tau}}{(s+\frac{\psi}{\tau})^2 - (\frac{\sqrt{\psi^2-1}}{\tau})^2}\right]$$

$$\boxed{Y(t) = kA\left[1 - e^{-(\psi/\tau)t}\left(\cosh wt + \frac{\psi}{\sqrt{\psi^2-1}} \sinh wt\right)\right]}$$

where $w = \frac{\sqrt{\psi^2-1}}{\tau}$



Terms Used to Describe an Underdamped System

Second order system response for a step change

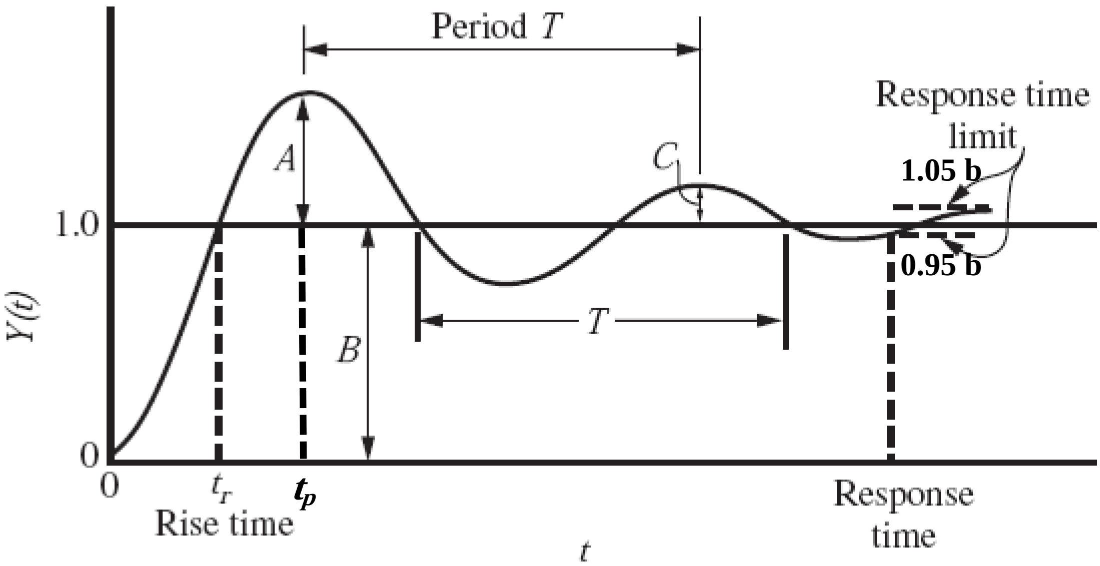


Figure (8-3) Terms used to describe an underdamped second-order response.

1. Overshoot (OS)

Overshoot is a measure of how much the response exceeds the ultimate value (new steady-state value) following a step change and is expressed as the ratio $\frac{A}{B}$ in the Fig. (8-3).

$$OS = exp\frac{-\pi\psi}{\sqrt{1-\psi^2}\ )}$$

$$OS\ \% = 100 \times OS$$

2. Decay ratio (DR)

The decay ratio is defined as the ratio of the sizes of successive peaks and is given by $\frac{C}{A}$ in Fig. (8-3).where *c* is the height of the second peak

$$DR = exp\frac{-2\pi\psi}{\sqrt{1-\psi^2}} = (OS)^2$$

3. Rise time (t_r)

This is the time required for the response to first reach its ultimate value and is labeled in Fig. (8-3).

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\psi^2}}{\psi}}{w}$$

4. Response time

This is the time required for the response to come within ±5 percent of its ultimate value and remain there. The response time is indicated in Fig. (8-3).

5. Period of oscillation (T)

The radian frequency (radians/time) is the coefficient of t in the sine term; thus,

$$T = \frac{2\pi\tau}{\sqrt{1-\psi^2}}$$

6. Natural period of oscillation

If the damping is eliminated (ψ=0), the system oscillates continuously without attenuation in amplitude. Under these "natural" or undamped condition, the radian frequency is $\frac{1}{\tau}$. This frequency is referred to as the natural frequency w_n.

$$w_n = \frac{1}{\tau}$$

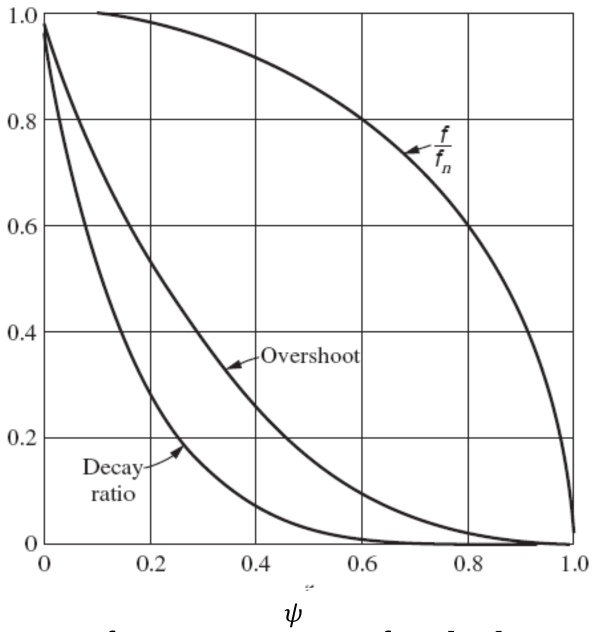
The corresponding natural cyclical frequency f_n and period T_n are related by the expression:-

$$f_n = \frac{1}{T_n} = \frac{1}{2\pi\tau}$$ Thus, τ has the significance of the undamped period.

7- Time to First Peak(t_p) :

Is the time required for the output to reach its first maximum value.

$$t_p = \frac{\pi}{w} = \frac{\pi\tau}{\sqrt{1-\psi^2}}$$



Figure(8-4) Characteristics of a step response of underdamped second-order system.

1-Over shoot

$wt + \phi = \phi + n\pi$

$$\boxed{t = \frac{n\pi}{w}}$$ max or min n=1, 2, 3 ……

If n=0, 2, 4 , 6, ……………. $\therefore$ min

If n=1, 3, 5, 7, ……………. $\therefore$ max

1st max when n=1

$$t = \frac{n\pi}{w} = \frac{\pi}{w}$$

$$y(t) = kA[1 - \frac{1}{\sqrt{1-\psi^2}} e^{\frac{-\psi\pi}{\tau w}} \sin(w\frac{\pi}{w} + \phi)]$$

$$y_{max} = kA[1 - \frac{1}{\sqrt{1-\psi^2}} e^{\frac{-\psi\pi}{\sqrt{1-\psi^2}}} (-\sin\phi)]$$

For Underdampded system

$$\cos\phi = -\psi \quad , \quad \sin\phi = \sqrt{1-\psi^2} \quad , \quad \tan\phi = \frac{\sqrt{1-\psi^2}}{-\psi}$$

$$\therefore y_{max} = kA[1 + \frac{1}{\sqrt{1-\psi^2}} e^{\frac{-\psi\pi}{\sqrt{1-\psi^2}}} (\sqrt{1-\psi^2})]$$

$$y_{max} = kA[1 + e^{\frac{-\psi\pi}{\sqrt{1-\psi^2}}}]$$

$$Overshoot = \frac{kA[1 + e^{\frac{-\psi\pi}{\sqrt{1-\psi^2}}}] - kA}{kA}]$$

$$\boxed{Overshoot = \exp\frac{-\psi\pi}{\sqrt{1-\psi^2})}}$$

$$Overshoot = \frac{A}{B} = \frac{max - B}{B}$$

$$Y(t) = kA[1 - \frac{1}{\sqrt{1-\psi^2}} e^{\frac{-\psi t}{\tau}} (\sinh(wt + \phi)]$$

where

$$w=\frac{\sqrt{1-\psi^2}}{\tau} \qquad , \qquad \phi=tan^{-1}\frac{\sqrt{1-\psi^2}}{\psi}$$

Overshoot could be optained by getting $\frac{dy}{dt}=0$

$$Y(t)=kA[1-e^{(-\psi/\tau)t}(r\,sin(wt+\phi))]$$

$$\frac{dY(t)}{dt}=0=0-\frac{1}{\sqrt{1-\psi^2}}[e^{\frac{-\psi t}{\tau}}cos(wt+\phi)w+sin(wt+\phi)e^{\frac{-\psi t}{\tau}}(\frac{-\psi}{\tau})]$$

$$0=-e^{\frac{-\psi t}{\tau}}[\frac{\sqrt{1-\psi^2}}{\tau}cos(wt+\phi)-\frac{\psi}{\tau}sin(wt+\phi)]$$

$$\frac{\sqrt{1-\psi^2}}{\tau}cos(wt+\phi)=\frac{\psi}{\tau}sin(wt+\phi)]$$

$$\frac{sin(wt+\phi)}{cos(wt+\phi)}=\frac{\sqrt{1-\psi^2}}{\psi}$$

$$tan(wt+\phi)=\frac{\sqrt{1-\psi^2}}{\psi}$$

$$(wt+\phi)=tan^{-1}\frac{\sqrt{1-\psi^2}}{\psi}$$

2-Decay Ratio

$$Decay\ \ ratio=\frac{C}{A}$$

$Decay\ \ ratio=\frac{C}{A}$ (The ratio of amount above the ultimate value of two sucessive peaks).

$t=\frac{n\pi}{w}$ for n=3 then $t=\frac{3\pi}{w}$

First peak at n=1 $y_{max}=kA[1+e^{\frac{-\pi\psi}{\sqrt{1-\psi^2}}}]$

Second peak at n=3 $y_{max}=kA[1+e^{\frac{-3\pi\psi}{\sqrt{1-\psi^2}}}]$

$$Decay\ Ratio = \frac{kA[1+e^{\frac{-3\pi\psi}{\sqrt{1-\psi^2}}}]-kA}{kA[1+e^{\frac{-\pi\psi}{\sqrt{1-\psi^2}}}]-kA} = \frac{e^{\frac{-3\pi\psi}{\sqrt{1-\psi^2}}}}{e^{\frac{-\pi\psi}{\sqrt{1-\psi^2}}}} = e^{\frac{-2\pi\psi}{\sqrt{1-\psi^2}}}$$

$$\boxed{Decay\ Ratio = \exp\frac{-2\pi\psi}{\sqrt{1-\psi^2}}}$$

3. Rise time ***(t_r)***

It is the time required for the response to first tauch the ultimate line.

$$y(t) = kA[1-\frac{1}{\sqrt{1-\psi^2}}e^{\frac{-\psi t}{\tau}}\sin(tw+\phi)]$$

At t_r $y(t)=kA$

$$kA = kA[1-\frac{1}{\sqrt{1-\psi^2}}e^{\frac{-\psi t}{\tau}}\sin(t_r w+\phi)]$$

$$0=\sin(t_r w+\phi)]$$

$$tr = \frac{\sin^{-1}(0)-\phi}{w}$$

$$tr = \frac{n\pi-\phi}{w} = \frac{n\pi-\tan^{-1}\frac{\sqrt{1-\psi^2}}{\psi}}{w} =$$

$$\boxed{tr = \frac{\pi-\tan^{-1}\frac{\sqrt{1-\psi^2}}{\psi}}{w}} \quad \text{for} \quad n=1$$

4-Period of oscillation (T)

$$w = Radian\ frequency = \frac{\sqrt{1-\psi^2}}{\tau}$$

$$w=2\pi f \quad also \quad T=\frac{1}{f}$$

$$\boxed{f=\frac{\sqrt{1-\psi^2}}{2\pi\tau}}$$

$$\boxed{\therefore T = \frac{2\pi\tau}{\sqrt{1-\psi^2}}}$$

5. Natural period of oscillation (T_n).

The system free of any damping forψ=0

$$w, radian\ \ of\ \ frequency = \frac{\sqrt{1-\psi^2}}{\tau} \quad \Rightarrow \quad w_n = \frac{1}{\tau}\ \ for\ \ \psi = 0$$

$$w_n = 2\pi f_n \qquad \frac{1}{\tau} = 2\pi f_n \qquad \therefore f_n = \frac{1}{2\pi\tau}$$

$$w_n = \frac{2\pi}{w_n} = \frac{2\pi\tau}{\sqrt{1-\psi^2}}$$

$$\boxed{T_n = 2\pi\tau} \text{ for } \psi=0 \qquad \boxed{\therefore \frac{f}{f_n} = \sqrt{1-\psi^2}}$$

6-Response time(t_s)

The time required for the response to reach (±5%) of its ultimate value and remain there.

7- Time to First Peak(t_p)

Is the time required for the output to reach its first maximum value.

$$t = \frac{n\pi}{w}$$

First peak is reached when n=1

$$t_p = \frac{n\pi}{w} = \frac{\pi}{w} = \frac{\pi\tau}{\sqrt{1-\psi^2}}$$

2- Impulse Response

If impulse δ(t) is applied to second order system then transfer response can be written as.

$$Y(s) = \frac{k}{\tau^2 s^2 + 2\psi\tau s + 1} X(s)$$

$$X(s) = Area = A$$

$$Y(s) = \frac{k}{\tau^2 s^2 + 2\psi\tau s + 1}.A$$

$$Y(s) = \frac{kA/\tau^2}{s^2 + \frac{2\psi}{\tau}s + \frac{1}{\tau^2}} = \frac{kA/\tau^2}{s^2 + \frac{2\psi}{\tau}s + \frac{1}{\tau^2} + (\frac{\psi}{\tau})^2 - (\frac{\psi}{\tau})^2}$$

$$=\frac{kA/\tau^2}{s^2+\frac{2\psi}{\tau}s+(\frac{\psi}{\tau})^2+\frac{1}{\tau^2}-(\frac{\psi}{\tau})^2}=\frac{kA/\tau^2}{(s+\frac{\psi}{\tau})^2+\frac{1-\psi^2}{\tau^2}}$$

i) ψ>1

$$Y(s)=\frac{kA/\tau^2}{(s+\frac{\psi}{\tau})^2+\frac{1-\psi^2}{\tau^2}}=\frac{kA/\tau^2}{(s+\frac{\psi}{\tau})^2-\left(\frac{\sqrt{\psi^2-1}}{\tau}\right)^2}=\frac{\frac{kA}{\tau^2}\frac{\tau}{\sqrt{\psi^2-1}}\frac{\sqrt{\psi^2-1}}{\tau}}{(s+\frac{\psi}{\tau})^2-\left(\frac{\sqrt{\psi^2-1}}{\tau}\right)^2}$$

$$Y(t)=\frac{kA}{\tau\sqrt{\psi^2-1}}e^{\frac{-\psi t}{\tau}}\ sinh\,wt$$

$$w=\frac{\sqrt{\psi^2-1}}{\tau}$$

ii)ψ<1

$$Y(s)=\frac{kA/\tau^2}{(s+\frac{\psi}{\tau})^2+\frac{1-\psi^2}{\tau^2}}=\frac{kA/\tau^2}{(s+\frac{\psi}{\tau})^2+\left(\frac{\sqrt{1-\psi^2}}{\tau}\right)^2}=\frac{\frac{kA}{\tau^2}\frac{\tau}{\sqrt{1-\psi^2}}\frac{\sqrt{1-\psi^2}}{\tau}}{(s+\frac{\psi}{\tau})^2+\left(\frac{\sqrt{1-\psi^2}}{\tau}\right)^2}$$

$$Y(t)=\frac{kA}{\tau\sqrt{1-\psi^2}}e^{\frac{-\psi t}{\tau}}\ sin\,wt$$

$$w=\frac{\sqrt{1-\psi^2}}{\tau}$$

iii)ψ=1

$$Y(s)=\frac{kA/\tau^2}{(s+\frac{\psi}{\tau})^2+\frac{1-\psi^2}{\tau^2}}=\frac{kA/\tau^2}{(s+\frac{1}{\tau})^2+\frac{1-1^2}{\tau^2}}=\frac{kA/\tau^2}{(s+\frac{1}{\tau})^2}$$

$$Y(t)=\frac{kA}{\tau^2}te^{-t/\tau}$$

Homework

1) For each of the second-order systems that follow, find ψ, τ, Tp, Tr ,OS and %OS.

$$a) \qquad Y(s) = \frac{16}{s^2 + 3s + 16}$$

$$b) \qquad Y(s) = \frac{0.04}{s^2 + -0.02s + 0.04}$$

2) The transfer function for a thermometer in a CSTR reactor is given by.

$$\frac{T_{thermometer}(s)}{T_{Reactor}(s)} = \frac{16}{(3s+1)(10s+1)}$$

Estimate the following.

a) find ψ, τ
b) The rise time.
c) The peak time.
d) The 2% settling time.
e) The percentage overshoot.

3) A second-order system is described by the differential equation:

$$\frac{d^2Y(t)}{dt^2} + 5\frac{dY(t)}{dt} + 25Y(t) = 25\,u(t)$$

a) Write down the transfer function Y(s)/U(s) of the system, where U(s) and Y(s) are the Laplace transforms of u(t) and y(t), respectively.
b) Obtain the damping ratio ψ and the natural frequency *Wn* of the system.
c) Calculate the rise time and percent overshoot of the system.
d) Evaluate y(t) for a unit-step input u(t).