Vectors: Triple Products

The Scalar Triple Product

The scalar triple product, as its name may suggest, results in a scalar as its result. It is a means of combining three vectors via cross product and a dot product. Given the vectors

$$\begin{aligned} \mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k} \\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k} \\ \mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k} \end{aligned}$$

a scalar triple product will involve a dot product and a cross product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

It is necessary to perform the cross product before the dot product when computing a scalar triple product,

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

since $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ one can take the dot product to find that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A_1) \begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - (A_2) \begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + (A_3) \begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

which is simply

Important Formula 3.1.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

The usefulness of being able to write the scalar triple product as a determinant is not only due to convenience in calculation but also due to the following property of determinants

Note Exchanging any two adjacent rows in a determinant changes the sign of the original determinant.

Thus,

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

Formula 3.1.

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

Example . Given,

$$\begin{aligned} \mathbf{A} &= 2\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{B} &= -\,\mathbf{i} + \mathbf{j} \\ \mathbf{C} &= 2\,\mathbf{i} + 2\,\mathbf{j} \end{aligned}$$

Find

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

Solution:
Method 1:

Begin by finding

$$\begin{aligned} \mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} \\ &= \big((1)(0) - (0)(2)\big)\mathbf{i} - \big((-1)(0) - (0)(2)\big)\mathbf{j} + \big((-1)(2) - (1)(2)\big)\mathbf{k} \\ &= 0\,\mathbf{i} + 0\,\mathbf{j} - 4\,\mathbf{k}. \end{aligned}$$

Take the dot product with **A** to find

$$\begin{aligned} \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= (2)(0) + (3)(0) + (-1)(-4) \\ &= 4 \end{aligned}$$

Method 2:
Evaluate the determinant

$$\begin{aligned} \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} &= (2)\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - (3)\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} \\ &= (2)\big((1)(0) - (0)(0)\big) - (3)\big((-1)(0) - (0)(2)\big) + (-1)\big((-1)(2) - (1)(2)\big) \\ &= 4 \end{aligned}$$

Example Prove that

Important Formula 3.2.

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

Solution:
Notice that there are no brackets given here as the only way to evaluate the scalar triple products is to perform the cross products before performing the dot products[a]. Let

$$\begin{aligned}\mathbf{A}&=A_1\,\mathbf{i}+A_2\,\mathbf{j}+A_3\,\mathbf{k}\\ \mathbf{B}&=B_1\,\mathbf{i}+B_2\,\mathbf{j}+B_3\,\mathbf{k}\\ \mathbf{C}&=C_1\,\mathbf{i}+C_2\,\mathbf{j}+C_3\,\mathbf{k}\end{aligned}$$

now,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}=-\begin{vmatrix}C_1 & C_2 & C_3\\ B_1 & B_2 & B_3\\ A_1 & A_2 & A_3\end{vmatrix}=\begin{vmatrix}C_1 & C_2 & C_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}=\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

The Vector Triple Product

The vector triple product, as its name suggests, produces a vector. It is the result of taking the cross product of one vector with the cross product of two other vectors.

Important Formula 3.3 (Vector Triple Product).

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Proving the vector triple product formula can be done in a number of ways. The straight-forward method is to assign

$$\begin{aligned}
\mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k} \\
\mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k} \\
\mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}
\end{aligned}$$

and work out the various dot and cross products to show that the result is the same. Here we shall however go through a slightly more subtle but less calculation heavy proof.

Note 3.2. The vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ must be in the same plane as $\mathbf{B}$ and $\mathbf{C}$. This is due to fact that the vector that results from the cross product is perpendicular to both the vectors whose product has just been taken. Since one can say that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is on the same plane as $\mathbf{B}$ and $\mathbf{C}$ it follows that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha\mathbf{B} + \beta\mathbf{C}$$

where α and β are scalars.

This is the desired result as returning to our definitions of **A**, **B**and **C** in this basis,

$$\begin{aligned}
\mathbf{A} &= A_1\mathbf{i}' + A_2\mathbf{j}' + A_3\mathbf{k}' \\
\mathbf{B} &= B_1\mathbf{i}' \\
\mathbf{C} &= C_1\mathbf{i}' + C_2\mathbf{j}'
\end{aligned}$$

one finds that,

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= A_1B_1 \\
\mathbf{A} \cdot \mathbf{C} &= A_1C_1 + A_2C_2
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (A_2C_2 + A_1C_1)B_1\mathbf{i}' - A_1B_1(C_1\mathbf{i}' + C_2\mathbf{j}') \\
&= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.
\end{aligned}$$

Summary of Vector Rules

Here we list most of the main results concerning vectors,

$\mathbf{A}\cdot\mathbf{A} = A^2 \equiv \mid \mathbf{A} \mid^2$	$\mathbf{A}\times\mathbf{A} = \mathbf{0}$
$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$	$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$
$\mathbf{A}\cdot(\alpha\mathbf{B}) = \alpha(\mathbf{A}\cdot\mathbf{B})$	$\mathbf{A}\times(\alpha\mathbf{B}) = \alpha(\mathbf{A}\times\mathbf{B})$
$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$	$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

Example 3.4.1 (Manipulating vectors without evaluation).
Prove that

(i) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})$

(ii) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D})$

Solution:

(i) let $\mathbf{U} = \mathbf{A}\times\mathbf{B}$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= \mathbf{U}\times(\mathbf{C}\times\mathbf{D})\\ &= (\mathbf{U}\cdot\mathbf{D})\mathbf{C} - (\mathbf{U}\cdot\mathbf{C})\mathbf{D}\\ &= (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}\\ &= (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\mathbf{D}\\ &= \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}).\end{aligned}$$

(ii) let $\mathbf{V} = (\mathbf{C}\times\mathbf{D})$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= -(\mathbf{C}\times\mathbf{D})\times(\mathbf{A}\times\mathbf{B})\\ &= -\mathbf{V}\times(\mathbf{A}\times\mathbf{B})\\ &= -\{(\mathbf{V}\cdot\mathbf{B})\mathbf{A} - (\mathbf{V}\cdot\mathbf{A})\mathbf{B}\}\\ &= (\mathbf{C}\times\mathbf{D}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\times\mathbf{D}\cdot\mathbf{B})\mathbf{A}\\ &= (\mathbf{C}\cdot\mathbf{D}\times\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{D}\times\mathbf{B})\mathbf{A}\\ &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D}).\end{aligned}$$

Example *If* $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{i}+m\mathbf{j}+4\mathbf{k}$ *are coplanar, find the value of* m .

Proof. Since the given three vectors are coplanar, we have

$$\begin{vmatrix}2 & -1 & 3\\ 3 & 2 & 1\\ 1 & m & 4\end{vmatrix} = 0 \implies m = -3$$

Example *Show that the four points (6, 7, 0), (16, 19, 4), (0, 3, 6), (2, 5,10) lie on a same plane.*

Proof. Let A = (6, 7, 0), B = (16, 19, 4), C = (0, 3, 6), D = (2, 5,10) . To show that the four points A, B, C, D lie on a plane, we have to prove that the three vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.
Now,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (16\mathbf{i} - 19\mathbf{j} - 4\mathbf{k}) - (6\mathbf{i} - 7\mathbf{j}) = (10\mathbf{i} - 12j - 4\mathbf{k})$$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-6\mathbf{i} + 10\mathbf{j} - 6\mathbf{k})$ and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-4\mathbf{i} + 2\mathbf{j} + 10\mathbf{k})$.
we have

$$[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix} = 0.$$

H.W.

Find the area of the triangle with the vertices P(0,1,4), Q(-5,9,2), and R(7,2,8):

H.W.

Use the scalar triple product to show that the vectors **a** = ⟨1, 4, –7⟩, **b** = ⟨2, –1, 4⟩, and **c** = ⟨0, –9, 18⟩ are coplanar.

Example

Find the volume of the parallelepiped spanned by the vectors $\mathbf{a} = (-2, 3, 1)$, $\mathbf{b} = (0, 4, 0)$, and $\mathbf{c} = (-1, 3, 3)$.

Solution: The volume is the absolute value of the scalar triple product of the three vectors.

The triple product is

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \begin{vmatrix} -1 & 3 & 3 \\ -2 & 3 & 1 \\ 0 & 4 & 0 \end{vmatrix} \\ &= -1(0-4) - 3(0-0) + 3(-8+0) \\ &= 4 - 24 = -20\end{aligned}$$

Hence the volume is $|-20| = 20$.