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حيدر

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Sequances and infinite Series

1) Sequances

المتتابعات

A sequance is a function whose domain is The set of Positive numbers, for Example

$a(1)=1, \quad a(2)=\frac{1}{2}, \quad a(3)=\frac{1}{3}, \; ----, a(n)=\frac{1}{n}, \; ----$

$a(n)$ is Called The nth term of The sequance, we write a_n instead of $a(n)$, also the sequance, $a_1, a_2, a_3, a_4, ---, a_n, ---$ is denoted by

$$\{a_n\}_{n=1}^{\infty}$$

Ex write The first five terms of :-

1) $\left\{\frac{2n-1}{3n+2}\right\}$, here $a_n=\frac{2n-1}{3n+2}$

Sol $\frac{1}{5}, \frac{3}{8}, \frac{5}{11}, \frac{17}{14}, \frac{9}{17}$

2) $\left\{\frac{1-(-1)^n}{n^3}\right\}$, here $a_n=\frac{1-(-1)^n}{n3}$

Sol $2, 0, \frac{2}{3^3}, 0, \frac{2}{5^3}$

Ex write the nth term of The Sequances :-

1) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \Rightarrow$ ans $\left\{1-\frac{1}{n}\right\}$

2) $0, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4}, \ldots \Rightarrow$ ans $\left\{\frac{\ln n}{n}\right\}$

3) $0, \frac{1}{2^2}, \frac{2}{3^2}, \frac{3}{4^2}, \frac{4}{5^2}, \ldots \Rightarrow$ ans $\left\{\frac{n-1}{n^2}\right\}$

4) $2, 1, \frac{2^3}{3^2}, \frac{2^4}{4^2}, \frac{2^5}{5^2}, \ldots \Rightarrow$ ans $\left\{\frac{2^n}{n^2}\right\}$

"مهم جداً"

Definition :- If limite $\lim\limits_{n\to\infty} a_n = L, (L \neq \infty)$ then the sequance $\{a_n\}$ is said to be **Converge** to L. if no such limit exists then The Sequance $\{a_n\}$ is **diverge**

Ex

1) The sequance $\{\frac{1}{n}\}$ converge to Zero since

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

2) The sequance $\{n^2\}$ divarge since $\lim\limits_{n\to\infty} n^2 = \infty$

3) The Sequance $\{1+(-1)^n\}$ diverge since $\lim_{n\to\infty}[1+(-1)^n]$ does not exists.

Theorm :- let $\{a_n\}$ and $\{b_n\}$ are two sequances such That $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

1) $\lim_{n\to\infty}(a_n \mp b_n) = \lim_{n\to\infty} a_n \mp \lim_{n\to\infty} b_n = A \mp B$

2) $\lim_{n\to\infty}(a_n \cdot b_n) = \left[\lim_{n\to\infty} a_n\right] \cdot \left[\lim_{n\to\infty} b_n\right] = A \cdot B$

3) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$ if $B \neq 0$

4) $\lim_{n\to\infty}(k\,a_n) = k \lim_{n\to\infty} a_n = k \cdot A$ where $k = $ const.

5) $\lim_{n\to\infty} a_n^k = \left[\lim_{n\to\infty} a_n\right]^k = A^k$

6) $\lim_{n\to\infty} k^{a_n} = k^{\lim_{n\to\infty} a_n} = k^A$, if A^k and k^A exists

Theorem :-

let $\{a_n\}$, $\{b_n\}$ and $\{C_n\}$ be three sequances such That $a_n \leq b_n \leq C_n$, for all n.
if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} C_n = L$, Then $\{b_n\}$ Converge to L

Ex Determine which of The following sequances Converge or divarge

1) $\{\sqrt{n+1} - \sqrt{n}\}$

sol

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[\frac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$ it converge to Zero

2) $\left\{\frac{3n^2 - 5n}{5n^2 + 2n - 6}\right\}$

sol

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 - 5n}{5n^2 + 2n - 6} = \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} - \frac{5n}{n^2}}{\frac{5n^2}{n^2} + \frac{2n}{n^2} - \frac{6}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3 - \frac{5}{n}}{5 + \frac{2}{n} - \frac{6}{n^2}} = \frac{3}{5}$$ it Converge to $\frac{3}{5}$.

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h.w

3) $\left\{\dfrac{3n^2+4n}{2n-1}\right\}$,

h.w

4) $\left\{\left(\dfrac{2n-3}{3n-7}\right)^4\right\}$

h.w

5) $\left\{\dfrac{2n^5-4n^2}{3n^7+n^2-10}\right\}$

h.w

6) $\left\{\dfrac{2^n}{5n}\right\}$

h.w

7) $\left\{\dfrac{\ln n}{e^n}\right\}$

8) $\left\{(1+n)^{\frac{1}{n}}\right\}$

<u>limits we need</u>

<u>ضروري</u>

1) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

2) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3) $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$, $(x>0)$

4) $\lim\limits_{n\to\infty} x^n = 0$, $(-1<x<1)$

5) $\lim\limits_{n\to\infty} \left(1+\dfrac{x}{n}\right)^n = e^x$, (any x)

6) $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$, (any x)

h.w ⑥

Ex Determine which of The Sequane Converge or dduarge. Find The limit of each Sequance That Converge.

1) $\left\{\frac{1-n}{n^2}\right\}$, 2) $\left\{\frac{n}{2^n}\right\}$, 3) $\left\{\left(\frac{1}{3}\right)^n\right\}$, 4) $\left\{\frac{1}{n!}\right\}$

5) $\left\{\frac{(-1)^{(n+1)}}{2n-1}\right\}$, 6) $\left\{2+(-1)^n\right\}$, 7) $\left\{8^{\frac{1}{n}}\right\}$

8) $\left\{\frac{1}{10n}\right\}$, 9) $\left\{\frac{1+(-1)^n}{n}\right\}$, 10) $\left\{1+\frac{(-1)^n}{n}\right\}$

11) $\left\{(-1)^n\left(1-\frac{1}{n}\right)\right\}$, 12) $\left\{\frac{2n+1}{1-3n}\right\}$, 13) $\left\{\frac{n^2-n}{2n^2+n}\right\}$

14) $\left\{\sqrt{\frac{2n}{n+1}}\right\}$, 15) $\left\{\frac{\sin n}{n}\right\}$, 16) $\left\{\sqrt[2n]{n}\right\}$

Infinite Series

اذا اردنا ان نجمع ما لانهاية من الارقام

$$a_1, a_2, a_3, a_4, \text{-----}, a_n, \text{-----}$$

$$\therefore \quad a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

Def

$\sum_{n=1}^{\infty} a_n$ is Converge to L

اذا كانت قابلة للجمع ، هناك طريقة للجمع هي :

أولاً: نجد S (الجمع) :

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

ثانياً: نأخذ النهاية

$$\lim_{n \to \infty} S_n = L \qquad \therefore \sum_{n=1}^{\infty} a_n = L$$

if The limit is not exist

$$\therefore \quad L = \infty$$

$$\text{or} \quad L = \begin{cases} \oplus \\ \ominus \end{cases}$$

نفس الجواب في حالة
رقمين مختلفين
هنا تسمى diverge

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Ex

حدد فيما اذا كانت هذه المتوالية هي

Converge or diverge

واذا كانت converge احسب المجموع

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots$$

اذا كان المقام هو حاصل ضرب كميات فأول خطوة هو تجزئة المقام

$$a_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{n(n+1)}$$

$$1 = A(n+1) + Bn = An + A + Bn$$

$$1 = (A+B)n + A$$

$$\therefore \quad A = \boxed{1} \quad , \quad A + B = 0 \Rightarrow B = -A = \boxed{-1}$$

$$\therefore \quad a_n = \frac{1}{n} - \frac{1}{n+1}$$

الان نحسب S_n :

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$$S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$

الكل تنشطب يبقى الاول والاخير فقط

$$\therefore \quad \boxed{S_n = 1 - \frac{1}{n+1}}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 - \frac{1}{\infty} = 1 - 0 = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \quad \underline{\text{or}} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converge to } 1$$

③

Geometric Series

وصيغتها:

$$a + ar + ar^2 + ar^3 + \dots + ar^n = \sum_{n=0}^{\infty} ar^n$$

$$= \sum_{n=1}^{\infty} ar^{n-1}$$

وهي بالنسبة للجمع ومجموعها موجود وهو

1) Converge to $\frac{a}{1-r}$ if $-1 < r < 1$

2) Diverge if $r \geq 1$, $r \leq -1$

Ex

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots$$

it is geometric series

$a = 1$, $r = \frac{2}{3}$

$\therefore r = \frac{2}{3} < 1 \Rightarrow$ it is Converge

بالنسبة للجمع

$$\therefore \frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = 3$$

$$\therefore \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \boxed{3}$$

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Ex $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n = \frac{3}{2} + \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^2 + \cdots$

it is geometric Series

$a = \frac{3}{2}$, $r = \frac{3}{2}$

$r = \frac{3}{2} > 1$, it is diverge

غير قابلة للجمع

$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n = \infty$

Ex $0.3333\cdots$

Sol

$0.33333\cdots = 0.3 + 0.03 + 0.003 + \cdots$

$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = 3\left[\frac{1}{10} + \left(\frac{1}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)\left(\frac{1}{10}\right)^2 + \cdots\right]$

it is geometric Series

$\therefore a = \frac{1}{10}$, $r = \frac{1}{10} \Rightarrow r = \frac{1}{10} < 1$ is Converge

$\therefore \frac{a}{1-r} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{0.1}{0.9} = \frac{1}{9}$

$\therefore 0.3333\cdots = 3\left[\frac{1}{9}\right] = \frac{3}{9} = \frac{1}{3}$

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اختبارات فيما اذا كانت المتوالية قابلة للجمع أم لا :

* Test for convergens and divergens

اول الاختبارات هو :-

1) The nth Term Test

هنا خطوات العمل :

1) if $\lim\limits_{n \to \infty} a_n \neq 0$

$$\sum_{n=1}^{\infty} a_n = \text{diverge}$$

ناخذ النهاية لـ a_n
فاذا كانت النهاية غير صفر لا يساوي صفر
معناه ان الدالة Diverge

Ex $\sum\limits_{n=1}^{\infty} \dfrac{2n}{n+1}$

sol

$$\lim_{n \to \infty} \frac{2n}{n+1} = \lim_{n \to \infty} \frac{2n}{n+1} = \lim_{n \to \infty} \frac{2}{1+\frac{1}{n}} = 2 \neq 0$$

by the nth Term Test , it is diverge

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2) integration Test

اختبار التكامل

if $\sum\limits_{n=1}^{\infty} a_n \Rightarrow$ Ex $a_n = \dfrac{2n}{n+1}$

أولاً نحول a_n الى $f(x)$ ثم نجد التكامل

$\therefore$ we find $f(x) = \dfrac{2x}{x+1}$

$$\therefore \boxed{\int_1^{\infty} f(x)\,dx}$$

- اذا كانت نتيجة التكامل = K اي عدد حقيقي

$\therefore \sum a_n = \text{converg}$

- اذا كانت نتيجة التكامل = ∞

$\therefore \sum a_n = \text{diverge}$

Ex (6) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Sol

$$a_n = \frac{1}{n(\ln n)^2} \Rightarrow f(x) = \frac{1}{x(\ln x)^2}$$

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_2^{b} \frac{dx}{x(\ln x)^2}$$

$$= \lim_{b\to\infty} \int_2^{b} \frac{1}{x}(\ln x)^{-2}\,dx = \lim_{b\to\infty} \left[\frac{(\ln x)^{-1}}{-1}\right]_2^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{\ln x}\right]_2^b = \lim_{b\to\infty} \left[\frac{-1}{\ln b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

$$\therefore \int_2^{\infty} f(x)\,dx = \frac{1}{\ln 2} = c.v$$

it is Converge

P_Series :

وهي بالشكل التالي :-

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\cdots=\sum_{n=1}^{\infty}\frac{1}{n^p}$$

$\boxed{P>1}$ وهي Converge عندما يكون

$\boxed{P\leq 1}$ وتكون diverge عندما يكون

Ex Test The series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{(n)^{\frac{1}{2}}}$$

بالمقارنة مع الصيغة اعلاه نجد انها P_Series

$\because P<1 \Rightarrow$ it is diverge