①

Differential equations

<u>def</u> Differential equations is a function which contains $x, y, \dot{y}, \ddot{y}, \dddot{y}, \ldots, \overset{(n)}{y}$; and can be expressed mathematically as $F(x, y, \dot{y}, \ddot{y}, \ldots, \overset{(n)}{y}) = 0$

<u>def</u> the type of differential equation (d.e) is either:-

- ordinary differential equation (o.d.e) if all derivatives are total derivatives

<u>Ex</u> $$x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} + \sin x = e^{xy}$$

<u>or</u>

- partial differential equation (p.d.e) if there exists a function of partial derivative.

<u>Ex</u> $$\frac{\partial^2 y}{\partial x^2} + \sin x \frac{\partial y}{\partial x} + x^2 = 0$$

<u>def</u> the order of a d.e is the order of the highest derivative

<u>Ex</u> $$\frac{d^2y}{dx^2} + x^2 \frac{d^3y}{dx^3} + \sin(xy) = e^x \Rightarrow$$ 3rd order o.d.e

<u>Ex</u> $$y^{(5)} + x^2 y^2 = \sin x$$ 5th order o.d.e

<u>def</u> the degree of d.e is the degree (exponent) of the highest derivative.

②

Ex $(\dot{y})^4 + (\ddot{y})^5 + (\dddot{y})^2 = \sin x$

This is a 3rd order, 2nd degree (o.d.e)

Ex $(\ddot{y})^6 + 6\dot{y} = 20$

This is a 2nd order, 6th degree o.d.e.

def the solution of d.e. is to find the function y which satisfy the d.e. in this chapter. the solution of the following two types of d.e. will be considered.

① First order differential equation $\{ f(x, y, \dot{y}) = 0 \}$

Ⓐ variable separable

Ⓑ linear first order

② Second order differential equation $\{ f(x, y, \dot{y}, \ddot{y}) = 0 \}$

ⓐ linear homogeneous second order with constant coefficient.

ⓑ linear non-homogeneous second order with constant coefficient.

* (A) Variable separable d.e (first order)

consider the following d.e

$$P(x) + Q(y)\frac{dy}{dx} = 0$$

if the x variable can be separabled from the y vari

such as :-

$$P(x)\,dx + Q(y)\,dy = 0$$

then this equation is called " separable d.e " .

The solution can be defined by integrating both sides o

$$\int P(x)\,dx + \int Q(y)\,dy = C$$

Ex solve the following d.e

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$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$(x+1)\frac{dy}{dx} = x(y^2+1)$$

sol $(x+1)\,dy = x(y^2+1)\,dx$

$$\frac{dy}{(y^2+1)} = \frac{x}{(1+x)}\,dx \Rightarrow \int \frac{dy}{(y^2+1)} = \int \frac{x\,dx}{(1+x)}$$

$$\tan^{-1} y = \int\left(1 - \frac{1}{x+1}\right)dx = x - \ln(1+x) + c$$

note $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

Ex Solve $y\,dx = 2(xy + x)\,dy$

sol. $y\,dx = 2x(y+1)\,dy$

$$\therefore \int \frac{dx}{\cancel{2}x} = \int \frac{2(y+1)}{y}\,dy$$

$$\therefore \ln|x| = 2y + 2\ln|y| + c$$

Ex $x(2y-3)\,dx + (x^2+1)\,dy = 0$

sol

$$\frac{x}{x^2+1}\,dx + \frac{dy}{2y-3} = 0$$

$$\frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{2\,dy}{2y-3} = 0$$

$$\frac{1}{\cancel{2}} \ln|x^2+1| + \frac{1}{\cancel{2}} \ln|2y+3| = c_1 \qquad * \cancel{2}$$

if $c_1 = \ln c$

$$\ln[(x^2+1)(2y-3)] = \ln c$$

$$\therefore (x^2+1)(2y-3) = c$$

Ex Solve
h.w

$$y\sqrt{2x^2+3}\,dy + x\sqrt{4-y^2}\,dx = 0$$

Ex solve
h.w

$$\frac{dy}{dx} = e^{x+y}$$

Ex solve
h.w

$$x e^{y}\,dy + \frac{(x^2+1)}{y}\,dx = 0$$

Equation redueable to separable d.e

$$\boxed{\dot{y} = g\left(\frac{y}{x}\right)}$$

Ex

$$\dot{y} = \left(\frac{y}{x}\right)^2 \quad ; \quad \dot{y} = \sin\frac{y}{x} + 2\left(\frac{x}{y}\right)^3$$

let $\frac{y}{x} = u \Rightarrow y = xu \Rightarrow \dot{y} = u + x\dot{u}$

$\therefore u + x\dot{u} = g(u) \Rightarrow x\dot{u} = g(u) - u$

$\therefore x\frac{du}{dx} = g(u) - u \Rightarrow \int \frac{du}{g(u)-u} = \int \frac{dx}{x}$ ---- etc

Ex **Solve** :- $(x-3y)\,dx = (3x+2y)\,dy$

sol $\dfrac{dy}{dx} = \dfrac{x-3y}{3x+2y} \Rightarrow \dfrac{dy}{dx} = \dfrac{1-3(y/x)}{3+2(y/x)}$

let $\dfrac{y}{x} = u \Rightarrow y = ux \Rightarrow \dot{y} = u + x\dot{u}$

$\therefore\ u + x\dot{u} = \dfrac{1-3u}{3+2u}$

$x\dot{u} = \dfrac{1-3u}{3+2u} - u$

$x\,\dfrac{du}{dx} = \dfrac{1-6u+2u^2}{3-2u}$

$\therefore\ \displaystyle\int \frac{dx}{x} = -\int \frac{3+2u}{2u^2+6u-1}\,du$

$\ln|x| = -\dfrac{1}{2}\ln|2u^2+6u-1| + c$

Ex - $\dot{y} = x^2 + 2xy + y^2 + 2x + 2y$ h.w

(11)

linear first order d.e :-

the first order d.e $f(x, y, \dot{y}) = 0$ is called a linear first order d.e. if it has the following form:-

$$\frac{dy}{dx} + P(x)\, y = Q(x) \quad \text{---} (1)$$

the solution of equation (1) is formed by multiplying both sides of the d.e by the (integrating factor, $I(x)$).

steps of the method of solution

1) put the d.e in the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

2) find $I(x)$ such that

$$I(x) = e^{\int P(x)\, dx}$$

3) Rewrite the d.e as

$$dy + y\, P(x)\, dx = Q(x)\, dx$$

4) multiply both sides by $I(x)$

$$I(x)\,dy + y\,P(x)\,I(x)\,dx = Q(x)\,I(x)\,dx$$ (5

5) Integrate both sides

$$\int [I(x)\,dy + y\,P(x)\,I(x)\,dx] = y\,I(x)$$

6) the solution is,

$$\boxed{y\,I(x) = \int Q(x)\,I(x)\,dx}$$

<u>Proof</u>

we have to prove

$$\int [I(x)\,dy + y\,P(x)\,I(x)\,dx] = I(x).y$$

1) $\frac{d}{dx}[I(x).y] = I(x)\frac{dy}{dx} + y\,P(x)\,I(x)$

$$\frac{d}{dx}[I(x).y] = y\,\dot{I}(x) + I(x).\frac{dy}{dx}$$

2) from the above we have

$$\dot{I}(x) = I(x)\,P(x)$$

3) sub. in step 1

$$\frac{d}{dx}[I(x).y] = y\,P(x)\,I(x) + I(x)\frac{dy}{dx}$$

$$d[I(x).y] = y\,P(x)\,I(x)\,dx + I(x)\,dy$$

hence $I(x).y = \int [y\,P(x)\,I(x)\,dx + I(x)\,dy]$

linear

Ex $\quad \frac{dy}{dx} + y = e^{x}$

sol $\quad P(x) = 1 \; , \quad Q(x) = e^{x}$

$$I(x) = e^{\int P(x)dx} = e^{\int dx} = e^{x}$$

$$\therefore \; dy + y\,dx = e^{x}\,dx$$

$$e^{x}dy + y e^{x}dx = e^{2x}dx$$

$$y e^{x} = \int e^{2x}\,dx$$

$$y e^{x} = \frac{1}{2} e^{2x} + c$$

$$y = \frac{1}{2} e^{x} + c e^{-x}$$

to verify :-

$$y = \frac{1}{2} e^{x} - c e^{-x}$$

subs in d.e $\Rightarrow$ h.w

<u>EX</u> $x\frac{dy}{dx} - 3y = x^2$

<u>Sol</u> $\frac{dy}{dx} - \frac{3}{x} y = x$

$\therefore P(x) = \frac{-3}{x}, \quad Q(x) = x$

$I(x) = e^{\int P(x)dx} = e^{-\int \frac{3}{x} dx} = e^{-3\ln|x|} = e^{\ln x^{-3}} = x^{-3}$

$\therefore dy - \frac{3}{x} y dx = x dx$

$x^{-3} dy - \frac{3x^{-3}}{x} y dx = x^{-3} x dx$

$x^{-3} y = \int x^{-2} dx = -x^{-1} + c$

<u>EX</u> Solve $\cos x \frac{dy}{dx} + \sin x \cdot y = \sin x \cos x$

<u>Sol</u> $\frac{dy}{dx} + \frac{\sin x}{\cos x} \cdot y = \sin x$

$P(x) = \frac{\sin x}{\cos x}, \quad Q(x) = \sin x$

$I(x) = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\int \frac{-\sin x}{\cos x} dx} = e^{-\ln|\cos x|}$

$= e^{\ln(\cos x)^{-1}} = (\cos x)^{-1} = \frac{1}{\cos x}$

$\therefore dy + y \frac{\sin x}{\cos x} dx = \sin x \, dx$

$\frac{1}{\cos x} dy + y \frac{\sin x}{(\cos x)^2} dx = \frac{\sin x}{\cos x} dx$

$\frac{1}{\cos x} \cdot y = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + c$

to verify $\Rightarrow$ h.w

Ex Solve $(e^{x}+1)\frac{dy}{dx}+ye^{x}=xe^{-2x}+xe^{-x}+5$