2. Systems and Classification of Systems

In this section, we classify systems into six basic categories:
(i) linear and non-linear systems;
(ii) time-invariant and time-varying systems;
(iii) systems with and without memory;
(iv) causal and non-causal systems;
(v) invertible and non-invertible systems;
(vi) stable and unstable systems.

In the following discussion, we make use of the following notations:

CT system $x(t) \rightarrow y(t)$;
DT system $x[k] \rightarrow y[k]$;

2.2.1 Linear and non-linear systems

A CT system with the following set of inputs and outputs:

$$x_1(t) \rightarrow y_1(t) \text{ and } x_2(t) \rightarrow y_2(t)$$

is linear if it satisfies the additive and the homogeneity properties described below:

additive property: $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$; (2.28)
homogeneity property: $\alpha x_1(t) \rightarrow \alpha y_1(t)$; (2.29)

or: A CT system with the following sets of inputs and outputs:

$$x_1(t) \rightarrow y_1(t) \text{ and } x_2(t) \rightarrow y_2(t)$$

is linear if

$$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t) \quad (2.30)$$

for any arbitrary set of values for α and β, and for all possible combinations of inputs and outputs.

Likewise, a DT system with

$$x_1[k] \rightarrow y_1[k] \text{ and } x_2[k] \rightarrow y_2[k],$$

is linear if

$$\alpha x_1[k] + \beta x_2[k] \rightarrow \alpha y_1[k] + \beta y_2[k] \quad (2.31)$$

for any arbitrary set of values for α and β, and for all possible combinations of inputs and outputs.

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Example 2.10: Consider the CT systems with the following input–output relationships:

(a) differentiator $y(t) = \frac{dx(t)}{dt}$ (2.33)

(b) exponential amplifier $x(t) \rightarrow e^{x(t)}$ (2.34)

(c) amplifier $y(t) = 3x(t)$ (2.35)

(d) amplifier with additive bias $y(t) = 3x(t) + 5$ (2.36)

Determine whether the CT systems are linear.

Solution

(a) From Eq. (2.33), it follows that

$$x_1(t) \rightarrow \frac{dx_1(t)}{dt} = y_1(t)$$

and

$$x_2(t) \rightarrow \frac{dx_2(t)}{dt} = y_2(t)$$

which yields

$$\alpha x_1(t) + \beta_1 x_2(t) \rightarrow \frac{d}{dt}\{\alpha x_1(t) + \beta_1 x_2(t)\} = \alpha\frac{dx_1(t)}{dt} + \beta\frac{dx_2(t)}{dt}.$$

Since

$$\alpha\frac{dx_1(t)}{dt} + \beta\frac{dx_2(t)}{dt} = \alpha y_1(t) + \beta y_2(t),$$

the differentiator as represented by Eq. (2.33) is a linear system.

(b) From Eq. (2.34), it follows that

$$x_1(t) \rightarrow e^{x_1(t)} = y_1(t)$$

and

$$x_2(t) \rightarrow e^{x_2(t)} = y_2(t),$$

giving

$$\alpha x_1(t) + \beta x_2(t) \rightarrow e^{\alpha x_1(t) + \beta x_2(t)}.$$

Since

$$e^{\alpha x_1(t) + \beta x_2(t)} = e^{\alpha x_1(t)} \cdot e^{\beta x_2(t)} = [y_1(t)]^{\alpha} + [y_2(t)]^{\beta} \neq \alpha y_1(t) + \beta y_2(t),$$

the exponential amplifier represented by Eq. (2.34) is not a linear system.

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(c) From (2.35), it follows that

$$x_1(t) \to 3x_1(t) = y_1(t)$$

and

$$x_2(t) \to 3x_2(t) = y_2(t),$$

giving

$$\alpha x_1(t) + \beta x_2(t) \to 3\{\alpha x_1(t) + \beta x_2(t)\} = 3\alpha x_1(t) + 3\beta x_2(t)$$
$$= \alpha y_1(t) + \beta y_2(t).$$

Therefore, the amplifier of Eq. (2.35) is a linear system.

(d) From Eq. (2.36), we can write

$$x_1(t) \to 3x_1(t) + 5 = y_1(t)$$

and

$$x_2(t) \to 3x_2(t) + 5 = y_2(t),$$

giving

$$\alpha x_1(t) + \beta x_2(t) \to 3[\alpha x_1(t) + \beta x_2(t)] + 5.$$

Since

$$3[\alpha x_1(t) + \beta x_2(t)] + 5 = \alpha y_1(t) + \beta y_2(t) - 5,$$

the amplifier with an additive bias as specified in Eq. (2.36) is not a linear system.

Example 2.11

Consider two DT systems with the following input–output relationships:

(a) differencing system $\quad y[k] = 3(x[k] - x[k-2]);$ (2.37)
(b) sinusoidal system $\quad y[k] = sin(x[k]).$ (2.38)

Determine if the DT systems are linear.

Solution

(a) From Eq. (2.37), it follows that:

$$x_1[k] \to 3x_1[k] - 3x_1[k-2] = y_1[k]$$

and

$$x_2[k] \to 3x_2[k] - 3x_2[k-2] = y_2[k],$$

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giving

$$\alpha x_1[k] + \beta x_2[k] \Rightarrow 3\alpha x_1[k] - 3\alpha x_1[k-2] + 3\beta x_2[k] - 3\beta x_2[k-2].$$

Since

$$3\alpha x_1[k] - 3\alpha x_1[k-2] + 3\beta x_2[k] - 3\beta x_2[k-2] = \alpha y_1[k] + \beta y_2[k],$$

the differencing system, Eq. (2.37), is linear.

- To illustrate the linearity property graphically, we consider two DT input signals $x_1[k]$ and $x_2[k]$ shown in the two top-left subplots in Figs. 2.11(a) and (c).
- The resulting outputs $y_1[k]$ and $y_2[k]$ for the two inputs applied to the differencing system, Eq. (2.37), are shown in the two top-right stem subplots in Figs. 2.11(b) and (d), respectively.
- A linear combination, $x_3[k] = x_1[k] + 2x_2[k]$, of the two inputs is shown in the bottom-left subplot in Fig. 2.11(e).
- The resulting output $y_3[k]$ of the system for input signal $x_3[k]$ is shown in the bottom-right subplot in Fig. 2.11(f).
- By looking at the subplots, it is clear that the output $y_3[k] = y_1[k] + 2y_2[k]$.
- In other words, the output $y_3[k]$ can be determined by using the same linear combination of outputs $y_1[k]$ and $y_2[k]$ as the linear combination used to obtain $x_3[k]$ from $x_1[k]$ and $x_2[k]$.

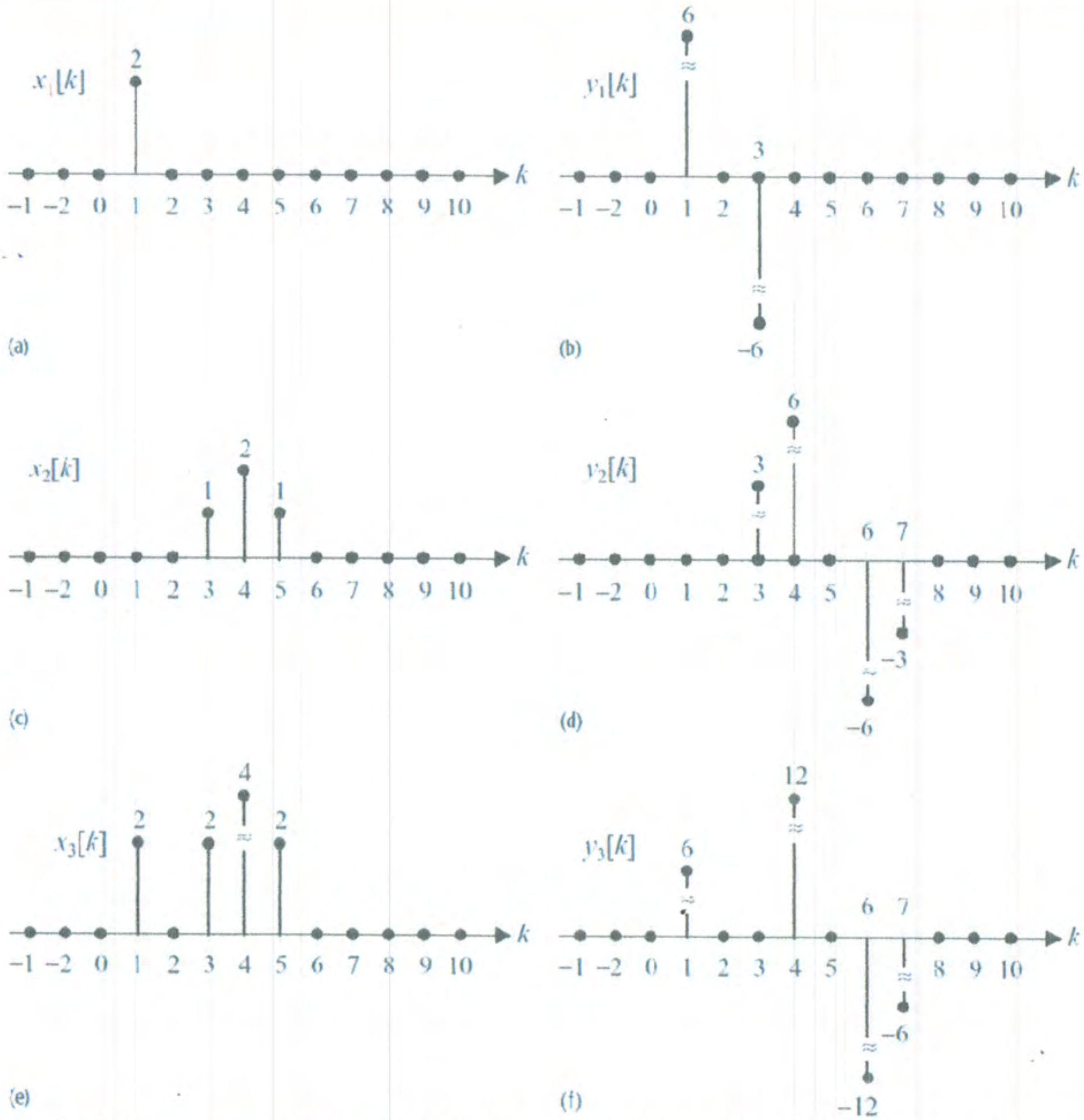

Fig. 2.11. Input–output pairs of the linear DT system specified in Example 2.2(a). Parts (a)–(f) are discussed in the text.

(b) From Eq. (2.38), it follows that:

$$x_1[k] \rightarrow \sin(x_1[k]) = y_1[k], \quad x_2[k] \rightarrow \sin(x_2[k]) = y_2[k],$$

giving

$$\alpha x_1[k] + \beta x_2[k] \rightarrow \sin(\alpha x_1[k]) + \sin(\beta x_2[k]) \neq \alpha y_1[k] + \beta y_2[k];$$

therefore, the sinusoidal system in Eq. (2.38) is not linear.

Example 2.12

Consider the AM system with input–output relationship given by

$$s(t) = [1 + 0.2m(t)]\cos(2\pi \times 10^8 t). \quad (2.39)$$

Determine if the AM system is linear.

Solution

From Eq. (2.39), it follows that:

$$m_1(t) \rightarrow [1 + 0.2m_1(t)]\cos(2\pi \times 10^8 t) = s_1(t)$$

and

$$m_2(t) \rightarrow [1 + 0.2m_2(t)]\cos(2\pi \times 10^8 t) = s_2(t)$$

giving

$$\alpha m_1(t) + \beta m_2(t) \rightarrow [1 + 0.2\{\alpha m_1(t) + \beta m_2(t)\}]\cos(2\pi \times 10^8 t) \neq \alpha s_1(t) + \beta s_2(t).$$

Therefore, the AM system is not linear.

2.2.2 Time-varying and time-invariant systems

A system is said to be time-invariant (TI) if a time delay or time advance of the input signal leads to an identical time-shift in the output signal.

A CT system with $x(t) \rightarrow y(t)$ is time-invariant if

$$x(t - t_0) \rightarrow y(t - t_0) \quad (2.40)$$

for any arbitrary time-shift t_0.

Likewise, a DT system with $x[k] \rightarrow y[k]$ is time-invariant if

$$x[k - k0] \rightarrow y[k - k0] \quad (2.41)$$

for any arbitrary discrete shift k_0.

Example 13: Consider two CT systems represented mathematically by the following input–output relationship:

(i) system I $\quad y(t) = \sin(x(t));$ $\quad$ (2.42)

(ii) system II $\quad y(t) = t\sin(x(t)).$ $\quad$ (2.43)

Determine if systems (i) and (ii) are time-invariant.

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Solution:

(i) From Eq. (2.42), it follows that:

$$x(t) \rightarrow \sin(x(t)) = y(t)$$

and

$$x(t - t_0) \rightarrow \sin(x(t - t_0)) = y(t - t_0).$$

Since $\sin[x(t - t_0)] = y(t - t_0)$, system I is time-invariant.

- We demonstrate the time-invariance property of system I graphically in Fig. 2.13,
- where a time-shifted version x(t − 1) of input x(t) produces an equal shift of one time unit in the original output y(t) obtained from x(t).

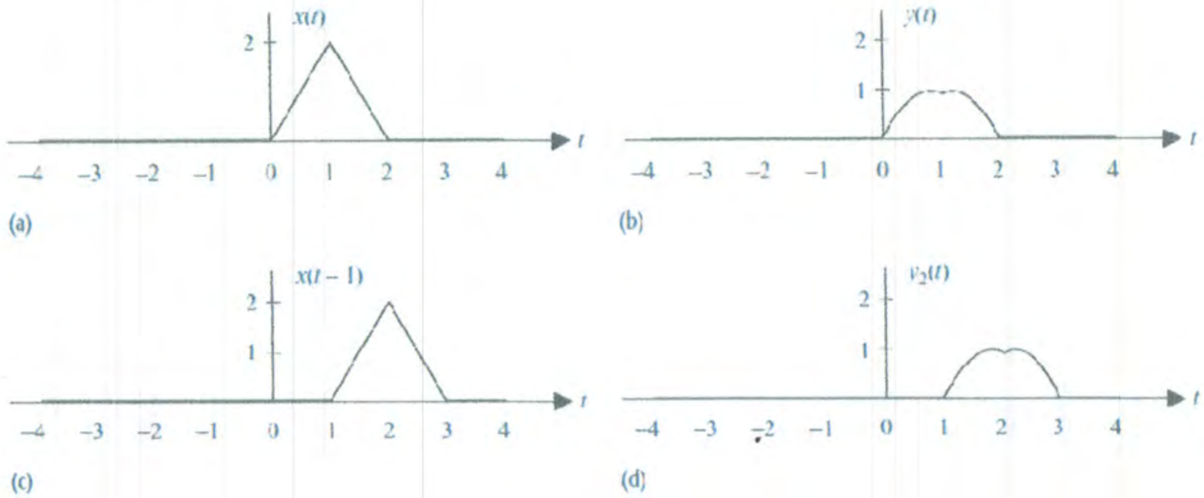

Fig. 2.13. Input–output pairs of the CT time-invariant system specified in Example 2.13(i). (a) Arbitrary signal x(t). (b) Output of system for input signal x(t). (c) Signal x(t − 1). (d) Output of system for input signal x(t − 1). Note that except for a time-shift, the two output signals are identical.

(ii) From Eq. (2.43), it follows that:

$$x(t) \rightarrow t\sin(x(t)) = y(t).$$

If the time-shifted signal $x(t - t_0)$ is applied at the input of Eq. (2.43), the new output is given by

$$x(t - t_0) \rightarrow t\sin(x(t - t_0)).$$

The shifted output $y(t - t_0)$ is given by

$$y(t - t_0) = (t - t_0)\sin(x(t - t_0)).$$

Since $t\sin[x(t - t_0)] \neq y(t - t_0)$, system II is not time-invariant.

The time-invariance property of system II is demonstrated in Fig. 2.14, where we observe that a right shift of one time unit in input x(t) alters the shape of the output y(t).

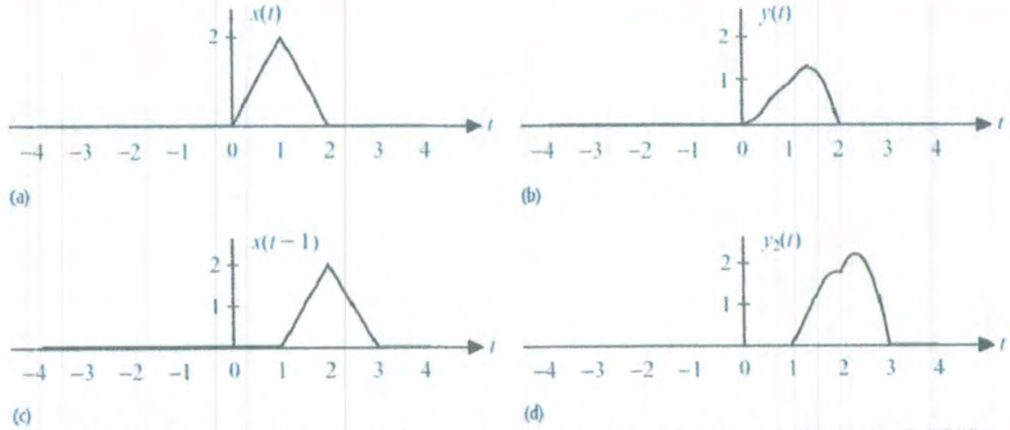

Fig. 2.14. Input–output pairs of the time-varying system specified in Example 2.13(ii). (a) Arbitrary signal x(t). (b) Output of system for input signal x(t). (c) Signal x(t − 1). (d) Output of system for input signal x(t − 1). Note that the output for time-shifted input x(t − 1) is different from the output y(t) for the original input x(t).

Example 2.14: Consider two DT systems with the following input–output relationships:

(i) system I $y[k] = 3(x[k] - x[k-2]);$ (2.44)

(ii) system II $y[k] = k\,x[k].$ (2.45)

Determine if the systems are time-invariant.

Solution:

(i) From Eq. (2.44), it follows that:

$$x[k] \rightarrow 3(x[k] - x[k-2]) = y[k]$$

$$x[k-k_0] \rightarrow 3(x[k-k_0] - x[k-k_0-2]) = y[k-k_0]$$

Therefore, the system in Eq. (2.44) is a time-invariant system.

(ii) From Eq. (2.45), it follows that:

$$x[k] \rightarrow kx[k] = y[k]$$

and

$$x[k-k_0] \rightarrow kx[k-k_0] \neq y[k-k_0] = (k-k_0)x[k-k_0].$$

Therefore, system II is not time-invariant.

In Fig. 2.15, we plot the outputs of the DT system in Eq. (2.45) for input x[k], shown in Fig. 2.15(a) and a shifted version x[k − 2] of the input, shown in Fig. 2.15(c). The resulting outputs are plotted, respectively, in Figs. 2.15(b) and (d). As expected, the Fig. 2.15(d) is not a delayed version of Fig. 2.15(b) since the system is time-variant.

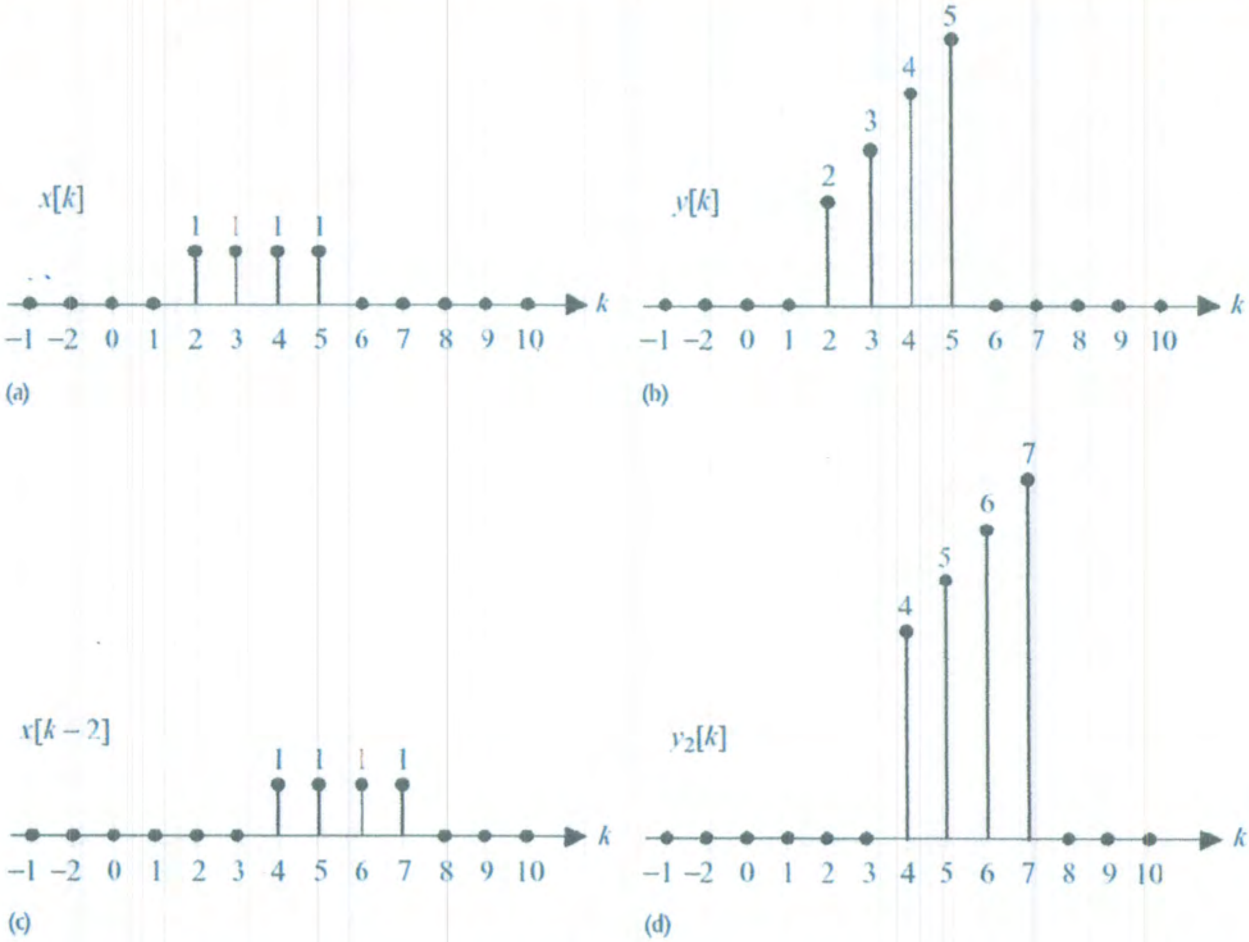

Fig. 2.15. Input–output pairs of the DT time-varying system specified in Example 2.14(ii). The output $y_2[k]$ for the time-shifted input $x_2[k] = x[k-2]$ is different in shape from the output y[k] obtained for input x[k]. Therefore the system is time-variant. Parts (a)–(d) are discussed in the text.

2.2.3 Systems with and without Memory

- A CT system is said to be without memory (***memoryless or instantaneous***) if it's output $y(t)$ at time $t = t_0$ depends only on the values of the applied input $x(t)$ at the same time $t = t_0$.

- On the other hand, if the response of a system at $t = t_0$ depends on the values of the input $x(t)$ in the past or in the future of time $t = t_0$, it is called a dynamic system, or a system with memory.

- Likewise, a DT system is said to be ***memoryless*** if its output $y[k]$ at instant $k = k_0$ depends only on the value of its input $x[k]$ at the same instant $k = k_0$. Otherwise, the DT system is said to have memory.

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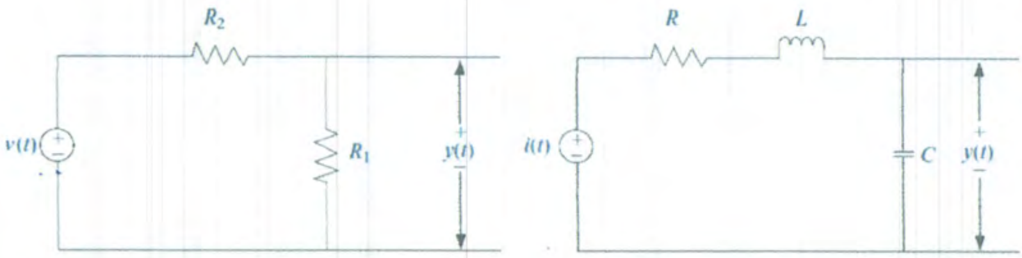
Example 2.15

Determine if the two electrical circuits shown are memoryless.



(a) (b)

Solution

The relationship between the input voltage $v(t)$ and the output voltage $y(t)$ across resistor R_1 in the electrical circuit of Fig. 2.16(a) is given by

$$y(t) = \frac{R_1}{R_1 + R_2} v(t). \qquad (2.46)$$

For time $t = t_0$, the output $y(t_0)$ depends only on the value $v(t_0)$ of the input $v(t)$ at $t = t_0$. The electrical circuit shown in Fig. 2.16(a) is, therefore, a memoryless system.

The relationship between the input current $i(t)$ and the output voltage $y(t)$ in Fig. 2.16(b) is given by

$$y(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau. \qquad (2.47)$$

To compute the output voltage $y(t_0)$ at time t_0, we require the value of the current source for the time range $(-\infty, t_0]$, which includes the entire past. Therefore, the electrical circuit in Fig. 2.16(b) is not a memoryless system.

Another examples of systems are shown in the following table:

Table 2.1. Examples of CT and DT systems with and without memory

Continuous-time		Discrete-time	
Memoryless systems	Systems with memory	Memoryless systems	Systems with memory
$y(t) = 3x(t) + 5$	$y(t) = x(t - 5)$	$y[k] = 3x[k] + 7$	$y[k] = x[k - 5]$
$y(t) = \sin\{x(t)\} + 5$	$y(t) = x(t + 2)$	$y[k] = \sin(x[k]) + 3$	$y[k] = x[k + 3]$
$y(t) = e^{x(t)}$	$y(t) = x(2t)$	$y[k] = e^{x[k]}$	$y[k] = x[2k]$
$y(t) = x^2(t)$	$y(t) = x(t/2)$	$y[k] = x^2[k]$	$y[k] = x[k/2]$

2.2.4 Causal and Non-causal systems

A CT system is causal if the output at time t_0 depends only on the input $x(t)$ for $t \leq t_0$.
A DT system is causal if the output at time instant k_0 depends only on the input $x[k]$ for $k \leq k_0$.

- Note that ***all memoryless systems are causal systems*** because the output at any time instant depends only on the input at that time instant.
- Systems with memory can either be causal or non-causal.
- Causality is a required condition for the system to be physically realizable.
- A non-causal system is a predictive system and cannot be implemented physically.

CT systems		DT systems	
Causal	Non-causal	Causal	Non-causal
$y(t) = x(t-5)$	$y(t) = x(t+2)$	$y[k] = 3x[k-1] + 7$	$y[k] = x[k+3]$
$y(t) = \sin\{x(t-4)\} + 3$	$y(t) = \sin\{x(t+4)\} + 3$	$y[k] = \sin(x[k-4]) + 3$	$y[k] = \sin(x[k+4]) + 3$
$y(t) = e^{x(t-2)}$	$y(t) = x(2t)$	$y[k] = e^{x[k-2]}$	$y[k] = x[2k]$
$y(t) = x^2(t-2)$	$y(t) = x(t/2)$	$y[k] = x^2[k-5]$	$y[k] = x[k/2]$
$y(t) = x(t-2) + x(t-5)$	$y(t) = x(t-2) + x(t+2)$	$y[k] = x[k-2] + x[k-8]$	$y[k] = x[k+2] + x[k-8]$

Example 2.16

(i) CT time-delay system $y(t) = x(t-2) \Rightarrow$ causal system;
(ii) CT time-forward system $y(t) = x(t+2) \Rightarrow$ non-causal system;
(iii) DT time-delay system $y[k] = x[k-2] \Rightarrow$ causal system;
(iv) DT time-advance system $y[k] = x[k+2] \Rightarrow$ non-causal system;
(v) DT linear system $y[k] = x[k-2] + x[k+10] \Rightarrow$ non-causal system.

2.2.6 Stable and Unstable systems

Before defining the stability criteria for a system, we define the bounded property for a signal.

- A CT signal x(t) or a DT signal x[k] is said to be bounded in magnitude if

CT signal $$|x(t)| \leq B_x < \infty \text{ for } t \in (-\infty, \infty); \quad (2.48)$$
DT signal $$|x[k]| \leq B_x < \infty \text{ for } k \in (-\infty, \infty), \quad (2.49)$$

where B_x is a finite number.

*A system is referred to as **bounded-input, bounded-output (BIBO)** stable if an arbitrary bounded-input signal always produces a bounded-output signal.*

CT system $$|y(t)| \leq B_y < \infty \text{ for } t \in (-\infty, \infty); \quad (2.50)$$
DT system $$|y[k]| \leq B_y < \infty \text{ for } k \in (-\infty, \infty). \quad (2.51)$$

Example 2.17: Determine if the following CT systems are stable.

(i) Incrementally linear system: $y(t) = 50x(t) + 10.$ (2.52)

(ii) Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ (2.53)

Solution: (i)

Assume $|x(t)| \leq B_x$ for all t. Based on Eq. (2.52), it follows that:

$$y(t) \leq 50B_x + 10 = B_y \text{ for all } t .$$

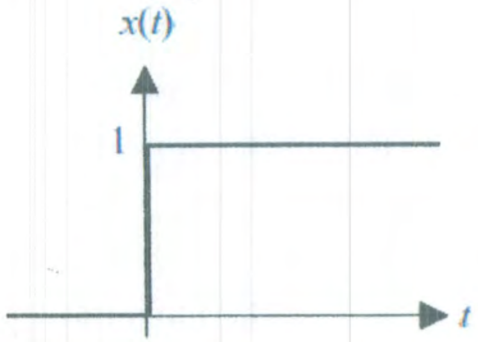
- As the magnitude of $y(t)$ does not exceed $50B_x + 10$, which is a finite number, the incrementally linear system given in Eq. (2.52) is a stable system.

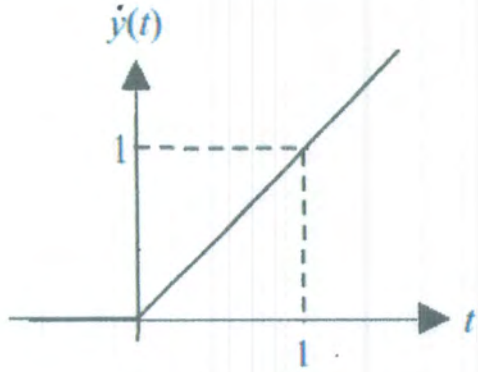
(ii) This system integrates the input signal from $= -\infty$ to t .

Assume that a unit-step function x(t) = u(t) is applied at the input of the integrator. The output of the system is given by

$$y(t) = tu(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0. \end{cases}$$

- Signal $y(t)$ is plotted. It is observed that $y(t)$ increases steadily for $t > 0$ and that there is no upper bound of $y(t)$. Hence, the integrator is not a BIBO stable system.





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Example 2.18: Determine if the following DT systems are stable.

(i) $y[k] = 50\sin(x[k]) + 10.$ (2.54)
(ii) $y[k] = e^{x[k]}.$ (2.55)
(iii) $y[k] = \sum_{m=-2}^{2} x[k-m]$ (2.56)
(iv) $y[k] = \sum_{m=-\infty}^{k} x[m]$ (2.57)

Solution:
(i) $y[k] = 50\sin(x[k]) + 10.$

- Note that $\sin(x[k])$ is bounded between $[-1, 1]$ for any arbitrary choice of $x[k]$.
- The output $y[k]$ is therefore bounded within the interval $[-40, 60]$. Therefore, system (i) is stable.

(ii) $y[k] = e^{x[k]}.$

Assume $|x[k]| \le B_x$ for all . Based on Eq. (2.52), it follows that:

$y[k] \le e^{B_x} = B_y$ for all k.

- Therefore, system (ii) is stable.

(iii) $y[k] = \sum_{m=-2}^{2} x[k-m]$ The output is expressed as follows:

$$y[k] = x[k-2] + x[k-1] + x[k] + x[k+1] + x[k+2].$$

If $|x[k]| \le B_x$ for all k, then $|y[k]| \le 5B_x$ for all k. Therefore, the system is stable.

(iv) $y[k] = \sum_{m=-\infty}^{k} x[m]$

- The output is calculated by summing an infinite number of input signal values.
- Hence, there is no guarantee that the output will be bounded even if all the input values are bounded.
- System (iv) is, therefore, not a stable system.

Problems:

2.3 Figure P2.3 shows the schematic of a square-law demodulator used in the demodulation of an AM signal. Demodulation is the process of extracting the information-bearing signal from the modulated signal. The input–output relationship of the non-linear device is approximated by (assuming $v_1(t)$ is small)

$$v_2(t) = c_1 v_1(t) + c_2 v_1^2(t),$$

where c_1 and c_2 are constants, and $v_1(t)$ and $v_2(t)$ are, respectively, the input and output signals.

(i) Show that the demodulator is a non-linear device.

(ii) Determine whether the non-linear device is (a) time-invariant, (b) memoryless, (c) invertible, and (d) stable.

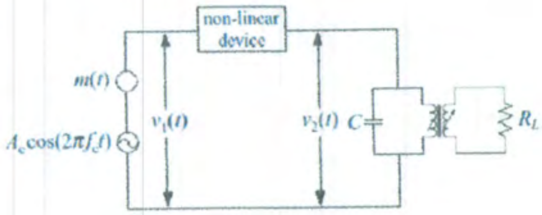

Fig. P2.3. AM demodulator. The input signal is represented by $v_1(t) = A_c\cos(2\pi f_c t) + m(t)$, where $A_c\cos(2\pi f_c t)$ is the carrier and $m(t)$ is the modulating signal.

2.9 The following CT systems are described using their input–output relation-ships between input x(t) and output y(t). Determine if the CT systems are (a) linear, (b) time-invariant, (c) stable, and (d) causal.

(i) $y(t) = x(t-2)$;

(ii) $y(t) = x(2t-5)$;

(iii) $y(t) = x(2t) - 5$;

(iv) $y(t) = t\,x(t+10)$;

(v) $y(t) = \begin{cases} 2 & x(t) \geq 0 \\ 0 & x(t) < 0; \end{cases}$

(vi) $y(t) = \begin{cases} 0 & t < 0 \\ x(t) - x(t-5) & t \geq 0; \end{cases}$

(vii) $y(t) = 7x^2(t) + 5x(t) + 3$;

(viii) $y(t) = \mathrm{sgn}(x(t))$;

(ix) $y(t) = \int_{-t_0}^{t_0} x(\lambda)\,d\lambda + 2x(t)$;

(x) $y(t) = \int_{-\infty}^{t_0} x(\lambda)\,d\lambda + \frac{dx}{dt}$;

(xi) $\frac{d^4 y}{dt^4} + 3\frac{d^3 y}{dt^3} + 5\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + y(t) = \frac{d^2 x}{dt^2} + 2x(t) + 1.$

2.10 The following DT systems are described using their input–output relation-ships between input x[k] and output y[k]. Determine if the DT systems are (a) linear, (b) time-invariant, (c) stable, and (d) causal.

(i) $y[k] = ax[k] + b$;

(ii) $y[k] = 5x[3k - 2]$;

(iii) $y[k] = 2^{x[k]}$;

(iv) $y[k] = \sum_{m=-\infty}^{k} x[m]$;

(v) $y[k] = \sum_{m=k-2}^{k+2} x[m] - 2|x[k]|$;

(vi) $y[k] + 5y[k-1] + 9y[k-2] + 5y[k-3] + y[k-4]$
$= 2x[k] + 4x[k-1] + 2x[k-2]$.

(vii) $y[k] = 0.5x[6k - 2] + 0.5x[6k + 2]$.

2.11 For an LTIC system, an input $x(t)$ produces an output $y(t)$ as shown in Fig. P2.11. Sketch the outputs for the following set of inputs:

(i) $5x(t)$;
(ii) $0.5x(t-1) + 0.5x(t+1)$;
(iii) $x(t+1) - x(t-1)$;
(iv) $\frac{dx(t)}{dt} + 3x(t)$.

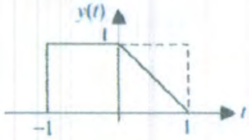

Fig. P2.11. CT output $y(t)$ for Problem 2.11.

2.12 For a DT linear, time-invariant system, an input $x[k]$ produces an output $y[k]$ as shown in Fig. P2.12. Sketch the outputs for the following set of inputs:

(i) $4x[k-1]$;
(ii) $0.5x[k-2] - 0.5x[k+2]$;
(iii) $x[k+1] - 2x[k] + x[k-1]$;
(iv) $x[-k]$.

Fig. P2.12. DT output $y[k]$ for Problem 2.12.