Convolution Integral

Lecture # 7

* Finding the effect of a signal on another signal is called Convolution.

* The convolution between two signals $f_1(t)$ and $f_2(t)$,

$$f_1(t) \circledast f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \qquad (70)$$

* Convolution integral is commutative:-

$$f_1(t) \circledast f_2(t) = f_2(t) \circledast f_1(t) \qquad (71)$$

* Convolution integral is Distributive:

$$f_1(t) \circledast \left[f_2(t) + f_3(t) \right] = f_1(t) \circledast f_2(t) + f_1(t) \circledast f_3(t) \qquad (72)$$

* Convolution integral is Associative:

$$f_1(t) \circledast \left[f_2(t) \circledast f_3(t) \right] = \left[f_1(t) * f_2(t) \right] * f_3(t) \qquad (73)$$

*Convolution can be shifted :-

if $f_1(t) \circledast f_2(t) = g(t)$ then

$$f_1(t) \circledast f_2(t-t_0) = g(t-t_0) \quad \text{———} (74)$$

$$f_1(t-t_0) \circledast f_2(t) = g(t-t_0) \quad \text{———} (75)$$

$$f_1(t-t_0) \circledast f_2(t-t_1) = g(t-t_0-t_1) \quad \text{———} (76)$$

*Convolution with $\delta(t)$:-

$$f(t) \circledast \delta(t) = f(t) \quad \text{———} (77)$$

Hence :- Convolution of Any signal with $\delta(t)$ is the signal itself.

System Impulse Response

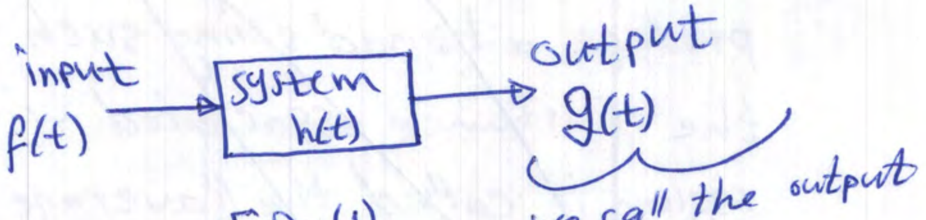


Fig.(1)

* In Fig.(1), the system can be Described mathematically as,

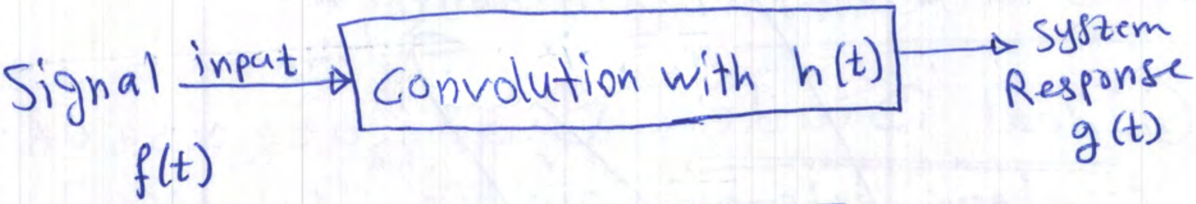
$$\boxed{g(t) = f(t) * h(t)} \quad \text{———} (77)$$

* Thus, to get the effect of any system on a signal, all we need is to do convolution.

* In other words, Convolution integral is an operation which gives the effect of a system or signal on a system or a signal.

Signal (1) $\circledast$ Signal (2) = effect of 1 on 2
convolution
effect of 2 on 1

Signal $\circledast$ system = effect of signal on system /or
convolution
effect of system on signal



$\therefore$ $\boxed{\text{System response} = \text{input} \circledast \text{system}}$

* As mentioned previously :- convolution of any signal with $\delta(t)$ is the signal itself, then to get the system's function (system transfer function) the system $h(t)$ must the input to the unknown system $h(t)$ must be $\delta(t)$.

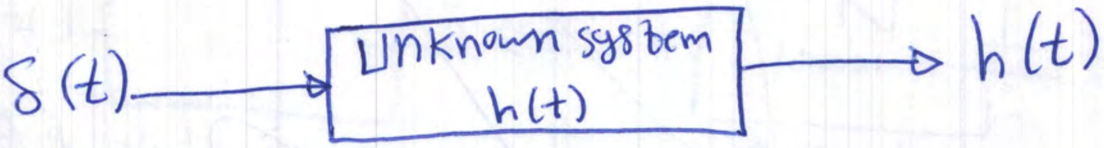


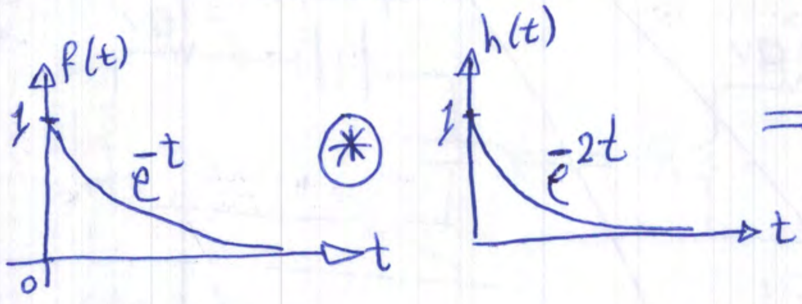
Fig. (2)

* In Fig. (2), the output $h(t)$ is called the system impulse response (which is the transfer function of the system).

EX. For a system with impulse response $h(t) = e^{-2t}u(t)$, determine the response $g(t)$ for the input $e^{-t}u(t)$.

Solution

$$e^{-t}u(t) \longrightarrow \boxed{h(t) = e^{-2t}u(t)} \longrightarrow g(t)$$



$= g(t) = f(t) \circledast h(t)$

* both functions starts at zero, $t \geq 0$, then

$$g(t) = f(t) \circledast h(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau \qquad t \geq 0$$

we have $f(\tau) = e^{-\tau}u(\tau)$ & $h(t-\tau) = e^{-2(t-\tau)}u(t-\tau)$

then $g(t) = \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau = \int_0^t e^{-\tau} e^{-2t} e^{2\tau}\, d\tau \quad t \geq 0$

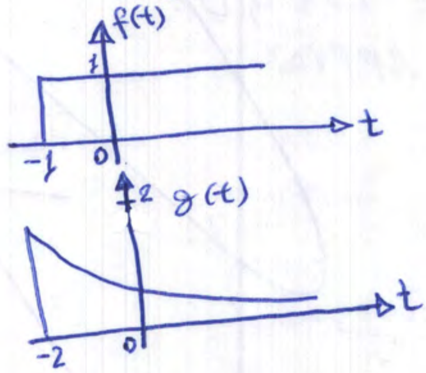
$$= e^{-2t}\int_0^t e^{\tau}\, d\tau = e^{-2t}\left[e^{\tau}\right]_0^t = e^{-2t}\left[e^{t} - 1\right] \quad t \geq 0$$

$$= e^{-2t}e^{t} - e^{-2t} = e^{-t} - e^{-2t} \qquad t \geq 0$$

$\therefore \boxed{g(t) = \left[e^{-t} - e^{-2t}\right]u(t)}$

Graphical Convolution

* Lets understand graphical convolution by an example,

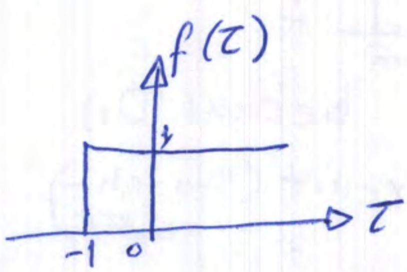


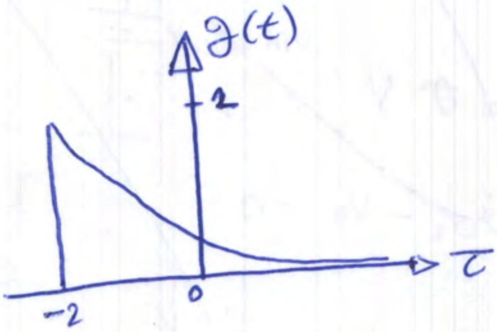
Let $f(t) = u(t+1)$

$g(t) = 2e^{-(t+2)}$

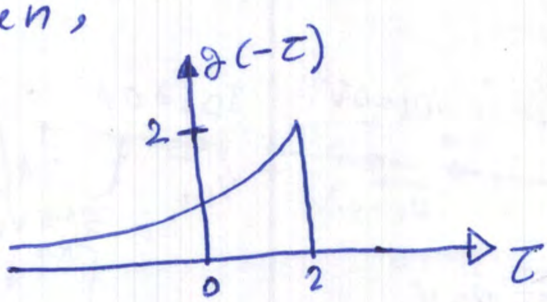
$c(t) = f(t) \circledast g(t)$

Step 1 replace each t with τ,





$$c(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$ then,



Step 2 plot $g(-\tau)$

Step 3 shift $g(-\tau)$ by t as follows:

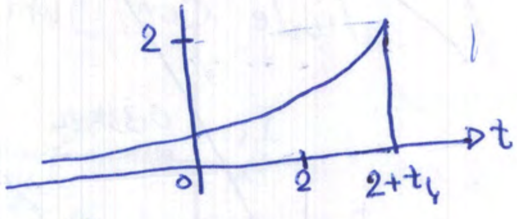
* assume $g(-\tau) = \phi(\tau)$
* now $\phi(\tau)$ shifted by t seconds is $\phi(\tau - t)$

$\therefore$ $\phi(\tau - t) = g(-(\tau - t)) = g(t-\tau)$

$\circledast$ if $+t \longrightarrow$ shift to the right

$\circledast$ if $-t \longrightarrow$ shift to the left

Thus, we start to shift to the right by t_1 to get $g(t_1-\tau)$, $t=t_1>0$



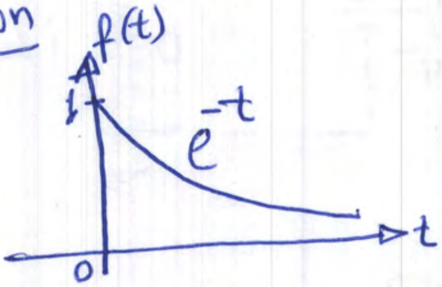
Step 4: Move $g(\tau)$ (scanning) over $f(\tau)$.

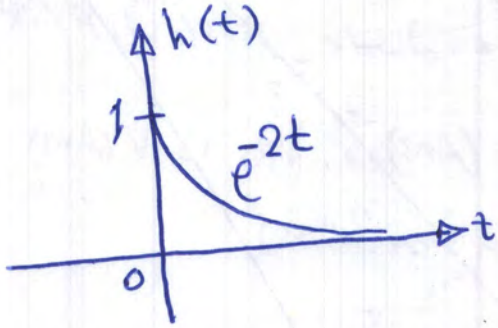
– The area under the product of $f(\tau)$ and $g(t_1-\tau)$ (the shifted frame) is $c(t_1)$, the value of the convolution at $t=t_1$.

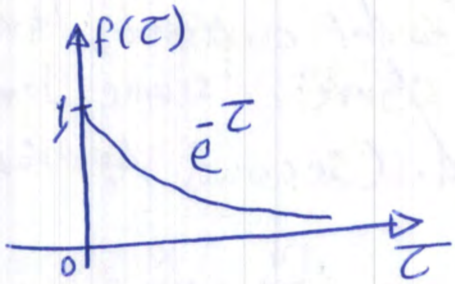
step 5: Repeat this procedure, shifting the frame by different values (positive & negative) to obtain $c(t)$ for all values of t.

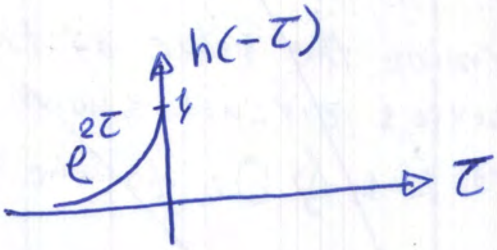
EX. Determine graphically $y(t) = f(t) \circledast h(t)$ for $f(t) = e^{-t}u(t)$ and $h(t) = e^{-2t}u(t)$.

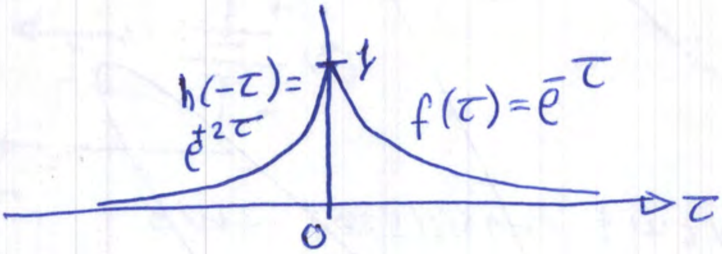
Solution



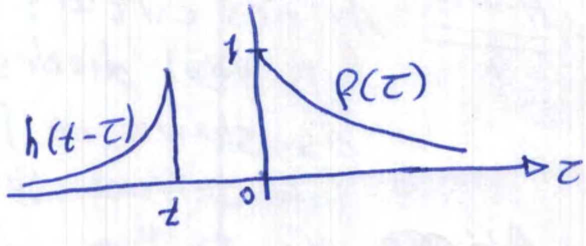








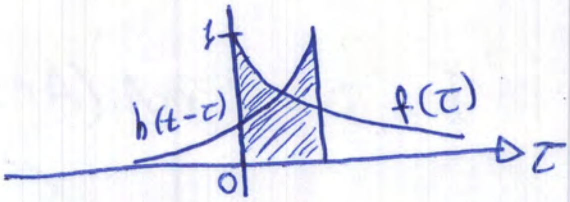
* In this case, there is no overlap:



$$f(\tau)h(t-\tau) = 0$$

$$\therefore y(t) = 0, \quad t < 0$$

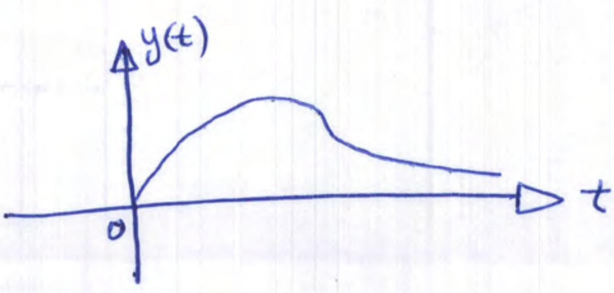
* In this case, there is overlap in the region $0 \leq \tau \leq t$ (shaded area)



$$y(t) = \int_0^t f(\tau)h(t-\tau)\,dt \quad t \geq 0$$

$$y(t) = \int_0^t e^{-\tau}e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_0^t e^{\tau}\,d\tau = e^{-t} - e^{-2t} \quad t \geq 0$$

$$\therefore y(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$



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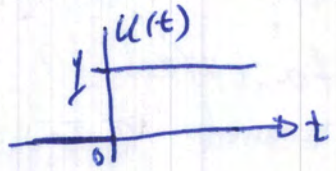
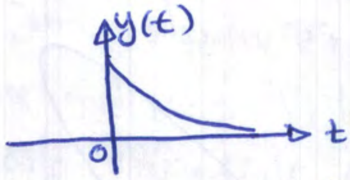
EX.1 convolve $x(t) = u(t)$ with $y(t) = e^{-t} u(t)$.

solution

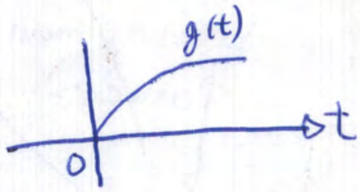




let $g(t) = u(t) \circledast e^{-t} u(t) = x(t) \circledast y(t)$

$$g(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = \int_{0}^{t} 1\, e^{-(t-\tau)}\, d\tau$$

$$= e^{-t} \int_{0}^{t} e^{\tau}\, d\tau = e^{-t}\left[e^{t} - 1\right] = 1 - e^{-t}$$



EX.2 given $r(t) = t \quad t \geq 0$ & $x(t) = e^{-\alpha t} u(t)$, find the convolution $g(t) = r(t) \circledast x(t)$.

solution

$$g(t) = \int_{-\infty}^{\infty} r(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} \tau\, e^{-\alpha(t-\tau)}\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} \tau\, e^{-\alpha(t-\tau)}\, d\tau \quad t \geq 0$$

$$= e^{-\alpha t} \int_{0}^{t} \tau\, e^{\alpha \tau}\, d\tau \quad t \geq 0 \quad \left\{\text{integrate it using by parts method}\right\}$$

$$= \frac{t}{\alpha} - \frac{1}{\alpha^2}\left(1 - e^{-\alpha t}\right) \quad t \geq 0$$

$$\therefore\ g(t) = \left[\frac{t}{\alpha} - \frac{1}{\alpha^2}\left(1 - e^{-\alpha t}\right)\right] u(t).$$

EX:
convolve $u(t)$ with itself.

solution

$$y(t) = u(t) \circledast u(t) = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = \int_{0}^{t} d\tau = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$$

OR $y(t) = t\, u(t)$.

EX. find the effect of the system $h(t) = \frac{1}{t+1} u(t)$ on the input signal $x(t) = u(t)$.

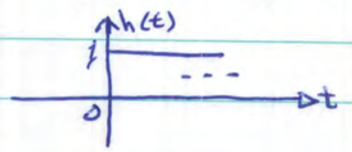
solution

* input signal $x(t) = u(t)$ (input is D.C.)

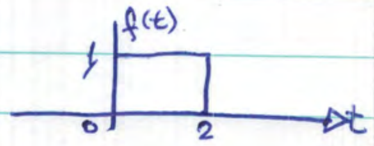
$$g(t) = \int_{-\infty}^{\infty} u(t-\tau) \frac{1}{\tau+1}\, d\tau = \int_{0}^{t} \frac{1}{\tau+1}\, d\tau = \overline{\ln(t+1)\, u(t)}.$$

EX. How the system $h(t) = u(t)$ will response if its input is $f(t) = u(t) - u(t-2)$?

solution

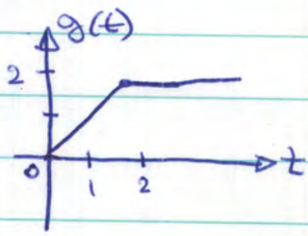


$$g(t) = \int_{-\infty}^{\infty} [u(\tau) - u(\tau-2)]\, u(t-\tau)\, d\tau$$



* if $\tau < 0$ and $\tau > 2 \rightarrow g(t) = 0$

$\tau > 0 \longrightarrow g(t) = 0$



* if $0 < t < 2 \rightarrow g(t) = \int_0^t d\tau = t$

* if $t > 2 \longrightarrow g(t) = \int_0^2 d\tau = 2$

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2 Continuous-Time Convolution

The input, $x(t)$, and output, $y(t)$, of a continuous-time LTI system are related by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{8}$$

where $h(t)$ is the impulse response of the system. Consider a system with impulse response and input shown in Fig. 4 and given by

$$\begin{aligned} h(t) &= e^{2t}u(1-t) && (9)\\ x(t) &= u(t) - 2u(t-2) + u(t-5). && (10) \end{aligned}$$

Substituting these into (8) yields a complicated looking expression,

$$y(t) = \int_{-\infty}^{\infty} \left[u(t-\tau) - 2u(t-2-\tau) + u(t-5-\tau)\right] e^{2\tau}u(1-\tau)d\tau. \tag{11}$$

This integration problem can be simplified by considering various intervals for t. Flipped and shifted versions of the impulse response are shown in Fig. 5 for these ranges of t. When $t < 1$ the curves only overlap between $t-5$ and t, see Fig. 5a, so this limits the integration interval. Furthermore, $x(t)$ is a constant value from $t-5$ to $t-2$ and a different constant value from $t-2$ to t, so separating the integral into two terms will greatly simplify our work.

$$\begin{aligned} y(t) &= \int_{t-5}^{t} \left[u(t-\tau) - 2u(t-2-\tau) + u(t-5-\tau)\right] e^{2\tau}d\tau. \\ &= \int_{t-5}^{t-2} (-1)e^{2\tau}d\tau + \int_{t-2}^{t} e^{2\tau}d\tau \\ &= -\frac{1}{2}\left[e^{2(t-2)} - e^{2(t-5)}\right] + \frac{1}{2}\left[e^{2t} - e^{2(t-2)}\right] \\ y(t) &= \frac{1}{2}\left[1 - 2e^{-4} + e^{-10}\right]e^{2t} \qquad t < 1 && (12) \end{aligned}$$

When $t > 1 > t-2$ (i.e. $1 < t < 3$), the leading edge of $x(t-\tau)$ has shifted out beyond the end of $h(\tau)$, see Fig. 5b. Therefore, the upper limit on the integration becomes the end of $h(\tau)$.

$$\begin{aligned} y(t) &= \int_{t-5}^{t-2} -e^{2\tau}d\tau + \int_{t-2}^{1} e^{2\tau}d\tau \\ &= -\frac{1}{2}\left[e^{2(t-2)} - e^{2(t-5)}\right] + \frac{1}{2}\left[e^{2} - e^{2(t-2)}\right] \\ y(t) &= \frac{1}{2}\left[-2e^{-4} + e^{-10}\right]e^{2t} + \frac{1}{2}e^{2} \qquad 1 < t < 3 && (13) \end{aligned}$$

When $t-5 < 1 < t-2$ (i.e. $3 < t < 6$), the leading section $x(t-\tau)$ has shifted out beyond the end of $h(\tau)$, so the second integral is zero and the upper limit on the first integral is the end of $h(\tau)$, see Fig. 5c.

$$y(t) = \int_{t-5}^{1} -e^{2\tau}d\tau = \frac{1}{2}\left[e^{2(t-5)} - e^{2}\right] \qquad 3 < t < 6 \tag{14}$$

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When $t-5>1$ (i.e. $t>6$), the curves no longer overlap and

$$y(t) \;=\; 0. \qquad\qquad t>6 \tag{15}$$

Putting (12) through (15) together yields the final answer,

$$y(t) \;=\; \begin{cases} \frac{1}{2}\left[1-2e^{-4}+e^{-10}\right]e^{2t} & t \;\le\; 1 \\ \frac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}+\frac{1}{2}\,e^{2} & 1 \;<\; t \;\le\; 3 \\ \frac{1}{2}\left[e^{2(t-5)}-e^{2}\right] & 3 \;<\; t \;\le\; 6 \\ 0 & 6 \;<\; t \end{cases} \tag{16}$$

We can use unit step functions to rewrite (16) as

$$\begin{aligned} y(t) \;&=\; \tfrac{1}{2}\left[1-2e^{-4}+e^{-10}\right]e^{2t}u(1-t)+\left(\tfrac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}+\tfrac{1}{2}\,e^{2}\right)\left[u(3-t)-u(1-t)\right] \\ &\quad +\tfrac{1}{2}\left[e^{2(t-5)}-e^{2}\right]\left[u(6-t)-u(3-t)\right] \\ &=\; \left(\tfrac{1}{2}\left[1-2e^{-4}+e^{-10}\right]e^{2t}-\tfrac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}-\tfrac{1}{2}\,e^{2}\right)u(1-t) \\ &\quad +\left(\tfrac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}+\tfrac{1}{2}\,e^{2}-\tfrac{1}{2}\left[e^{2(t-5)}-e^{2}\right]\right)u(3-t) \\ &\quad +\tfrac{1}{2}\left[e^{2(t-5)}-e^{2}\right]u(6-t) \end{aligned}$$

$$y(t) \;=\; \tfrac{1}{2}\left(e^{2t}-e^{2}\right)u(1-t)+\left(-e^{2t-4}+e^{2}\right)u(3-t)+\tfrac{1}{2}\left(e^{2t-10}-e^{2}\right)u(6-t) \tag{17}$$

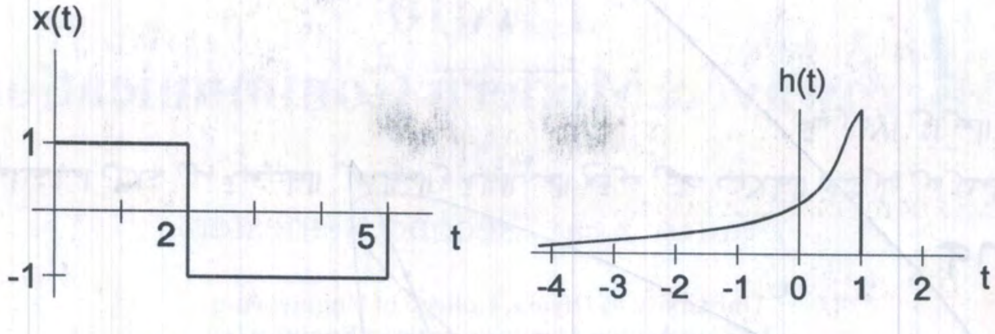
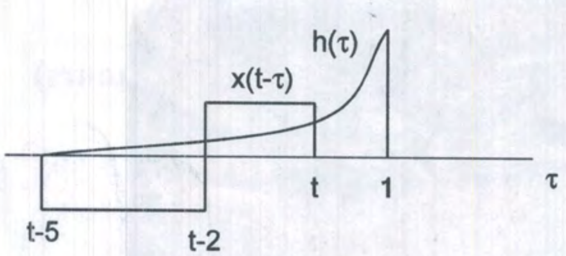
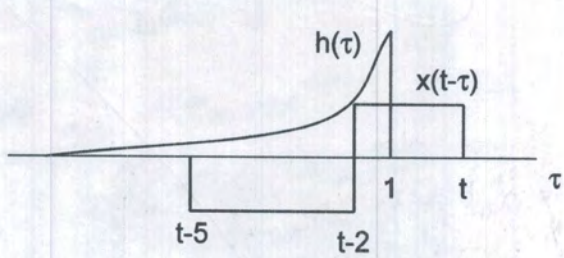


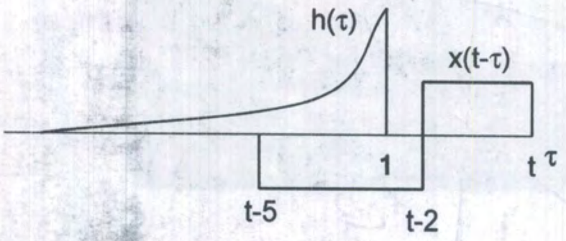
Figure 4: Input signal and system impulse response.



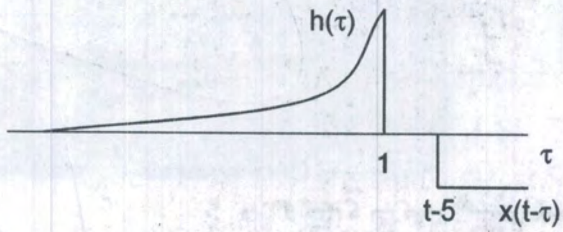
(a) Case where $t < 1$



(b) Case where $t - 2 < 1 < t$



(c) Case where $t - 5 < 1 < t - 2$



(d) Case where $1 < t - 5$

Figure 5: Overlapping curves for various values of t.

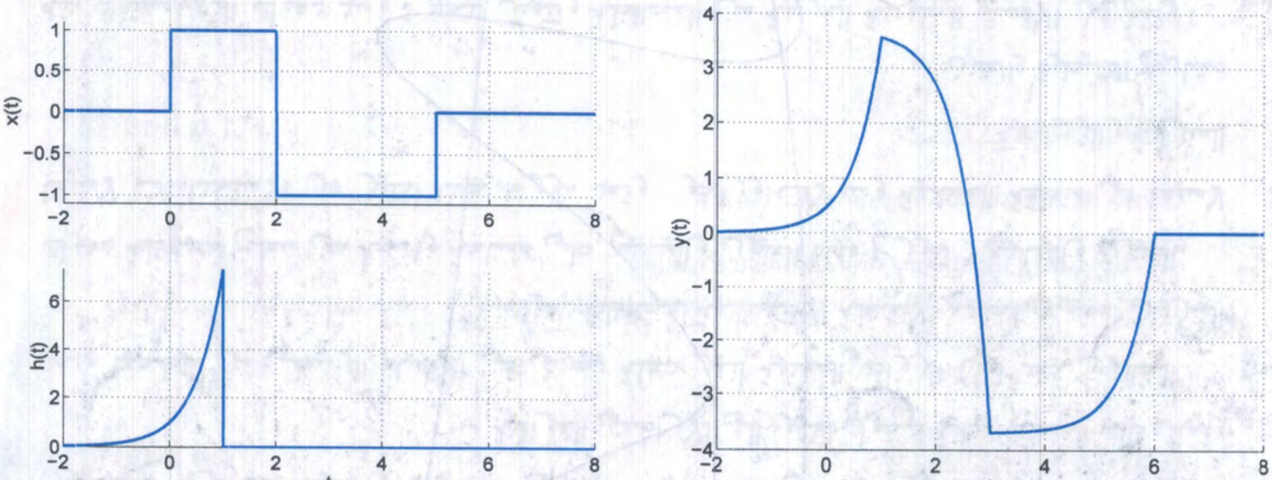


Figure 6: Continuous-time convolution example.

Discrete Time Convolution

given $x[n]$ and system $h[n]$, the convolution of $x[n]$ with $h[n]$ is $y[n] = x[n] * h[n]$

$$\boxed{y[n] = \sum_{k=-\infty}^{k=\infty} x[k]\, h[n-k]}$$

note that the independent variable is $\underline{k}$ not $\underline{n}$

* The same properties of the CT convolution are applicable to the DT convolution.

Ex. 1 Find the total response when the input signal is $x[n] = [\frac{1}{2}]^n u[n]$ and the impulse response is given by $h[n] = (\frac{1}{3})^n u[n]$.

solution
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^k u[k] \left(\tfrac{1}{3}\right)^{n-k} u[n-k]$$

$$= \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{3}\right)^{n-k}$$

k starts at $\underline{0}$ because $u[k]=0$ for $k<0$ and $u[n-k]=0$ for $k>n$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{3}{2}\right)^k = \left(\tfrac{1}{3}\right)^n \frac{1-\left(\frac{3}{2}\right)^{n+1}}{1-\left(\frac{3}{2}\right)}$$

$$= (-2)\left(\tfrac{1}{3}\right)^n u[n] + 3\left(\tfrac{1}{2}\right)^n u[n]$$

EX. 2 Find the system response $h[n] = 1 \quad -2 \le n \le 2$ to the input signal $x[n] = 1 \quad 0 \le n \le 4$.

solution

A simple convolution method is by writing the functions as polynomials.

$$x[n] = [\underline{1}, 1, 1, 1, 1] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

$$h[n] = [1, 1, \underline{1}, 1, 1] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]$$

$$y[n] = x[n] * \{\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]\}$$

$$y[n] = x[n] * \delta[n+2] + x[n] * \delta[n+1] + x[n] * \delta[n] + x[n] * \delta[n-1] + x[n] * \delta[n-2]$$

Since $x[n] * \delta[n \pm c] = x[n \pm c]$

$$\therefore y[n] = x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2] \text{ , substitute } x[n]$$

$$x[n] = \{\delta[n+2]$$

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$$y[n] = x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2]$$

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

we need $\underset{①}{x[n+2]}$, $\underset{②}{x[n+1]}$, $\underset{③}{x[n]}$, $\underset{④}{x[n-1]}$, $\underset{⑤}{x[n-2]}$

① $x[n+2] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]$

② $x[n+1] = \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$

③ $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$

④ $x[n-1] = \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$

⑤ $x[n-2] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + \delta[n-6]$

Summation of them is $y[n]$

the convolution

$$y[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 4\delta[n-1] + 5\delta[n-2] + 4\delta[n-3] + 3\delta[n-4] + 2\delta[n-5] + \delta[n-6]$$

or

$$y[n] = [1, 2, \underline{3}, 4, 5, 4, 3, 2, 1]$$

Ex. 3 Find the convolution between $x[n]=[4,0,\underline{3},2,1]$ and $h[n]=[2,\underline{1},3]$.

solution $x[n]=4\delta[n+2]+0\delta[n+1]+3\delta[n]+2\delta[n-1]+\delta[n-2]$

$$h[n]=2\delta[n+1]+\delta[n]+3\delta[n-1].$$

$$y[n]=x[n]*h[n]=x[n]*\{2\delta[n+1]+\delta[n]+3\delta[n-1]\}.$$

$$=2x[n]*\delta[n+1]+x[n]*\delta[n]+3x[n]*\delta[n-1]$$

$$=2x[n+1]+x[n]+3x[n-1]$$

we need $x[n]$, $x[n+1]$, $x[n-1]$

$$x[n]=4\delta[n+2]+0\delta[n+1]+3\delta[n]+2\delta[n-1]+\delta[n-2]$$

$$x[n+1]=4\delta[n+3]+0\delta[n+2]+3\delta[n+1]+2\delta[n]+\delta[n-1]$$

$$x[n-1]=4\delta[n+1]+0\delta[n]+3\delta[n-1]+2\delta[n-2]+\delta[n-3]$$

$$y[n]=2x[n+1]+x[n]+3x[n-1]$$

$$=8\delta[n+3]+0\delta[n+2]+6\delta[n+1]+4\delta[n]+2\delta[n-1]+$$

$$4\delta[n+2]+0\delta[n+1]+3\delta[n]+2\delta[n-1]+\delta[n-2]+$$

$$12\delta[n+1]+0\delta[n]+9\delta[n-1]+6\delta[n-2]+3\delta[n-3]$$

$$y[n]=8\delta[n+3]+4\delta[n+2]+18\delta[n+1]+7\delta[n]+13\delta[n-1]+7\delta[n-2]+3\delta[n-3]$$

$\therefore$

$$y[n]=[8,4,18,\underline{7},13,7,3]$$

Homework

H.1 Find the total system response of the system $h[n] = \left[\frac{1}{4}\right]^n u[n]$ to the input signal $x[n] = 4\left[\frac{1}{n}\right] u[n]$ where $n = 0, 1, 2, 3, 4$, or $0 \le n \le 4$.

H.2 Calculate the convolution between the two signals drawn in Figure H.2.

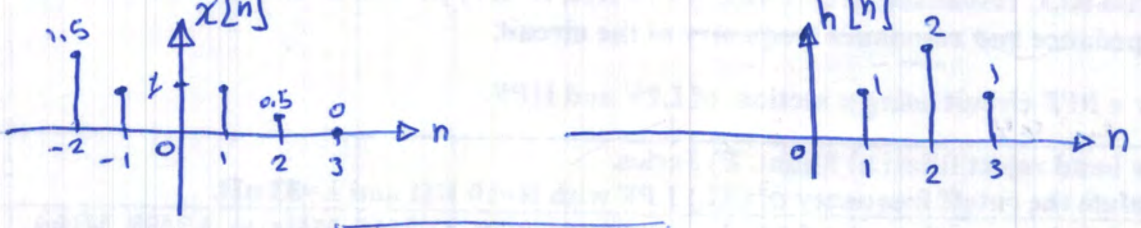

Figure H.2

H.3 Convolve $x[n] = u[n]$ $-1 \le n \le 2$ with itself.

H.4 Convolve the signal $x[n]$ with the system impulse response $h[n]$, in other words, find the system effect $h[n]$ on the signal $x[n]$.