Dr. Montadar Abas Taher 2019-2020

Lecture #8.2

Fourier Transform

* We have studied Fourier series and we knew it deals with periodic signals (power signals).
* Fourier transform deals with aperiodic signals.
* If the signal is aperiodic, we assume it repeats itself at ∞, then

$$f(t) \xleftrightarrow{F.T.} F(\omega)$$

where $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$ [Foreword]

and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad \text{[inverse]}$$

Another form of Fourier transform is

$$F(f) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi f t}\, dt \qquad [\text{Foreword}]$$

and

$$f(t) = \int_{-\infty}^{\infty} F(f)\, e^{j2\pi f t}\, df \qquad [\text{inverse}]$$

* in this course, mostly, we will use the second form.

* Fourier transform plays an important and significant role in communication systems.
* By making use of Fourier transform properties, differenet values can be obtained, such as simplifying problems and easy designes.

* However, Fourier transform, transforms time-domain Signals to Frequency-domain Signals.

EX.1 Find the Fourier transform of $e^{-at} \quad t \geq 0$.

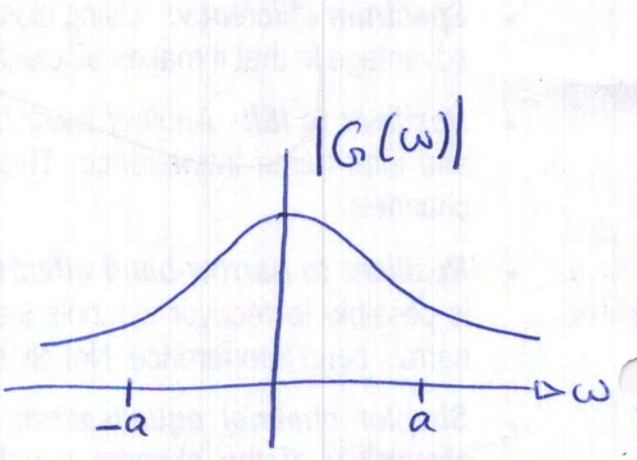
solution $g(t) = e^{-at} \quad t \geq 0$

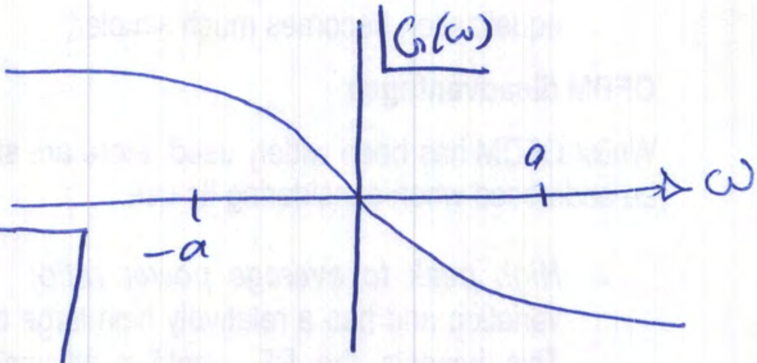
$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt = \frac{-1}{a+j2\pi f}\, e^{-(a+j2\pi f)t} \Big|_{0}^{\infty}$$

$$= \frac{-1}{a+j2\pi f}\left[0 - 1\right]$$

$$\therefore \boxed{G(f) = \frac{1}{a+j2\pi f}}$$





* Generally:

$$\boxed{e^{\mp at} \xleftrightarrow{F.T.} \frac{1}{a \pm j\omega}}$$

EX. 2 Find Fourier transform of $g(t) = \delta(t-5)$ and Sketch the magnitude and phase spectrums.

Solution

$$G(f) = \int_{-\infty}^{\infty} \delta(t-5)\, e^{-j2\pi f t}\, dt$$

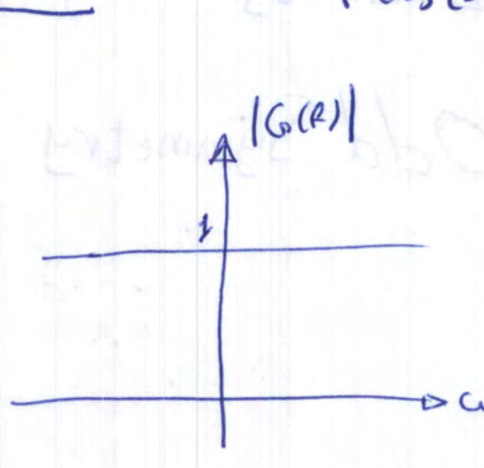
* we know that $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0)$, then

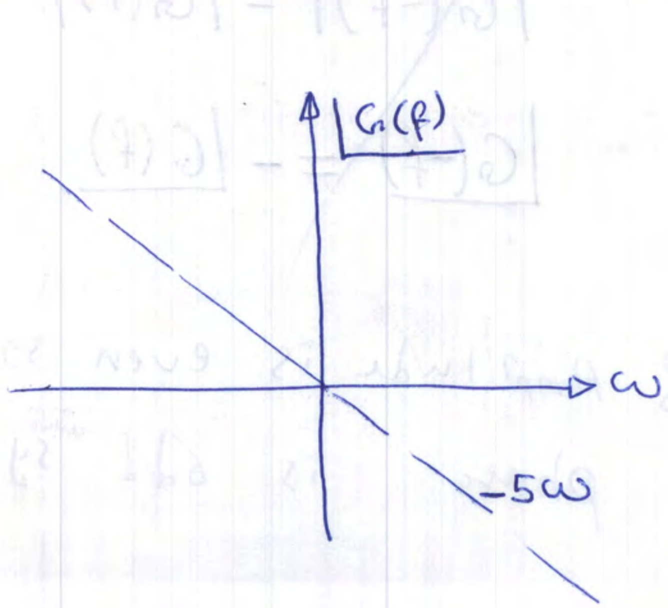
$$G(f) = \int_{-\infty}^{\infty} \delta(t-5)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f 5}$$

Thus $\boxed{G(f) = e^{-j\omega 5}} \Rightarrow e^{-j5\omega} = \cos(5\omega) - j\sin(5\omega)$

$$|G(f)| = |e^{-j5\omega}| = 1 \quad \text{for all } \omega$$

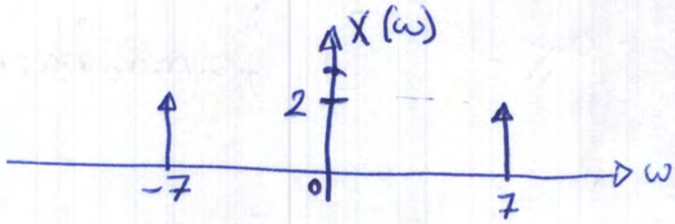
$$\angle G(f) = \tan^{-1}\left(\frac{-\sin(5\omega)}{\cos(5\omega)}\right) = -5\omega$$





EX. 3 Find the inverse Fourier transform of the signal shown below.

Solution



From the Figure:

$$X(\omega) = 2\,\delta(\omega+7) + 2\,\delta(\omega-7)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega = \frac{2}{2\pi}\int_{-\infty}^{\infty} \delta(\omega+7)\, e^{j\omega t}\, d\omega + \frac{2}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-7)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{\pi} e^{-j7t} + \frac{1}{\pi} e^{j7t} = \frac{2}{\pi}\cos(7t).$$

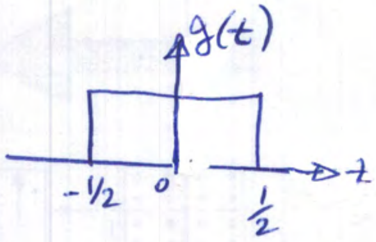
EX. 4 Find $F\{\delta(t)\}$.

Solution $g(t) = \delta(t)$

$$G(f) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1$$

EX.4 Find the Fourier transform of $g(t) = 2\,\mathrm{rect}(t-2)$.

Solution $g(t) = 2\,\mathrm{rect}(t-2)$



$$G(f) = 2\int_{-1/2}^{1/2} \mathrm{rect}(t-2)\, e^{-j2\pi f t}\, dt$$

Let $t-2 = \tau \rightarrow dt = d\tau$ and $t = \tau + 2$

$$G(f) = 2\int_{-1/2}^{1/2} \mathrm{rect}(\tau)\, e^{-j2\pi f(\tau+2)}\, d\tau = \int_{-1/2}^{1/2} e^{-j2\pi f\tau}\, e^{-j2\pi f 2}\, d\tau$$

$$= 2e^{-j2\pi f 2}\int_{-\infty}^{\infty} e^{-j2\pi f\tau}\, d\tau = 2e^{-j2\pi f 2}\left[\int_{-\infty}^{\infty}\Big(\underbrace{\cos(2\pi f\tau)}_{\text{even}} - j\overbrace{\underbrace{\sin(2\pi f\tau)}_{\text{odd}}}^{\text{zero}}\Big)\, d\tau\right]$$

$$= 2e^{-j2\pi f 2}\; 2\int_{0}^{1/2} \cos(2\pi f\tau)\, d\tau = \frac{2*2\, e^{-j2\pi f 2}}{2\pi f}\sin(2\pi f\tau)\Big|_{0}^{1/2}$$

$$= \frac{4e^{-j2\pi f 2}}{2\pi f}\sin\left(\frac{2\pi f}{2}\right) = \frac{2e^{-j2\pi f 2}}{\pi f}\sin(\pi f)$$

$$= 2e^{-j2\pi f 2}\,\frac{\sin(\pi f)}{\pi f} = 2e^{-j2\omega}\,\mathrm{sinc}(f)$$

Ex.5 Find the Fourier transform of $\cos(2\pi f_0 t)$.

solution

$$g(t) = \cos(2\pi f_0 t) = \cos(\omega_0 t)$$

$$G(f) = \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j\omega t} dt$$

$\Rightarrow$ Since $e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$

$$G(f) = \int_{-\infty}^{\infty} \cos(\omega_0 t)\cos(\omega t)\, dt - j \underbrace{\int_{-\infty}^{\infty} \cos(\omega_0 t)\sin(\omega t)\, dt}_{\text{zero}}$$

$$= \int_{-\infty}^{\infty} \cos(2\pi f_0 t)\cos(2\pi f t)\, dt$$

* if $f = f_0 \Rightarrow \cos(2\pi f)\cos(2\pi f) = \cos^2(2\pi f)$

* if $f = -f_0 \Rightarrow \cos(2\pi f)\cos(-2\pi f) = \cos^2(2\pi f)$

Hence: only when $f_0 = \mp f$ the integration is non-zero and equals to $\frac{1}{2}$, then

$$G(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Fourier Transform Pairs

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#	Signal	Time Domain Signal	Frequency Domain Signal
1	Rectangle	$A\,\mathrm{rct}\left(\frac{t}{T}\right)$	$AT\,\mathrm{sinc}(fT)$
2	Triangle	$A\,\mathrm{tri}\left(\frac{t}{T}\right)$	$AT\,\mathrm{sinc}^2(fT)$
3	Gaussian	$e^{-\pi t^2}$	$e^{-\pi f^2}$
4	Constant	A	$A\delta(f)$ or $2\pi A\delta(\omega)$
5	Impulse	$\delta(t \pm c)$	$e^{\pm j2\pi cf}$
6	Complex Exponential	$e^{\pm j2\pi f_o t}$	$\delta(f \mp f_o)$ or $2\pi\delta(\omega \mp \omega_o)$
7	Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$ Or $\pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$
8	Sine	$\sin(2\pi f_o t)$	$\frac{1}{2j}[\delta(f - f_o) - \delta(f + f_o)]$ Or $\frac{\pi}{j}[\delta(\omega - \omega_o) - \delta(\omega + \omega_o)]$
9	Unit Step	$u(t)$	$\frac{1}{j2\pi f} + \frac{\delta(f)}{2}$ or $\frac{1}{j\omega} + \pi\delta(\omega)$
10	Signum	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$ or $\frac{2}{j\omega}$
11	Decay (Real) Exponential	$e^{\pm \alpha t}$	$\frac{1}{\alpha \mp j2\pi f}$ or $\frac{1}{\alpha \mp j\omega}$
12		$e^{\pm \alpha \lvert t \rvert}$	$\frac{2\alpha}{\alpha^2 \mp \omega^2}$