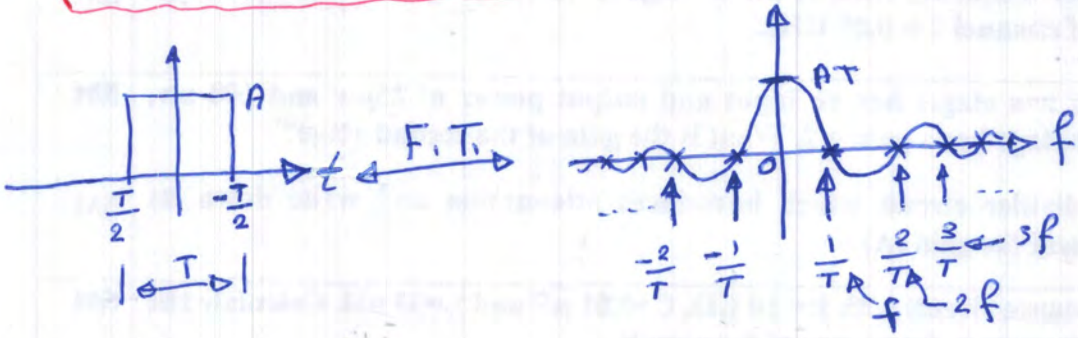
# Lecture 8.3

Remember the following Fourier transform pairs:

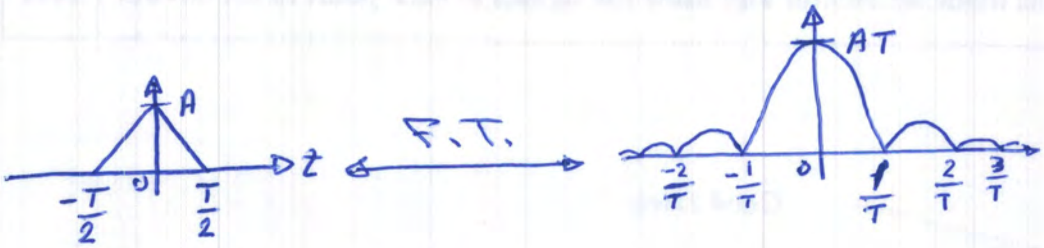
① Rectangular or Box or Boxer or Gate function $g(t) = A\,rct\left(\frac{t}{T}\right)$

$$\boxed{A\,rct\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\text{sinc}(fT)}$$




② Triangular function $tri\left(\frac{t}{T}\right) = \Lambda\left(\frac{t}{T}\right)$

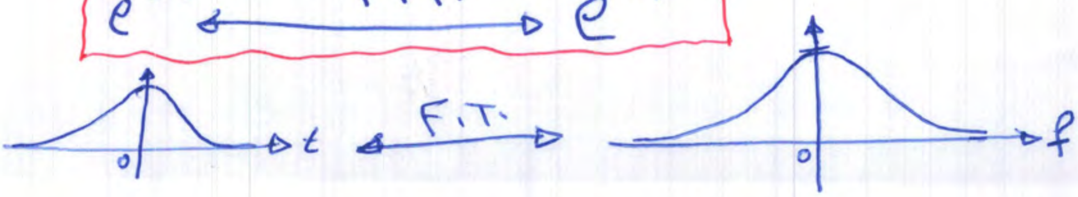
$$\boxed{A\,tri\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\text{sinc}^2(fT)}$$




$$A\,tri\left(\frac{t}{T}\right) = \begin{cases} 1-2|t| & |t| < \frac{T}{2} \\ 0 & |t| \geq \frac{T}{2} \end{cases}$$

③ Gaussian function $g(t) = e^{-\pi t^2}$

$$\boxed{e^{-\pi t^2} \xleftrightarrow{F.T.} e^{-\pi f^2}}$$



④ The constant function $g(t) = A$

$$\boxed{A \xleftrightarrow{F.T.} A\,\delta(f) \quad \text{or} \quad 2\pi A\,\delta(\omega)}$$


⑤ The Impulse function $\delta(t)$ or $g(t) = \delta(t \pm c)$

$$\boxed{\delta(t \pm c) \xleftrightarrow{F.T.} e^{\pm j2\pi cf}}$$

⑥ The complex exponential (sinusoidal)

$$\boxed{e^{\pm j2\pi f_0 t} \xleftrightarrow{F.T.} \delta(f \mp f_0)}$$

sign reversal

OR

$$\boxed{e^{\pm j\omega_0 t} \xleftrightarrow{F.T.} 2\pi\,\delta(\omega \mp \omega_0)}$$

⑦ The Cosine function $g(t)=\cos(2\pi f_0 t)$

$$g(t)=\cos(2\pi f_0 t)=\frac{1}{2}\Big(\underbrace{e^{j2\pi f_0 t}}_{\delta(f-f_0)}+\underbrace{e^{-j2\pi f_0 t}}_{\delta(f+f_0)}\Big)$$

OR

$$g(t)=\cos(2\pi A t)=\frac{1}{2}\Big(\underbrace{e^{j2\pi A t}}_{\delta(f-A)}+\underbrace{e^{-j2\pi A t}}_{\delta(f+A)}\Big)$$

$$\boxed{\cos(2\pi f_0 t)\xleftrightarrow{\text{F.T.}}\frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]}$$

OR

$$\boxed{\cos(2\pi f_0 t)\xleftrightarrow{\text{F.T.}}\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]}$$

⑧ The sine function $g(t)=\sin(2\pi f_0 t)$

$$\boxed{\sin(2\pi f_0 t)\xleftrightarrow{\text{F.T.}}\frac{1}{2j}\left[\delta(f-f_0)-\delta(f+f_0)\right]}$$

or

$$\boxed{\sin(2\pi f_0 t)\xleftrightarrow{\text{F.T.}}\frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]}$$

(9) The unit step Function $u(t)$

$$\boxed{u(t) \xleftrightarrow{F.T.} \frac{1}{j2\pi f} + \frac{\delta(f)}{2}}$$

OR

$$\boxed{u(t) \xleftrightarrow{F.T.} \frac{1}{j\omega} + \pi\,\delta(\omega)}$$

(10) Signum or sgn or sign function $sgn(t)$

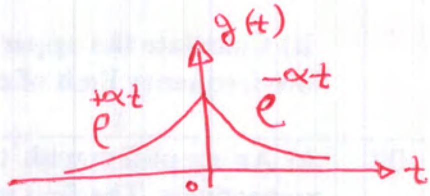
$$sgn(t) = \frac{|t|}{t} = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\boxed{sgn(t) \xleftrightarrow{F.T.} \frac{1}{j\pi f}}$$

$$\boxed{sgn(t) \xleftrightarrow{F.T.} \frac{2}{j\omega}}$$

## (11) Decay Exponential Function

Be careful !!! Note that the decay exponential function is not sinusoidal, it has no j in the exponent, it is a real function $\boxed{g(t) = e^{\pm\alpha t}}$




$$\boxed{e^{\pm\alpha t} \xleftrightarrow{F.T.} \frac{1}{\alpha \mp j2\pi f}}$$

OR

$$\boxed{e^{\pm\alpha t} \xleftrightarrow{F.T.} \frac{1}{\alpha \mp j\omega}}$$

Even more :

$$\boxed{e^{\pm\alpha|t|} \xleftrightarrow{F.T.} \frac{2\alpha}{\alpha^2 \mp \omega^2}}$$

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## Fourier Transform Theorems : without the theorems or the properties, Fourier transform is useless. Just Like a car without fuel.

1) Linearity { superposition }

For any constant (real or complex) $a_1, a_2 ---$

$$\boxed{a_1 g_1(t) + a_2 g_2(t) + \cdots \xleftrightarrow{F.T.} a_1 G_1(f) + a_2 G_2(f) + \cdots}$$

2) Complex Conjugate : if $g(t)$ is complex, then

$$\boxed{\begin{array}{l} g(t) \xleftrightarrow{F.T.} G(f) \\ g^*(t) \xleftrightarrow{F.T.} G^*(-f) \end{array}} \Longleftarrow \text{Complex } g(t)$$

if $g(t)$ is real, then $g(t) = g^*(t)$

$$\boxed{\begin{array}{l} g(t) \xleftrightarrow{F.T.} G(f) \\ g^*(t) \longleftrightarrow G^*(-f) \end{array} \quad \text{or } G(f) = G^*(-f)} \Longleftarrow \text{real } g(t)$$

## Property 3 : Duality theorem

if $$g(t) \xleftrightarrow{FT} G(f)$$

then $$G(t) \longleftrightarrow g(-f)$$

This relation can helps, very much, to solve or to find the time-domain function from the frequency-domain, which is sometimes very difficult integration, and vice versa.

## Property 4 : Time Scaling

if $g(t) \xleftrightarrow{FT} G(f)$

then $g(\alpha t) \xleftrightarrow{FT} \dfrac{1}{|\alpha|} G\left(\dfrac{f}{\alpha}\right)$

Thus : Time compression $\longleftrightarrow$ frequency Expanssion

Time expanssion $\longleftrightarrow$ frequency compression

## Property 5: Time Shifting (Delay)

* if the signal is time-shifted in time-domain, then the corresponding magnitude spectrum did not affected, but there will be a phase shift in the frequency domain.

Hence $g(t) \xleftrightarrow{F.T.} G(f)$

$$g(t \mp a) \longleftrightarrow G(f)\, e^{\mp j2\pi f a}$$

## Property 6: Frequency Shifting

* Frequency shifting OR Modulation Theorem

$$g(t) \longleftrightarrow G(f)$$

$$g(t)\, e^{\mp j2\pi f_c t} \longleftrightarrow G(f \pm f_c)$$

# Property 7 : Differentiation

$$g(t) \xleftrightarrow{F.T.} G(f)$$

$$\boxed{\frac{d^n}{dt^n} g(t) \xleftrightarrow{FT} (j2\pi f)^n G(f)}$$

NOTE : The physical meaning of differentiation

* Differentiation enhances the high frequency components of a signal.
* In other words, differentiation accentuates time variations.

# Property 8 : Integration

$$g(t) \xleftrightarrow{FT} G(f)$$

$$\int_{-\infty}^{t} g(\tau)\, d\tau \xleftrightarrow{FT} \frac{1}{j2\pi f} G(f)$$

* Integration smoothes out the high-frequency components. In other words, suppression of the high-frequency components of the signal.

## Property 9: Convolution

$$g(t) \xleftrightarrow{FT} G(f)$$

$$h(t) \xleftrightarrow{FT} H(f)$$

then $\quad g(t) \circledast h(t) \xleftrightarrow{FT} G(f)\,H(f)$

and

$$g(t)\,h(t) \xleftrightarrow{FT} G(f) \circledast H(f)$$

Thus: In general

$$\text{Convolution} \xleftrightarrow{FT} \text{Multiplication}$$

$$\text{Multiplication} \xleftrightarrow{FT} \text{Convolution}$$

## Property 10 : Area under $g(t)$.

area under curve stands for average value or the D.C. value.

$$\int_{-\infty}^{\infty} g(t)\,dt = G(0)$$

## Property 11 : Area under $G(f)$

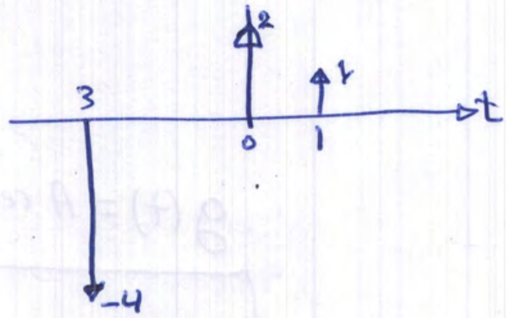
$$g(0) = \int_{-\infty}^{\infty} G(f)\,df$$

## Property 12 : Rayleigh's Energy Theorem

$$\int_{-\infty}^{\infty} |g(t)|^2\,dt = \int_{-\infty}^{\infty} |G(f)|^2\,df$$

**Ex. 1** Find the Fourier transform of $g(t) = \delta(t-1) + 2\delta(t) - 4\delta(t+3)$.

**Solution** $g(t) = \delta(t-1) + 2\delta(t) - 4\delta(t+3)$




we know $\delta(t-t_0) \xleftrightarrow{FT} e^{-j2\pi f t_0}$

and using the linearity property

$$\therefore \; G(f) = e^{-j2\pi f} + 2 - 4\, e^{j2\pi f 3}$$

**Ex. 2** Find the Fourier transform of $g(t) = A\,\text{rect}\left(\frac{t-7}{T}\right) + \delta(t+7)$

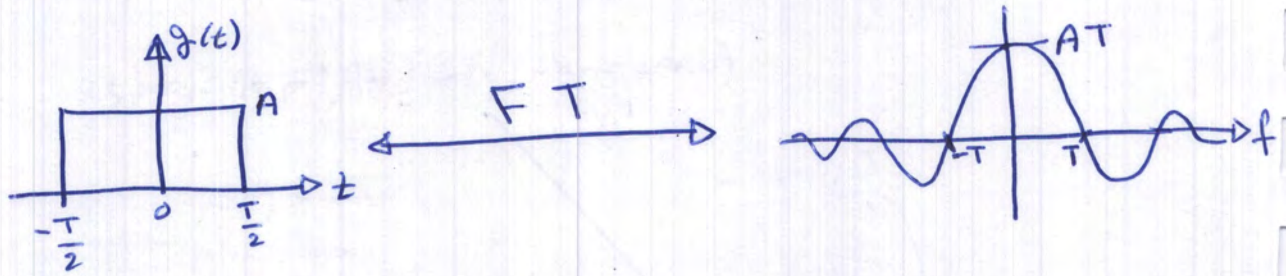
**Solution** $g(t) = A\,\text{rect}\left(\frac{t-7}{T}\right) + \delta(t+7)$

using linearity property

$$G(f) = FT\left\{A\,\text{rect}\left(\frac{t-7}{T}\right)\right\} + FT\left\{\delta(t+7)\right\}$$

$$= AT\,\text{sinc}(fT)\, e^{-j14\pi f} + e^{j14\pi f}$$

**EX. 3** given the signal and its frequency-domain sketches shown below. Find the Fourier transform of $A\,\mathrm{sinc}(2Wt)$



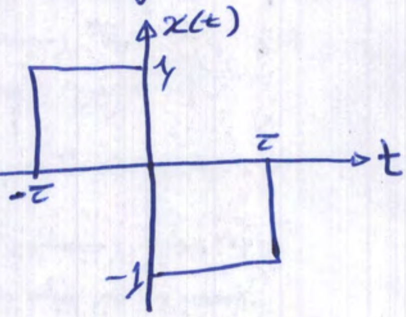

**Solution** $g(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{FT} G(f) = AT\,\mathrm{sinc}(fT)$

Using duality theorem:

$$G(t) \longleftrightarrow g(-f)$$

$$A\,\mathrm{sinc}(2Wt) \longleftrightarrow \frac{A}{2W}\,\mathrm{rect}\left(\frac{-f}{2W}\right) = \frac{A}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right).$$

**EX. 4** Determine the Fourier transform of the signal shown below.




**Solution** $x(t) = \mathrm{rect}\left(\frac{t+\frac{\tau}{2}}{\tau}\right) - \mathrm{rect}\left(\frac{t-\frac{\tau}{2}}{\tau}\right)$

$$X(f) = \tau\,\mathrm{sinc}(f\tau)\,e^{j2\pi f\frac{\tau}{2}} - \tau\,\mathrm{sinc}(f\tau)\,e^{-j2\pi f\frac{\tau}{2}}$$

$$= \tau\,\mathrm{sinc}(f\tau)\left[e^{j2\pi f\frac{\tau}{2}} - e^{-j2\pi f\frac{\tau}{2}}\right]$$

$$= \frac{2j}{\pi f}\sin^2(\pi f\tau).$$

**EX. 6** $x(t)$ is a time-domain signal. If $x(t)$ shifted in frequency by $e^{j2\pi f_0 t}$, find the Fourier transform of the above signal combination.

**Solution**: This is frequency shift or translation property.

$$\int_{-\infty}^{\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi (f-f_0)}\, dt$$

So, $FT\left\{x(t)\, e^{j2\pi f_0 t}\right\} = X(f-f_0)$

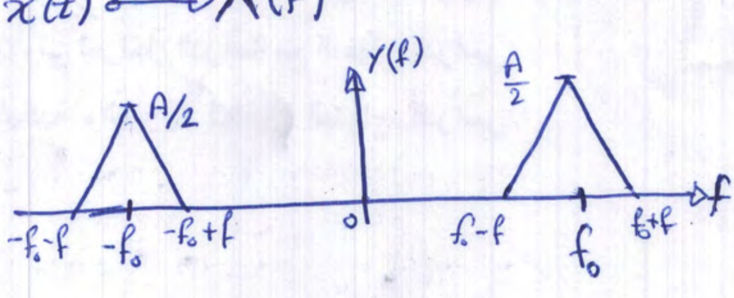
**EX. 7** Find the Fourier transform of $x(t)\cos(2\pi f_0 t)$.

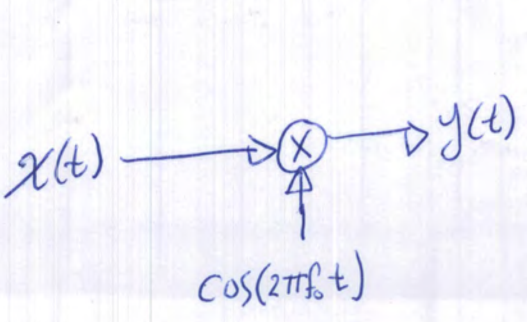
**solution** we know $\cos(2\pi f_0 t) = \frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t}$

then $FT\left\{\frac{1}{2}x(t)\, e^{j2\pi f_0 t} + \frac{1}{2}x(t)\, e^{-j2\pi f_0 t}\right\} = \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$

For more explanation assume $x(t) \xrightarrow{FT} X(f) \Rightarrow$


then $FT\left\{x(t)\cos(2\pi f_0 t)\right\} \Rightarrow$




} A simple modulator

# Fourier Transform of Periodic Signals

The Fourier Series of a periodic signal is

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{j2\pi n f_0 t}$$

where $X_n = \frac{1}{T}\int_T x(t)\, e^{-jn2\pi f_0 t}\, dt$

Thus $x(t) = \cdots X_{-1} e^{-j2\pi f_0 t} + X_0 + X_1 e^{j2\pi f_0 t} + X_2 e^{j2\pi f_0 t} + \cdots$

By using Linearity property of Fourier transform, then

$$X(f) = FT\left\{\sum_{n=-\infty}^{\infty} \underbrace{X_n}_{\text{these are constants}} e^{j2\pi n f_0 t}\right\} = \sum_{n=-\infty}^{\infty} X_n \underbrace{FT\left\{e^{j2\pi n f_0 t}\right\}}_{\delta(f-nf_0)}$$

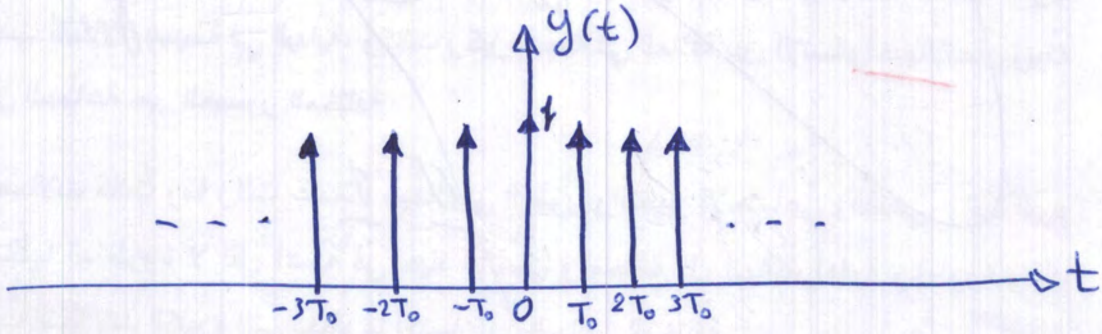
$$\therefore \quad X(f) = \sum_{n=-\infty}^{\infty} X_n\, \delta(f-nf_0)$$

# Fourier Transform of Impulse Train

A train of impulses can be written as

$$\delta_T(t) = y(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

where $T_0$ is the fundamental period.




Since $y(t)$ is periodic, then it has a Fourier series with coefficients given as:

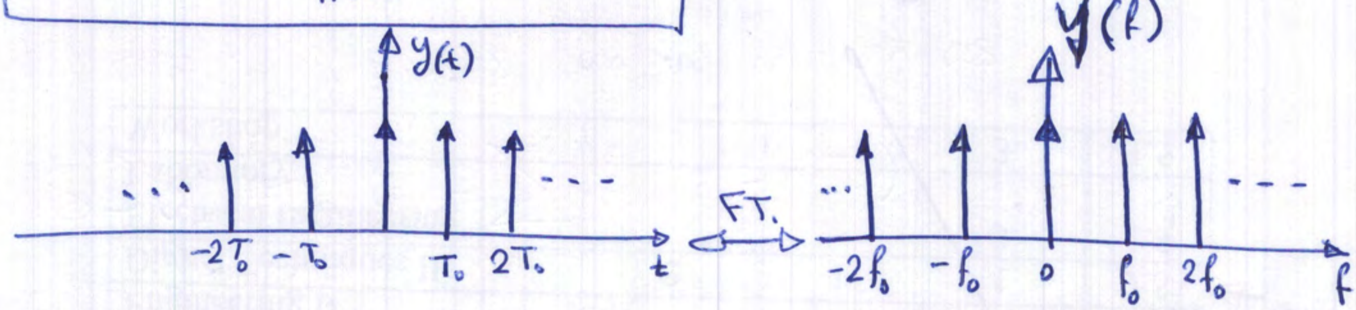
$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-j2\pi f_0 n t}\, dt = \frac{1}{T_0}$$

$$D_n = \frac{1}{T_0} = F_0 \quad \text{for any } n$$

Then the Fourier transform of $y(t)$ is

$$Y(f) = f_0 \sum_{n=-\infty}^{\infty} F.T.\left\{ e^{j2\pi n f_0 t} \right\} = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

$$\boxed{Y(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$$




∴ In general

$$\boxed{\sum_{k=-\infty}^{\infty} \delta(t - kT_0) \xleftrightarrow{FT} f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$$

impulse train in time-domain $\xleftrightarrow{FT}$ impulse train in frequency domain

EX.1 Find the Fourier transform of a periodic signal of pulse $p(t)$ of period $T_s$ using the Convolution theorem.

Solution The periodic pulse signal can be expressed as

$$x(t) = \left[\sum_{n=-\infty}^{\infty} \delta(t - nT_0)\right] \circledast p(t) = \sum_{n=-\infty}^{\infty} p(t - nT_0)$$

where $p(t)$ represents one period of $x(t)$, with period $T_0$.

From the convolution theorem:

$$X(f) = FT\left\{\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_0) \circledast p(t)\right]\right\}$$

$$= FT\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT_0)\right\} P(f)$$

$$= f_0 P(f) \sum_{k=-\infty}^{\infty} \delta(f - kf_0)$$

$$= f_0 \sum_{k=-\infty}^{\infty} P(kf_0)\, \delta(f - kf_0)$$

In general:

$$\boxed{\sum_{n=-\infty}^{\infty} p(t - nT_0) \xrightarrow{FT} \sum_{k=-\infty}^{\infty} f_0 P(kf_0)\, \delta(f - kf_0)}$$