Sampling Theory

The continuous time signal $x(t)$ can be sampled instantinuously as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \quad \text{——} (1)$$

where T_s is the ~~sampling~~ interval.

* After Sampling :

① Is it possible to recover $x(t)$ from $x_\delta(t)$?

② How is $x(t)$ recovered from $x_\delta(t)$?

* To answer these two questions, we have to understand the uniform Sampling theorem.

Before we go ahead, Lets have Some important Knowledge about signals and their types.

* In real world, there are continuous-time signals and discrete-time signals

* Due to the recent advance developement in digital technology:-

  - Inexpensive,
  - Light weight,
  - programmable, and
  - easily reproducible

  discrete-time systems are available.

→ therefore, the processing of discrete-time signals is more flexible and is also preferable to processing than the continuous-time ones.

* Thus, we should be able to convert a continuous-time signals into discrete-time signals.

* Sampling process, is a tool that can be used to convert the continuous-time signal to a discrete-time signal and vice-versa.

Sampling Theory :

A band limited signal of bandwidth W Hz sampled at frequency f_s, can be reconstructed from the sampled version if $f_s \geqslant 2W$.

Proving Sampling Theory :

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s) \quad \text{——— (1)}$$

$T_s = \frac{1}{f_s}$ = sampling interval.

Since we know $\delta(t-nT_s)$ is zero everywhere except at $t = nT_s$

$$\therefore\ x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \quad \text{——— (2)}$$

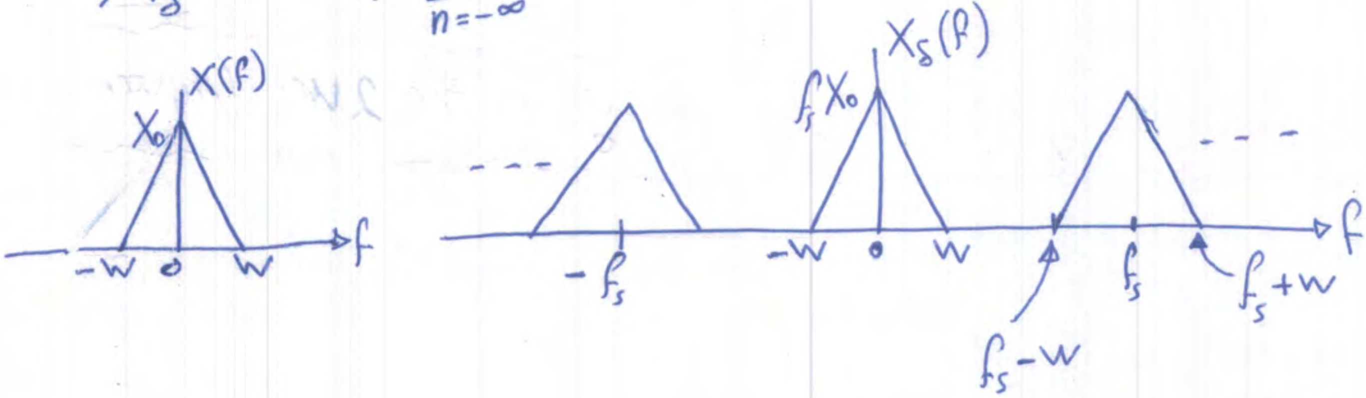
$$X_\delta(f) = X(f) * \left[f_s \sum_{n=-\infty}^{\infty}\delta(f-nf_s)\right] \quad \text{——— (3)}$$

Since $X(f) * \delta(f - nf_s) = \int_{-\infty}^{\infty} X(u)\,\delta(f-u-nf_s)\,du$

$$\therefore \quad X(f) * \delta(f - nf_s) = X(f - nf_s) \qquad (4)$$

by the sifting property of the delta function,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad (5)$$



$\therefore$ Sampling produces a periodic repetition of $X(f)$ in the frequency domain with a spacing f_s.

① If $f_s < 2W$, the separate terms overlap, $x(t)$ can not be recovered from $x_\delta(t)$. (aliasing)

② If $f_s = 2W$, the separate terms touch each other, $x(t)$ can be recovered from $x_\delta(t)$ using a sharp LPF.

③ If $f_s > 2W$, $x(t)$ can be recovered from $x_\delta(t)$ using LPF.

End of Prove

* The reconstruction filter should has a bandwidth B in the range

$$\boxed{W < B < (f_s - W)}$$

where W : Bandwidth of the signal $x(t)$

f_s : Sampling frequency

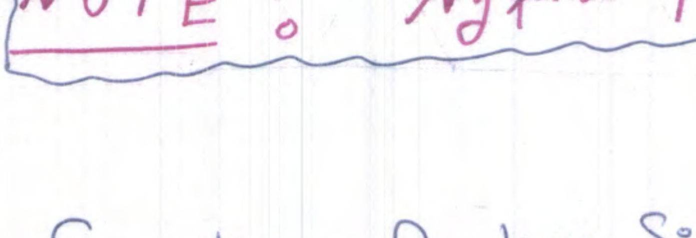
NOTE : Nyquist frequency is the $2W$ frequency.

Sampling a Bandpass Signal

if a signal has a spectrum W Hz and upper frequency f_u Hz, then a rate f_s at which the signal can be sampled is $2f_u/m$ Hz, where m is the largest integer not exceeding f_u/W.

$W = 3 - 1.75 = 1.25$ Hz

Ex./



$f_u = 3$ Hz

$m \leq \left\lceil \frac{f_u}{W} \right\rceil = \frac{3}{1.25} = 2.4$

$\therefore\ m = 2$

$\therefore\ f_s = \frac{2 f_u}{m} = \frac{2 * 3}{2} = 3$ Hz.

note that if the signal is a Lowpass signal then, $f_s = 6$ Hz.

EX. A continuous time signal $x(t) = 8\cos(200\pi t)$. Determine:-

① Minimum sampling rate, (Nyquist rate to avoid aliasing).

② If sampling frequency $f_s = 400$ Hz, what is the discrete-time signal $x(n)$ or $x(nT_s)$ obtained after sampling?

③ If sampling frequency $f_s = 150$ Hz, what is the discrete-time signal $x(n)$ or $x(nT_s)$ obtained after sampling?

④ what is the frequency $0 < f < \frac{f_s}{2}$ that yields samples identical to those obtained in part ③?

Solution: $x(t) = 8\cos(2\pi 100 t) \rightarrow f = 100$ Hz.

① $f_s = 2f = 2 * 100 = 200$ Hz.

② $f_s = 400 \text{ Hz} \rightarrow x(nT_s) = x\left(\frac{n}{f_s}\right) = 8\cos\left(2\pi n \frac{100}{400}\right) = 8\cos\left(\frac{2\pi n}{4}\right) = 8\cos\left(\frac{\pi n}{2}\right)$.

$\therefore x(n) = x(nT_s) = 8\cos\left(\frac{n\pi}{2}\right)$.

③ $f_s = 150\text{Hz} \rightarrow x(nT_s) = x\left(\frac{n}{f_s}\right) = 8\cos\left(2\pi n \frac{\cancel{100}^{2}}{\cancel{150}_{3}}\right) = 8\cos\left(\frac{4\pi n}{3}\right) = 8\cos\left(\frac{6\pi n}{3} - \frac{2\pi n}{3}\right)$

$x(n) = 8\cos\left[\left(2\pi - \frac{2\pi}{3}\right)n\right] = 8\cos\left(\frac{2\pi n}{3}\right)$

④ $f_s = 150\text{ Hz} \rightarrow x(nT_s) = x\left(\frac{n}{f_s}\right) = 8\cos(2\pi f \cancel{t}^{\frac{n}{f_s}}) = 8\cos(2\pi f n/150)$

From part ③ $\frac{\cancel{2\pi n}}{3} = \frac{\cancel{2\pi} f \cancel{n}}{150} \rightarrow f = \frac{150}{3} = 50$ Hz

$\therefore x_4(t) = 8\cos(100\pi t)$.

NOTE:- $x(t) = 8\cos(2\pi 100 t)$ if $f_s = 150$Hz then alis frequency $f_a = 150 - 100 = 50$Hz will appears.

EX. An analog signal is expressed by the equation $x(t) = 3\cos(50\pi t) + 10\sin(300\pi t) + \cancel{10\sin(300\pi t)} - \cos(100\pi t)$. Calculate the Nyquist rate for this signal.

Solu.

$$x(t) = 3\cos(50\pi t) + 10\sin(300\pi t) - \cos(100\pi t)$$

$$x(t) = 3\cos(\omega_1 t) + 10\sin(\omega_2 t) - \cos(\omega_3 t)$$

$$\omega_1 = 2\pi f_1 = 50\pi \rightarrow f_1 = 25\ Hz$$

$$\omega_2 = 2\pi f_2 = 300\pi f_2 \rightarrow f_2 = 150\ Hz$$

$$\omega_3 = 2\pi f_3 = 100\pi f_3 \rightarrow f_3 = 50\ Hz$$

Largest frequency is $f_2 = 150\ Hz$

$$\therefore f_s = 2 f_2 = 300\ Hz.$$

EX. Find the Nyquist rate and the Nyquist interval for the signal

$$x(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t).$$

Solu. since $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$\therefore x(t) = \frac{1}{2\pi}\cdot\frac{2}{2}\cos(4000\pi t)\cos(1000\pi t)$$

$$= \frac{1}{4\pi}\left[2\cos(4000\pi t)\cos(1000\pi t)\right]$$

$$= \frac{1}{4\pi}\left[\cos(4000\pi t + 1000\pi t) + \cos(4000\pi t - 1000\pi t)\right]$$

$$= \frac{1}{4\pi}\left[\cos(5000\pi t) + \cos(3000\pi t)\right]$$

$$\therefore \omega_1 = 2\pi f_1 = 5000\pi f_1 \longrightarrow f_1 = 2500\ Hz$$

$$\omega_2 = 2\pi f_2 = 3000\pi f_2 \longrightarrow f_2 = 1500\ Hz$$

$$\therefore f_s = 2 f_1 = 5000\ Hz$$

$$T_s = \frac{1}{f_s} = \frac{1}{5000} = 0.2 * 10^{-3}\ sec.$$

Reconstruction Filter (LPF) :

* reconstruction filter is a low-pass-filter (LPF), which is also known as interpolation filter.

* The process of reconstructing the signal from the discret-time version is called interpolation.

- Assuming an ideal LPF :

$$\boxed{H(f) = H_o \,\Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_o}} \qquad W \leq B \leq f_s - W$$

$$\therefore \; Y(f) = f_s \, H_o \, X(f) \, e^{-j2\pi f t_o}$$

and the time-domain becomes :-

$$\boxed{y(t) = f_s \, H_o \, x(t - t_o)}$$ — reconstructed signal.

OR
$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s)$$

$$y(t) = 2BH_o \sum_{n=-\infty}^{\infty} x(nT_s)\, \text{sinc}\left[2B(t - t_o - nT_s)\right]$$

But $B = \frac{f_s}{2}$, $H_o = T_s$, and let $t_o = 0$ for simplicity

$$y(t) = \sum_{n} x(nT_s)\, \text{sinc}(f_s t - n)$$