"Engineering Analysis"

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"Matrix"

1- Deffinition

Matrix is rectangular array of element arranged in horizontal rows and vertical columns.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where:-

a_{ij} represents that element appearing in the ith row and jth column.

A matrix is square if it has the same number of (rows and columns)

2- Matrix Addition

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

and

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

Ex:- show that $A + B = B + A$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+1 & 6+2 \\ 7+3 & 8+4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

3- Scalar and Matrix Multiplication

if λ is numbe or a function of t then λA is defined to the matrix obtained by multiplying every element of A by λ

$$\lambda A = \lambda [a_{ij}] = [\lambda a_{ij}]$$

Ex:- $\lambda = 3$ and $A = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$; find λA ?

$$\lambda A = 3 \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 3\times 2 & 3\times 1 \\ 3\times 4 & 3\times(-2) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 12 & -6 \end{bmatrix}$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$

the the product AB is defined to be the matrix $C = [c_{ij}]$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

then

$$AB = C$$

and

$$AB \neq BA$$

$$A(BC) = (AB)C$$

$$A(B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$; find AB and BA

$$AB = \begin{bmatrix} 1\times5 + 2\times7 & 1\times6 + 2\times8 \\ 3\times5 + 4\times7 & 3\times6 + 4\times8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5\times1 + 6\times3 & 5\times2 + 6\times4 \\ 7\times1 + 8\times3 & 7\times2 + 8\times4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$AB \neq BA$$

4- Powers of squre Matrix

if n is apositive integer and A is a squre matrix, then

$$A^n = \underbrace{AA \cdots\cdots A}_{n \text{ times}}$$

ie

$$A^2 = AA$$
$$A^3 = AAA$$

and By definition $A^0 = I$

where $I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{bmatrix}$

and $AI = IA = A$

Ex.1- let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find A^2 and AI?

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times3 & 1\times2+2\times4 \\ 3\times1+4\times3 & 3\times2+4\times4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$AI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Ex.2:- let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ find $(2A-B)^2$?

$$2A-B = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + (-1)\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -1 & 0 \end{bmatrix}$$

$$(2A-B)^2 = (2A-B)(2A-B) = \begin{bmatrix} -3 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3\times-3 + (-2)\times(-1) & -3(-2) + (-2\times0) \\ -1(-3) + 0\times(-1) & -1\times(-2) + 0\times0 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 3 & 2 \end{bmatrix}$$

5- "Differentiation and Integration Matrices":-

the derivative of $A = [a_{ij}]$ is the matrix obtained by differentating each element of A ; that is

$$\frac{dA}{dt} = \left[\frac{da_{ij}}{dt}\right]$$

and similarly, the integration of A either definte or indefinite is obtained by integrating each element of A

$$\int_a^b A\,dt = \left[\int_a^b a_{ij}\,dt\right] \quad \text{and} \quad \int A\,dt = \left[\int a_{ij}\,dt\right]$$

Ex1- let $A = \begin{bmatrix} t^2+1 & e^{2t} \\ \sin t & 45 \end{bmatrix}$ find $\frac{dA}{dt}$ and $\int A$

$$\frac{dA}{dt} = \begin{bmatrix} \frac{d}{dt}(t^2+1) & \frac{d}{dt}(e^{2t}) \\ \frac{d}{dt}(\sin t) & \frac{d}{dt}(45) \end{bmatrix} = \begin{bmatrix} 2t & e^{2t} \\ \cos t & 0 \end{bmatrix}$$

$$\int A = \begin{bmatrix} \int (t^2+1) & \int e^{2t} \\ \int \sin t & \int 45 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t^3+t+C_1 & \frac{1}{2}e^{2t}+C_2 \\ -\cos t + C_3 & 45t + C_4 \end{bmatrix}$$

Ex2:- Find $\frac{dx}{dt}$ if $x \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

$$\frac{dx}{dt} = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}$$

Ex3:- find $\int_0^1 x\,dt$ if $x \begin{bmatrix} 1 \\ e^t \\ 0 \end{bmatrix}$

$$\int_0^1 x\,dt = \begin{bmatrix} \int_0^1 1\,dt \\ \int_0^1 e^t\,dt \\ \int_0^1 0\,dt \end{bmatrix} = \begin{bmatrix} 1 \\ e-1 \\ 0 \end{bmatrix}$$

6- "Matrix Analysis"

(A): Singular and non-singular Matrices

let $A(n,n)$ be a square matrix, then A is singular matrix if $\det(A) = 0$ else A is non singular

Ex1:- let $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ is this matrix singular ?

$\det(A) = |A| = 1\times 8 - 2\times 4 = 0$

hence A is the singular matrix.

Ex2:- let $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$\det(B) = |B| = 3\times 2 - 4\times 1 = 2 \neq 0$

$\therefore$ B is a non singular matrix

Note that:- any Matrix has inverse matrix if this matrix is non singular maatrix

Ex3: let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A) = 1\times[(1\times 1) - 0\times 0] - 0[\cancel{(}0\times 1 - 0\times 0)] + 0[(0\times 0 - 1\times 0)]$

$= 1 \neq 0$

$\therefore$ the matrix A is non singular.

Ex4:- find the value of x if the matrix A is singular?

$\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$

sol:-

$\because$ the matrix A is singular i.e $\det(A) = 0$

$4(5-x) - 2(x+1) = 0 \Rightarrow 10 - 2x - x - 1 = 0 \Rightarrow x = 3$

the the matrix A will be

$A = \begin{bmatrix} 5-3 & 3+1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$

(B) Consistent system

- A system of equations is said to consistent if its solution exist (i.e the system is nonsingular)
- A system of equations is said to be inconsistent if its solution not exist (i.e the system is singular).

Ex1 Examine the consistency of the system of equation

$$5x - y + 4z = 5$$
$$2x + 3y + 5z = 2$$
$$5x - 2y + 6z = -1$$

solution. the equations can be expressed as $AX = B$

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} \quad ; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

$$\det(A) = 5(18+10) - (-1)(12-25) + 4(-4-15) = 51 \neq 0$$

$\therefore$ A is nonsingular matrix, So the system is consistent

Ex2:- Examine the consistency of the system of equations.

$$x + y - z = 1$$
$$x - y + z = 1$$
$$2x - 2y + 2z = 2$$

Solution the equations can be expressed AS $AX = B$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix} ; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\det(A) = 1(-2+2) - 1(2-2) + (-1)(-2+2) = 0$$

A is singular matrix, So the system of equations ~~are~~ inconsistent

© Rank of a Matrix

The rank of a matrix may be defined as the number of non-zero rows for the given matrix. Let us follow the following steps to find the rank of the matrix.

① **Step (1)**

Suppose that the Matrix (A) has a diminsion $(n \times n)$ if the determinant of the matrix is non-zero then the rank of the matrix is (n), otherwise go to next step.

② **Step (2)**

If the determinant of matrix is zero then take any matrix has the diminsion $(n-1) \times (n-1)$ and find the determinant if non-zero, then the matrix has rank $(n-1)$ and so on.

Ex 1:- find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

solution

$$\det(A) = 1 * (2\times2 - 1\times1) - 2(1\times2 - 1\times2) + 3(1\times1 - 2\times2)$$
$$= -6$$

$\therefore$ the determinant of matrix is non-zero, so the matrix has rank (3)

Ex 2:- find the rank of matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 4 & 6 \end{bmatrix}$

solution

$$\det(B) = 1(12-4) - 2(6-2) + 3(4-4) = 8 - 8 = 0$$

Now, take any 2×2 sub matrix (B) then find the determinant

$$B_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = 0; \quad B_2 = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = -4; \quad B_3 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0, \quad B_4 = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix} = 8$$

$\because$ B_2 and B_4 non-zero matrix, then the rank of matrix B is (2)

Ex:-3- find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

solution

$\det(A) = 1(12-12) - 2(6-6) + 3(4-4) = 0$

then the sub matrix are

$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$; $A_2 = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = 0$; $A_3 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = 0$; $A_4 = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix} = 0$

the determinant of any 2x2 matrises are zero, So the rank of matrix is (1) because of found an integer number in the matrix.

Notes

1- Rank of a null matrix is always zero (null matrix mean all of the number in the matrix are zero, So the rank of this matrix is (0)

2- for non-zero matrix $\rho(A) \geq 1$ ($\rho(A)$ is rank of matrix)

3- For a square matrix the rank is n if its determinant $\neq 0$

4- if the matrix has different dimentions then $\rho(A)$ is equal to min (m x n)

Ex:- find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

solution

$\det(A) = 1$, So the $\rho(A) = 3$

(D) Solution of set of equations using inverse matrix method

Ex1:- Solve the following equations using matrix invesion method.

$$X_1 + 2X_2 + 2X_3 = 5$$
$$3X_1 - 2X_2 + X_3 = -6$$
$$2X_1 + X_2 - X_3 = -1$$

Solution

$$AX = B$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix} ; \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} ; \quad B = \begin{bmatrix} 5 \\ -6 \\ -1 \end{bmatrix}$$

then $X = A^{-1}B$

Now, find A^{-1}

$$A^{-1} = \frac{C^T}{|A|}$$

step (1) find det (A)

$$\det(A) = |A| = 1(2-1) - 2(-3-2) + 2(3+4) = 25$$

step (2) find C^T

cofactor of (A) = C =

or adj(A) = C^T

$$\begin{vmatrix} +\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \end{vmatrix}$$

$$\Rightarrow C = \begin{vmatrix} 1 & 5 & 7 \\ 4 & -5 & 3 \\ 6 & 5 & -8 \end{vmatrix}$$

$$C^T = \begin{vmatrix} 1 & 4 & 6 \\ 5 & -5 & 5 \\ 7 & 3 & -8 \end{vmatrix}$$

step (3)

$$\Rightarrow A^{-1} = \frac{C^T}{|A|} = \begin{bmatrix} \frac{1}{25} & \frac{4}{25} & \frac{6}{25} \\ \frac{5}{25} & \frac{-5}{25} & \frac{5}{25} \\ \frac{7}{25} & \frac{3}{25} & \frac{-8}{25} \end{bmatrix}$$

step 4

$$A^{-1} \cdot B = X$$

$$\begin{bmatrix} \frac{1}{25} & \frac{4}{25} & \frac{6}{25} \\ \frac{5}{25} & \frac{-5}{25} & \frac{5}{25} \\ \frac{7}{25} & \frac{3}{25} & \frac{-8}{25} \end{bmatrix} \begin{bmatrix} 5 \\ -6 \\ -1 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{5}{25} - \frac{24}{25} - \frac{6}{25} \\ \frac{25}{25} + \frac{30}{25} - \frac{5}{25} \\ \frac{35}{25} - \frac{18}{25} + \frac{8}{25} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$X_1 = -1$, $X_2 = 2$ and $X_3 = 1$

<u>EX 2:-</u> find the vector of X for the following equations

$$x_1 - x_2 - x_3 + x_4 = 0$$
$$x_1 + x_2 + x_3 - 2x_4 = -2$$
$$2x_1 - 2x_2 - x_3 + x_4 = -1$$
$$x_1 + x_2 + x_3 + x_4 = 10$$

solution

$$AX = B$$

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -2 \\ 2 & -2 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \; ; \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \; ; \; B = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 10 \end{bmatrix}$$

$$\det(A) = 1 \begin{bmatrix} 1 & 1 & -2 \\ -2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= 1\left[1(-1-1) - 1(-2-1) - 2(-2+1)\right] + 1\left[1(-1-1) - 1(2-1) - 2(2+1)\right]$$
$$-1\left[1(-2-1) - 1(2-1) - 2(2+2)\right] - 1\left[1(-2+1) - 1(2+1) + 1(2+2)\right]$$
$$= 6$$

$$C = \begin{bmatrix} +\begin{bmatrix} 1 & 1 & -2 \\ -2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} & +\begin{bmatrix} 1 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ -\begin{bmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} & +\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} & +\begin{bmatrix} 1 & -1 & -1 \\ 2 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ +\begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} & +\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ -\begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix} & +\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} & +\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 9 & -12 & 0 \\ 2 & 2 & -2 & -2 \\ 0 & -6 & 6 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{C^T}{|A|} = \frac{1}{6} \begin{bmatrix} 3 & 2 & 0 & 1 \\ 9 & 2 & -6 & 1 \\ -12 & -2 & 6 & 2 \\ 0 & -2 & 0 & 2 \end{bmatrix}$$

$$A^{-1} . B = X$$

$$\begin{bmatrix} \frac{3}{6} & \frac{2}{6} & 0 & \frac{1}{6} \\ \frac{9}{6} & \frac{2}{6} & \frac{-6}{6} & \frac{1}{6} \\ \frac{-12}{6} & \frac{-2}{6} & \frac{6}{6} & \frac{2}{6} \\ 0 & \frac{-2}{6} & 0 & \frac{2}{6} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ -1 \\ 10 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

$$X_1 = \frac{3}{6} \times 0 + \frac{2}{6}(-2) + 0 + \frac{10}{6} = \frac{6}{6} = 1$$

$$X_2 = \frac{9}{6} \times 0 + \frac{2}{6}(-2) + \frac{(-6)}{6}(-1) + \frac{10}{6} = \frac{12}{6} = 2$$

$$X_3 = \frac{-12}{6} \times 0 + \frac{(-2)}{6}(-2) + \frac{6}{6}(-1) + \frac{2}{6}(10) = \frac{18}{6} = 3$$

$$X_4 = 0 \times 0 + \frac{(-2)}{6}(-2) + 0 \times (-1) + \frac{2}{6}(10) = \frac{24}{6} = 4$$

$$\therefore X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Ex 3:- find the value of X_1 and X_2 for the following equations.

$$4X_1 + 7X_2 = 2$$

$$2X_1 + 6X_2 = 5$$

Solution

$$AX = B$$

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \; ; \; X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \; ; \; B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$A^{-1} \cdot B = X$$

to find A^{-1}

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$det(A) = 4 \times 6 - 2 \times 7 = 10$$

$$adj(A) = \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

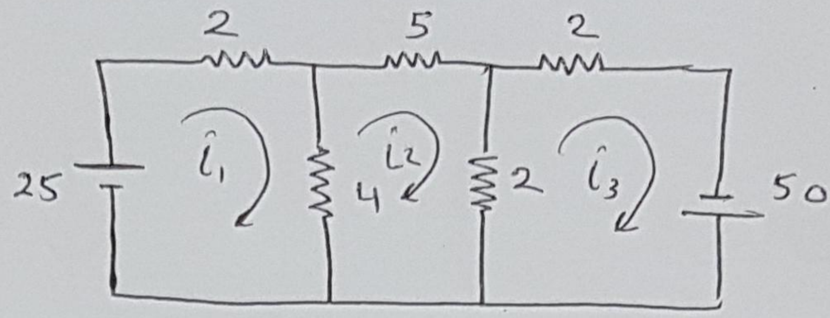
$$\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$X_1 = 1.2 - 3.5 = -2.3$$

$$X_2 = -0.4 + 2 = 1.6$$

$$X = \begin{bmatrix} -2.3 \\ 1.6 \end{bmatrix}$$

Ex:- find (I) using matrix method to solve the circuit as shown below :



Solution:-

$$6i_1 - 4i_2 - 0i_3 = 25$$

$$-4i_1 + 11i_2 - 2i_3 = 0$$

$$0i_1 - 2i_2 + 4i_3 = 50$$

$$AI = B$$

$$A = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 11 & -2 \\ 0 & -2 & 4 \end{bmatrix} \; ; \quad B = \begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -4 & 0 \\ -4 & 11 & -2 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix}$$

$$I = A^{-1} B$$

$$\det(A) = 6(44-4) + 4(-16-0) + 0 = 176$$

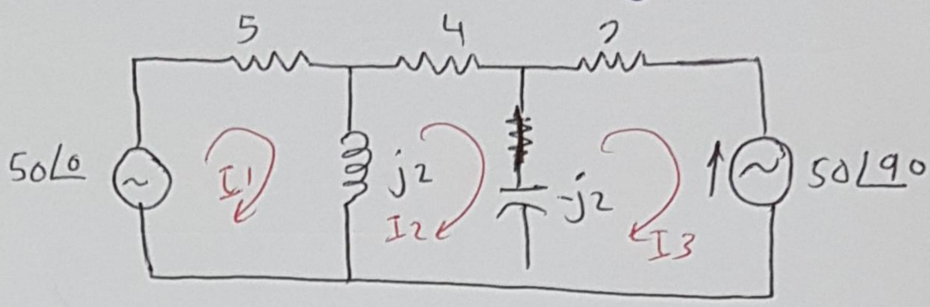
$$C = \begin{bmatrix} 240 & 16 & 8 \\ 16 & 24 & 12 \\ 8 & 12 & 50 \end{bmatrix} \xRightarrow{(adj A)} = C^T = \begin{bmatrix} 240 & 16 & 8 \\ 16 & 24 & 12 \\ 8 & 12 & 50 \end{bmatrix}$$

$$A^{-1} = \frac{(adj\,A)}{\det(A)} = \begin{bmatrix} 0.2273 & 0.0909 & 0.0455 \\ 0.0909 & 0.1364 & 0.0682 \\ 0.0455 & 0.0682 & 0.2841 \end{bmatrix}$$

$$\therefore \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix} = \begin{bmatrix} 7.9545 \\ 5.6818 \\ 15.3409 \end{bmatrix} \text{ Amper.}$$

EX2:- Find the currents I_1, I_2 and I_3 using Matrix Methods

solution:-



$$(5+j2)I_1 - j2I_2 - 0 = 50$$

$$-j2I_1 + 4I_2 + j2I_3 = 0$$

$$0 + j2I_2 + (2-2j)I_3 = -50j$$

$$A = \begin{bmatrix} 5+2j & -2j & 0 \\ -2j & 4 & 2j \\ 0 & 2j & 2-2j \end{bmatrix} \; ; \; B = \begin{bmatrix} 50 \\ 0 \\ -50j \end{bmatrix} \; ; \; I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}.$$

$$AI = B.$$

$$I = A^{-1}B = A^{-1}\begin{bmatrix} 50 \\ 0 \\ -50j \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 8.4906 - 4.7075j \\ -3.3019 - 0.9434j \\ 10.3774 - 11.3208j \end{bmatrix}$$

(E) Row transformation Method

this operation to find inverse matrix

As we know $I = A^{-1} A$

Ex:- find the inverse matrix by Row transformation Methods'

for the Matrix $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

solution:-

$$A = I A$$

$$\Rightarrow I = A^{-1} A$$

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow R_1 - R_2 = R_1 \Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow R_2 - 2R_1 = R_2 \Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow R_3 - 3R_1 = R_3 \Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow R_2 \Rightarrow \frac{1}{2} R_2 \Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow R_1 \Rightarrow R_1 + R_2 \Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ -3 & 3 & 1 \end{bmatrix} \cdot A$$

$$R_3 \Rightarrow R_3 + 2R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ -5 & 6 & 1 \end{bmatrix} \cdot A$$

$$R_3 \to \frac{R_3}{4} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ \frac{-5}{4} & \frac{6}{4} & \frac{1}{4} \end{bmatrix} \cdot A$$

$$R_1 \to R_1 + \frac{1}{2} R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -1 & \frac{3}{2} & 0 \\ \frac{-5}{4} & \frac{6}{4} & \frac{1}{4} \end{bmatrix} \cdot A$$

$$R_2 \to R_2 - \frac{1}{2} R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ \frac{-5}{4} & \frac{6}{4} & \frac{1}{4} \end{bmatrix} \cdot A$$

$$I = A^{-1} A$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ \frac{-5}{6} & \frac{6}{4} & \frac{1}{4} \end{bmatrix}$$

Ex.2 find the inverse matrix for $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

Solution:

$$A = I\,A$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A$$

$$\xrightarrow[R_3 \Rightarrow 6R_1 + R_3]{R_2 \Rightarrow -R_1 + R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} . A$$

$$\xrightarrow[R_3 \Rightarrow 4R_2 + R_3]{R_1 \Rightarrow R_2 + R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} . A$$

$$R_3 \Rightarrow -R_3 \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix} . A$$

$$\xrightarrow[R_2 \Rightarrow R_3 + R_2]{R_1 \Rightarrow R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix} . A$$

$$I = A^{-1} . A$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

(F) Gaussian Elemination

Ex:- use Gussian elimination method to solve the following equations

$$2x_1 - x_2 + 2x_3 = 10$$
$$x_1 - 2x_2 + x_3 = 8$$
$$3x_1 - x_2 + 2x_3 = 11$$

Solution :-

$$A\cancel{X} = B$$

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -1 & 2 \end{bmatrix} ; \quad B = \begin{bmatrix} 10 \\ 8 \\ 11 \end{bmatrix} ; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Augmeted Matrix is $[A ; B]$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 10 \\ 1 & -2 & 1 & 8 \\ 3 & -1 & 2 & 11 \end{array}\right]$$

first step we can chang R_1 by R_2 to becomes

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 2 & -1 & 2 & 10 \\ 3 & -1 & 2 & 11 \end{array}\right] \xrightarrow[R_3 \Rightarrow -3R_1 + R_3]{R_2 \Rightarrow -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 3 & 0 & -6 \\ 0 & 5 & -1 & -13 \end{array}\right]$$

$$R_2 \Rightarrow \frac{1}{3}R_2 \left[\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 5 & -1 & -13 \end{array}\right] \Rightarrow R_3 \Rightarrow -5R_2 + R_3 \left[\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix} \Rightarrow -x_3 = -3 \Rightarrow x_3 = 3$$

$$x_2 = -2$$

$$x_1 - 2x_2 + x_3 = 8$$

$$X = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \Longleftarrow \quad x_1 + 4 + 3 = 8$$

$$x_1 = 1$$

Ex2 Use Gassian elimination Method to solve the following set of eqn.

$$x_1 + x_2 - x_3 = 9$$

$$0 \cdot x_1 + x_2 + 3x_3 = 3$$

$$-x_1 + 0 \cdot x_2 - 2x_3 = 2$$

Solution:-

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix} ; \quad B = \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ -1 & 0 & -2 & 2 \end{array}\right] \Rightarrow R_3 \Rightarrow R_1 + R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -3 & 11 \end{array}\right]$$

$$R_3 - R_2 \Rightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & 8 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 8 \end{bmatrix}$$

$$-6x_3 = 8$$

$$\Rightarrow x_3 = \frac{8}{-6} = \frac{-4}{3}$$

$$x_2 + 3x_3 = 3 \longrightarrow x_2 - \frac{4}{\cancel{3}} \times \cancel{3} = 3 \qquad x_2 = 7$$

$$x_1 + x_2 - x_3 = 9 \Rightarrow x_1 + 7 + \frac{4}{3} = 9 \Rightarrow x_1 = \frac{2}{3}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 7 \\ \frac{-4}{3} \end{bmatrix}$$

Ex:- use Gaussian elimination to find the value of X-vector for the following set of equations

solution

$$x_2 + x_3 - 2x_4 = -3$$

$$x_1 + 2x_2 - x_3 = +2$$

$$2x_1 + 4x_2 + x_3 - 3x_4 = -2$$

$$x_1 - 4x_2 - 7x_3 - x_4 = -19$$

solution:-

$[A \vdots B]$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & +2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \Rightarrow \text{change } R_1 \text{ by } R_2 \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & +2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

$-2R_1 + R_3 \Rightarrow R_3$

$-R_1 + R_4 \Rightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & +2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right]$$

$6R_2 + R_4 \Rightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right]$$

$\frac{1}{3} R_3 \Rightarrow R_3$

$-\frac{1}{13} R_4 \Rightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \quad ; \quad AX = B \Rightarrow$$

$\Rightarrow x_4 = 3$

$x_3 - x_4 = -2 \Rightarrow x_3 - 3 = -2 \Rightarrow x_3 = 1$

$x_2 + x_3 - 2x_4 = -3 \Rightarrow x_2 + 1 - 2*3 = -3 \Rightarrow x_2 = 2$

$x_1 + 2x_2 - x_3 = 2 \Rightarrow x_1 + 2*2 - 1 = 2 \Rightarrow x_1 = -1$

$$\Rightarrow X = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

(G) Gauss - Jordan Elemination :-

Ex:- use Gauss - Jordan elimination to solve the system below

$$\begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & -6 \\ 5 & -3 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ -9 \end{bmatrix}$$

Solution:-

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & -3 \\ -3 & 2 & -6 & 7 \\ 5 & -3 & 8 & -9 \end{array}\right] \Rightarrow R_1 \Rightarrow R_1 + R_2 \left[\begin{array}{ccc|c} -1 & 1 & -3 & 4 \\ -3 & 2 & -6 & 7 \\ 5 & -3 & 8 & -9 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} -3R_1 + R_2 \Rightarrow R_2 \\ 5R_1 + R_3 \Rightarrow R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 1 & -3 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 2 & -7 & 11 \end{array}\right] \begin{array}{l} R_1 \Rightarrow R_1 + R_2 \\ R_3 \Rightarrow 2R_2 + R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\Rightarrow 3R_3 + R_2 \Rightarrow 2 \left[\begin{array}{ccc|c} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} -R_1 \Rightarrow R_1 \\ -R_2 \Rightarrow R_2 \\ -R_3 \Rightarrow R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\Rightarrow \quad \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = -1 \end{array}$$

H.w :- solve the same Example by Guss-elimint method.

(H) Eigen Value and Eigen Vector

To solve eigen Values λ_i, and the corresponding eigen vectors x_i of a nxn Matrix A. do the following.

1. Multiply a I matrix by λ
2. $A - \lambda I$
3. find the determint of $A - \lambda I = 0$ to find λ_i
4. Solve for the corresponding vector to each λ

EX (1) Find Eigen Value and Eigen vector of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

Solution:-

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0 \Rightarrow 2 - 3\lambda + \lambda^2 - 6 = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda + 1)(\lambda - 4) = 0 \Rightarrow \lambda_1 = -1 \quad \& \quad \lambda_2 = 4$$

to find eigen vector corresponding to each λ

for $\lambda_1 = -1$

$$\begin{bmatrix} 1-\lambda_1 & 3 \\ 2 & 2-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2X_1 + 3X_2 = 0 \Rightarrow X_1 = \frac{-3}{2} X_2$

let $X_2 = 1 \Rightarrow X_1 = -\frac{3}{2}$

$$\Rightarrow P_1 = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

for $\lambda_2 = +4$

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3X_1 + 3X_2 = 0 \Rightarrow 3X_1 = 3X_2$ or $X_1 = X_2$

let $X_2 = 1 \Rightarrow X_1 = 1$

$$P_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} -\frac{3}{2} & 1 \\ 1 & 1 \end{bmatrix}$$

Ex2:- find Eigen value and Eigen vector of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

Solution.

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 1 \\ -1 & 2-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(2-\lambda)\left[(2-\lambda)(2-\lambda) - 0\right] - 0 + 1\left[0 - (2-\lambda)\right]$$

$$(2-\lambda)\left[4 - 4\lambda + \lambda^2\right] - (2-\lambda) = 0$$

$$8 - 8\lambda + 2\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 2 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \quad \times -1$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \text{——} \; (1)$$

let $\lambda = 1$

$1 - 6 + 11 - 6 = 0$ there fore $\lambda_1 = +1$

Now to find another λi, we can divided this equation (1) by $(\lambda - 1)$

$$\begin{array}{r|l} & \lambda^2 - 5\lambda + 6 \\ \hline \lambda - 1 & \lambda^3 - 6\lambda^2 + 11\lambda - 6 \\ & -\lambda^3 - \lambda^2 \\ \hline & 0 - 5\lambda^2 + 11\lambda \\ & - \quad -5\lambda^2 + 5\lambda \\ \hline & 0 + 6\lambda - 6 \\ & 6\lambda - 6 \\ \hline & 0 \quad 0 \end{array}$$

$$(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda - 2)(\lambda - 3) \Rightarrow \lambda_2 = 2 \;\&\; \lambda_3 = 3$$

to find Eigen vector for each λ_i

for $\lambda_1 = 1$

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ -1 & 2-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ replace λ by $\lambda_1 = 1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$$X_1 + X_3 = 0 \quad \text{—} ①$$

$$-X_1 + X_2 + 3X_3 = 0 \quad \text{—} ②$$

from ① $X_3 = -\cancel{X}1$

let $X_1 = 1 \Rightarrow X_3 = -1$

$-1 + X_2 - 3\cancel{X}1 = 0 \Rightarrow X_2 = 4$

$$\Rightarrow P_1 = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

for $\lambda_2 = 2$

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$X_3 = 0$

$X_1 = 0$

$-X_1 + 3X_3 = 0$ let $X_2 = -1$

$$P_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

for $\lambda_3 = 3$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-x_1 + x_3 = 0 \Rightarrow x_1 = x_3$

let $x_1 = 1 \Rightarrow x_3 = 1$

$-x_1 - x_2 + 3x_3 = 0$

$-1 - x + 3 = 0 \Rightarrow x_2 = 2$

$$P_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

the matrix $P = \begin{bmatrix} 1 & 0 & 1 \\ 4 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

Ex3:- find Eigen Value & Eigen Vector of the matrix A

$$A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:-

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 6-\lambda & 2 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}\right) = 0$$

$(6-\lambda)\left[(3-\lambda)(-1-\lambda) - 0\right] - 2\left[2(-1-\lambda) - 0\right] + 0 = 0$

$(6-\lambda)\left[-3 - 3\lambda + \lambda + \lambda^2\right] - 2\left[-2 - 2\lambda\right] = 0$

$-18 - 18\lambda + 6\lambda + 6\lambda^2 + 3\lambda + 3\lambda^2 - \lambda^2 - \lambda^3 + 4 + 4\lambda = 0$

$-\lambda^3 + 8\lambda^2 - 5\lambda - 14 = 0 \qquad X - 1$

$\lambda^3 - 8\lambda^2 + 5\lambda + 14 = 0$

$-1 - 8 - 5 + 14 = 0 \qquad \lambda_1 = -1$

$$\begin{array}{r|l} & \lambda^2 - 9\lambda + 14 \\ \hline (\lambda + 1) & \lambda^3 - 8\lambda^2 + 5\lambda + 14 \\ & \lambda^3 + \lambda^2 \\ \hline & 0 \; - 9\lambda^2 + 5\lambda \\ & - \;\; -9\lambda^2 - 9\lambda \\ \hline & 0 + 14\lambda + 14 \\ & 14\lambda + 14 \\ \hline & 0 \quad 0 \end{array}$$

$(\lambda^2 - 9\lambda + 14) = 0$

$(\lambda - 2)(\lambda - 7)$

$\lambda_2 = 2$ and $\lambda_3 = 7$

to find eigen vector for each λ_i

for $\lambda_1 = -1$

$$\begin{bmatrix} 7 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$7x_1 + 2x_2 = 0$

$2x_1 + 4x_2 = 0$

only for $x_1 = 0$ & $x_2 = 0$ that set of equation equal to zero

then let $x_3 = 1$

$$P_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda_2 = 2$

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1 + 2x_2 = 0 \Rightarrow x_2 = -2x_1$ let $x_1 = 1 \Rightarrow x_2 = -2$

$0x_1 + 0x_2 - 3x_3 = 0 \Rightarrow x_3 = 0$

$$P_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

for $\lambda_3 = 7$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-8x_3 = 0 \Rightarrow x_3 = 0$

$2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2x_2$ let $x_2 = 1$

$\Rightarrow x_1 = 2$

$$P_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad P = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

H.W: find eigen value and eigen vector of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(I) Application of eigen Value and Eigen Vector

Diagonlization Matrix and Multiplication Matrix

to find diagonal matrix, do the following

1- find eigen value and corresponding eigen vector

2- each eigenvector of the matrix arrange as a matrix (P)

3- find matrix inverse of P

4- diagonal of matrix $D = P^{-1} A P = \lambda I$

5- multiplication Matrix $A^n = P D^n P^{-1}$ where n is the number

EX 1:- find the diagonal matrix D and A^4 for the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$; if the has the matrix $P = \begin{bmatrix} -\frac{3}{2} & 1 \\ 1 & 1 \end{bmatrix}$

Solution

$$D = P^{-1} A P$$

Now, find P^{-1}

$$P^{-1} = \frac{\text{adj } P}{\det P} = \frac{\begin{vmatrix} 1 & -1 \\ -1 & -1.5 \end{vmatrix}}{-1.5 - 1} = \frac{\begin{vmatrix} 1 & -1 \\ -1 & -1.5 \end{vmatrix}}{-2.5} = \begin{bmatrix} -0.4 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

$$D = P^{-1} A P = \begin{bmatrix} -0.4 & 0.4 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1.5 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.4 \\ 1.6 & 2.4 \end{bmatrix} \begin{bmatrix} -1.5 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = \lambda I$$

to find A^4

$$A^4 = P D^4 . P^{-1}$$

$$= \begin{bmatrix} 1.5 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (4)^4 \end{bmatrix} \begin{bmatrix} -0.4 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 256 \\ 1 & 256 \end{bmatrix} \begin{bmatrix} -0.4 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 103 & 153 \\ 102 & 154 \end{bmatrix}$$

EX:2 find D matrix ~~and~~ A^3 for the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$ if has the P matrix $$\begin{bmatrix} 1 & 0 & 1 \\ 4 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution:- to find D must be find P^{-1}

$$P^{-1} = \frac{adj(P)}{det(P)}$$

$$= \frac{\begin{vmatrix} -1 & 0 & 1 \\ -6 & 2 & 2 \\ -2 & 0 & -1 \end{vmatrix}}{-2} = \begin{bmatrix} 0.5 & 0 & 0.8 \\ 3 & -1 & -1 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 0.5 & 0 & 0.8 \\ 3 & -1 & -1 \\ 0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 4 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 6 & -2 & -2 \\ 1.5 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 4 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \lambda I$$

$$A^3 = P D^3 P^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 4 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & -0.5 \\ 6 & -2 & -2 \\ 1.5 & 0 & 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 27 \\ 4 & -8 & 54 \\ -1 & 0 & 27 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & -0.5 \\ 6 & -2 & -2 \\ 1.5 & 0 & 1.5 \end{bmatrix}$$

$$\begin{bmatrix} 41 & 0 & -41 \\ 35 & 16 & 95 \\ 40 & 0 & 41 \end{bmatrix}$$

Ex3:- find D and A^4 for the matrix A

$$A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ has } P = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D = P^{-1} A P$$

$$\Rightarrow P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0.2 & -0.4 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 1 \\ 0.2 & -0.4 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$A^4 = P D^4 P^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 2401 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0.2 & -0.4 & 0 \\ 0.4 & 0.2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1924 & 954 & 0 \\ 954 & 493 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$