Cayley - Hamilton theorm

Let $A_{n\times n}$ Matrix has an c/cs eqn. $P(\lambda)$, then $P(A)=0$

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

$$(A-\lambda I)=P(\lambda) = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 4$$

$$\Rightarrow P(A) = A^2 - 3A - 4I$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Application of Cayley - Hamilton theorm :-

Ex:- use Cayley-Hamilton to find A^{-1} of $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

First step:- find $(\lambda I - A)$

$$\lambda I - A = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{bmatrix}$$

$$\lambda^3 - (\text{trace } A)\lambda^2 + \left(\begin{matrix}\text{minors of the terms} \\ \text{on the leading diagonal of } A\end{matrix}\right)\lambda - \det(A) = 0$$

$$\lambda^3 - (1+2+4)\lambda^2 + \left(\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\right)\lambda - \left(1\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}\right) = 0$$

$$\lambda^3 - 7\lambda^2 + (9+6+0)\lambda - (9-6+6) = 0$$

$$\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

by using Cayley-Hamilton theorem

$$A^3 - 7A^2 + 15A - 9I = 0 \quad * A^{-1}$$

$$A^3A^{-1} - 7A^2A^{-1} + 15AA^{-1} - 9IA^{-1} = 0$$

$$A^2AA^{-1} - 7AAA^{-1} + 15AA^{-1} - 9IA^{-1} = 0$$

$$A^2I - 7AI + 15I - 9IA^{-1} = 0$$

$$A^2 - 7A + 15I - 9A^{-1} = 0$$

$$9A^{-1} = A^2 - 7A + 15I$$

$$9A^{-1} = \begin{bmatrix} 1 & 8 & 8 \\ 4 & 5 & -4 \\ -4 & 4 & 13 \end{bmatrix} - 7\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix} + 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -6 & -6 \\ -3 & 6 & 3 \\ 3 & -3 & 0 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{9}\begin{bmatrix} 9 & -6 & -6 \\ -3 & 6 & 3 \\ 3 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix}$$

Ex 2:- verify C.H.T for $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and find A^4 and A^{-1}

step ① $\lambda I - A = 0$

$$\begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = \lambda^3 - \lambda^2 + 5\lambda + 5 = 0$$

step ②

$$A^3 - A^2 - 5A + 5I = 0$$

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, to find A^4

$$A^3 - A^2 - 5A + 5I = 0 \qquad * A$$

$$A^4 - A^3 - 5A^2 + 5AI = 0$$

$$A^4 = A^3 + 5A^2 - 5A$$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to find A^{-1}

$$A^3 - A^2 - 5A + 5I = 0 \quad * A^{-1}$$

$$A^3A^{-1} - A^2A^{-1} - 5AA^{-1} + 5IA^{-1} = 0$$

$$A^2I - AI - 5I + 5A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{5}\left[-A^2 + A + 5I\right]$$

$$= \frac{1}{5}\left\{\begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$