Department of Communications Engineering, College of Engineering, University of Diyala

Digital Communication I

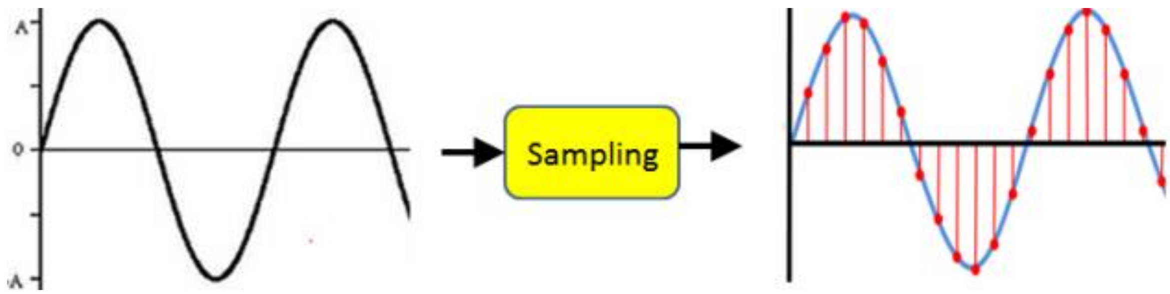
Lecture #3

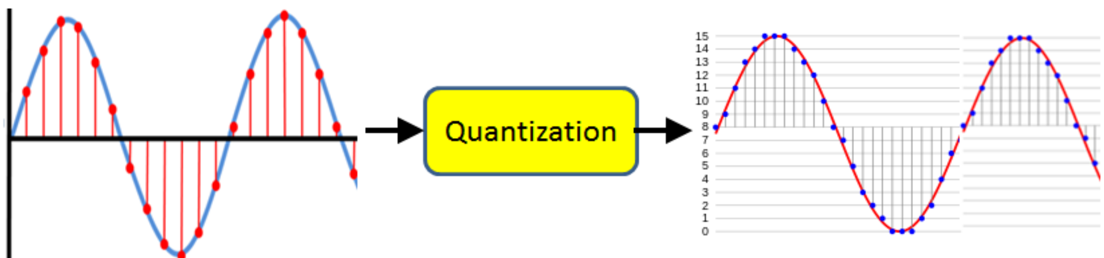
Pulse Code Modulation (PCM): PCM, Noise in Pulse Code Modulation.

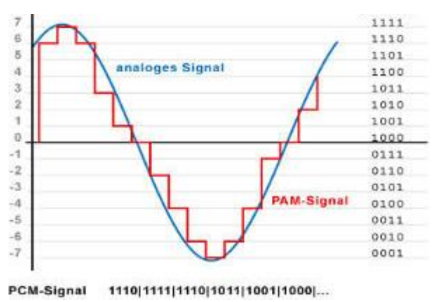
- To transport an information-bearing signal (modulating signal or message signal) from one point to another point over a communication channel, we can use ***digital*** or ***analog*** techniques.
- Digital communication offers several important advantages as compared to analog communication:
  - Ruggedness to channel noise and external interference unmatched by any analog communication system.
  - Flexible operation of the system.
  - Integration of diverse sources of information into a common format.
  - Security of information in the course of its transmission from source to the destination.
- However, to handle the transmission of analog message signals (i.e., voice and video signals) by digital means, the signal has to undergo an ***analog-to-digital conversion***.
- Digital signal is in the form of a train or stream of binary digits **0** and **1**. Thus, with waveform coding techniques, we enter into the world of ***digital communication***
- In PCM, the message signal is sampled and amplitude of each sample is approximated (rounded off) to the nearest one of a finite set of discrete levels.
- This will enable us to represent both time and amplitude in discrete form.
- Hence, it is possible to transmit the message signal by means of a digital (coded) waveform.
- In the pulse digital modulation, the time and the pulse amplitude occur in discrete form and digital coded form respectively.
- Pulse digital modulation is therefore a scheme, which converts the analog signal to its corresponding digital form.
- It is for this reason that the analog-to-digital conversion is sometimes known as pulse digital modulation.

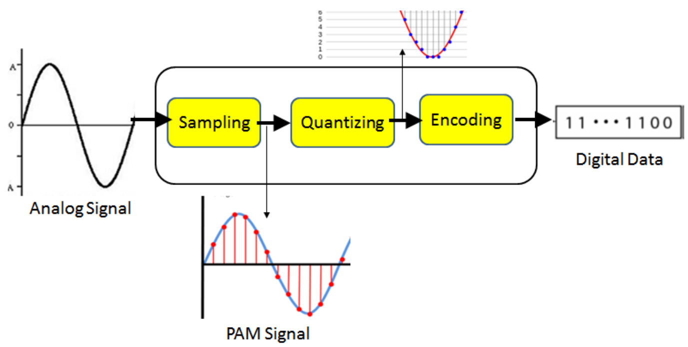
Pulse Code Modulation (PCM) consists of three main parts:

1. **Sampler:** to convert the continuous-time to discrete-time signal.
2. **Quantizer:** to convert the analog amplitude to finite range of values (discrete amplitude).
3. **Encoder:** to convert the discrete amplitudes to binary digits (10111000101…).



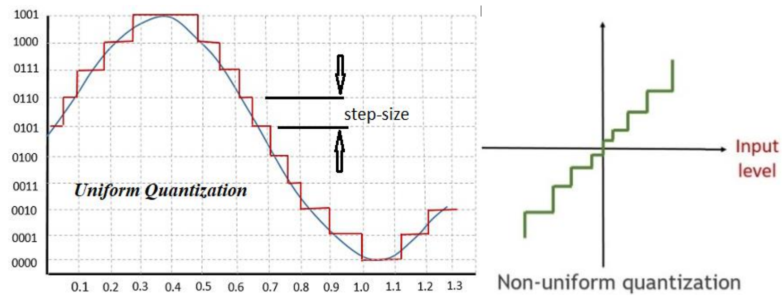






Quantizer:

- The quantization process can be classified into two types:
  - **Uniform quantization:** A uniform quantizer is that type of quantizer in which the '***step-size*** ' remains same throughout the input range.
  - **Non-uniform quantization:** A non-uniform quantizer is that type of quantizer in which the '***step-size***' *varies according to the input signal values*.



In uniform quantizer, there are two types of quantizers: **midtread** and **midrise** as shown in the following figure.

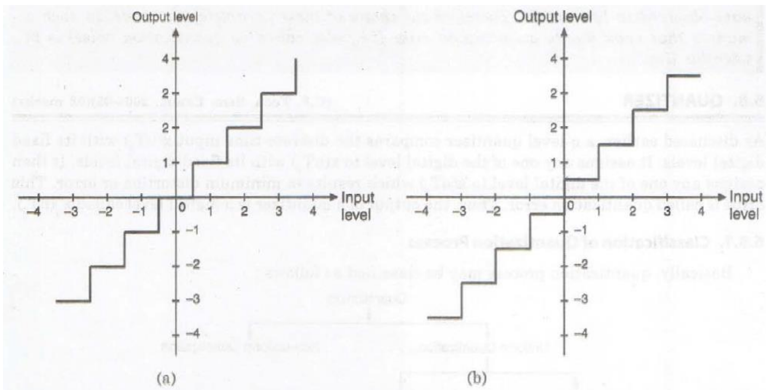
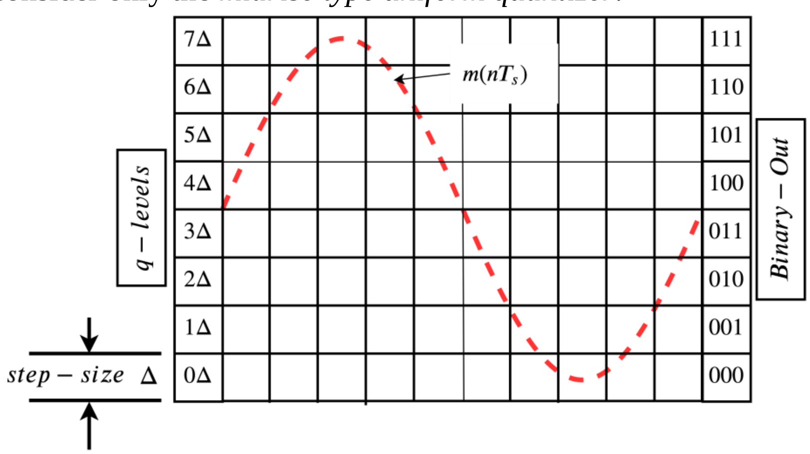


Fig. 6.8. Two types of Uniform quantization: (a) Midtread, and (b) Midrise.

- We will consider only the *midrise type uniform quantizer*.



- The quantizer will assign quantities to each sampled signal,
- The number of quantities are the allowable levels $q = \frac{2[m(nT_s)]_{max}}{\Delta}$
- In other words, the step-size is $\Delta = \frac{m_{max} - (-m_{min})}{q} = \frac{2m_{max}}{q}$ or it is called the analog to digital converter resolution.
- Assume the signal $m(nT_s)$ normal to maximum value of **+1** and minimum value of **-1**, then the step-size becomes $\Delta = \frac{2}{q}$
- The code-word length (number of binary digits of each quantized-sample) is $\text{v} = \log_2(\text{q})$

- Or in other words, number of levels $q = 2^v$.
- You know that the number of samples per one second is F_s then the number of bits per one second is total number of bits in one second, this known as signaling rate ***r***, or it is also known as bit rate, thus $r = vF_s$ *bps*.
- From Nyquist-Shannon theorem, bandwidth needed for PCM transmission is given by half of the signaling rate $BW_{PCM} \geq \frac{r}{2}$
- Then the bandwidth in terms of f_m is $BW_{PCM} \geq vf_m$Hz. Proof!!!

Now, let m_q indicates the quantization level, and ***i*** is an index corresponding to the binary code, then

$$i = round\left(\frac{m - m_{min}}{\Delta}\right)$$

Then,

$$m_q = m_{min} + i\Delta \quad where\ i = 0,1 \cdots q - 1$$

Example 1:

Assuming that a 3-bit ADC channel accepts analog input ranging from 0 to 5 volts, determine

a) The number of quantization levels;
b) The step size of the quantizer or resolution;
c) The quantization level when the analog voltage is 3.2 volts;
d) The binary code produced by the ADC.

Solution:

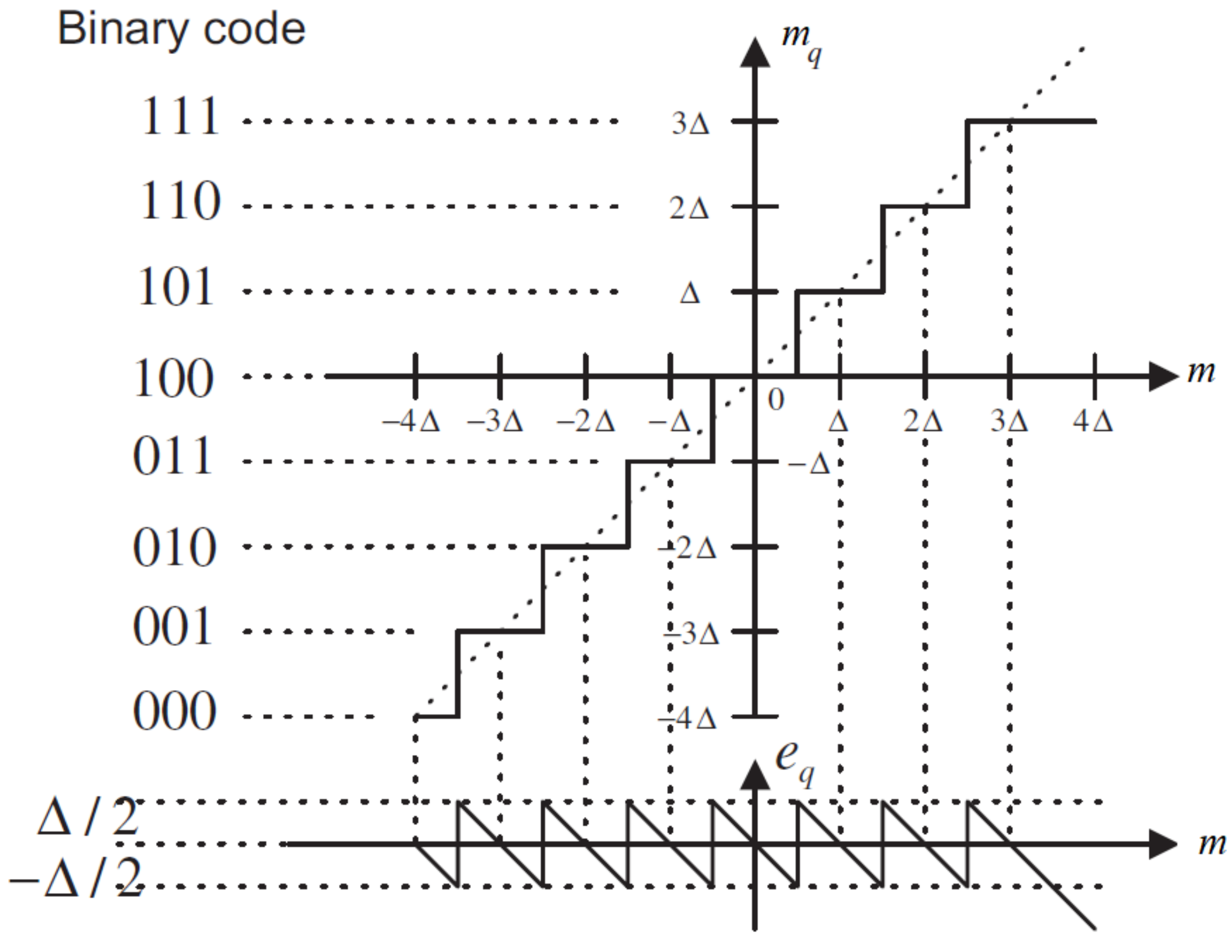
Since the range is from 0 to 5 volts and a 3-bit ADC is used, we have
$m_{min} = 0\ volt\ m_{max} = 5\ volets$ and $v = 3\ bits$

a) Number of quantization levels $q = 2^v = 2^3 = 8$ bits
b) Step-size $\Delta = \frac{5-0}{8} = 0.625$ volts
c) $i = round\left(\frac{m-m_{min}}{\Delta}\right) = round\left(\frac{3.2-0}{0.625}\right) = round(5.12) = 5$

$$m_q = m_{min} + i\Delta = 0 + 5\Delta = 5 \times 0.625 = 3.125\ volts$$

d) The binary code is determined according to the index $i = 5$ which corresponds to binary number = 101

Input Signal Sub Range	Quantization Level m_q	Binary Code
$0\Delta \leq m < 0.5\Delta$	0Δ	000
$0.5\Delta \leq m < 1.5\Delta$	1Δ	001
$1.5\Delta \leq m < 2.5\Delta$	2Δ	010
$2.5\Delta \leq m < 3.5\Delta$	3Δ	011
$3.5\Delta \leq m < 4.5\Delta$	4Δ	100
$4.5\Delta \leq m < 5.5\Delta$	5Δ	101
$5.5\Delta \leq m < 6.5\Delta$	6Δ	110
$6.5\Delta \leq m < 7.5\Delta$	7Δ	111



Quantization Error: quantizing error is due to the difference between the quantized and the samples signals,

$$\varepsilon_q = m_q(nT_s) - m(nT_s)$$

From which,

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right|$$

In other form,

$$-\frac{\Delta}{2} \leq \varepsilon_{max} \leq \frac{\Delta}{2}$$

- We know that the input $m(nT_s)$ to a linear or uniform quantizer has continuous amplitude in the range $-m_{max}$ to $+m_{max}$.
- Further, we know that the total amplitude range $= 2m_{max}$

Example 2: Using Example 1, determine the quantization error when the analog input is 3.2 volts.

Solution: Using $\varepsilon_q = m_q(nT_s) - m(nT_s)$, we obtain

$$\varepsilon_q = m_q - m = 3.125 - 3.2 = -0.075\ volt$$

Note that the quantization error is less than the half of the step size, that is,

$$|e_q| = 0.075 < \Delta/2 = 0.3125 \text{ volt}$$

In practice, we can empirically confirm that the quantization error appears in uniform distribution when the step size is much smaller than the dynamic range of the signal samples and we have a sufficiently large number of samples.

Based on the theory of probability and random variables, ***the power of quantization noise is related to the quantization step*** and given by

$$Normalized\ Noise\ Power = \frac{\Delta^2}{12}$$

Signal to Quantization Noise Ratio for Linear Quantization:

$$\frac{S}{N} = \frac{Normalized\ Signal\ Power}{Normalized\ Noise\ Power} \quad (1)$$

We know

$$Normalized\ Noise\ Power = \frac{\Delta^2}{12} \quad (2)$$

Also,

$$\Delta = \frac{2m_{max}}{2^v} \quad (3)$$

Assuming the Normalized Signal Power = P then, substituting (2) and (3) in (1),

$$\text{SNR} = \frac{P}{\frac{4m_{max}^2}{2^{2v}} \times \frac{1}{12}} = \frac{3P}{m_{max}^2} \cdot 2^{2v} \quad (4)$$

- *This expression shows that signal to noise power ratio of quantizer increases exponentially with increasing bits per sample v*
- Also, if the destination signal power '*P*' is normalized, i.e., $P \leq 1$, then

$$\text{SNR} \leq 3 \times 2^{2v} \quad (5)$$

- In terms of dB,

$$(\text{SNR})_{dB} \leq (4.8 + 6v)dB \quad (6)$$

Example 6.1: Derive an expression for signal to quantization noise ratio for a PCM system which employs linear (i.e., uniform) quantization technique. Given that input to the PCM system is a sinusoidal signal.

Solution: let us assume that the modulating signal is a sinusoidal voltage having a peak amplitude equal to A_m. Also let this signal covers the complete excursion of representation levels. Then the power of this signal will be,

$P = \frac{V^2}{R}$ where $V = rms\ value, i.e., V = \left[\frac{A_m}{\sqrt{2}}\right]^2$ therefore, we have $P = \frac{A_m^2}{2} \cdot \frac{1}{R}$ and in case $R = 1\Omega$ the power P is normalized, $P = \frac{A_m^2}{2}$

We know that, signal to qunntization noise ratio is, SNR $= \frac{3P}{m_{max}^2} \cdot 2^{2v}$ and $m_{max} = A_m$, then

$$\frac{S}{N} = \frac{3 \times \frac{A_m^2}{2} \times 2^{2v}}{A_m^2} = \frac{3}{2} \times 2^{2v} = 1.5 \times 2^{2v}$$

$$(\mathrm{SNR})_{dB} = 1.76 + 2v \times 10 \times 0.3$$

$$(\mathrm{SNR})_{dB} = 1.8 + 6v\ (for\ sinusoidal\ signal\ only)$$

EXAMPLE 6.2. A Television signal having a bandwidth of 10.2 MHz is transmitted using binary PCM system. Given that the number of quantization levels is 512. Determine:

(i) Code word length

(ii) Transmission bandwidth

(iii) Final bit rate

(iv) Output signal to quantization noise ratio.

Solution: Given that the bandwidth is 4.2 MHz. This means that highest frequency component will have frequency of 4.2 MHz, i.e.,

$$f_m = 4.2 \text{ MHz}$$

Also, given that Quantization levels, q = 512

(i) We know that the number of bits and quantization levels are related in binary PCM as under:

$$q = 2^v$$

i.e.,
$$512 = 2^v$$

or
$$\log_{10} 512 = v \log_{10} 2$$

or
$$v = \frac{\log_{10} 512}{\log_{10} 2}$$

Simplifying, we get, $v = 9$ bits

Hence, the code word length is 9 bits. **Ans.**

(ii) We know that the transmission channel bandwidth is given as,

$$BW \geq vf_m \geq 9 \times 4.2 \times 10^6 \text{ Hz} \geq 37.8 \text{ MHz} \quad \textbf{Ans.}$$

(iii) The final bit rate is equal to signalling rate.

We know that the signalling rate is given as,

$$r = vf_s \quad \text{...(i)}$$

Here, sampling frequency is given as $f_s \geq 2f_m$

Thus,
$$f_s \geq 2 \times 4.2 \text{ MHz} \quad \text{since } f_m = 4.2 \text{ MHz}$$

or
$$f_s \geq 8.4 \text{ MHz}$$

Substituting this value of 'f_s' in equation (i) for signalling rate, we get

or
$$r = 9 \times 8.4 \times 10^6 \text{ bits/sec} = 75.6 \times 10^6 \text{ bits/sec} \quad \textbf{Ans.}$$

The transmission bandwidth may also be obtained as,

$$BW \geq \frac{1}{2} r \geq \frac{1}{2} \times 75.6 \times 10^6 \text{ bits/sec}$$

or BW ≥ 37.8 MHz which is same as the value obtained earlier.

(iv) The output signal to noise ratio is expressed as

$$\left(\frac{S}{N}\right) dB \leq (4.8 + 6v) \text{ dB}$$

But v = 9

Therefore,
$$\left(\frac{S}{N}\right) dB \leq 4.8 + 6 \times 9$$

or
$$\left(\frac{S}{N}\right) dB \leq 58.8 \text{ dB} \quad \textbf{Ans.}$$

EXAMPLE 6.3. The bandwidth of an input signal to the PCM is restricted to 4 kHz. The input signal varies in amplitude from – 3.8 V to + 3.8 V and has the average power of 30 mW. The required signal to noise ratio is given as 20 dB. The PCM modulator produces binary output. Assuming uniform quantization,

(i) Find the number of bits required per sample.

Solution: The given value of signal to noise ratio is 20 dB.

This means that.

$$\left(\frac{S}{N}\right) dB = 10\log_{10}\left(\frac{S}{N}\right) = 20 \text{ dB}$$

Hence,
$$\frac{S}{N} = 100$$

(i) We know that the signal to quantization noise ratio is given as,

$$\frac{S}{N} = \frac{3P\cdot 2^{2v}}{x_{max}^2}$$

Here, we are given

$$x_{max} = 3.8 \text{ V}$$
$$P = 30 \text{ mW}$$

and
$$\frac{S}{N} = 100$$

Therefore,
$$100 = \frac{3\times 30\times 10^{-3}\times 2^{2v}}{(3.8)^2}$$

Solving, we get $v = 6.98 \text{ bits} = 7 \text{ bits}$ **Ans.**

EXAMPLE 6.4. The information in an analog signal voltage waveform is to be transmitted over a PCM system with an accuracy of ± 0.1% (full scale). The analog voltage waveform has a bandwidth of 100 Hz and an amplitude range of – 10 to + 10 volts.

(i) Find the minimum sampling rate required.

(ii) Find the number of bits in each PCM word.

(iii) Find minimum bit rate required in the PCM signal.

(iv) Find the minimum absolute channel bandwidth required for the transmission of the PCM signal. **(U.P. Tech., Sem. Exam., 2004-05)**

Solution: Here an accuracy is given as ± 0.1%. This means that the quantization error must be ± 0.1% or the maximum quantization error must be ± 0.1%.

Thus, $\varepsilon_{max} = \pm 0.1\% = \pm 0.001$

We know that the maximum quantization error for an uniform quantizer is expressed as,

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right|$$

or
$$\left|\frac{\Delta}{2}\right| = 0.001$$

Therefore,

Step size $\Delta = 2 \times 0.001 = 0.002$

We know that the step size, number of quantization levels and maximum value of the signal are related as

$$\Delta = \frac{2\,x_{max}}{q} \qquad \text{...(i)}$$

Here, given $|x_{max}| = 10$ volts

Substituting, values of Δ and x_{max} in equation (i), we get

$$0.002 = \frac{2\times10}{q} \quad \text{or} \quad q = \frac{20}{0.002} = 10{,}000$$

Hence, the number of levels are 10,000.

(i) The maximum frequency in the signal is given as 100 Hz, i.e.,

$$f_m = 100 \text{ Hz}$$

By sampling theorem minimum sampling frequency should be,

$$f_s \geq 2\,f_m \geq 2\times100 \geq 200 \text{ Hz}$$ **Ans.**

(ii) We know that minimum 10,000 levels should be used to quantize the signal. If binary PCM is used, then number of bits for each samples may be calculated as under, i.e.,

$$q = 2^v$$

Here, q = number of levels

v = bits in PCM,

Thus, $10{,}000 = 2^v$

$$\log_{10} 10{,}000 = v \log_{10} 2$$

or

$$v = \frac{\log_{10} 10{,}000}{\log_{10} 2} = 13.288 = 14 \text{ bits}$$ **Ans.**

(iii) The bit rate or signalling rate is expressed as,

$$r \geq v\,f_s \geq 14\times200 \geq 2800 \text{ bits per second.}$$ **Ans.**

(iv) The transmission bandwidth for PCM is expressed as,

$$BW \geq \frac{1}{2}r \geq \frac{1}{2}\times2800 \geq 1400 \text{ Hz}$$ **Ans.**

EXAMPLE 6.6. A PCM system uses a uniform quantizer followed by a 7-bit binary encoder. The bit rate of the system is equal to 50×10^6 bits/sec.

(i) What is the maximum message signal bandwidth for which the system operates satisfactorily?

(ii) Calculate the output signal to quantization noise ratio when a full load sinusoidal modulating wave of frequency 1 MHz is applied to the input.

(U.P. Tech., Sem. Exam., 2005-2006)

Solution: (i) Let us assume that the message bandwidth be f_m Hz. Therefore sampling frequency should be,

$$f_s \geq 2 f_m$$

The number of bits given as v = 7 bits

We know that the signalling rate is given as,

$$r \geq v \cdot f_s$$

or $$r \geq 7 \times 2 f_m$$

Substituting value for r, we get

$$50 \times 10^6 \geq 14 f_m$$

or $$f_m \leq 3.57 \text{ MHz} \quad \textbf{Ans.}$$

Thus, the maximum message bandwidth is 3.57 MHz.

(ii) The modulating wave is sinusoidal. For such signal, the signal to quantization noise ratio is expressed as,

$$\left(\frac{S}{N}\right) dB = 1.8 + 6v$$

Substituting the value of v, we get

$$\left(\frac{S}{N}\right) dB = 1.8 + 6 \times 7 = 43.8 \text{ dB} \quad \textbf{Ans.}$$

EXAMPLE 6.7. The information in an analog waveform with maximum frequency $f_m = 3$ kHz is to be transmitted over an M-level PCM system where the number of quantization levels is M = 16. The quantization distortion is specified not to exeed 1% of peak to peak analog signal.

(i) What would be the maximum number of bits per sample that should be used in this PCM system?

(ii) What is the minimum sampling rate and what is the resulting bit transmission rate?

Solution: (i) Since the number of quantization levels given here are M = 16,

$$q = M = 16$$

We know that the bits and levels in binary PCM are related as,

$$q = 2^v$$

Here, v = number of bits in a codeword

Thus, $$16 = 2^v \quad \text{or} \quad v = 4 \text{ bits.} \quad \textbf{Ans.}$$

(ii) Again since $$f_m = 3 \text{ kHz}$$

By sampling theorem, we know that

$$f_s \geq 2 f_m$$

Thus, $$f_s \geq 2 \times 3 \text{ kHz} \geq 6 \text{ kHz} \quad \textbf{Ans.}$$

Hence, the minimum sampling rate is 6 kHz

Also, bit transmission rate or signaling rate is given as,

$$r \geq v f_s \geq 4 \times 6 \times 10^3 \quad \text{or} \quad r \geq 24 \times 10^3 \text{ bits per second}$$

EXAMPLE 6.8. A signal having bandwidth equal to 3.5 kHz is sampled, quantized and coded by a PCM system. The coded signal is then transmitted over a transmission channel of supporting a transmission rate of 50 k bits/sec. Determine the maximum signal to noise ratio that can be obtained by this system.

The input signal has peak-to-peak value of 4 volts and rms value of 0.2 V.

Solution: The maximum frequency of the signal is given as 3.5 kHz,

i.e., $$f_m = 3.5 \text{ kHz}$$

Therefore sampling frequency will be

$$f_s \geq 2f_m \geq 2 \times 3.5 \text{ kHz} \geq 7 \text{ kHz}$$

We know that the signalling rate is given by

$$r \geq v f_s$$

Substituting values of $r = 50 \times 10^3$ bits/sec and $f_s \geq 7 \times 10^3$ Hz in above equation, we get

$$50 \times 10^3 \geq v \cdot 7 \times 10^3$$

Simplifying, we get

$$v \leq 7.142 \text{ bits} \cong 8 \text{ bits}$$

The rms value of the signal is 0.2 V. Therefore the normalized signal power will be,

$$\text{Normalized signal power } P = \frac{(0.2)^2}{1} *$$

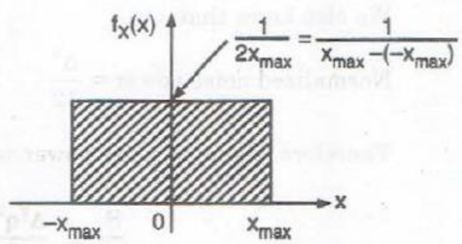
i.e., $$P = 0.04 \text{ W}$$

Further, the maximum signal to noise ratio is given by,

$$\frac{S}{N} = \frac{3P \cdot 2^{2v}}{x_{max}^2}$$

Substituting the values of $P = 0.04$, $v = 8$ and $x_{max} = 2$ in above equation, we have

$$\frac{S}{N} = \frac{3 \times 0.04 \times 2^{2 \times 8}}{4} = 1966.08 \cong 33 \text{ dB}$$ **Ans.**



EXAMPLE 6.10. Given an audio signal consisting of the sinusoidal term given as

$$x(t) = 3\cos(500\pi t)$$

(i) Determine the signal to quantization noise ratio when this is quantized using 10 bit PCM.

(ii) How many bits of quantization are needed to achieve a signal to quantization noise ratio of atleast 40 dB?

Solution: Here, given that $x(t) = 3\cos(500\pi t)$

This is sinusoidal signal applied to the quantizer.

(i) Let us assume that peak value of cosine wave defined by x(t) covers the complete range of quantizer.

i.e., $A_m = 3V$ covers complete range

In Example 6.6, we have derived signal to noise ratio for a sinusoidal signal. It is expressed as

$$\left(\frac{S}{N}\right) dB = 1.8 + 6v$$

Since here 10 bit PCM is used, i.e.,

$$v = 10$$

Thus,

$$\left(\frac{S}{N}\right) dB = 1.8 + 6 \times 10 = 61.8 \text{ dB} \quad \textbf{Ans.}$$

(ii) For sinusoidal signal, again, let us use the same relation

i.e.,

$$\left(\frac{S}{N}\right) dB = 1.8 + 6v \text{ dB}$$

To get signal to noise ratio of at least 40 dB we can write above equation as,

$$1.8 + 6v \geq 40 \text{ dB}$$

Solving this, we get $v \geq 6.36 \text{ bits} \approx 7 \text{ bits}$

Hence, at least 7 bits are required to get signal to noise ratio of 40 dB. **Ans.**

EXAMPLE 6.11. A 7 bit PCM system employing uniform quantization has an overall signaling rate of 56 k bits per second. Calculate the signal to quantization noise that would result when its input is a sine wave with peak amplitude equal to 5 Volt. Find the dynamic range for the sine wave inputs in order that the signal to quantization noise ratio may be less than 30 dBs. What is the theoretical maximum frequency that this system can handle?

(U.P. Tech., Sem. Exam., 2005-07)

Solution: The number of bits in the PCM system are

$$v = 7 \text{ bits}$$

Assume that 5 V peak to peak voltage utilizes complete range of quantizer. Then, we can find the signal to quantization noise ratio as,

$$\left(\frac{S}{N}\right) dB = 1.8 + 6v \text{ dB} = 1.8 + 6 \times 7 = 43.8 \text{ dB}$$

We know that the signalling rate is given as,

$$r = v f_s$$

Substituting $r = 56 \times 10^3$ bits/second and $v = 7$ bits in above equation, we obtain

$$56 \times 10^3 = 7 \cdot f_s$$

Simplifying, we get

Sampling frequency, $f_s = 8 \times 103$ Hz

Further, using sampling theorem we have,

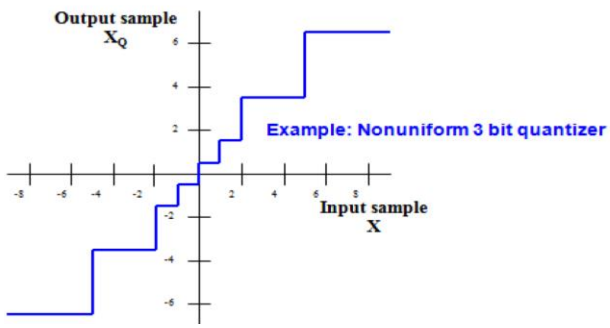
$$f_s \geq 2 f_m$$

Thus, maximum frequency that can be handled is given by,

$$f_m \leq \frac{f_s}{2} \leq \frac{8000}{2} \leq 4000 \text{ Hz} \leq 4 \text{ kHz} \quad \textbf{Ans.}$$

Non-Uniform Quantization

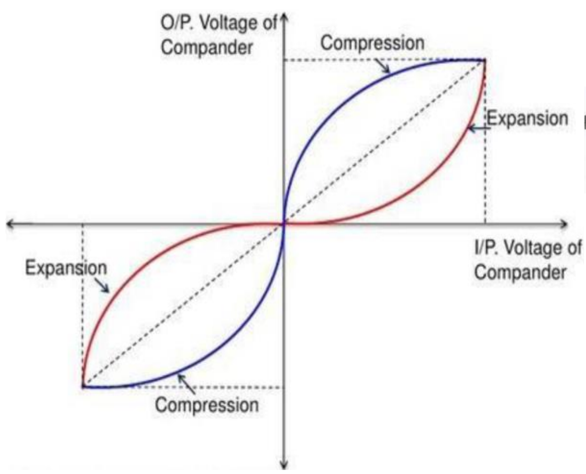
In non-uniform quantization, the step size is not fixed. It varies according to certain law or as per input signal amplitude. The following fig shows the characteristics of Non uniform quantizer.



- Note that the step-size is small at low input signal levels.
- Hence: quantization error is small at these inputs.
- Therefore: signal to quantization noise power ratio is improved at low signal levels.
- Step-size is higher at high input levels,
- Hence: signal to noise power ratio remains almost same throughout the dynamic range of quantizer.

Companding PCM System:

- Non-uniform quantizers are difficult to make and expensive.
- An alternative is to first pass the speech signal through nonlinearity before quantizing with a uniform quantizer.
- The nonlinearity causes the signal amplitude to be ***compressed***.
    - The input to the quantizer will have a more uniform distribution.
- At the receiver, the signal is ***expanded*** by an inverse to the nonlinearity.
- The process of *compressing* and *expanding* is called ***Companding***.



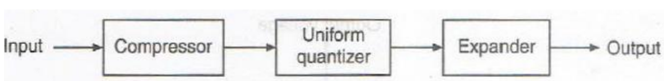


Fig. 6.18. A companding model

- The most known companders are μ-Law and A-Law companders.
- The $\boldsymbol{\mu}$**-law** algorithm (sometimes written "mu-law", often approximated as "u-law") is a companding algorithm, primarily used in 8-bit PCM digital telecommunication systems in *North America and Japan*.
- The ***A*-law**, used in regions where digital telecommunication signals are carried on E-1 circuits, e.g. Europe.
- Companding algorithms reduce the dynamic range of an audio signal. In analog systems, this can increase the signal-to-noise ratio (SNR) achieved during transmission; in the digital domain, it can reduce the quantization error (*hence increasing signal to quantization noise ratio*). These SNR increases can be traded instead for reduced bandwidth for equivalent SNR.

The $\boldsymbol{\mu}$-Law Compander

$$z(x) = \operatorname{sgn}(x) \frac{\ln(1+\mu|x|)}{\ln(1+\mu)}$$

Where

$$-1 \leq x \leq 1$$

and μ is the compression parameter (μ =255 for the U.S. and Japan), and x is the normalized integer to be compressed

the inverse or the above compression is the expanssion:

$$z^{-1} = \operatorname{sgn}(z) \frac{1}{\mu}\left[(1+\mu)^{|z|} - 1\right]$$

$$-1 \leq z \leq 1$$

The *A*-Law Compander

$$z(x) = \begin{cases} \operatorname{sgn}(x) \dfrac{A|x|}{1+\ln(A)} & 0 \leq |x| \leq \dfrac{1}{A} \\ \operatorname{sgn}(x) \dfrac{1+\ln(A|x|)}{1+\ln(A)} & \dfrac{1}{A} \leq |x| \leq 1 \end{cases}$$

where A is the compression parameter (A=87.6 in Europe), and x is the normalized integer to be compressed.

The expansion of A-Law companding is:

$$F^{-1}(y) = \begin{cases} \operatorname{sgn}(y) \dfrac{|y|[1+\ln(A)]}{A} & 0 \leq |y| \leq \dfrac{1}{1+\ln(A)} \\ \operatorname{sgn}(y) \dfrac{e^{(|y|[1+\ln(A)]-1)}}{A + A\ln(A)} & \dfrac{1}{1+\ln(A)} \leq |y| \leq 1 \end{cases}$$

MISCELLANEOUS SOLVED EXAMPLES

EXAMPLE 6.12. A binary channel with bit rate r = 36000 bits per second (b/s) is available for PCM voice transmission. Evaluate the appropriate values of the sampling rate f_s, the quantizing level q, and the number of binary digits v. Assume f_m = 3.2 kHz.

Solution: Here, we require that

$$f_s \geq 2f_m = 6400$$

and

$$vf_s \leq r = 36000$$

Therefore, we have, $v \leq \frac{r}{f_s} \leq \frac{36000}{6400} = 5.6$

Hence, we have $v \cong 5$,

and also, $q = 2^v = 2^5 = 32$,

and $f_s = \frac{36000}{5} = 7200 \text{ Hz} = 7.2 \text{ kHz}$ **Ans.**

EXAMPLE 6.13. An analog signal is sampled at the Nyquist rate f_s and quantized into q levels. Find the time duration τ of 1 bit of the binary-encoded signal.

Solution: Let v be number of bits per sample. Then we have

$$v = \log_2 q$$

where $\log_2 q$ indicates the next higher integer to be taken if $\log_2 q$ is not integer value i.e, vf_s binary pulses must be transmitted per second.

Thus, we have

$$\tau = \frac{1}{vf_s} = \frac{T_s}{v} = \frac{T}{[\log_2 q]}$$

where T_s is the Nyquist interval. **Ans.**

EXAMPLE 6.14. The output signal-to-quantizing-noise ratio $(SNR)_0$ in a PCM system is defined as the ratio of average signal power to average quantizing noise power. For a full-scale sinusoidal modulating signal with amplitude A, prove that

$$(SNR)_o = \left(\frac{S}{N_q}\right)_o = \frac{3}{2}q^2 \quad \text{...(i)}$$

or

$$\left(\frac{S}{N_q}\right)_{odB} = 10 \log\left(\frac{S}{N_q}\right)_o = 1.76 + 20 \log q \quad \text{...(ii)}$$

where q is the number of quantizing levels. **(U.P. Tech., Sem. Exam., 2002-2003)**

Solution: Since, here peak-to-peak excursion of the quantizer input is 2A. Therefore, the quantizer step size will be

$$\Delta = \frac{2A}{q}$$

Then, the average quantizing noise power is

then,

$$N_q = \langle q_e^2 \rangle = \frac{\Delta^2}{12} = \frac{A^2}{3q^2}$$

The output signal-to-quantizing-noise ratio of a PCM system for a full scale test tone is, therefore,

$$(SNR)_0 = \left(\frac{S}{N_q}\right)_o = \frac{A^2/2}{A^2/(3q^2)} = \frac{3}{2}q^2$$

Expressing this in decibels, we have

$$\left(\frac{S}{N_q}\right)_{0dB} = 10\log\left(\frac{S}{N_q}\right)_0 = 1.76 + 20 \log q \quad \textbf{Hence Proved.}$$

EXAMPLE 6.15. In a binary PCM system, the output signal-to-quantizing-noise ratio is to be held to a minimum value of 40 dB. Determine the number of required levels, and find the corresponding output signal-to-quantizing-noise ratio. (GATE Examination-1997)

Solution: In a binary PCM system, $q = 2^v$, where v is the number of binary digits. Then, we have

$$\left(\frac{S}{N_q}\right)_{0dB} = 1.76 + 20 \log 2^v = 1.76 + 6.02\, v \text{ dB} \quad \text{...(i)}$$

Now, since $\left(\frac{S}{N_q}\right)_{0dB} = 40$ dB

Therefore, $\left(\frac{S}{N_q}\right)_0 = 10{,}000$

Thus, we have, $q = \sqrt{\frac{2}{3}\left(\frac{S}{N_q}\right)_0} = \sqrt{\frac{2}{3}(10000)} = [81.6] \cong 82$

and the number of binary digits v is

$$v = [\log_2 82] = [6.36] \cong 7$$

Then, the number of levels required is $q = 2^7 = 128$, and corresponding output signal-to-quantizing noise ratio will be

$$\left(\frac{S}{N_q}\right)_{0dB} = 1.76 + 6.02 \times 7 = 43.9 \text{ dB} \quad \textbf{Ans.}$$

Important Point: *Equation (i) indicates that each bit in the code word of a binary PCM system contributes 6 dB to the output signal-to-quantizing noise ratio. In fact, this is called the 6 dB rule.*

EXAMPLE 6.16. Consider an audio signal with spectral components limited to the frequency band of 300 to 3300 Hz. A PCM signal is generated with a sampling rate of 8000 samples/s. The required output signal-to-quantizing-noise ratio is 30 dB.

(i) What is the minimum number of uniform quantizing levels needed, and what is the minimum number of bits per sample needed?

(ii) Calculate the minimum system bandwidth required.

(iii) Repeat parts (i) and (ii) when a μ-law compander is used with μ = 255.

Solution: (i) Here, we have

$$\left(\frac{S}{N_q}\right)_{0dB} = 1.76 + 20 \log q \geq 30$$

$$\log q \geq \frac{1}{20}(30 - 1.76) = 1.412$$

or $q \geq 25.82$

Thus, the minimum number of uniform quantizing levels required is 26.

$$v = [\log_2 q] = [\log_2 26] = [4.7] \cong 5 \text{ bits per sample}$$

The minimum number of bits per sample is 5.

(ii) The minimum required system bandwidth will be

$$f_{PCM} = \frac{v}{2} f_s = \frac{5}{2}(8000)$$

$$f_{PCM} = 20{,}000 \text{ Hz} = 20 \text{ kHz}$$

we have $\left(\frac{S}{N_q}\right)_{0dB} = 20 \log q - 10.1 \geq 30$

$$\log q \geq \frac{1}{20}(30 + 10.1) = 20.005$$

or $q \geq 101.2$

Thus, the minimum number of quantizing levels needed is 102.

Also, $v = [\log_2 q] = [6.67] \cong 7$

The minimum number of bits per sample is 7.

The minimum bandwidth required for this case will be

$$f_{PCM} = \frac{v}{2} f_s = \frac{7}{2}(8000) = 28000 \text{ Hz} = 28 \text{ kHz } \textbf{Ans.}$$

EXAMPLE 6.17. Bandwidth of the input to pulse code modulator is restricted to 4 kHz. The input varies from – 3.8 V to 3.8 V and has the average power of 30 mW, the required signal to quantization noise power ratio is 20 dB. The modulator produces binary output. Assume uniform quantization Calculate the number of bits required per sample.

Solution: Given that $\left(\frac{S}{N_q}\right)_{odB} = 20 \text{ dB}$

or $$10\log\left(\frac{S}{N_q}\right)_o = 20 \text{ dB}$$

or $$\left(\frac{S}{N_q}\right)_o = 100$$

Quantizer step size, $\Delta = \frac{2A}{L}$

where $L = 2^n$, n is the number of binary digits.

then, average quantizing power is,

$$N_q = \langle q_e^2 \rangle = \frac{\Delta^2}{12} = \frac{A^2}{3L^2}$$

then, $$\left(\frac{S}{N_q}\right)_o = \frac{\text{Average signal power}}{\text{Average quantizing power}}$$

or $$100 = \frac{30\times10^{-3}}{A^2/3L^2}$$

or $$L = \sqrt{\frac{30\times10^{-3}}{3\times100\times(3.8)^2}} = 126.67$$

or $$2^n = 128$$

Hence, n = 7 = number of bits required per sample.

EXAMPLE 6.18. A low pass signal of 3 kHz bandwidth and amplitude over – 5 Volts to + 5 Volts range is sampled at Nyquist rate and converted to 8-bit PCM using uniform quantization. The mean squared value of message signal is 2 Volt-squared. Determine the following :

(i) The normalized power for quantization noise.

(ii) The bit transmission rate.

(iii) The signal to quantization noise ratio in dB.

(iv) Derive the expressions used in (i) and (iii).

Solution : Given that

$$f_m = 3 \text{ kHz}, \quad v = 8$$

It is given that uniform quantization is used.

Also, $\overline{x^2(t)}$ = Mean square value of message signal is 2 Volt

(i) Normalized power for quantization noise (N_q) is given by

$$N_q = \frac{\Delta^2}{12} \text{ where } \Delta = \text{Step size} \quad \text{... (i)}$$

But $$\Delta = \frac{V_H - V_L}{q} = \frac{V_H - V_L}{2^v}$$

Therefore, $$\Delta = \frac{5-(-5)}{2^8} = \frac{10}{256} = 39.06 \text{ mV} \quad \text{... (ii)}$$

Substituting above value of Δ in equation (i), we obtain

$$N_q = \frac{(39.06 \times 10^{-3})^2}{12} = 127.15 \times 10^{-6} \text{ W} \quad \textbf{Ans.}$$

Now, let us calculate the bit transmission rate (r).

The bit transmission rate or signalling rate is the number of bits transmitted by the PCM system per second.

Therefore, $r = v f_s$

As the signal is sampled at Nyquist rate, $f_s = 2 f_m$.

$\therefore$ $r = 8 \times 2 f_m$

we have

$$r = 16 \times 3 \text{ kHz} = 48 \text{ K bits/sec.} \quad \textbf{Ans.}$$

(iii) The signal to quantization noise ratio in dB may be calculated as under :

$$\text{The normalized signal power } P = \frac{\text{Mean square value of signal}}{1\,\Omega}$$

$$\therefore \quad P = \frac{2\text{Volt}^2}{1\Omega} = 2 \text{ Watt} \quad \text{... (iii)}$$

Therefore, $$(SNR)_q = \frac{P}{N_q} = \frac{2}{127.15 \times 10^{-6}} = 15728.64$$

and $(SNR)_q$ in dB $= 10 \log_{10} (15728.64)$

or $(SNR)_q = 41.96$ dB. **Ans.**

EXAMPLE 6.19. For a full scale sinusoidal modulating signal with peak value A, show that, output signal to quantization noise ratio in binary PCM system is given by,

$$\frac{S}{N} = 1.76 + 20 \log M \text{ dB}$$

where M = Number of quantization levels.

A compact disc recording system samples each of the two stereo signals with a 16 bit A/D converter at 44.1 Kb/sec.

(i) Determine output S/N ratio for a full scale sinusoid.

(ii) The bit stream of digitized data is augmented by addition of error correcting bits, clock extraction bits etc. and these additional bits represent 100% overhead. Determine output bit rate of CD system.

(iii) The CD can record an hour's worth of music. Determine number of bits recorded on CD.

Solution : There are two stereo channels.

$$v = 16, \; f_s = 44.1 \text{ Kbits/sec.}$$

(i) Output signal to noise ratio for full scale sinusoid is given by

$$\left(\frac{S}{N_q}\right) = 1.76 + 6N = 1.76 + (6 \times 16) = 97.76 \text{ dB}$$

(ii) Now, let us evaluate the output bit rate of the CD system.

The bit rate for each of two stereo channels = $v\ f_s$

Therefore, the bit rate of two channels = $2\ v\ f_s$

$$= 2 \times 16 \times 44.1 \times 10^3 = 1.4112 \text{ Mbits/sec.}$$

Including the additional 100% overhead, the output bit rate will be

$$2 \times 1.4112 \times 10^6 \text{ b/s} = 2.822 \text{ Mbits/sec.} \quad \textbf{Ans.}$$

(iii) Next, we calculate the number of recorded on CD.

The CD can record an hour's worth of music.

Therefore, the number of bits recorded on CD = bit rate × Number of seconds/hour

$$= 2.822 \times 10^6 \times 3600 = 10.16 \times 10^9 \text{ bits or } 10.16 \text{ gigabits} \quad \textbf{Ans.}$$

EXAMPLE 6.20. The bandwidth of TV video plus audio signal is 4.5 MHz. If this signal is converted into PCM bit stream with 1024 quantization levels, determine number of bits/sec of the resulting signal. Assume that the signal is sampled at the rate 20% above Nyquist rate.

Solution : Given that $f_m = 4.5$ MHz, $q = 1024$

f_s = 20% above the Nyquist rate = $1.2 \times 2\ f_m = 1.2 \times 2 \times 4.5$ MHz.

or $f_s = 10.8$ MHz.

Let us calculate the number of bits/sec.

$$r = v\ f_s.$$

But, we do not know the value of v.

We know that

$$q = 2^v$$

$$\therefore \quad 2^v = 1024$$

$$v = 10$$

Therefore, bits/sec = 10 × 10.8 MHz = 108 M bits/sec. **Ans.**

EXAMPLE 6.21. If a voice frequency signal is sampled at the rate of 32,000 samples/sec and characterized by peak value of 2 Volts, determine the value of step size to avoid slope overload. What is quantization noise power N_q and corresponding SNR ? Assume bandwidth of signal as 4 kHz.

Solution : Given that f_s = 32,000 samples/sec.

Peak value of the signal A = 2 V.

Bandwidth BW = 4 kHz.

(i) Step size Δ to avoid slope overload can be calculated as under :

To avoid slope overload the following condition must be satisfied :

$$A \leq \frac{\Delta}{2\pi f_m T_s} = \frac{\Delta f_s}{2\pi f_m}$$

Substituting the values, we obtain

$$2 \leq \frac{\Delta \times 32000}{2\pi \times 4 \times 10^3} \quad \text{or} \quad \Delta \leq \frac{2 \times 2\pi \times 4 \times 10^3}{32000}$$

or $\Delta \geq 1.57$ Volt **Ans.**

(ii) Next, we find the quantization noise power (N_q).

The quantization noise power for a delta modulator is given by

$$N_q = \frac{\Delta^2}{3} = \frac{(1.57)^2}{3} = 0.822 \text{ W}$$

(iii) We know that the signal to noise ratio is given by

$$\text{SNR} = \frac{3 f_s^3}{8\pi^2 f_m^2 \text{BW}} = \frac{3 \times (32 \times 10^3)^3}{8\pi^2 \times (4 \times 10^3)^2 \times 4 \times 10^3} = 19.45 \quad \textbf{Ans.}$$

EXAMPLE 6.22. A compact disc (CD) records audio signals digitally by PCM. Assume audio signal's bandwidth to be 15 kHz. If signals are sampled at a rate 20% above Nyquist rate for practical reasons and the samples are quantized into 65,536 levels. Determine bits/sec required to encode the signal and minimum bandwidth required to transmit encoded signal. (U.P. Tech, Sem. Exam., 2006-07)

Solution : Given that $f_m = 15$ kHz ; $f_s = 1.2 \times 2 f_m = 2.4 \times 1.5$ kHz $= 36$ kHz ; $q = 65{,}536$

Signalling rate (r) can be calculated as under :

We know that $q = 2^v$

$\therefore$ $v = \log_2 q$

or $$v = \frac{\log_{10}(65{,}536)}{\log_{10} 2} = 16$$

Now, signalling rate $r = v f_s = 16 \times 36$ kHz $= 576$ Kbits/sec. Ans.

Hence, the signaling rate r is 576 Kbits/sec.

(ii) Minimum bandwidth can be calculated as under :

$$BW = \frac{1}{2}\text{ (signaling rate)} = \frac{576}{2} \text{ Kbits/sec.}$$

Therefore, minimum bandwidth, $BW_{min} = 288$ kHz. Ans.

SHORT QUESTIONS WITH ANSWERS

Q.1. What is a pulse digital modulation scheme?

Ans. It is the modulation in which the message and also the carrier are in discrete form.

These are classified as:

Pulse code modulation and Delta modulation.

Q.2. What are the advantages of digital representation of analog signals?

Ans. Some of the advantages of a digital signal over analog signal are:

(i) Ruggedness to transmission noise and interference

(ii) Efficient regeneration of the coded signal along the transmission path

(iii) The possibility of a uniform format for different kinds of baseband signals.

Q.3. Define Pulse Code Modulation.

Ans. It is the process in which the message signal is sampled and the amplitude of each sample is rounded off to the nearest one of a finite set of allowable values.

Q.4. Discuss Noise effect in PCM.

Ans. The performance of a PCM system is influenced by two major sources of noise.

(i) **Transmission Noise:** It is introduced anywhere transmitter output and the receiver input. It is also named as channel Noise.

(ii) **Quantizing Noise:** This is introduced in the transmitter and is carried along to the receiver output.

REVIEW QUESTIONS

1. With the help of neat diagrams, explain the transmitter and receiver of pulse code modulation.
2. Explain what is uniform (linear) quantization ?
3. Explain the quantization error and derive an expression for maximum signal to noise ratio in PCM system that uses Linear quantization.
4. Derive the relations for signaling rate and transmission bandwidth in PCM system.

NUMERICAL PROBLEM

1. In the binary PCM system , find out the minimum number of bits reqired so that quantizing noise is less than ± k per cent of the analog level. [**Ans.** $V \leq \log2\ (50/k)$]

II. Multiple Choice Questions

1. Companding is used
   (a) to overcome quantizing noise in PCM
   (b) in PCM transmitters, to allow amplitude limiting in the receivers
   (c) to protect small signals in PCM from quantizing distortion.
   (d) in PCM receivers, to overcome impulse noise.
2. The biggest disadvantages of PCM is
   (a) its inability to handle analog signals
   (b) the high error rate which its quantizing noise introduces
   (c) its incompatibility with TDM
   (d) the large bandwidths that are required for it.
3. Indicate which of the following pulse modulation systems is analog
   (a) PCM (b) Differential PCM
   (c) PWM (d) Delta modulation
4. Quantizing noise occurs in
   (a) time-division multiplexing (b) FDM
   (c) PCM (d) PWM
5. Quantizing noise can be reduced by increasing the number of samples per second. It is true,
   (a) yes, it is (b) no, it is not
   (c) not necessarily (d) none of these
6. In PCM a system, the quantization noise depends upon **(IETE, 1998)**
   (a) the number of quantization levels only
   (b) the sampling rate only
   (c) both the sampling rate and the number of quantization levels
   (d) none of the above is correct
7. The signal-to quantization noise ratio in PCM system depends upon **(IETE, 1997)**
   (a) sampling rate (b) number of quantization levels
   (c) message signal bandwidth (d) none of the above
8. Indicate which of the following systems is digital?
   (a) Pulse-position modulation (b) Pulse-code modulation
   (c) Pulse-width modulation (d) Pulse-frequency modulation
9. Quantizing noise occurs in
   (a) time-division multiplex (b) frequency-division multiplex
   (c) pulse-code modulation (d) pulse-width modulation
10. In order to reduce quantizing noise, one must
    (a) increase the number of standard amplitudes
    (b) send pulses whose sides are more nearly vertical
    (c) use an R.F. amplifier in the receiver
    (d) increase the number of samples per second
11. The biggest disadvantages of PCM is
    (a) its inability to handle analog signals (b) the high error rate which its quantizing noise reduces
    (c) its incompatibility with TDM (d) the large bandwidth that are required for it
12. Companding is used
    (a) to overcome quantizing noise in PCM
    (b) in PCM transmitters, to allow amplitude limiting in the receivers
    (c) to protect small signals in PCM from quantizing distortion
    (d) In PCM receiver, to overcome impulse noise

13. The main advantage of PCM system is
    (a) lower bandwidth (b) lower power (c) lower noise
14. Quantization noise is produced in
    (a) all pulse modulation system (b) PCM
    (c) all modulation system
15. One of the following systems is analog
    (a) PCM (b) delta (c) differential PCM (d) PAM
16. For an efficient communication in PCM system number of samples per second must at least be equal to twice the highest modulating frequency. Comment
    (a) Not necessary (b) A very important consideration
    (c) Who care (d) 80 – 50, true
17. In PCM system, output S/N increases
    (a) linearly with bandwidth (b) exponentially with bandwidth
    (c) inversely with bandwidth (d) none of these

For the answers of MCQ, see the book ☺☺☺☺☺☺☺