Department of Communications Engineering, College of Engineering, University of Diyala

Digital Communication I

Lecture # 9

Performance of Digital Modulation in Presence of Noise

- Remember that we measured the performance for analog modulation techniques in terms of signal quality, which was related to output signal-to-noise ratio ($\boldsymbol{SNR_{out}}$).
- For digital modulation techniques, the performance is measured in terms of output **BER**,
- BER represents the ***number of erroneous bits that the receiver expects per second***.
- For example, a $\boldsymbol{BER = 10^{-4}}$ means that we expect on average **1-bit error out of every 10,000 transmitted bits.**
- We say the system exhibits good performance if the $\boldsymbol{BER \leq 10^{-6}}$.

- *Remember* that we are using the ***A**dditive **W**hite **G**aussian **N**oise* (**AWGN**) mathematical model to describe the noise on a communication channel.
- Hence, the noise $\boldsymbol{n(t)}$ is considered as a **Gaussian** random process with zero average and a Variance $\boldsymbol{\sigma^2}$.
- The variance of the noise $\boldsymbol{\sigma^2}$ is its **average power**.

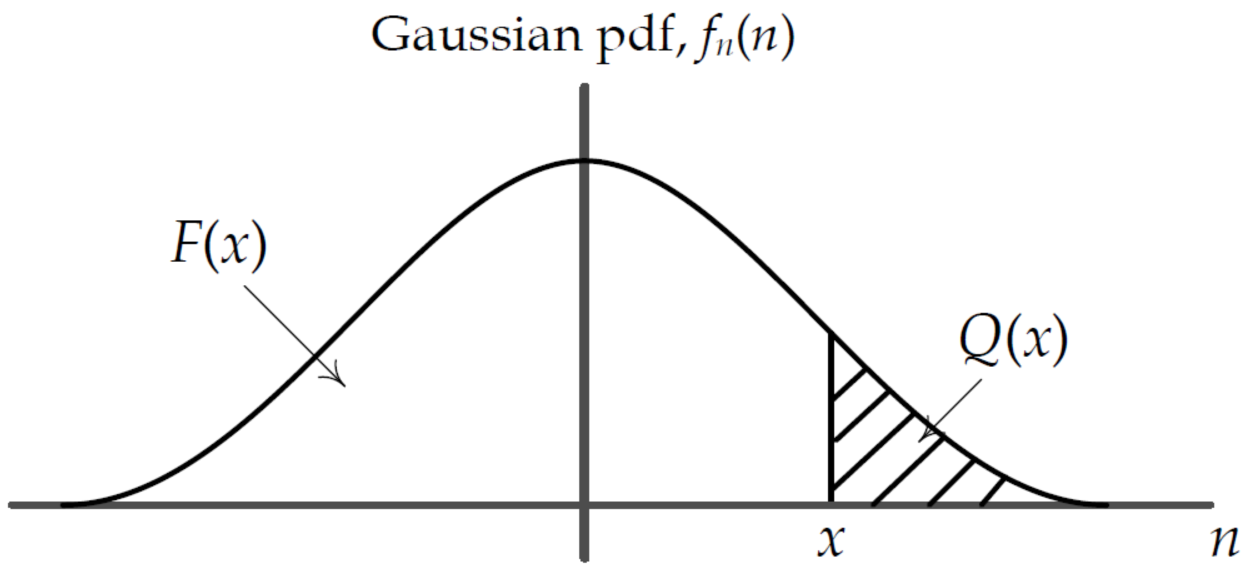
Standard Gaussian random variable $\boldsymbol{X}$ with zero-mean and unity variance, the *probability density function* (**pdf**) is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

For the purpose of our performance analysis, we will define the ***Quantile function*** $\boldsymbol{Q(x)}$ as the *complement of the cumulative distribution function* $\boldsymbol{F(x)}$ of the standard Gaussian random variable, i.e,

$$Q(x) = 1 - F(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-\alpha^2}{2}} d\alpha$$

The diagram below gives a visual representation for $Q(x)$ which represents the ***shaded area*** under the standard Gaussian density curve:



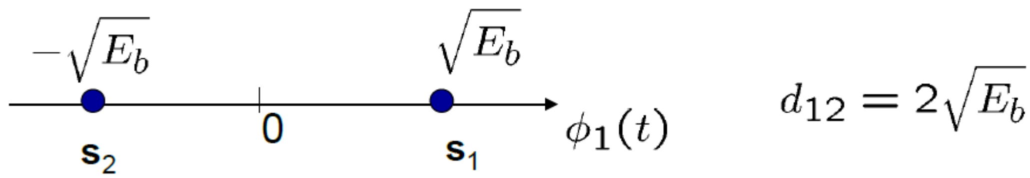
- Usually we use a table to lookup $\boldsymbol{Q}(\boldsymbol{x})$ values for specific $\boldsymbol{x}$ since the above integral has no closed form solution.

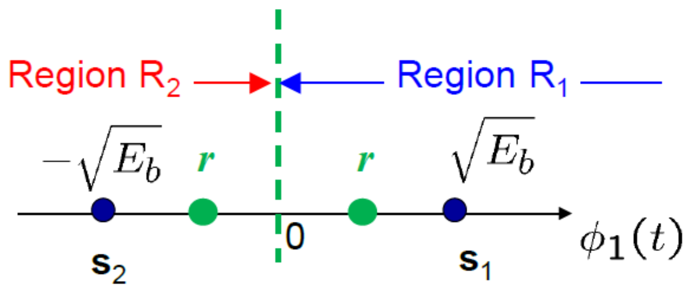
Table 1: Values of $Q(x)$ for $0 \leq x \leq 9$

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	0.5	2.30	0.010724	4.55	2.6823×10^{-6}	6.80	5.231×10^{-12}
0.05	0.48006	2.35	0.0093867	4.60	2.1125×10^{-6}	6.85	3.6925×10^{-12}
0.10	0.46017	2.40	0.0081975	4.65	1.6597×10^{-6}	6.90	2.6001×10^{-12}
0.15	0.44038	2.45	0.0071428	4.70	1.3008×10^{-6}	6.95	1.8264×10^{-12}
0.20	0.42074	2.50	0.0062097	4.75	1.0171×10^{-6}	7.00	1.2798×10^{-12}
0.25	0.40129	2.55	0.0053861	4.80	7.9333×10^{-7}	7.05	8.9459×10^{-13}
0.30	0.38209	2.60	0.0046612	4.85	6.1731×10^{-7}	7.10	6.2378×10^{-13}
0.35	0.36317	2.65	0.0040246	4.90	4.7918×10^{-7}	7.15	4.3389×10^{-13}
0.40	0.34458	2.70	0.003467	4.95	3.7107×10^{-7}	7.20	3.0106×10^{-13}
0.45	0.32636	2.75	0.0029798	5.00	2.8665×10^{-7}	7.25	2.0839×10^{-13}
0.50	0.30854	2.80	0.0025551	5.05	2.2091×10^{-7}	7.30	1.4388×10^{-13}
0.55	0.29116	2.85	0.002186	5.10	1.6983×10^{-7}	7.35	9.9103×10^{-14}
0.60	0.27425	2.90	0.0018658	5.15	1.3024×10^{-7}	7.40	6.8092×10^{-14}
0.65	0.25785	2.95	0.0015889	5.20	9.9644×10^{-8}	7.45	4.667×10^{-14}
0.70	0.24196	3.00	0.0013499	5.25	7.605×10^{-8}	7.50	3.1909×10^{-14}
0.75	0.22663	3.05	0.0011442	5.30	5.7901×10^{-8}	7.55	2.1763×10^{-14}
0.80	0.21186	3.10	0.0009676	5.35	4.3977×10^{-8}	7.60	1.4807×10^{-14}
0.85	0.19766	3.15	0.00081635	5.40	3.332×10^{-8}	7.65	1.0049×10^{-14}
0.90	0.18406	3.20	0.00068714	5.45	2.5185×10^{-8}	7.70	6.8033×10^{-15}
0.95	0.17106	3.25	0.00057703	5.50	1.899×10^{-8}	7.75	4.5946×10^{-15}
1.00	0.15866	3.30	0.00048342	5.55	1.4283×10^{-8}	7.80	3.0954×10^{-15}
1.05	0.14686	3.35	0.00040406	5.60	1.0718×10^{-8}	7.85	2.0802×10^{-15}
1.10	0.13567	3.40	0.00033693	5.65	8.0224×10^{-9}	7.90	1.3945×10^{-15}
1.15	0.12507	3.45	0.00028029	5.70	5.9904×10^{-9}	7.95	9.3256×10^{-16}
1.20	0.11507	3.50	0.00023263	5.75	4.4622×10^{-9}	8.00	6.221×10^{-16}
1.25	0.10565	3.55	0.00019262	5.80	3.3157×10^{-9}	8.05	4.1397×10^{-16}
1.30	0.0968	3.60	0.00015911	5.85	2.4579×10^{-9}	8.10	2.748×10^{-16}
1.35	0.088508	3.65	0.00013112	5.90	1.8175×10^{-9}	8.15	1.8196×10^{-16}
1.40	0.080757	3.70	0.0001078	5.95	1.3407×10^{-9}	8.20	1.2019×10^{-16}
1.45	0.073529	3.75	8.8417×10^{-5}	6.00	9.8659×10^{-10}	8.25	7.9197×10^{-17}
1.50	0.066807	3.80	7.2348×10^{-5}	6.05	7.2423×10^{-10}	8.30	5.2056×10^{-17}
1.55	0.060571	3.85	5.9059×10^{-5}	6.10	5.3034×10^{-10}	8.35	3.4131×10^{-17}
1.60	0.054799	3.90	4.8096×10^{-5}	6.15	3.8741×10^{-10}	8.40	2.2324×10^{-17}
1.65	0.049471	3.95	3.9076×10^{-5}	6.20	2.8232×10^{-10}	8.45	1.4565×10^{-17}
1.70	0.044565	4.00	3.1671×10^{-5}	6.25	2.0523×10^{-10}	8.50	9.4795×10^{-18}
1.75	0.040059	4.05	2.5609×10^{-5}	6.30	1.4882×10^{-10}	8.55	6.1544×10^{-18}
1.80	0.03593	4.10	2.0658×10^{-5}	6.35	1.0766×10^{-10}	8.60	3.9858×10^{-18}
1.85	0.032157	4.15	1.6624×10^{-5}	6.40	7.7688×10^{-11}	8.65	2.575×10^{-18}
1.90	0.028717	4.20	1.3346×10^{-5}	6.45	5.5925×10^{-11}	8.70	1.6594×10^{-18}
1.95	0.025588	4.25	1.0689×10^{-5}	6.50	4.016×10^{-11}	8.75	1.0668×10^{-18}
2.00	0.02275	4.30	8.5399×10^{-6}	6.55	2.8769×10^{-11}	8.80	6.8408×10^{-19}
2.05	0.020182	4.35	6.8069×10^{-6}	6.60	2.0558×10^{-11}	8.85	4.376×10^{-19}
2.10	0.017864	4.40	5.4125×10^{-6}	6.65	1.4655×10^{-11}	8.90	2.7923×10^{-19}
2.15	0.015778	4.45	4.2935×10^{-6}	6.70	1.0421×10^{-11}	8.95	1.7774×10^{-19}
2.20	0.013903	4.50	3.3977×10^{-6}	6.75	7.3923×10^{-12}	9.00	1.1286×10^{-19}
2.25	0.012224						

x	$Q(x)$	x	$Q(x)$
9.05	7.14841701126977e-20	12	1.77648211207770e-33
9.1	4.51659149143546e-20	12.05	9.69749658013116e-34
9.15	2.84667740846024e-20	12.1	5.28055876743362e-34
9.2	1.78974881201403e-20	12.15	2.86827981883687e-34
9.25	1.12246335913281e-20	12.2	1.55411978638962e-34
9.3	7.02228424044164e-21	12.25	8.39979606363358e-35
9.35	4.38238629906653e-21	12.3	4.52870695615878e-35
9.4	2.72815357134612e-21	12.35	2.43557140667725e-35
9.45	1.69415350248813e-21	12.4	1.30661798312467e-35
9.5	1.04945150753627e-21	12.45	6.99225823301256e-36
9.55	6.48481445307730e-22	12.5	3.73256429887781e-36
9.6	3.99722120572618e-22	12.55	1.98755015060812e-36
9.65	2.45778648347234e-22	12.6	1.05572255808867e-36
9.7	1.50749316881018e-22	12.65	5.59374113081704e-37
9.75	9.22341352493945e-23	12.7	2.95648536485212e-37
9.8	5.62928231137660e-23	12.75	1.55872628888120e-37
9.85	3.42719879411364e-23	12.8	8.19756171316302e-38
9.9	2.08137521949323e-23	12.85	4.30051225029159e-38
9.95	1.26091606702068e-23	12.9	2.25048589341516e-38
10	7.61985302416059e-24	12.95	1.17477031818436e-38
10.05	4.59337105561310e-24	13	6.11716439954992e-39
10.1	2.76210947176454e-24	13.05	3.17737014468950e-39
10.15	1.65681711680272e-24	13.1	1.64628830695088e-39
10.2	9.91362512256013e-25	13.15	8.50872242451975e-40
10.25	5.91717690736563e-25	13.2	4.38675271307426e-40
10.3	3.52306507892649e-25	13.25	2.25601633968582e-40
10.35	2.09242903757845e-25	13.3	1.15734162836906e-40
10.4	1.23966597958403e-25	13.35	5.92244602045211e-41
10.45	7.32626143174437e-26	13.4	3.02315773594524e-41
10.5	4.31900631780926e-26	13.45	1.53936150237680e-41
10.55	2.53985474308414e-26	13.5	7.81880730565805e-42
10.6	1.48990112729649e-26	13.55	3.96150704499510e-42
10.65	8.71825291966585e-27	13.6	2.00216722368154e-42
10.7	5.08891085502733e-27	13.65	1.00939282671681e-42
10.75	2.96308087809439e-27	13.7	5.07621481159796e-43
10.8	1.72101783947981e-27	13.75	2.54647631597403e-43
10.85	9.97126740930682e-28	13.8	1.27426314550685e-43
10.9	5.76286441383563e-28	13.85	6.36060456594702e-44
10.95	3.32238081981657e-28	13.9	3.16706826813090e-44
11	1.91065957449868e-28	13.95	1.57302697148055e-44
11.05	1.09607441199644e-28	14	7.79353681919280e-45
11.1	6.27219439321706e-29		
11.15	3.58031843131030e-29		
11.2	2.03866750354492e-29		
11.25	1.15796031856865e-29		
11.3	6.56089994090437e-30		
11.35	3.70813505418173e-30		
11.4	2.09059542173856e-30		
11.45	1.17572757000198e-30		
11.5	6.59577144611368e-31		
11.55	3.69102185890384e-31		
11.6	2.06039123615879e-31		
11.65	1.14729411076974e-31		
11.7	6.37267491568614e-32		
11.75	3.53094239588602e-32		
11.8	1.95155729316959e-32		
11.85	1.07595404604104e-32		
11.9	5.91735771630616e-33		
11.95	3.24626185772667e-33		

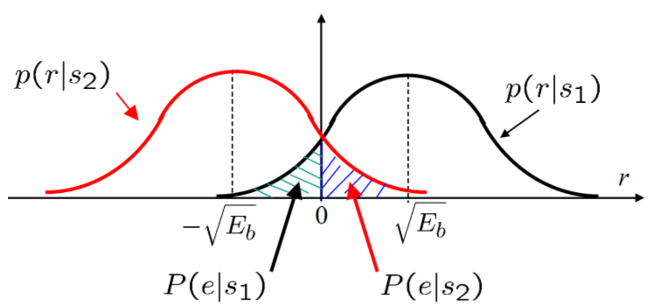
For The BPSK:





Decision rule:

- Guess signal $s_1(t)$ (or binary 1) was transmitted if the received signal point r falls in region R_1 $(r > 0)$
- Guess signal $s_2(t)$ (or binary 0) was transmitted otherwise $(r \leq 0)$



Since the signals $s_1(t)$ and $s_2(t)$ are equally likely to be transmitted, the **average probability of error** is

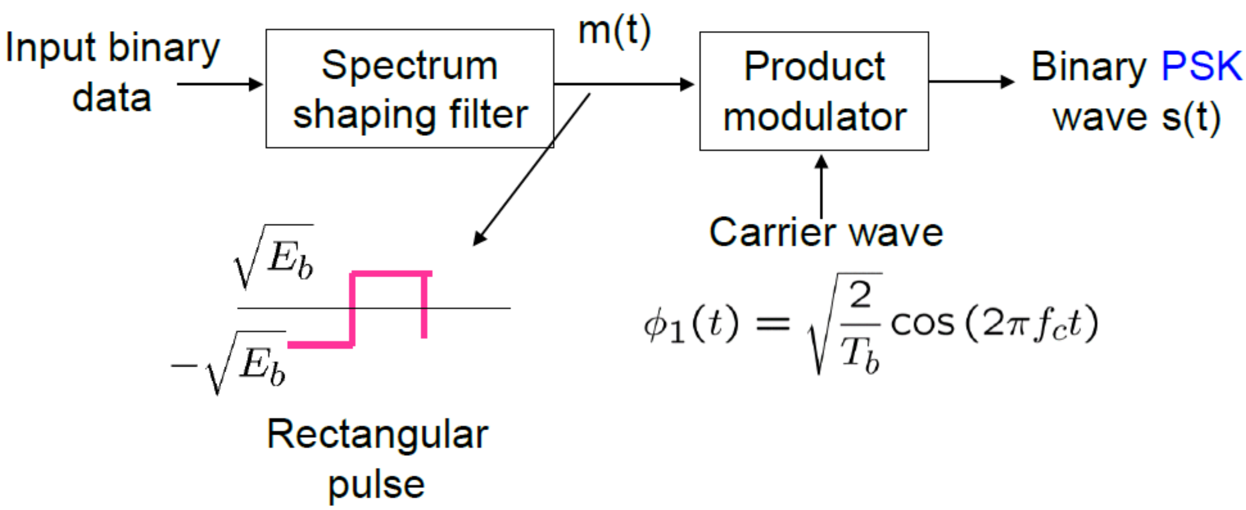
$$P_e = 0.5P(e|s_1) + 0.5P(e|s_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\Downarrow$$

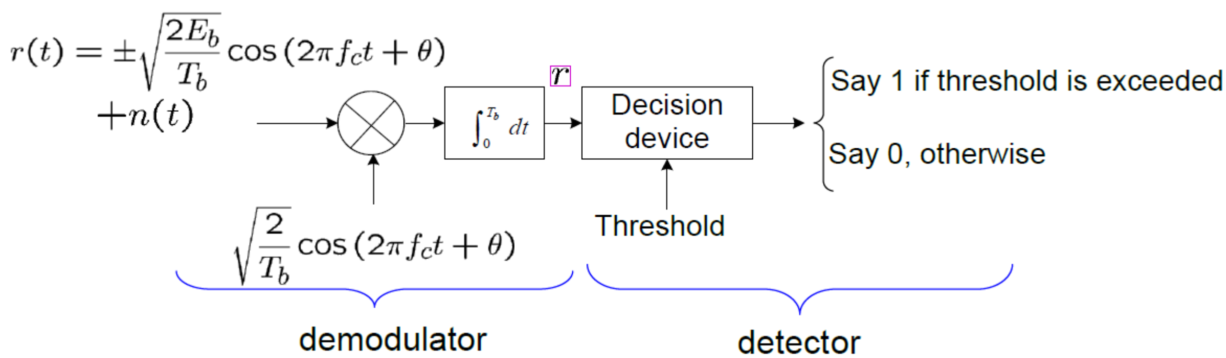
P_e depends on ratio $\frac{E_b}{N_0}$

This ratio is normally called bit energy to noise density ratio (or SNR/bit)

BPSK Transmitter

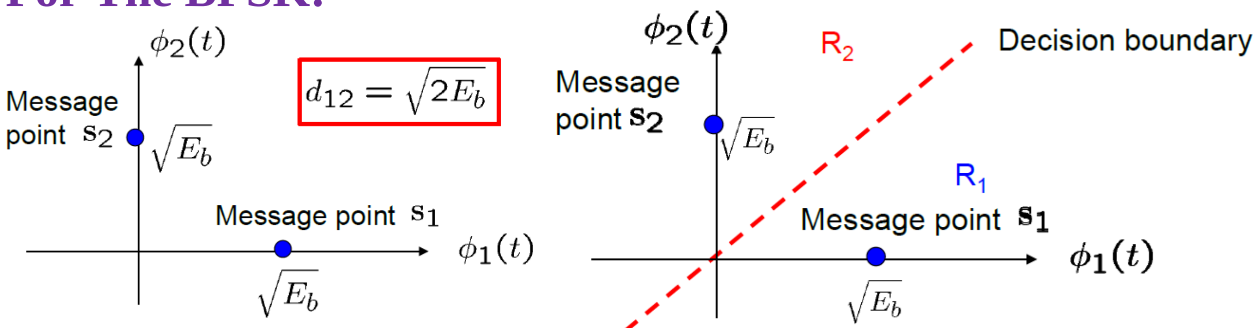


BPSK Receiver



- θ is the carrier-phase offset, due to propagation delay or oscillators at transmitter and receiver are not synchronous
- The detection is coherent in the sense of
  - Phase synchronization
  - Timing synchronization

For The BFSK:



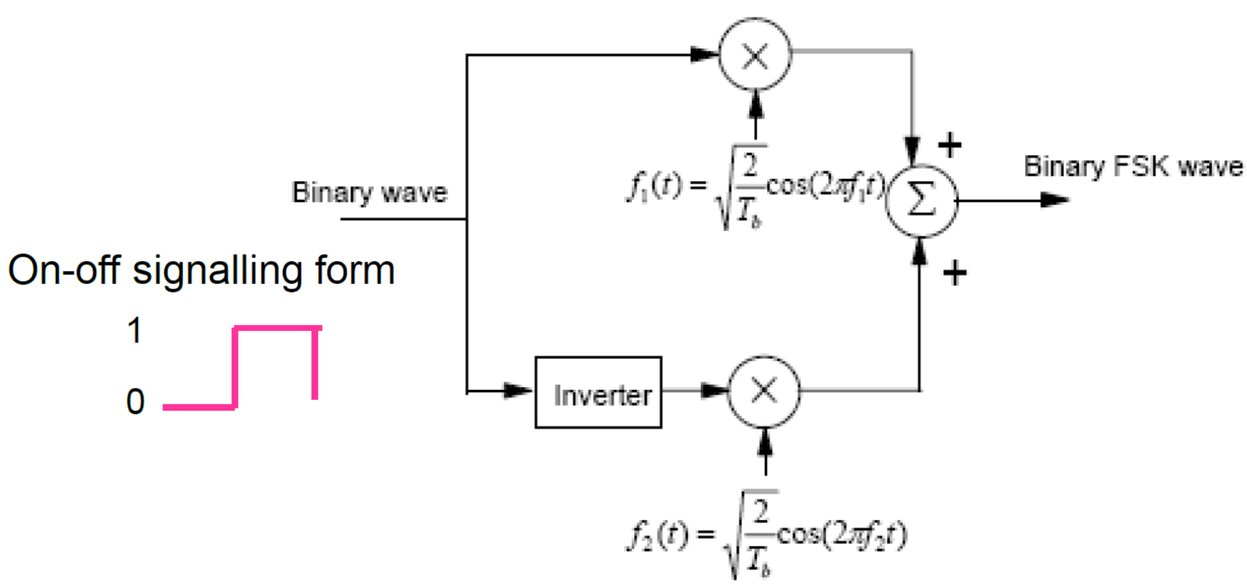
- The receiver decides in favor of s_1 if the observation vector r falls inside region R_1. This occurs when $r_1 > r_2$
- When $r_1 < r_2$, r falls inside region R_2 and the receiver decides in favor of s_2

The average probability of error for coherent binary FSK is

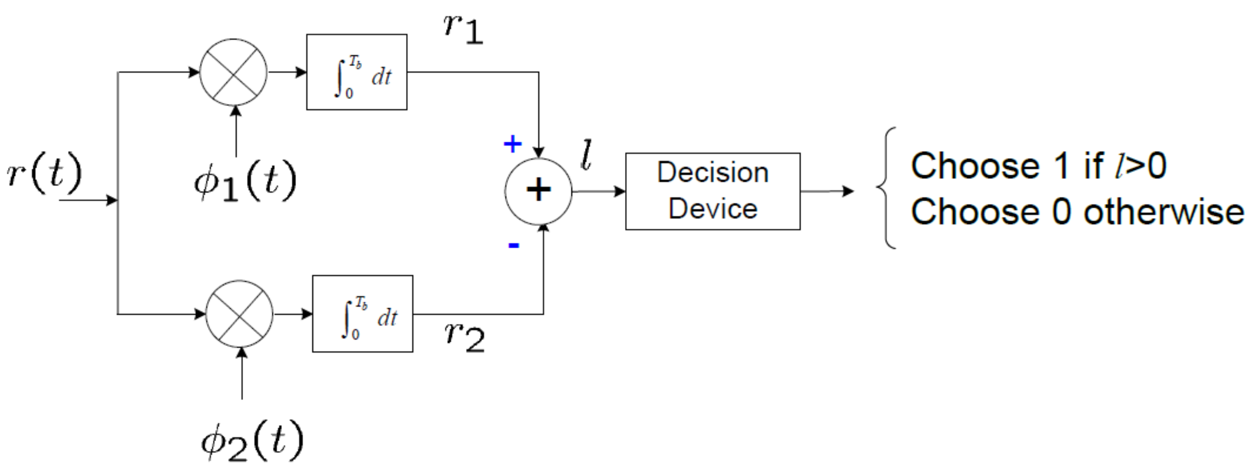
$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \Rightarrow \text{3 dB worse than BPSK}$$

To achieve the same P_e, BFSK needs 3dB more transmission power than BPSK

Binary FSK Transmitter



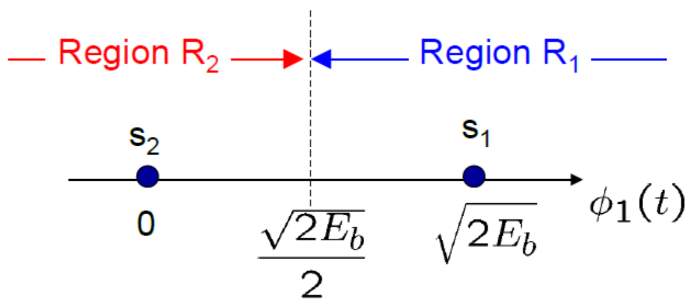
Coherent Binary FSK Receiver



For The Binary ASK

- Decision Region

$$d_{12} = \sqrt{2E_b}$$



- Average probability of error is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Identical to that of coherent binary FSK

Probability of Error and the Distance Between Signals

BPSK	BFSK	BASK
$d_{1,2} = 2\sqrt{E_b}$	$d_{1,2} = \sqrt{2E_b}$	$d_{1,2} = \sqrt{2E_b}$
$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

- In general,

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

For Optimum Detection

For the rest of this class, we will use the following notation:

- $\boldsymbol{v}$ = the number of bits sent per transmitted symbol [$\boldsymbol{v = log_2(M)}$].
- $\boldsymbol{E_s}$ = Average energy-per-transmitted-symbol in the modulated signal ($\boldsymbol{Joule}$).
- $\boldsymbol{E_b}$ = Average energy per transmitted-bit in the modulated signal ($Joule$) $\boldsymbol{E_b = E_s/v}$.
- $\boldsymbol{T_0}$ = Bit duration.
- T_{symb} = Symbol duration $\boldsymbol{T_{symb} = vT_0}$

Modulation with AWGN	Error Probability	
ASK	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	Coherent
	$P_e = \frac{1}{2}e^{-\frac{1}{2}\left(\frac{E_b}{N_0}\right)}$	Non-Coherent
***M*-ASK**	$P_s = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{3\log_2(M)}{M^2-1}\times\frac{2E_b}{N_0}}\right)$ $P_e = \frac{P_s}{\log_2(M)}$	
FSK	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	Coherent
	$P_e = \frac{1}{2}e^{-\frac{1}{2}\left(\frac{E_b}{N_0}\right)}$	Non-Coherent
***M*-FSK**	$P_e = \frac{M-1}{\log_2(M)}Q\left(\sqrt{\frac{E_s}{N_0}}\right)$	Coherent
	$P_e = \frac{M-1}{2\log_2(M)}e^{-\frac{E_s}{2N_0}}$	Non-Coherent
BPSK	$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	Coherent
	$P_e = \frac{1}{2}e^{-\left(\frac{E_b}{N_0}\right)}$	Non-Coherent
DPSK	$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	Coherent
	$P_e = \frac{1}{2}e^{-\left(\frac{E_b}{N_0}\right)}$	Non-Coherent
QPSK (4-PSK), $\frac{\pi}{4}$–QPSK, OQPSK	$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$	Coherent
DQPSK, $\frac{\pi}{4}$–DQPSK	$P_e = Q\left(\sqrt{1.1716\frac{E_b}{N_0}}\right)$	Coherent
***M*-PSK**	$P_e \approx \frac{2}{v}Q\left(\sqrt{2v\frac{E_b}{N_0}}\times\sin\left(\frac{\pi}{M}\right)\right)$	
***M*-QAM (Rectangular)**	$P_{bc} = \frac{4}{v}\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3v}{M-1}\frac{E_b}{N_0}}\right)$ $P_e = 1-(1-P_{bc})^2$	
MSK	$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	Coherent
	$P_e = \frac{1}{2}e^{-\frac{1}{2}\left(\frac{E_b}{N_0}\right)}$	Non-Coherent

Example: Find the BER for BPSK if we use an optimal detector (a matched filter). Assume the amplitude of the carrier is $A = 0.5V$, data rate is $2bps$, and $N_0 = 2 \times 10^{-2}$W/Hz.

Solution: In BPSK there is one symbol per bit (i.e., a total of two symbols that the modulated signal can assume). The two symbols can be written as:

$$s_1 = A\cos(\omega_c t) \qquad s_2 = -A\cos(\omega_c t) = A\cos(\omega_c t - \pi)$$

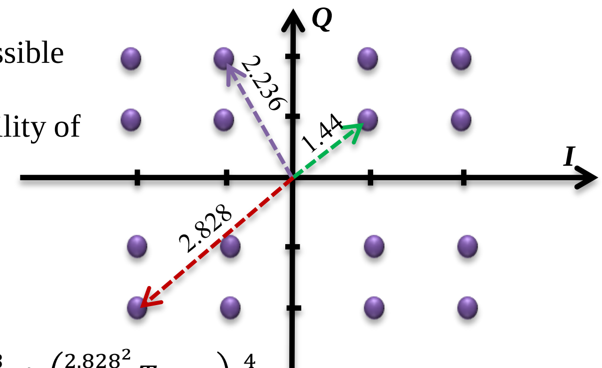
The energy-per symbol here is the same as the energy-per bit and is equal for both possible symbols. Hence, its average is:

$$E_b = E_s = \left(\frac{A^2}{2}T_{symb}\right)\Pr[1] + \left(\frac{A^2}{2}T_{symb}\right)\Pr[0] = \frac{A^2}{2}T_{symb} = \frac{A^2}{2}T_0 = \frac{A^2}{2}\frac{1}{f_0}$$. Hence,

$$BER = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2}{N_0 f_0}}\right) = Q\left(\sqrt{\frac{(0.5)^2}{2\times10^{-2}\times2}}\right) = Q\left(\sqrt{6.25}\right) = Q(2.5) = 6.21 \times 10^{-3}$$

Example: Find the BER for the 16-QAM constellation shown below if we use an optimal detector (a matched filter). Assume the data rate is 4bps, and $N_0 = 5 \times 10^{-2}$W/Hz

Solution: In this system there are 16 possible Symbols, which we assume to be equally Probable, i.e., each occures with a probability of 1/16. Hence, the energy-per-symbol is:



$$E_s = \left(\frac{1.414^2}{2}T_{symb}\right)\frac{4}{16} + \left(\frac{2.236^2}{2}T_{symb}\right)\frac{8}{16} + \left(\frac{2.828^2}{2}T_{symb}\right)\frac{4}{16}$$

$$E_s = [0.25 + 1.25 + 1]T_{symb} = 2.5T_{symb}$$

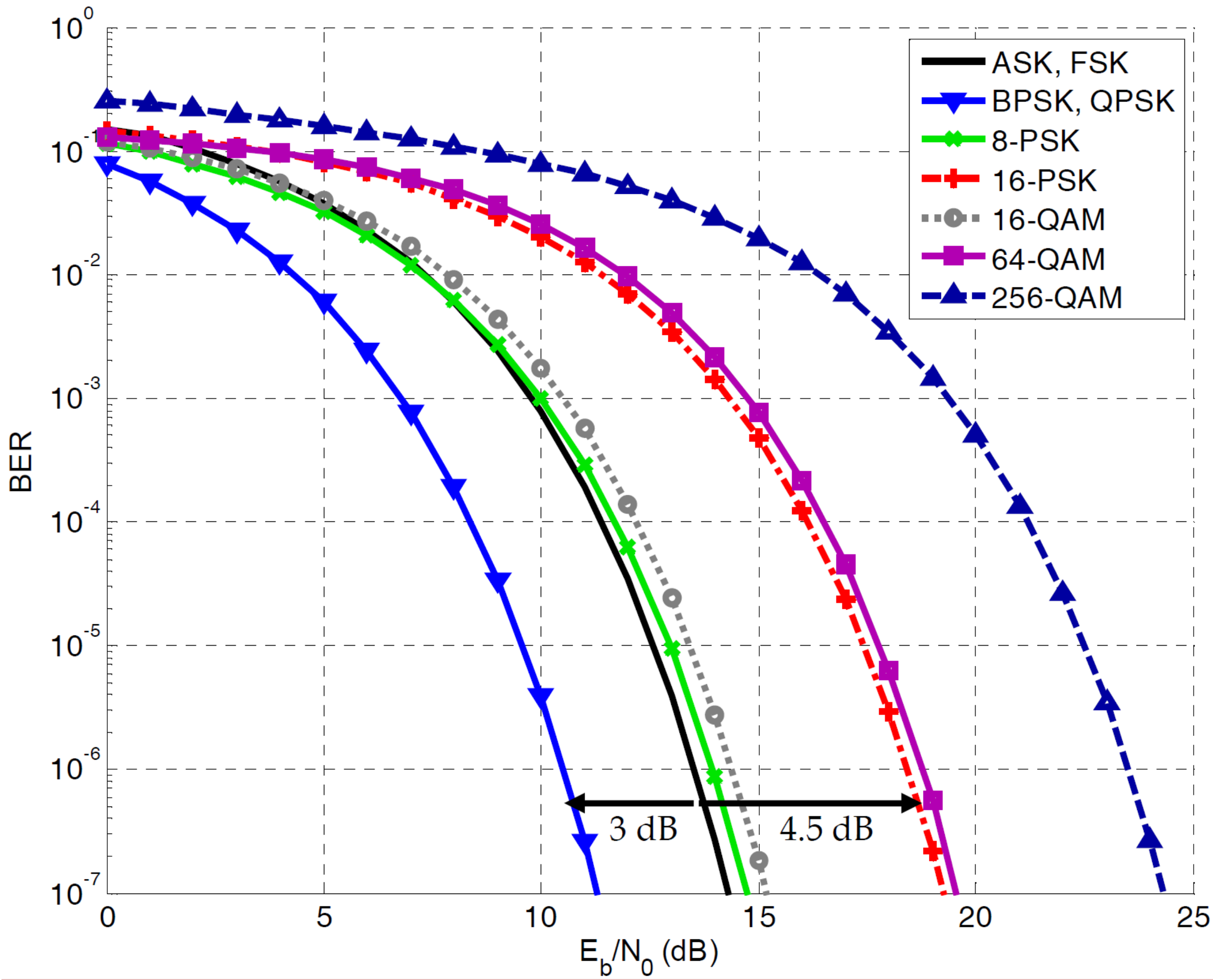
$$E_b = \frac{E_s}{v} = 2.5\left(\frac{T_{symb}}{v}\right) = 2.5T_0 = \frac{2.5}{f_0} = \frac{2.5}{4} = 0.625\,\boldsymbol{J}$$

$$P_{bc} = \frac{4}{v}\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3v}{M-1}\frac{E_b}{N_0}}\right) = \frac{4}{4}\left(1 - \frac{1}{\sqrt{16}}\right)Q\left(\sqrt{\frac{3\times4}{16-1}\times\frac{0.625}{0.05}}\right) = \frac{3}{4}Q\left(\sqrt{10}\right)$$

$$= \frac{3}{4}Q(3.162) = \frac{3}{4}\times 8\times10^{-4} = 6\times10^{-4}$$

$$BER = 1 - (1 - P_{bc})^2 = 1 - (1 - 6\times10^{-4})^2 = 1.2\times10^{-3}$$

Comparison of Digital Modulation Schemes



Comparing **BPSK** and **QPSK** with **ASK** and **FSK**, we notice that **BPSK** and **QPSK** provide smaller bit error rate for the same E_b/N_0. In other words, for the same bit error rate, we need less signal-to-noise ratio (E_b/N_0) to send **BPSK** and **QPSK**. This means that **BPSK** and **QPSK** have better immunity to noise than **ASK** and **FSK**.

Notice also that the performance of **BPSK** is the same as that for **QPSK**, while the performance of **8-PSK** and **16-PSK** are **worse** (i.e., they require **more** E_b/N_0 to achieve the same **BER**). This is an expected result because **8-PSK** and **16-PSK** have more constellation diagram points (**which are now closer and closer to each other**).

Also notice how **16-QAM** has a superior performance compared to **16-PSK**, which is to be expected ***because the constellation points are further apart*** in **16-QAM** compared to **16-PSK**.

Shannon's Limit:

- You see by now that the two main resources in communication systems are:
  - The **channel bandwidth** and
  - The **transmitted power** (or **SNR**).
- In a given communication channel, one resource may be more valuable than the other, and the communication scheme should be designed accordingly.

The limitation imposed on communication by the channel bandwidth and the SNR is dramatically highlighted by **Shannon's** noisy channel theorem, which applies for channels **contaminated** with **A**dditive **W**hite **G**aussian **N**oise (**AWGN**).

Shannon's equation states that:

$$C = B\log_2(1 + SNR) \qquad \frac{bits}{second}$$

Where:

- C is the rate of information transmission in bits per second,
- B is the channel bandwidth (in Hz), and
- SNR is the signal-to-noise ratio (unitless) on the channel.
- This rate $\boldsymbol{C}$ (known as the **channel capacity**) is *the maximum number of binary bits that can be transmitted per second with a probability of error arbitrarily close to zero*.
- In other words, it is impossible to transmit at a rate higher than $\boldsymbol{C}$ without incurring errors.
- **Shannon's** equation clearly brings out the limitation on the rate of communication imposed by $\boldsymbol{B}$ and $\boldsymbol{SNR}$. This is why bandwidth and SNR are the main quantities that we study in communication systems.
- Notice that if noise was zero (i.e., $\boldsymbol{SNR} = \infty$), or the bandwidth was infinite ($\boldsymbol{B} = \infty$), then we can transmit infinity information and communication would cease to be a problem.
- ✓ It should be remembered that **Shannon's** result represents the upper limit on the rate of communication over a channel and **can be achieved only with a systems of great complexity and a time delay in reception approaching infinity**.
- ✓ **Practical systems operate at rates below the Shannon rate.**

Applications of Digital Modulation Techniques:

The following are some current-day communication systems that use digital modulation:

1. **IEEE 802.11 (Wi-Fi)**: A very important Wireless Local Area Networking technology. Since Wi-Fi has many variants, it uses different modulation techniques such as: BPSK, QPSK, 16-QAM, 64-QAM and CCK (Complementary Code Keying) (CCK is an extension of QPSK).
2. **IEEE 802.16 (Wi-MAX):** A very important Wireless Metropolitan Area Network, and currently competes with ADSL for Internet delivery. Wi-MAX switches dynamically between different modulations schemes such as: BPSK, QPSK, 16-QAM, and 64-QAM. It uses these modulation schemes in combination with OFDM (**O**rthogonal **F**requency **D**ivision **M**ultiplexing) (OFDM is an extension of **FDM**).
3. **DVB (Digital Video Broadcasting)**: This is the European standard for digital television broadcasting. There are many variants within the standard: DVB-S (for satellite broadcasting) uses QPSK or 8-PSK; DVB-C (for cable) uses 16-QAM, 32-QAM, 64-QAM, 128-QAM or 256-QAM; and DVB-T (for terrestrial television broadcasting) uses 16-QAM or 64-QAM.
4. **DAB (Digital Audio Broadcasting)**: Future European standard for digital radio broadcasting, which should replace AM and FM radio broadcasting. DAB use DQPSK (Differential QPSK) (DQPSK is a variation of QPSK).
5. **ADSL**: Currently one of the main choices for connecting to the Internet. Uses adaptive QAM in a scheme called DMT (Discrete Multi-Tone modulation).