Discrete-Time Signals and Systems

In this lecture, we present in some detail the characteristics of discrete-time sinusoidal signals. These discrete-time signals are introduced in this lecture and are used as basic functions or building blocks to describe more complex signals. The major emphasis in this lecture is the characterization of discrete-time systems in general and the class of linear time-invariant (LTI) systems in particular. A number of important time-domain properties of LTI system s are defined and developed.

4.1 DISCRETE-TIME SIGNALS

As we discussed in lecture 1, a discrete-time signal *x{n)* is a function of an independent variable that is an integer. It is graphically represented as in Fig. 4.1. It is important to note that a discrete-time signal is *n o t defined* at instants between two successive samples. Also, it is incorrect to think that x(n) is equal to zero if n is not an integer. Simply, **the signal *x (n)* is n o t defined for non-integer values of *n.***

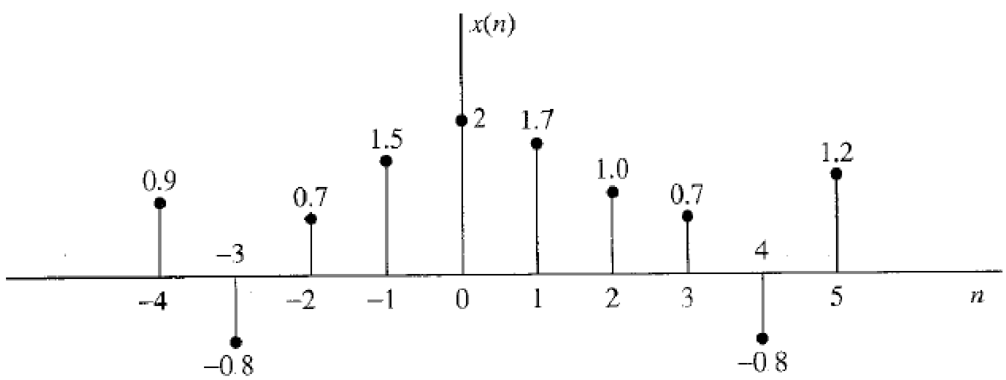


Figure 4-1 Graphical representation of a discrete-time signal.

Besides the graphical representation of a discrete-time signal or sequence as illustrated in Fig. 4.1. there are some alternative representations that are often more convenient to use. These are:

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

2. Tabular representation, such as

n	...	−2	−1	0	1	2	3	4	5	...
$x(n)$	...	0	0	0	1	4	1	0	0	...

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol ↑ is represented as

$$x(n) = \{\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\}$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\}$$

4.2 Block Diagram Representation *of* Discrete-Time Systems

It is useful at this point to introduce a block diagram representation of discrete-time systems. For this purpose we need to define some basic building blocks that can be interconnected to form complex systems.

An adder. Figure 4.2 illustrates a system (adder) that performs the addition of two signal sequences to form another (the sum) sequence, which we denote as $y(n)$. Note that it is not necessary to store either one of the sequences in order to perform the addition. In other words, the addition operation is *memoryless*.

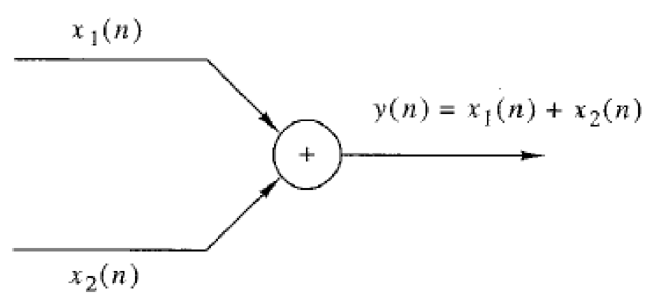


Figure 4.2. Graphical representation of an adder.

A constant multiplier. This operation is depicted by Fig. 4.3, and simply represents applying a scale factor on the input $x(n)$. Note that this operation is also memoryless.

Figure 4.3 Graphical representation of a constant multiplier.

A signal multiplier. Figure 4.4 illustrates the multiplication of two signal sequences to form another (the product) sequence, denoted in the figure as $y(n)$. As in the preceding two cases, we can view the multiplication operation as memoryless.

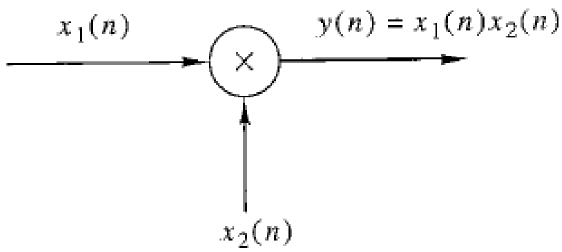


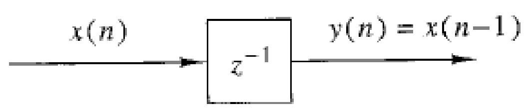
Figure 4.4
Graphical representation of a signal multiplier.

A unit delay element. The unit delay is a special system that simply delays the signal passing through it by one sample. Figure 4.5 illustrates such a system. If the input signal is $x(n)$, the output is $x(n-1)$. In fact, the sample $x(n-1)$ is stored in memory at time $n-1$ and it is recalled from memory at time n to form

$$y(n) = x(n-1)$$

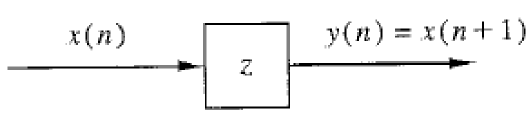
Thus this basic building block requires memory. The use of the symbol z^{-1} to denote the unit of delay will become apparent when we discuss the z-transform in Chapter 3.

Figure 4.5
Graphical representation of the unit delay element.



A unit advance element. In contrast to the unit delay, a unit advance moves the input $x(n)$ ahead by one sample in time to yield $x(n+1)$. Figure 4.6 illustrates this operation, with the operator z being used to denote the unit advance. We observe that any such advance is physically impossible in real time, since, in fact, it involves looking into the future of the signal. On the other hand, if we store the signal in the memory of the computer, we can recall any sample at any time. In such a non-real-time application, it is possible to advance the signal $x(n)$ in time.

Figure 4.6
Graphical representation of the unit advance element.



Example 1

Using basic building blocks introduced above, sketch the block diagram representation of the discrete-time system described by the input–output relation

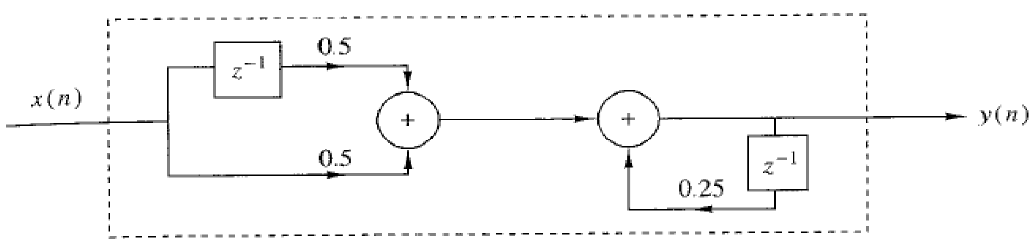
$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$

where $x(n)$ is the input and $y(n)$ is the output of the system.

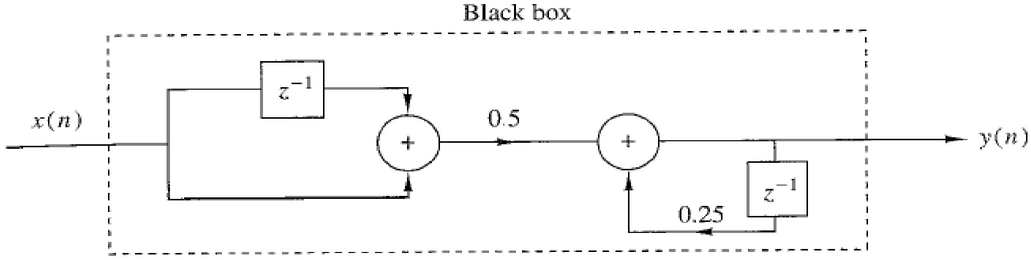
Solution. According to (A), the output $y(n)$ is obtained by multiplying the input $x(n)$ by 0.5, multiplying the previous input $x(n-1)$ by 0.5, adding the two products, and then adding the previous output $y(n-1)$ multiplied by $\frac{1}{4}$. Figure (4.7.a) illustrates this block diagram realization of the system. A simple rearrangement of (A),

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}[x(n) + x(n-1)]$$

leads to the block diagram realization shown in Fig. 4. 7(b).



(a)



(b)

Figure 4.7 Block diagram realizations of the system $y(n) = 0.25y(n-1) + 0.5x(n) + 0.5x(n-1)$.

4.3 Response Of A Discrete-Time LTI System And Convolution Sum

A. Impulse Response:

The impulse response (or unit sample response) h[n] of a discrete-time LTI system (represented by **T)** is defined to be the response of the system when the input is δ [n], that is,

$$h[n] = \mathbf{T}\{\delta[n]\}$$

B. Response to an Arbitrary Input:

the input x[n] can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Since the system is linear, the response **y[n]** of the system to an arbitrary input **x[n]** can be expressed as

$$\begin{aligned} y[n] &= \mathbf{T}\{x[n]\} = \mathbf{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} x[k]\mathbf{T}\{\delta[n-k]\} \end{aligned}$$

Since the system is time-invariant, we have

$$h[n-k] = \mathbf{T}\{\delta[n-k]\}$$

By substituting:

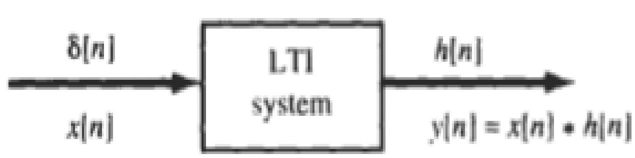
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad \dots \; k$$

C. Convolution Sum:

Equation **(k)** defines the **convolution** of two sequences **x[n]** and **h[n]** denoted by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad \dots \; j$$

Equation (j) is commonly called the convolution sum. Thus, we have the fundamental result that the output of any discrete-time LTI system is the convolution of the input x[n] with the impulse response h[n] of the system. The Figure below illustrates the definition of the impulse response h[n] and the relationship of Eq. (i).



D. Properties of the Convolution Sum:

Convolution is a linear operator and, therefore, has a number of important properties including the commutative, associative, and distributive properties. The definitions and interpretations of these properties are summarized below.

1 . Commutative

The commutative property states that the order in which two sequences are convolved is not important. Mathematically, the commutative property is

$$x[n] * h[n] = h[n] * x[n]$$

this property states that a system with a unit sample response h(n) and input x(n) behaves in exactly the same way as a system with unit sample response x(n) and an input h(n). This is illustrated in Figure 4.8(a).

2. Associative:

The convolution operator satisfies the associative property, which is

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

the associative property states that if two systems with unit sample responses h_1(n) and h_2(n) are connected in cascade as shown in Figure 4.8 (b), an equivalent system is one that has a unit sample response equal to the convolution of h_1(n) and h_2(n) :

$$h_{eq} = h_1(n) * h_2(n)$$

3. Distributive:

The distributive property of the convolution operator states that

$$x[n] * \{h_1[n]\} + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

this property asserts that if two systems with unit sample responses h_1(n) and h_2(n) are connected in parallel, as shown in Figure 4.8 (c), an equivalent system is one that has a unit sample response equal to the sum of h_1(n) and h_2(n)

$$h_{eq} = h_1(n) + h_2(n)$$

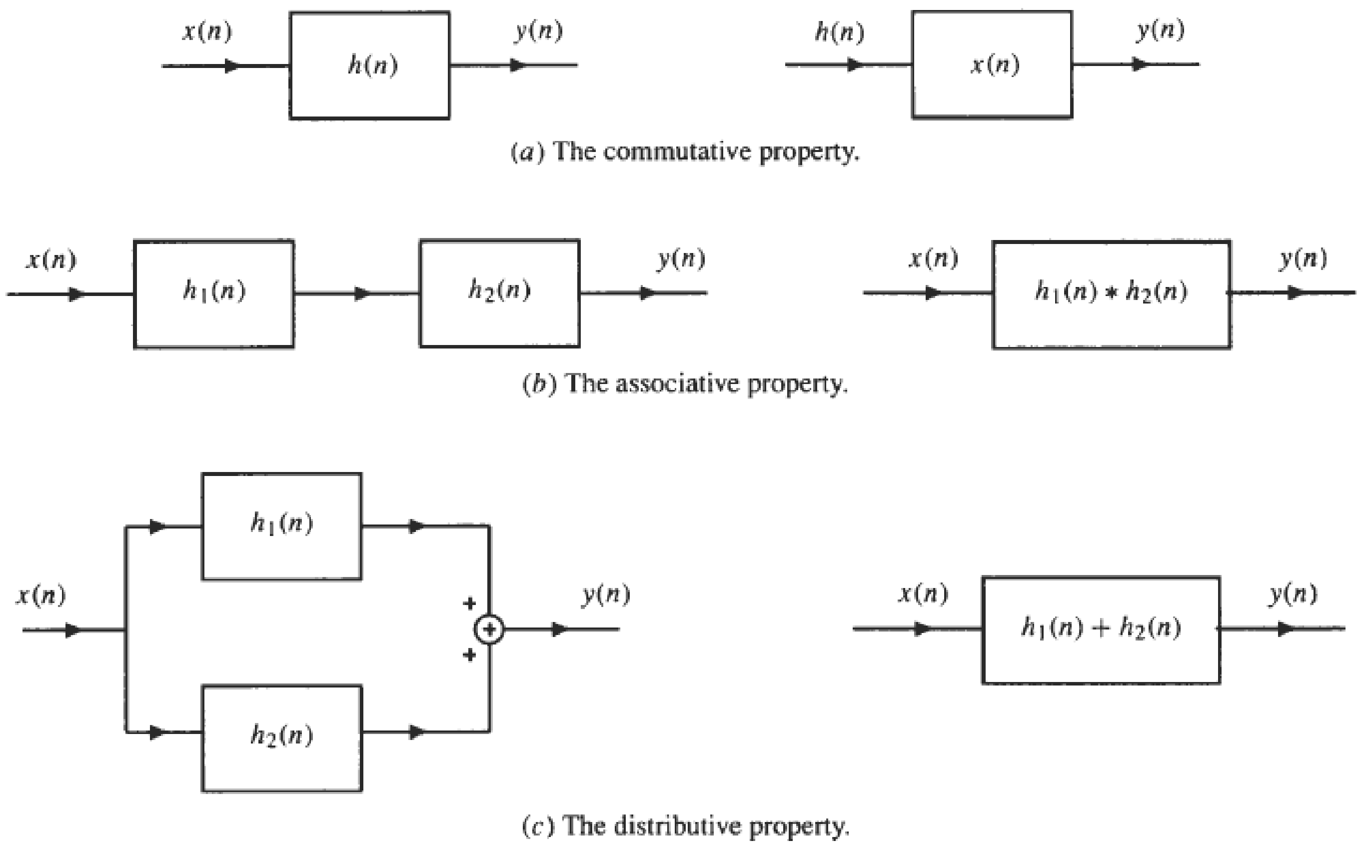


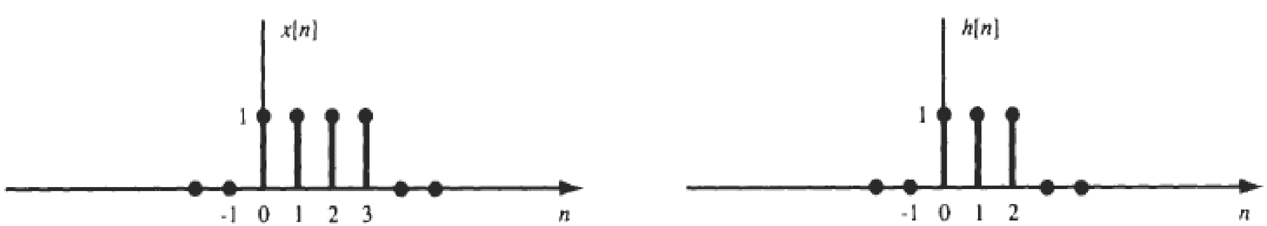
Figure (4.8) The interpretation of convolution properties

E. Performing Convolutions

There are several different approaches that may be used, and the one that is the easiest will depend upon the form and type of sequences that are to be convolved.

1. Analytical Technique

if we want to evaluate $y[n] = x[n] * h[n]$, where $x[n]$ and $h[n]$ are shown in Figure by an analytical technique



(a) Note that $x[n]$ and $h[n]$ can be expressed as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$
$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

$$\begin{aligned} x[n] * h[n] &= x[n] * \{\delta[n] + \delta[n-1] + \delta[n-2]\} \\ &= x[n] * \delta[n] + x[n] * \delta[n-1] + x[n] * \delta[n-2]\} \\ &= x[n] + x[n-1] + x[n-2] \end{aligned}$$

Thus,
$$\begin{aligned} y[n] = \delta[n] &+ \delta[n-1] + \delta[n-2] + \delta[n-3] \\ &+ \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] \\ &+ \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] \end{aligned}$$

or $$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$

or $$y[n] = \{1, 2, 3, 3, 2, 1\}$$

2.Graphical Approach

The steps involved in using the graphical approach are as follows:

- Plot both sequences, x(k) and h(k), as functions of k.
- Choose one of the sequences, say h(k), and time-reverse it to form the sequence h(-k).
- Shift the time-reversed sequence by n. [**Note**: If n > 0, this corresponds to a shift to the right (delay), whereas if n < 0**,** this corresponds to a shift to the left]
- Multiply the two sequences x(k) and h(n - k) and sum the product for all values of k. The resulting value will be equal to y(n). This process is repeated for all possible shifts, n.

Ex 1: To illustrate the graphical approach to convolution, let evaluate y(n) = x(n)*h(n) where x(n) and h(n) are the sequences shown in Figure 4.9 (a) and (b), respectively. To perform this convolution, we follow the steps listed above :

1. Because x(k) and h(k) are both plotted as a function of k in Figure 4.9 (a) and (b), we next choose one of the sequences to reverse in time.

In this example, we time-reverse h(k), which is shown in Figure 4.9 (c).

2. Forming the product, x(k)h(-k), and summing over k, we find that y(0) = 1.

3. Shifting h(k) to the right by one results in the sequence h(l - k) shown in Figure 4.9(d). Forming the product, x(k)h(l - k), and summing over k, we find that y(1) = 3.

4. Shifting h(l - k) to the right again gives the sequence h(2 - k) shown in Figure 4.9 (e). Forming the product, x(k)h(2 - k), and summing over k, we find that y(2) = 6.

5. Continuing in this manner, we find that y(3) = 5. y(4) = 3, and y(n) = 0 for n > 4.

6. We next take h(-k) and shift it to the left by one as shown in Figure 4.9 (f). Because the product, x(k)h(- 1 - k), is equal to zero for all k, we find that y(- 1) = 0. In fact, y(n) = 0 for all n < 0.

Figure 4.9 (g) shows the convolution for all n.

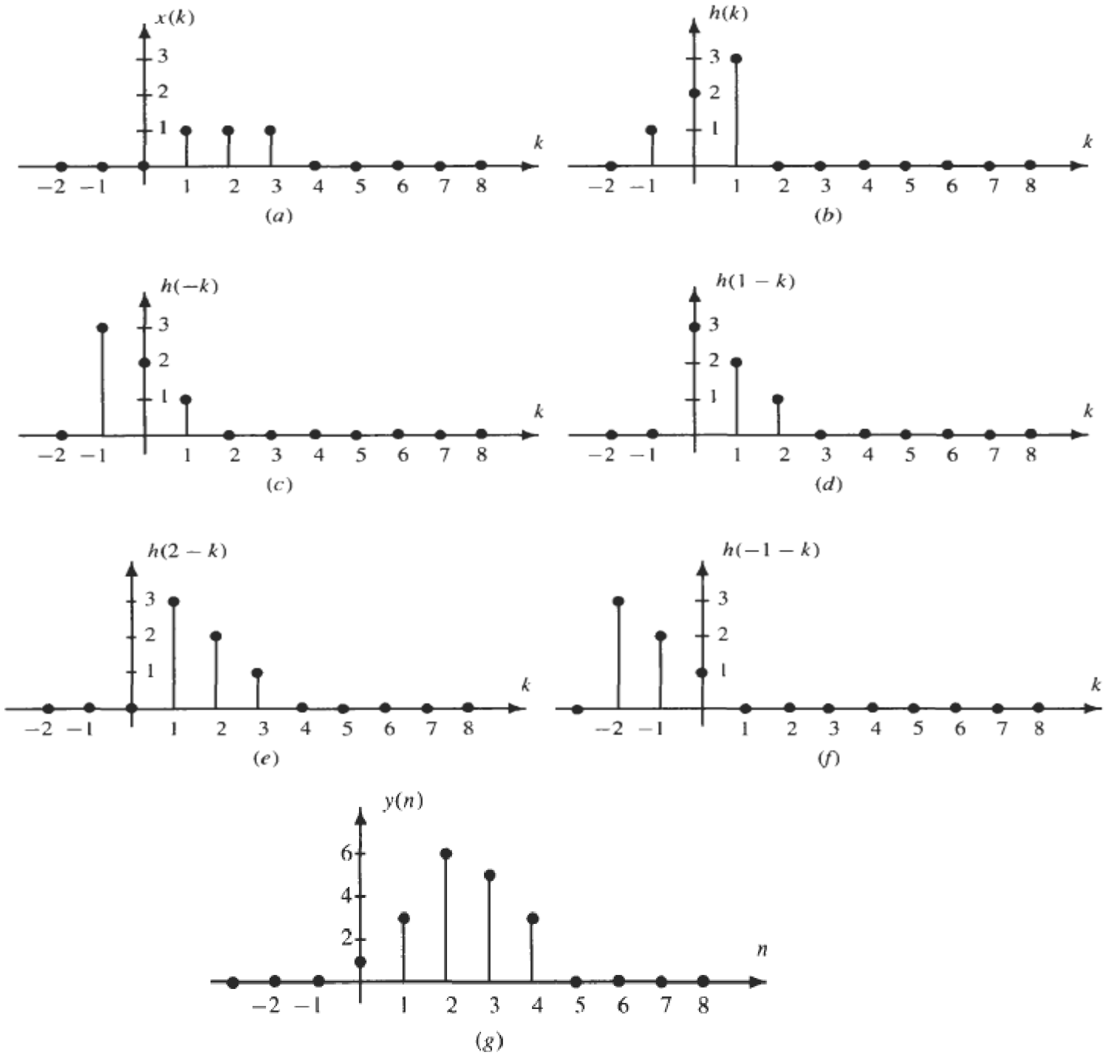
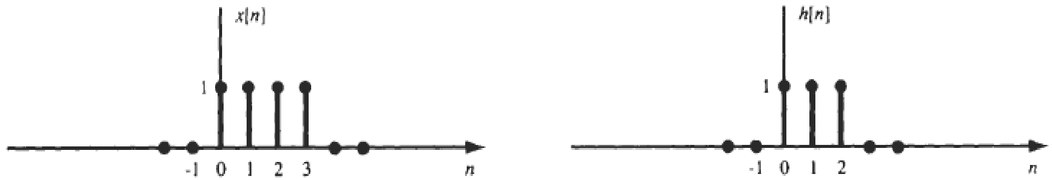


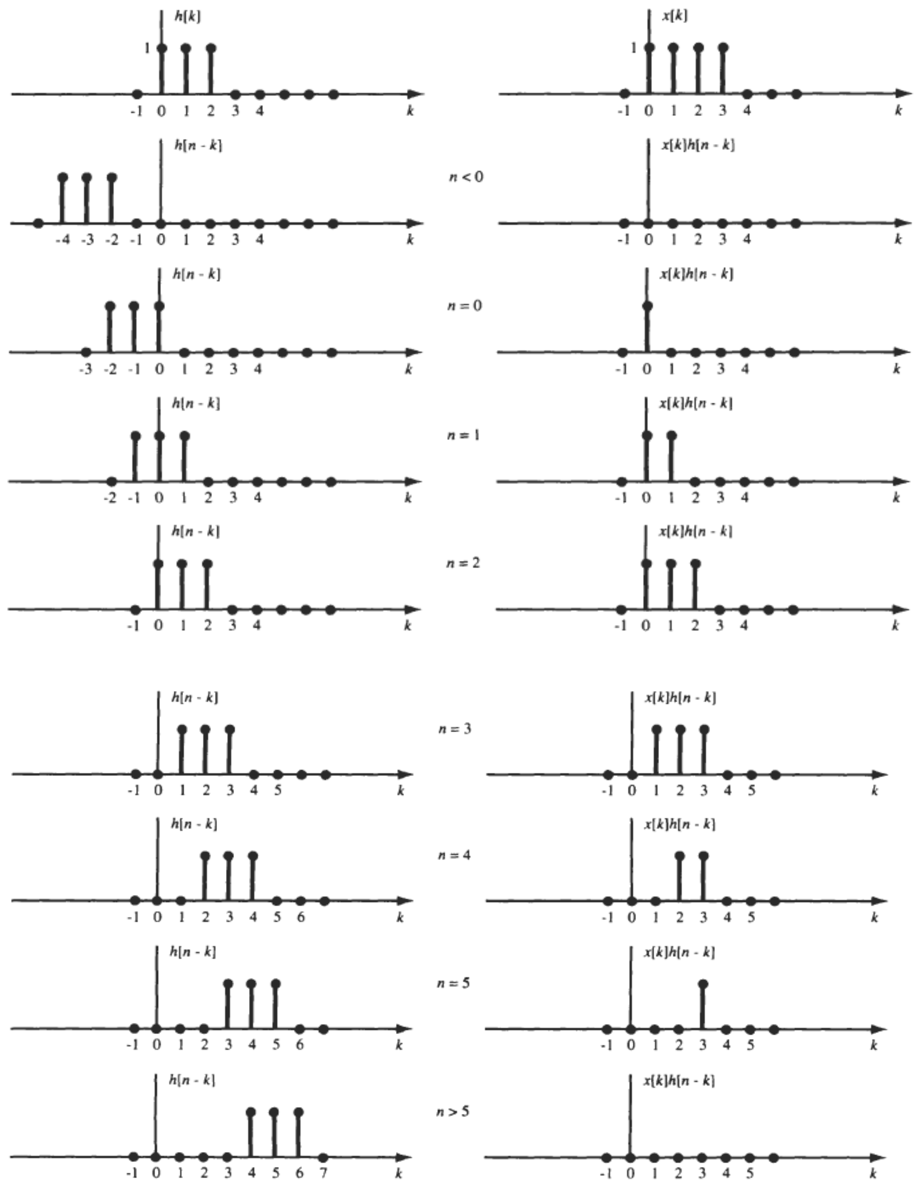
Figure 4.9: The graphical approach to convolution.

- **A useful fact to remember in performing the convolution of two finite length sequences is that if x(n) is of length L1 and h(n) is of length L2, y(n) = x(n) * h(n) will be of length: L = L1 + L2 – 1**

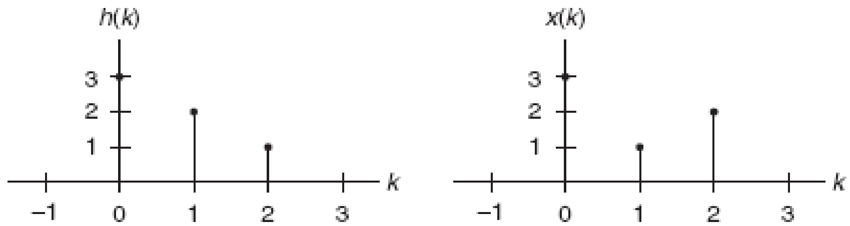
Ex 2: Evaluate *y[n]* = *x[n]* * *h[n]*, where *x[n]* and *h[n]* are shown in Fig. by a graphical method.



Sol:



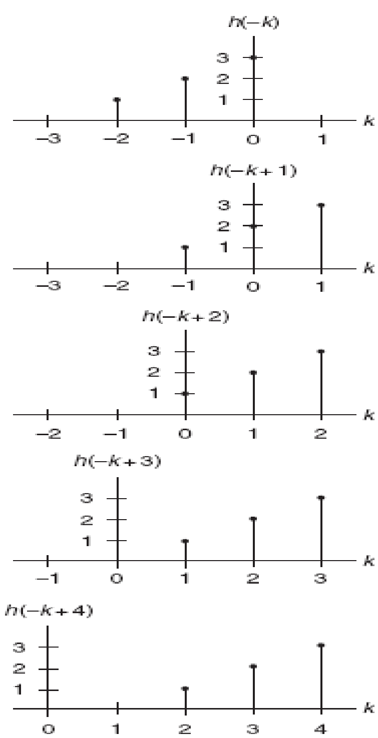
Example 3: Using the following sequences defined in Figure , evaluate the digital convolution. a. By the graphical method. b. By applying the formula directly.

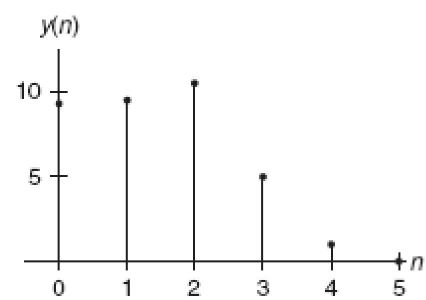


Sol:

a. To obtain $y(0)$, we need the reversed sequence $h(-k)$; and to obtain $y(1)$, we need the reversed sequence $h(1-k)$, and so on. Using the technique we have discussed, sequences $h(-k)$, $h(-k+1)$, $h(-k+2)$, $h(-k+3)$, and $h(-k+4)$ are achieved and plotted in Figure

sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$
sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$
sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$
sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$
sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$
sum of product of $x(k)$ and $h(5-k)$: $y(n) = 0$ for $n > 4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.

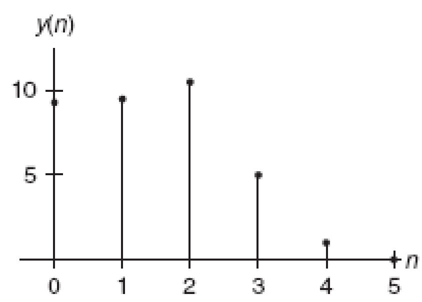




b. Applying Equation (i) with zero initial conditions leads to

y(n) = x(0)h(n) + x(1)h(n _ 1)+ x(2)h(n _ 2)

$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3\times 3 + 1\times 0 + 2\times 0 = 9,$$
$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3\times 2 + 1\times 3 + 2\times 0 = 9,$$
$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3\times 1 + 1\times 2 + 2\times 3 = 11,$$
$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3\times 0 + 1\times 1 + 2\times 2 = 5.$$
$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3\times 0 + 1\times 0 + 2\times 1 = 2,$$
$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3\times 0 + 1\times 0 + 2\times 0 = 0.$$



H.W: Using the sequence definitions

$$x(k) = \begin{cases} -2, & k = 0,1,2 \\ 1, & k = 3,4 \\ 0 & elsewhere \end{cases} \quad \text{and} \quad h(k) = \begin{cases} 2, & k = 0 \\ -1, & k = 1,2 \\ 0 & elsewhere, \end{cases}$$

evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

a. using the graphical method;

b. using the table method;

c. applying the convolution formula directly.

3.**Tabular Method**

example 3 can be solved by table method as shown in figure below

$y(0) = 9$

$y(1) = 9$

$y(2) = 11$

$y(3) = 5$

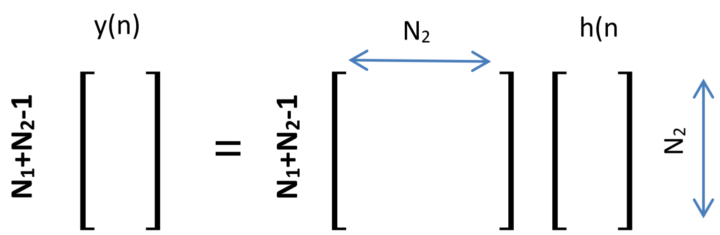
$y(4) = 2$

h(n) ↓

x(n) →

	3	2	1
3	9	6	3
1	3	2	1
2	6	4	2

4. Matrix by Vector method



Ex 4: If x(n) = [0.5 0.5 0.5], and h(n) = [3 2 1]

$$\begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.5 \\ 3 \\ 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

Ex 5: If h(n) = [1 $\underset{\uparrow}{-1}$ 2], and x(n) = [2 1 $\underset{\uparrow}{-1}$ 3]

Sol:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 2 \\ 3 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 6 \\ -5 \\ 6 \end{bmatrix}$$

- ***$O=O_1+O_2-1$=position of cursor in y(n), where O_1=cursor position in h(n) & O_2=cursor position in x(n)***

For previous example, O=2+3-1=4 ***cursor position in y(n)={2 -1 2 6 -5 6}***

5-Z-Transform Method

Y(z)=X(z) H(z)

4.4 Difference Equations and Impulse Responses

A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M), \quad \ldots b$$

Where $a_1, \ldots, a_N$ and $b_0, b_1, \ldots, b_M$ are the coefficients of the difference equation. Equation (b) can further be written as:

$$y(n) = -a_1 y(n-1) - \ldots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) \quad \ldots f$$

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j).$$

Notice that y(n) is the current output, which depends on the past output samples y(n − 1), . . . , y(n − N), the current input sample x(n), and the past input samples, x(n−1), . . . , x(n − M).

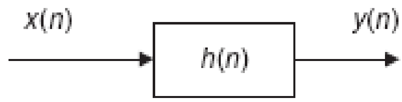
Example1: Given a linear system described by the difference equation

y(n) = x(n) + 0.5x(n − 1), Determine the nonzero system coefficients.

Solution: a. By comparing Equation (f), we have, $b_0 = 1$, and $b_1 = 0.5$

4.5 System Representation Using Its Impulse Response

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input $\delta(n)$ with zero initial conditions, depicted in Figure below Here $x(n) = \delta(n)$ and $y(n) = h(n)$.



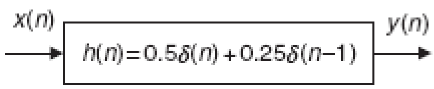
Example 2: Given the linear time-invariant system

$$y(n) = 0.5x(n) + 0.25x(n-1) \text{ with an initial condition } x(-1) = 0$$

a. Determine the unit-impulse response h(n).
b. Draw the system block diagram.
c. Write the output using the obtained impulse response.

Solution:

a. $h(n) = 0.5\ \delta(n) + 0.25\ \delta(n-1)$, where $h(0)= 0.5$, $h(1) = 0.25$ and $h(n) = 0$ elsewhere.
b.



c. $y(n) = h(0)\ x(n) + h(1)\ x(n-1)$

From this result, it is noted that if the difference equation without the past output terms, $y(n-1), \ldots, y(n-N)$, that is, the corresponding coefficients $a_1, \ldots, a_N$, are zeros, the impulse response h(n) has a finite number of terms. We call this a **finite impulse response (FIR) system**.

In general, we can express the output sequence of a linear time-invariant system from its impulse response and inputs as:

$$y(n) = \ldots + h(-1)\ x(n+1) + h(0)\ x(n) + h(1)\ x(n-1) + h(2)\ x(n-2) + \ldots \qquad (j)$$

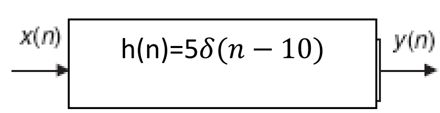
Equation (j) is called the ***digital convolution sum.***

Example 3: For each of the following linear systems, find the unit-impulse response, and draw the block diagram.

a. $y(n) = 5x(n-10)$
b. $y(n) = x(n) + 0.5x(n-1)$

Sol:

a. $h(n) = 5\ \delta(n-10)$, where $h(10) = 5$ and $h(n) = 0$ elsewhere.



H.W : Compute the convolution **y(n) = x(n) * h(n)** of the following signals

(1) $x(n) = \{1, 2, 4\}, h(n) = \{1, 1, 1, 1, 1\}$

(2) $x(n) = \{1, 2, -1\}, h(n) = x(n)$

(3) $x(n) = \{0, 1, -2, 3, -4\}, h(n) = \{\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\}$

(4) $x(n) = \{1, 2, 3, 4, 5\}, h(n) = \{1\}$

(5) $x(n) = \{\underset{\uparrow}{1}, -2, 3\}, h(n) = \{\underset{\uparrow}{0}, 0, 1, 1, 1, 1\}$

(6) $x(n) = \{\underset{\uparrow}{0}, 0, 1, 1, 1, 1\}, h(n) = \{1, \underset{\uparrow}{-2}, 3\}$

(7) $x(n) = \{\underset{\uparrow}{0}, 1, 4, -3\}, h(n) = \{\underset{\uparrow}{1}, 0, -1, -1\}$

(8) $x(n) = \{\underset{\uparrow}{1}, 1, 2\}, h(n) = u(n)$

(9) $x(n) = \{1, 1, \underset{\uparrow}{0}, 1, 1\}, h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$

(10) $x(n) = \{1, 2, \underset{\uparrow}{0}, 2, 1\} h(n) = x(n)$