# The Z-Transform And Its Application To The Analysis Of LTI Systems

**5.1 Definition of Z.T** The z-transform is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals. The z-transform of a *causal* sequence x(n), designated by X(z) or Z(x(n)), is defined as:

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n)z^{-n}$$
$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \dots \qquad (5.1)$$

Where, z is the complex variable. Here, the summation taken from n = 0 to n = ∞ is according to the fact that for most situations, the digital signal x(n) is the ***causal*** sequence, that is, x(n) = 0 for n ≤ 0. For non-causal system, the summation starts at n = -∞. Thus, the definition in Equation (5.1) is referred to as a one-sided z-transform or a unilateral transform. The region of convergence is defined based on the particular sequence x(n) being applied. The z-transforms for common sequences are summarized below:

| | Signal, $x(n)$ | z-Transform, $X(z)$ | ROC |
|---|---|---|---|
| 1 | $\delta(n)$ | $1$ | All $z$ |
| 2 | $u(n)$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3 | $a^n u(n)$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 4 | $na^n u(n)$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 5 | $-a^n u(-n-1)$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 6 | $-na^n u(-n-1)$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7 | $(\cos\omega_0 n)u(n)$ | $\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 8 | $(\sin\omega_0 n)u(n)$ | $\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 9 | $(a^n\cos\omega_0 n)u(n)$ | $\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 10 | $(a^n\sin\omega_0 n)u(n)$ | $\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$ | $\lvert z\rvert > \lvert a\rvert$ |

## 5.2 Properties of Z.T:

**1- Linearity**: The z-transform is a linear transformation, which implies

$$Z\,(a\,x_1(n) \pm b\,x_2(n)\,) = a\,X_1(Z) \pm b\,X_2(Z)$$

Where *a* and *b* are constants

**2- Shift theorem (without initial conditions):** Given X(z), the z-transform of a sequence x(n), the z-transform of x(n - m), the time-shifted sequence, is given by;

$$Z\{x(n\ -m)\} = Z^{-m}\ X(Z)$$

**3- Convolution**: Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n) \otimes x_2 = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k) = \sum_{k=-\infty}^{\infty} x_1(n-k)\,x_2(k)$$

Where $\otimes$ designates the linear convolution. In z-transform domain, we have

$$X(Z) = X_1(Z)\,.\,X_2(Z)$$

**4- Multiplication by exponential:**

$$Z\,\{a^n\ x(n)\} = X(Z)\ \Big|_{Z \to \frac{Z}{a}}$$

$$Z\{e^{\pm an}\,x(n)\} = X(Z)\,\Big|_{Z \to e^{\mp a}\,Z}$$

**5- Initial and final value theorems:**

$$\lim_{n \to 0} x(n) = \lim_{Z \to \infty} X(Z) = x(0) \quad \textit{initial value theorem}$$

$$\lim_{n \to \infty} x(n) = \lim_{Z \to 1} Z^{-1}\ (Z-1)\ \ X(Z) \quad \textit{final value theorem}$$

**6- Multiplication by n:**

$$Z\,\{n\ x(n)\} = -\,Z\,\frac{d}{dZ}\,X(Z)$$

Example 1:A finite sequence ***x [ n ]*** is defined as: $x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$

sol:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-2}^{3} x[n] z^{-n}$$

$$= x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$$= 5z^{2} + 3z - 2 + 4z^{-2} - 3z^{-3}$$

Example 2: Find the z-transform ***X (z)*** for each of the following sequences:

$$(a) \quad x[n] = (\tfrac{1}{2})^n u[n] + (\tfrac{1}{3})^n u[n]$$

$$(b) \quad x[n] = (\tfrac{1}{3})^n u[n] + (\tfrac{1}{2})^n u[-n-1]$$

Sol: (a)

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{z}{z - \frac{1}{2}}$$

$$\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{z}{z - \frac{1}{3}} \quad \longrightarrow \quad X(z) = \frac{z}{z - \frac{1}{2}} + \frac{z}{z - \frac{1}{3}} = \frac{2z\left(z - \frac{5}{12}\right)}{\left(s - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

(b)

$$\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{z}{z - \frac{1}{3}} \qquad |z| > \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[-n-1] \longleftrightarrow -\frac{z}{z - \frac{1}{2}} \qquad |z| < \frac{1}{2}$$

$$\longrightarrow \quad X(z) = \frac{z}{z - \frac{1}{3}} - \frac{z}{z - \frac{1}{2}} = -\frac{1}{6} \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

Example 3: Compute the convolution *x(n)* of the signals by using Z-transform

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

Sol:

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

$$x(n) = \{\underset{\uparrow}{1},$$

Example 4: Find $Z\{(n-2)\, a^{(n-2)} \cos[w(n-2)]\, u(n-2)$

The solution is:

$$= Z^{-2}\; Z\{ n\, a^{n} \cos wn\; u(n) \}$$

$$= Z^{-2}\; (-Z)\, \frac{d}{dZ}\, Z\{ a^{n} \cos wn\; u(n) \}$$

$$= -Z^{-1}\, \frac{d}{dZ}\, \frac{Z^2 - Z\cos w}{Z^2 - 2\,Z\cos w + 1}\,\Bigg|_{Z \to \frac{Z}{a}}$$

H.W: Compute the convolution of the following pair of signals by using the z-transform.

**(a)** $x_1(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$, $\quad x_2(n) = \{\underset{\uparrow}{1}, 1, 1\}$

**(b)** $x_1(n) = (\frac{1}{2})^n u(n)$, $\quad x_2(n) = (\frac{1}{3})^n u(n)$

**(c)** $x_1(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$, $\quad x_2(n) = \{4, 3, \underset{\uparrow}{2}, 1\}$

**(d)** $x_1(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$, $\quad x_2(n) = \{\underset{\uparrow}{1}, 1, 1\}$

## 5.3 Inverse of Z.T

$$x(n) = Z^{-1}\,\{X(Z)\}$$

**The inverse z-transform may be obtained by the following methods:**

1. Using properties.
2. Partial fraction expansion method.

3. Residue method.
4. Power series expansion (the solution is obtained by applying long division because the

denominator can't be analyzed. It is not accurate method compared with the above three methods)

**Example1**: Find x(n), using properties , if

$$X(z) = \frac{10z}{z^2 - z + 1}$$

**Solution:**

Since $X(z) = \frac{10z}{z^2 - z + 1} = \left(\frac{10}{\sin(a)}\right)\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}$,
by coefficient matching, we have

$$-2\cos(a) = -1.$$

Hence, $\cos(a) = 0.5$, and $a = 60^\circ$. Substituting $a = 60^\circ$ into the sine function leads to

$$\sin(a) = \sin(60^\circ) = 0.866.$$

Finally, we have

$$x(n) = \frac{10}{\sin(a)} Z^{-1}\left(\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \frac{10}{0.866}\sin(n \cdot 60^0)$$
$$= 11.547\sin(n \cdot 60^0).$$

**<u>Example(2):</u>** Find x(n) using partial fraction method , if:

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

**Solution:**

Eliminating the negative power of $z$ by multiplying the numerator and denominator by $z^2$ yields

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})}.$$
$$= \frac{z^2}{(z - 1)(z - 0.5)}$$

Dividing both sides by $z$ leads to

$$\frac{X(z)}{z} = \frac{z}{(z - 1)(z - 0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z - 1)} + \frac{B}{(z - 0.5)}.$$

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying $z$ on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

$$x(n) = 2u(n) - (0.5)^n u(n).$$

H.W: Determine the signal *x(n)* with z-transform $X(z) = \dfrac{3}{1 - \frac{10}{3}z^{-1} + z^{-2}}$

**Example 3:**

The output $y[n]$ of a discrete-time LTI system is found to be $2(\frac{1}{3})^n u[n]$ when the input $x[n]$ is $u[n]$.

(*a*) Find the impulse response $h[n]$ of the system.

(*b*) Find the output $y[n]$ when the input $x[n]$ is $(\frac{1}{2})^n u[n]$.

(*a*)

$$x[n] = u[n] \longleftrightarrow X(z) = \frac{z}{z-1} \qquad |z| > 1$$

$$y[n] = 2\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow Y(z) = \frac{2z}{z - \frac{1}{3}} \qquad |z| > \frac{1}{3}$$

Hence, the system function $H(z)$ is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(z-1)}{z - \frac{1}{3}} \qquad |z| > \frac{1}{3}$$

Using partial-fraction expansion, we have

$$\frac{H(z)}{z} = \frac{2(z-1)}{z(z - \frac{1}{3})} = \frac{c_1}{z} + \frac{c_2}{z - \frac{1}{3}}$$

where $c_1 = \dfrac{2(z-1)}{z - \frac{1}{3}}\bigg|_{z=0} = 6 \qquad c_2 = \dfrac{2(z-1)}{z}\bigg|_{z=1/3} = -4$

Thus,

$$H(z) = 6 - 4\frac{z}{z-\frac{1}{3}} \qquad |z| > \frac{1}{3}$$

Taking the inverse $z$-transform of $H(z)$, we obtain

$$h[n] = 6\delta[n] - 4\left(\tfrac{1}{3}\right)^n u[n]$$

(b) $$x[n] = \left(\frac{1}{2}\right)^n u[n] \longleftrightarrow X(z) = \frac{z}{z-\frac{1}{2}} \qquad |z| > \frac{1}{2}$$

Then, $$Y(z) = X(z)H(z) = \frac{2z(z-1)}{(z-\frac{1}{2})(z-\frac{1}{3})} \qquad |z| > \frac{1}{2}$$

Again by partial-fraction expansion we have

$$\frac{Y(z)}{z} = \frac{2(z-1)}{(z-\frac{1}{2})(z-\frac{1}{3})} = \frac{c_1}{z-\frac{1}{2}} + \frac{c_2}{z-\frac{1}{3}}$$

where $$c_1 = \frac{2(z-1)}{z-\frac{1}{3}}\bigg|_{z=1/2} = -6 \qquad c_2 = \frac{2(z-1)}{z-\frac{1}{2}}\bigg|_{z=1/3} = 8$$

Thus,

$$Y(z) = -6\frac{z}{z-\frac{1}{2}} + 8\frac{z}{z-\frac{1}{3}} \qquad |z| > \frac{1}{2}$$

Taking the inverse $z$-transform of $Y(z)$, we obtain

$$y[n] = \left[-6\left(\tfrac{1}{2}\right)^n + 8\left(\tfrac{1}{3}\right)^n\right]u[n]$$

H.W: Consider a discrete-time LTI system whose system function ***H(z)*** is given **by**

$$H(z) = \frac{z}{z-\frac{1}{2}} \qquad |z| > \frac{1}{2}$$

Find the output $y[n]$ to the input $x[n] = nu[n]$.