Frequency Response Of Linear Time-Invariant Systems

In this section we develop the characterization of linear time-invariant systems in the frequency domain . The basic excitation signals in this development are the complex exponentials and sinusoidal functions. The characteristics of the system are described by a function of the frequency variable *co* called the frequency response, which is the Fourier transform of the impulse response *h(n)* of the system . The frequency response function completely characterizes a linear time invariant system in the frequency domain .

6.1 Response to Complex Exponential and Sinusoidal Signals: The Frequency Response Function

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

In this input–output relationship, the system is characterized in the time domain by its unit sample response $\{h(n), -\infty < n < \infty\}$.

To develop a frequency-domain characterization of the system, let us excite the system with the complex exponential

$$x(n) = Ae^{j\omega n}, \qquad -\infty < n < \infty$$

By substituting x(n) in to y(n) we obtain the response

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)[Ae^{j\omega(n-k)}]$$

$$= A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n}$$

In fact, this term is the Fourier transform of the unit sample response *h{k)* of the system . Hence we denote this function as

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

With the definition in above equation, the response of the system to the complex exponential will be

$$y(n) = AH(\omega)e^{j\omega n}$$

Example 1:Determine the output sequence of the system with impulse response

$$h(n) = (\frac{1}{2})^n u(n)$$

when the input is the complex exponential sequence

$$x(n) = Ae^{j\pi n/2}, \qquad -\infty < n < \infty$$

Solution:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

At $\omega = \pi/2$ $\longrightarrow$ $H\left(\frac{\pi}{2}\right) = \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}}e^{-j26.6^\circ}$

and therefore the output is

$$y(n) = A\left(\frac{2}{\sqrt{5}}e^{-j26.6^\circ}\right)e^{j\pi n/2}$$

$$y(n) = \frac{2}{\sqrt{5}}Ae^{j(\pi n/2 - 26.6^\circ)}, \qquad -\infty < n < \infty$$

This example clearly illustrates that the only effect of the system on the input signal is to scale the amplitude by $2/\sqrt{5}$ and shift the phase by -26.6°. Thus the output is also a complex exponential of frequency $\pi/2$, amplitude $2A/\sqrt{5}$, and phase -26.6°.

In general, $H(\omega)$ is a complex-valued function of the frequency variable ω. Hence it can be expressed in polar form as

$$H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$$

where $|H(\omega)|$ is the magnitude of $H(\omega)$ and

$$\Theta(\omega) = \measuredangle H(\omega)$$

which is the phase shift imparted on the input signal by the system at the frequency ω.

For a linear time-invariant system with a real-valued impulse response, the magnitude and phase functions possess symmetry properties which are developed as follows. From the definition of $H(\omega)$, we have

$$\begin{aligned} H(\omega) &= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \\ &= \sum_{k=-\infty}^{\infty} h(k)\cos\omega k - j\sum_{k=-\infty}^{\infty} h(k)\sin\omega k \\ &= H_R(\omega) + jH_I(\omega) \\ &= \sqrt{H_R^2(\omega) + H_I^2(\omega)}\, e^{j\tan^{-1}[H_I(\omega)/H_R(\omega)]} \end{aligned}$$

where $H_R(\omega)$ and $H_I(\omega)$ denote the real and imaginary components of $H(\omega)$

Hence, the magnitude and phase of H(w) can be expressed in terms of H_R(w) and H_I(w) as:

$$H_R(\omega) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k$$

$$H_I(\omega) = -\sum_{k=-\infty}^{\infty} h(k)\sin\omega k$$

The symmetry properties satisfied by the magnitude and phase functions of $H(\omega)$, and the fact that a sinusoid can be expressed as a sum or difference of two complex-conjugate exponential functions, imply that the response of a linear time-invariant system to a sinusoid is similar in form to the response when the inp .t is a complex exponential. Indeed, if the input is

$$x_1(n) = Ae^{j\omega n}$$

the output is

$$y_1(n) = A|H(\omega)|e^{j\Theta(\omega)}e^{j\omega n}$$

On the other hand, if the input is

$$x_2(n) = Ae^{-j\omega n}$$

the response of the system is

$$y_2(n) = A|H(-\omega)|e^{j\Theta(-\omega)}e^{-j\omega n}$$
$$= A|H(\omega)|e^{-j\Theta(\omega)}e^{-j\omega n}$$

where, in the last expression, we have made use of the symmetry properties $|H(\omega)| = |H(-\omega)|$ and $\Theta(\omega) = -\Theta(-\omega)$. Now, by applying the superposition property of the linear time-invariant system, we find that the response of the system to the input

$$x(n) = \frac{1}{2}[x_1(n) + x_2(n)] = A\cos\omega n$$

is

$$y(n) = \frac{1}{2}[y_1(n) + y_2(n)]$$
$$y(n) = A|H(\omega)|\cos[\omega n + \Theta(\omega)]$$

Similarly, if the input is

$$x(n) = \frac{1}{j2}[x_1(n) - x_2(n)] = A\sin\omega n$$

the response of the system is

$$y(n) = \frac{1}{j2}[y_1(n) - y_2(n)]$$
$$y(n) = A|H(\omega)|\sin[\omega n + \Theta(\omega)]$$

It is apparent from this discussion that $H(\omega)$, or equivalently, $|H(\omega)|$ and $\Theta(\omega)$, completely characterize the effect of the system on a sinusoidal input signal of any arbitrary frequency. Indeed, we note that $|H(\omega)|$ determines the amplification ($|H(\omega)| > 1$) or attenuation ($|H(\omega)| < 1$) imparted by the system on the input sinusoid. The phase $\Theta(\omega)$ determines the amount of phase shift imparted by the system on the input sinusoid. Consequently, by knowing $H(\omega)$, we are able to determine the response of the system to any sinusoidal input signal. Sinc $H(\omega)$ specifies the response of the system in the frequency domain, it is called the *frequency response* of the system. Correspondingly, $|H(\omega)|$ is called the *magnitude response* and $\Theta(\omega)$ is called the *phase response* of the system.

If the input to the system consists of more than one sinusoid, the superposition property of the linear system can be used to determine the response. The following examples illustrate the use of the superposition property.

Example 2:Determine the response of the system in example 1 to input signal

$$x(n) = 10 - 5\sin\frac{\pi}{2}n + 20\cos\pi n, \qquad -\infty < n < \infty$$

Solution: the frequency response of the system is:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

The first term in the input signal is a fixed signal component corresponding to $\omega = 0$. Thu

$$H(0) = \frac{1}{1 - \frac{1}{2}} = 2$$

The second term in $x(n)$ has a frequency $\pi/2$. At this frequency the frequency response of the system is

$$H\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{5}}e^{-j26.6^\circ}$$

Finally, the third term in $x(n)$ has a frequency $\omega = \pi$. At this frequency

$$H(\pi) = \frac{2}{3}$$

Hence the response of the system to $x(n)$ is

$$y(n) = 20 - \frac{10}{\sqrt{5}}\sin\left(\frac{\pi}{2}n - 26.6^\circ\right) + \frac{40}{3}\cos\pi n, \qquad -\infty < n < \infty$$