Electric Circuits
Comm. Dep. /2^{nd} stage
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"1^{st} order cct"

"First order Circuits"

1 - The Source-free RC circuit

Consider a series combination of a resistor and an initially charged capacitor, as shown in figure. (1)

assume that at $t=0$

$$v(0) = V_0$$

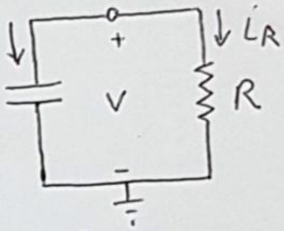


fig (1)

the energy stored in the capacitor

$$\omega(0) = \frac{1}{2} C V_0^2$$

Apply KCL

$$i_c + i_R = 0$$

by definition $i_c = C\frac{dv}{dt}$ & $i_R = \frac{v}{R}$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

this is a first-order differential equation, we rearrange the item as

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

$$\int \frac{dv}{v} = \int -\frac{1}{R}dt$$

$$\ln v = -\frac{1}{RC} + \ln A$$

where $\ln A$ is the integration constant, thus

$$\ln v - \ln A = \ln \frac{v}{A} = -\frac{1}{RC}$$

taking powers of e produces

$$v(t) = A e^{-\frac{t}{RC}}$$

but the initial condition $v(0) = A = V_0$, hence.

$$v(t) = V_0 e^{-\frac{t}{RC}}$$

$$v(t) = V_0 e^{-\frac{t}{\tau}}$$

where $RC = \tau$

and

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{\tau}}$$

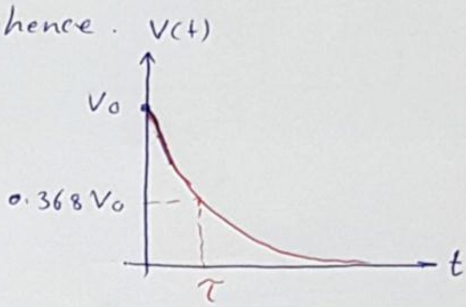
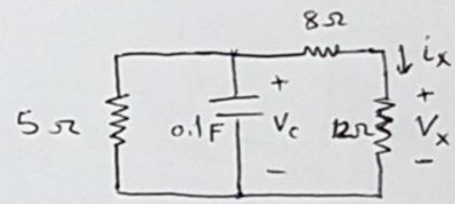


fig (2)

Ex1:- in fig(3) let $V_{c(0)} = 15\,V$; find v_c, v_x and i_x for $t > 0$

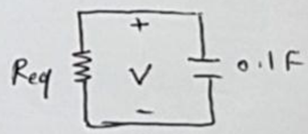
solution:-

① we need to make the circuit shown in fig(3) conform with the standered RC circuit in fig(1)



fig(3)

⇓



fig(4)

fig(4) represent the equivent cct. in fig(3)

$$\Rightarrow R_{eq} = (8 + 12) // 5$$

$$= \frac{20 \times 5}{20 + 5} = 4\,\Omega$$

$$\tau = R_{eq} \cdot C = 4 \times 0.1 = 0.4\,s$$

thus

$$v(t) = v_0 e^{-\frac{t}{\tau}} = 15 e^{-\frac{t}{0.4}} \Rightarrow V = v_c = 15 e^{-2.5t}\ V$$

by voltage division

$$v_x(t) = \frac{12}{12+8} v = 0.6(15 e^{-2.5t}) = 9 e^{-2.5t}\ V$$

$$i_x = \frac{v_x}{12} = \frac{9}{12} e^{-2.5t}\ A = 0.75 e^{-2.5t}\ A$$

Ex2:- for the fig. (5), the switch has been closed for a long time, and open at $t=0$. Find $v(t)$ for $t \geq 0$, calculate stored energy in the capacitor.

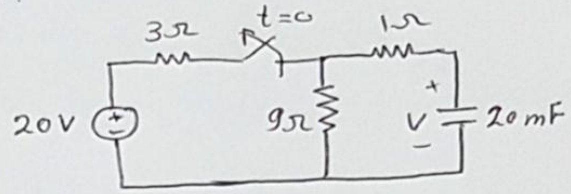


fig (5)

Solution:-

Now, there are two kind of Solution.

① for $t<0$, the switch is closed

$\Rightarrow$ the capacitor is open cct. to dc, fig(6) is illustrate this principle by voltage division the voltage a cross $R = 9\Omega$ represent $V_c(0)$

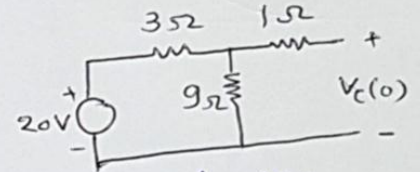


fig (6)

$$V_c(0) = \frac{9}{9+3}(20) = 15 \text{ V}$$

$$V_c(0) = V_o = 15 \text{ V}$$

② for $t>0$ the switch is open, and the circuit in fig (6) becomes as shown in fig(7)

$\Rightarrow$ Now, the Voltage a cross capacitor will become the source to the cct.

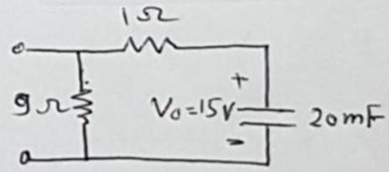


fig (7)

$$R_{eq} = 1 + 9 = 10\,\Omega$$

$$\tau = R_{eq} \cdot C = 10 * 20 * 10^{-3} = 0.2 \text{ S}$$

$$\Rightarrow V(t) = V_c(0)\, e^{-\frac{t}{\tau}} = 15\, e^{-\frac{t}{0.2}} \text{ V} = 15\, e^{-5t}$$

$$W_c(0) = \frac{1}{2} C V_c^2 = \frac{1}{2}\, 20 * 10^{-3} * 15^2 = 2.25 \text{ J}$$

2- The sourse- Free RL circuit.

At $t=0$ we assume that the inductor has an intial current I_o, or

$$i(0) = I_o$$

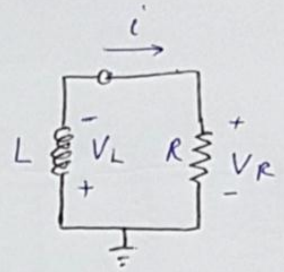


fig (8)

energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_o^2$$

Apply KVL

$$V_L + V_R = 0$$

$$L \frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0$$

Rearranging terms and integrating gives

$$\int_{I_o}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\ln i \Big|_{I_o}^{i(t)} = -\frac{Rt}{L} \Big|_0^t \Rightarrow \ln i(t) - \ln I_o = -\frac{Rt}{L} + 0$$

$$\ln \frac{i(t)}{I_o} = -\frac{R}{L} t$$

$$\Rightarrow i(t) = I_o e^{-\frac{Rt}{L}}$$

$$\Rightarrow i(t) = I_o e^{-\frac{t}{\tau}} \quad \text{where } \tau = \frac{L}{R}$$

$$v(t) = i(t) \cdot R = I_o R e^{-\frac{t}{\tau}}$$

$$P = v_R i = I_o^2 R e^{-\frac{2t}{\tau}}$$

$$w_R(t) = \int_0^t p\, dt = \int_0^t I_o^2 R e^{-\frac{2t}{\tau}} = -\frac{1}{2} \tau I_o^2 R e^{-\frac{2t}{\tau}} \Big|_0^t$$

$$= \frac{1}{2} L I_o^2 (1 - e^{-\frac{2t}{\tau}})$$

Ex 1:- The Switch in the circuit shown in fig (10) has been closed for along time. At $t=0$ the Switch is open

Calculate $i(t)$ for $t>0$

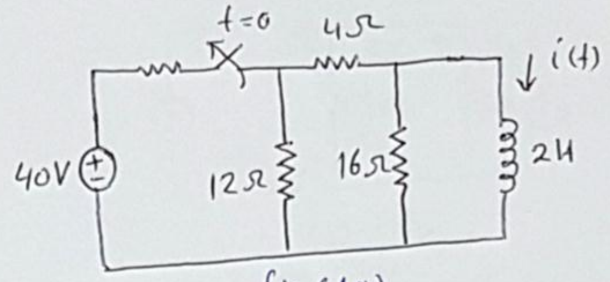


fig (10)

Solution

for $t<0$

the circuit as shown in fig (11)

the inductor became short

and from this, you can calculate intial current I_o

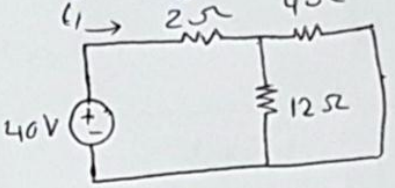


fig (11)

$$R_{eq} = 2 + 4 // 12$$

$$= 2 + \frac{4 \times 12}{4+12} = 5\,\Omega$$

$$i_1 = \frac{40}{5} = 8\text{ A}$$

and the current $i(t)$ by using curren division.

$$i(t) = \frac{12}{12+4} i_1 = 6\text{A} \quad t<0$$

$$i(0) = 6\text{ A}.$$

for $t>0$ (Switch open)

the circuit as shown in fig (12)

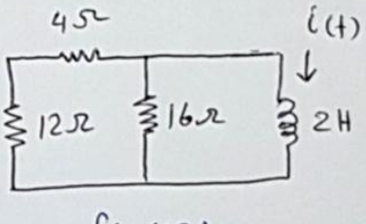


fig (12)

$$R_{eq} = (12+4) // 16 = 8\,\Omega$$

$$\Rightarrow \tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4} = 0.25\text{ S}$$

$$i(t) = i(0)\, e^{-\frac{t}{\tau}} = 6\, e^{-4t}$$

Ex2:- In the circuit shown in fig. (13) find i_o, V_o and i for all time. Assuming that the Swiching was open for a long time

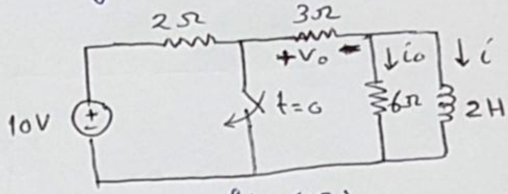


fig. (13)

Solution:-

for $t<0$ (switch is open)

fig (14) illustrate for $t<0$

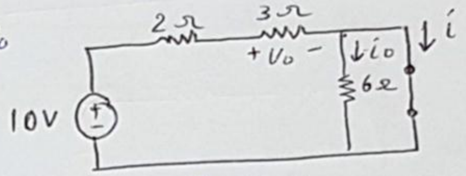


fig (14)

from this circut we find I_o and V_o

resistor (6) became short and

$$i(t) = \frac{10}{2+3} = 2\ A = I_o$$

$$v_o(t) = 3\, i(t) = 6V$$

for $t>0$ (switch is closed)

fig (15) illustrate for $t>0$

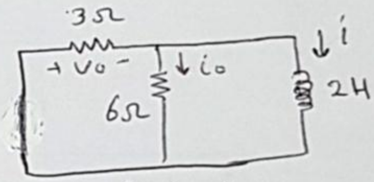


fig (15)

$$R_{th} = 3 // 6 = \frac{3\times 6}{3+6} = 2\ \Omega$$

$$\tau = \frac{L}{R_{th}} = \frac{2}{2} = 1$$

$$i(t) = i(0)\, e^{-t/\tau} = 2e^{-t}\ A$$

$$V_o = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ V$$

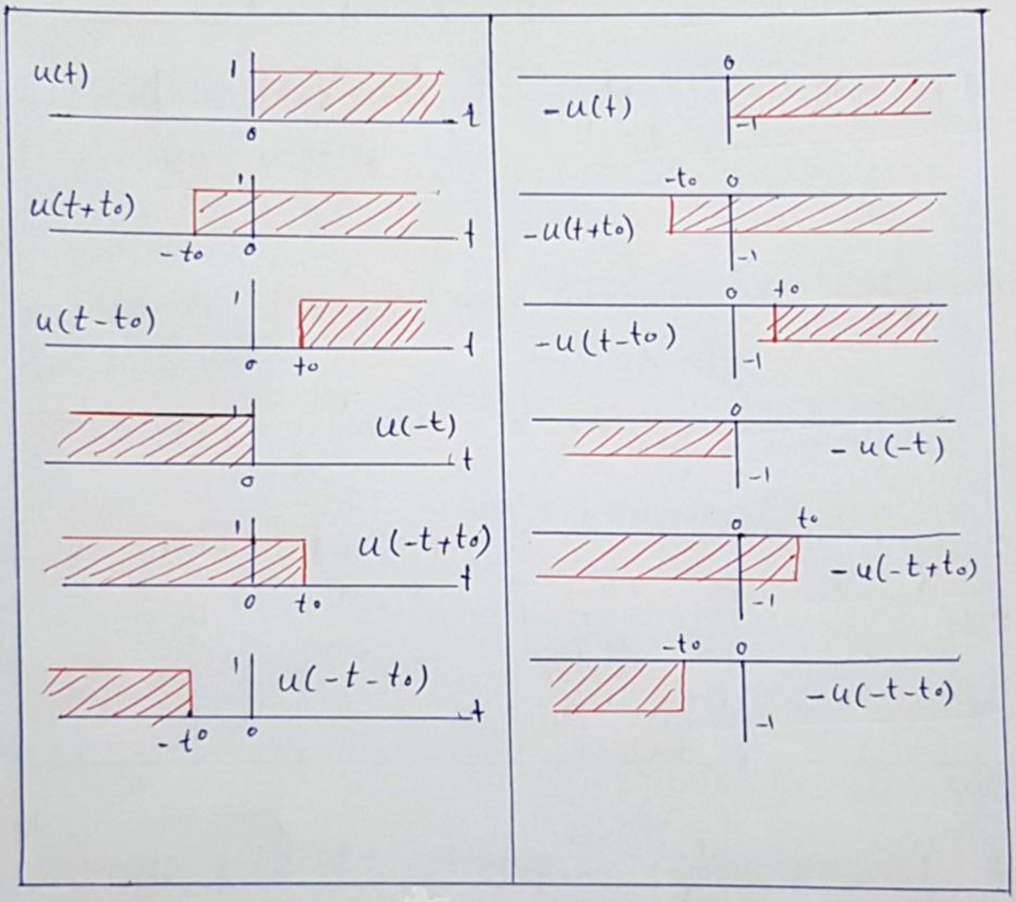
$$i_o(t) = \frac{V}{L} = -\frac{2}{3}e^{-t}\ A$$

3- Singularity functions

The Three most widly used singulavity functions in Circuit analysis are unit step ; unit impulse and unit ramp functions

① unit step function

in mathematical terms $u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$



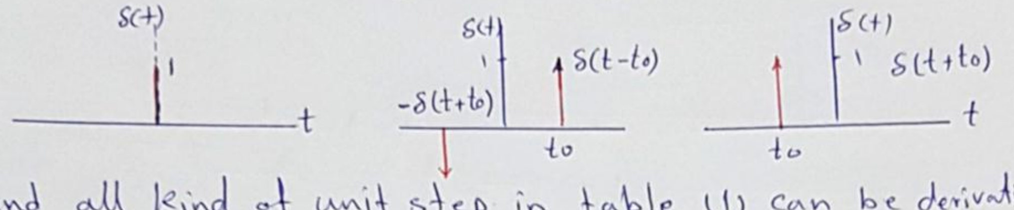
this tabe represent all type of unit step.

table (1)

② unit impulse

The derivative of a unit step function $u(t)$ is the unit impulse function $\delta(t)$ which we write as.

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & t<0 \\ \text{undefined} & t=0 \\ 0 & t>0 \end{cases}$$



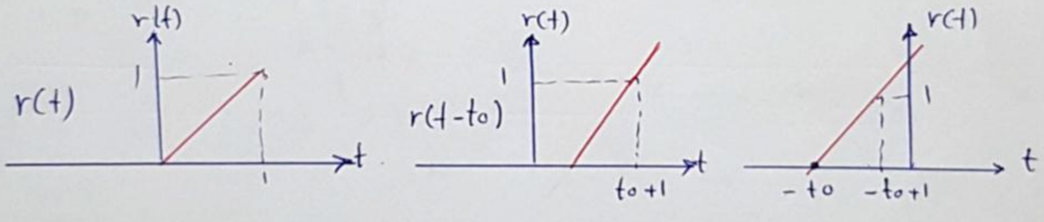
and all kind of unit step in table (1) can be derivative to get unit impulse.

③ unit ramp.

The integrate of a unit step function $u(t)$ is the unit ramp function $r(t)$

$$r(t) = \int_{-\infty}^{t} u(t)dt = t\,u(t)$$

$$r(t) = \begin{cases} 0 & t\leq 0 \\ t & t\geq 0 \end{cases}$$



and all kind of unit step in table (1) can be integrate to get unit ramp.