6- RLC circuit.

The series RLC circuit shown in fig (24) contains no sourse & KVL for closed Loop when switch is closed at $t=0$

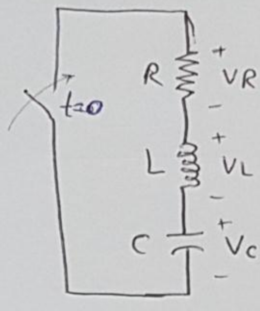


fig (24)

$$v_R + v_L + v_C = 0$$

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0$$

differentiating and dividing by L

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

It is important that this same (homogeneous D.E)

its like of this expression

$$i'' + \frac{R}{L}i' + \frac{1}{LC}i = 0$$

then the solution of this D.E is

$$i = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$A_1 e^{s_1 t}\left(s_1^2 + \frac{R}{L}s_1 + \frac{1}{LC}\right) + A_2 e^{s_2 t}\left(s_2^2 + \frac{R}{L}s_2 + \frac{1}{LC}\right) = 0$$

that is if S_1 and S_2 are the two roots of $\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)$

$$\Rightarrow \quad S_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \beta$$

$$S_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \beta$$

where $\alpha = R/2L$; $\omega_o = \frac{1}{\sqrt{LC}}$; $\beta = \sqrt{\alpha^2 - \omega_o^2}$

Now there are three cases

① overdamped case ($\alpha > \omega_0$)

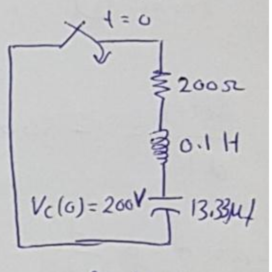
② critically damped case ($\alpha = \omega_0$)

③ underdamped case ($\alpha < \omega_0$)

Now we will discuss all of these cases

① overdamped case ($\alpha > \omega_0$)

here $i = e^{-\alpha t}(A_1 e^{\beta t} + A_2 e^{-\beta t})$ where α and β are real positive numbers



fig(25)

Ex:- the circuit as shown in fig(25) obtain the current transiend if switch is closed at $t=0$

Solution

for this circuit.

$$\alpha = \frac{R}{2L} = 10^3 \; s^{-1}$$

$$\omega_0^2 = \frac{1}{LC} = 7.5\times10^5 \; s^{-2}$$

$$\beta = \sqrt{\alpha^2 - \omega_0^2} = 500 \; s^{-1}$$

$$i = e^{-1000t}(A_1 e^{500t} + A_2 e^{-500t})$$

to find the constant A_1 & A_2

KVL at $t=0$

$$\cancel{(0)R}^{\text{zero}} + L\frac{di}{dt} + V_c(0) = 0$$

$$\frac{di}{dt} = \frac{-V_c(0)}{L} = \frac{-200}{0.1} = -2000 \; A/s$$

at $t=0$

$$i = A_1 + A_2 = 0$$

and $\frac{di}{dt} = -2000 = -500A_1 - 1500A_2 \Rightarrow A_1 = -2$ & $A_2 = 2$

$$\Rightarrow \quad i = -2e^{-500t} + 2e^{-1500t}$$

(17)

⑫ critically damped case ($\alpha = \omega_0$)

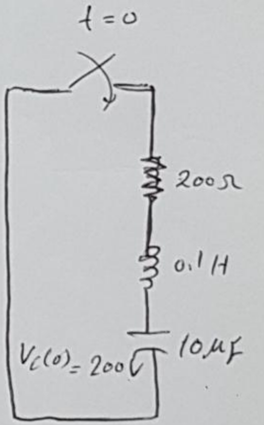
when $\alpha = \omega_0$

$\Rightarrow \beta = 0$ i.e $S_1 = S_2$

the solution of D.E becomes

$$i = e^{-\alpha t}(A_1 + A_2 t)$$

Ex1 :- the same circuit as shown in fig (25) but the value of $C = 10\,\mu F$



$$\alpha = \frac{R}{2L} = 10^3\ s^{-1} \qquad \omega_0 = \frac{1}{LC} = 10^3\ s^{-2}$$

$\beta = 0$

$$i = e^{-1000t}(A_1 + A_2 t)$$

$i(0) = 0 \qquad \left.\frac{di}{dt}\right|_0 = -2000\ A/s$

$i(0) = 0 \Rightarrow A_1 = 0$

and $A_2 = -2000$

$$i(t) = -2000\,t\,e^{-1000t}\ A$$

$i(0) = 0 = A_1 \Rightarrow A_1 = 0$

$$\frac{di}{dt} = -2000 = -1000\,e^{-1000t}(A_1 + A_2 t)$$

at $t = 0$

$$\Rightarrow -2000 = -1000\,e^{-1000t} \times 0 + \underbrace{A_2 t \times (-1000)\,e^{-1000t}}_{zero} + e^{-1000t} \cdot A_2$$

$-2000 = A_2$

$$i(t) = -2000\,t\,e^{-1000t}$$

(18)

③ Under-damped case ($\alpha < \omega_0$)

In this case β is a pure imaginary part $\beta = j|\beta|$
then the solution of D.E becomes

$$i = e^{-\alpha t}(A_1 \cos|\beta|t + A_2 \sin|\beta|t)$$

OR

$$i = e^{-\alpha t} A_3 \sin(|\beta|t + \phi)$$

wher A_3 and ϕ are two new constant.

<u>EX</u>- the same circuit as shown in fig (25) but the value of $C = 1\,\mu f$

$$i(0) = 0 \qquad \left.\frac{di}{dt}\right|_0 = -2000$$

$$\alpha = 1000\ s^{-1}$$

$$\beta = \sqrt{10^6 - 10^7} = j\,3000$$

$$i = e^{-1000t} A_3 \sin(3000t + \phi)$$

when $t = 0$ $\quad i = 0$ and $\phi = 0$ and $A_3 = -0.667$ A

$$i = -0.667\, e^{-1000t} \sin 3000t\ (A)$$

<u>hint</u>:- $\quad i = e^{-\alpha t}(A_1 \cos|\beta|t + A_2 \sin|\beta|t)$ --- ①

$\quad i = A_3 \sin(\beta t + \phi)$

by derivative of eq. ①

$$= e^{-\alpha t}(k_1 e^{j\beta t} + k_2 e^{-j\beta t})$$

$$= e^{-\alpha t}(k_1(\cos\beta t + j\sin\beta t) + k_2(\cos\beta t - j\sin\beta t))$$

$$= e^{-\alpha t}((k_1+k_2)\cos\beta t + j(k_1-k_2)\sin\omega t)$$

$$= e^{-\alpha t}(A_1\cos|\beta|t + A_2\sin|\beta|t) \quad ; \quad A_1 = k_1+k_2 \quad , \quad A_2 = j(k_1-k_2)$$

(19)

7 - Two mesh circuit :-

from fig (26) we get

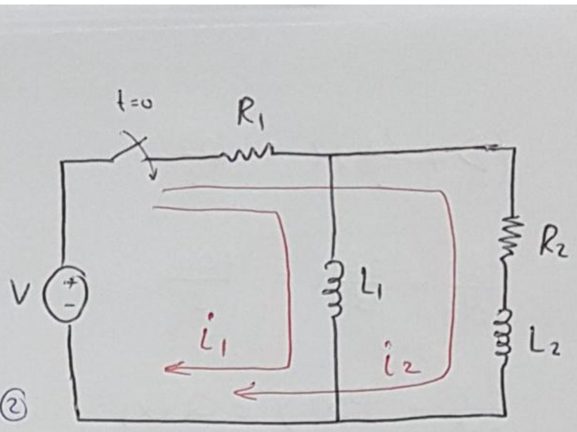


fig (26)

$$R_1 i_1 + L_1 \frac{di_1}{dt} + R_1 i_2 = V \quad \text{---} (1)$$

$$R_1 i_1 + (R_1 + R_2) i_2 + L_2 \frac{di_2}{dt} = V \quad \text{---} (2)$$

derivative eq-(1) we get

$$R_1 \frac{di_1}{dt} + L_1 \frac{d^2 i_1}{dt^2} + R_1 \frac{di_2}{dt} = 0 \quad \text{---} (3)$$

from eq. (2)

$$\frac{di_2}{dt} = \frac{V - (R_1 i_1 + (R_1 + R_2) i_2)}{L_2} \quad \text{---} (4)$$

from eq. (1)

$$i_2 = \frac{V - (R_1 i_1 + L_1 \frac{di_1}{dt})}{R_1} \quad \text{---} (5)$$

sub. eq.(4) & eq.(5) in to eq. (3) and rearrengment, we get

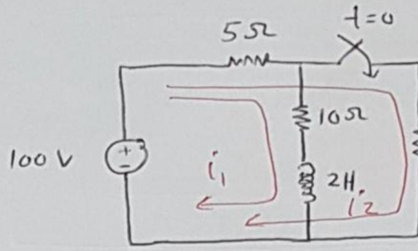
$$\frac{d^2 i_1}{dt} + \left(\frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2} \right) \frac{di_1}{dt} + \frac{R_1 R_2}{L_1 L_2} i_1 = \frac{V R_2}{L_1 L_2} \quad \text{---} (6)$$

the steady-state Solution of eq.(6) ; $i_1(\infty) = \frac{V}{R_1}$

$$S^2 + \left(\frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2} \right) S + \frac{R_1 R_2}{L_1 L_2} = 0$$

$$i(0) = 0 \quad ; \quad \frac{di_1}{dt} = \frac{V}{L_1}$$

Ex1- write the D.E for circuit (27) and solve for i_1 and i_2 when the switch is closed at $t=0$



fig(27)

$$15i_1 + 2\frac{di_1}{dt} + 5i_2 = 100 \quad \text{—— ①}$$

$$5i_1 + 15i_2 = 100 \quad \text{—— ②}$$

from eq. ② $\quad i_2 = \dfrac{100-5i_1}{15} = \dfrac{20-i}{3}$ —— ③

sub eq. ③ in ① we get

$$15i_1 + 2\frac{di_1}{dt} + \frac{100-5i_1}{3} = 100$$

$$45i_1 + 6\frac{di_1}{dt} + 100 - 5i_1 = 300$$

$$\frac{6\,di_1}{dt} + 40i_1 = 200$$

$$\frac{di_1}{dt} + \frac{20}{3}i = \frac{100}{3}$$

$$\left(D + \frac{20}{3}\right)i_1 = \frac{100}{3} \quad \text{—— ④}$$

solution of eq. ④ is

$$i_1 = C\,e^{-\frac{20}{3}t} + \frac{\frac{100}{3}}{\frac{20}{3}} = C\,e^{-\frac{20}{3}t} + 5 \quad \text{—— ⑤}$$

at $t=0$ $\quad i_1 = \dfrac{100}{15} = \dfrac{20}{3}$ (from eq. ①) —— (6)

sub eq. ⑥ in eq. ⑤ we get

$$\frac{20}{3} = C + 5 \Rightarrow C = \frac{20}{3} - 5 = \frac{5}{3}$$

$$i_1 = \frac{5}{3}e^{-\frac{20}{3}t} + 5 \quad \text{—— ⑦}$$

sub eq ⑦ in ③ we get

$$i_2 = \frac{20}{3} - \frac{5}{9}e^{-\frac{20}{3}} - \frac{5}{3} = 5 - \frac{5}{9}e^{-\frac{20}{3}t}$$

Tutorial

Ex 1 In the fig(28), the switch is closed at $t=0$, when the $6\,\mu f$ capacitor has charge $Q_0 = 300\,\mu C$. obtain the expression for the transient voltage V_R

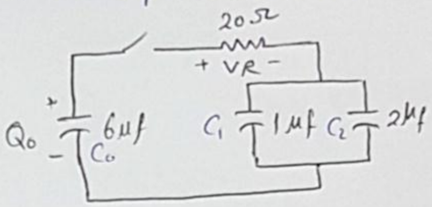


fig (28)

Solution:-

$$C_{eq} = C_1 // C_2 + C_0$$

$$= \frac{(C_1 + C_2) \times C_0}{(C_1 + C_2) + C_0} = \frac{3 \times 6}{3+6} = 2\,\mu f$$

$$\Rightarrow \tau = R\,C_{eq} = 20 * 2 = 40\,\mu s.$$

At $t=0$ KVL gives

$$V_R = \frac{Q_0}{C_0} = \frac{300}{6} = 50\,V = V_0$$

$$\therefore V(t) = V_0\, e^{-\frac{t}{\tau}} = 50\, e^{-\frac{t}{40}}\ V \quad \text{if } t \text{ in } \mu s$$

Ex:- In the fig(29) the switch is moved from position ① to position ② at $t=0$, obtain the current i_2 at $t = 34.7$ ms.

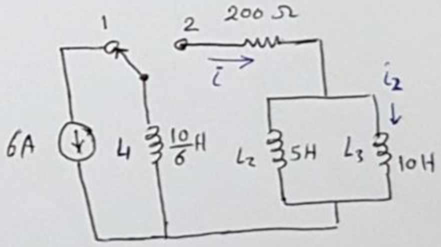


fig (29)

Solution:-

when the switch at the position ②

$$L_{eq} = L_1 + L_2 // L_3$$

$$= \frac{10}{6} + \frac{5 \times 10}{5+10} = 5\,H$$

$$\Rightarrow \tau = \frac{L_{eq}}{R} = \frac{5}{200} = 25\ ms.$$

$$\Rightarrow i = i_0\, e^{-\frac{t}{\tau}} = 6\, e^{-\frac{t}{25}}\ A$$

$$i_2 = \frac{5}{15}\, i = \frac{5}{15} \times 6\, e^{-\frac{t}{25}} = 2\, e^{-\frac{t}{25}}$$

$$i_2(34.7) = 2\, e^{-34.7/25} = 0.5\ A$$

(22)

Ex2 In fig (30) the switch is closed at $t=0$, obtain the current i and capacitor voltage V_c for $t>0$

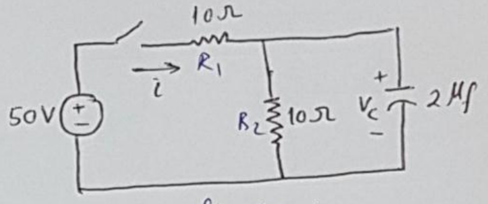


fig (30)

Solution:-

for $t>0$.

$$R_{eq} = R_1 // R_2 = \frac{10 \times 10}{10+10} = 5\Omega$$

$$\tau = R_{eq} * C = 5 * 2 = 10\ \mu s$$

at $t \to \infty$ then the capacitor is open

$$i(\infty) = \frac{50}{20} = 2.5\,A$$

$$V_c(\infty) = i * R_2 = 2.5 * 10 = 25\,V$$

$$V_c(t) = [V_c(0) - V_c(\infty)]\, e^{-\frac{t}{\tau}} + V_c(\infty)$$

$$= 25(1 - e^{-\frac{t}{10}})\ V \quad \text{if } t \text{ in } \mu s.$$

$$i_c = \frac{c\,dv}{dt} = 5\,e^{-\frac{t}{10}}\ A$$

the current in the parallel R_2 is

$$i_{10\Omega} = \frac{V_c}{10} = 2.5(1 - e^{-\frac{t}{10}})\ A$$

hence.

$$i = i_c + i_{(R_2)} = 2.5(1 + e^{-\frac{t}{10}})\ A.$$

Ex3 for the fig (31), obtain the current i_L for all values of t

Solution: for $t < 0$ 50 V source is 50V

$$i = \frac{50}{R_1+R_2} = \frac{50}{20} = 2.5\ A$$

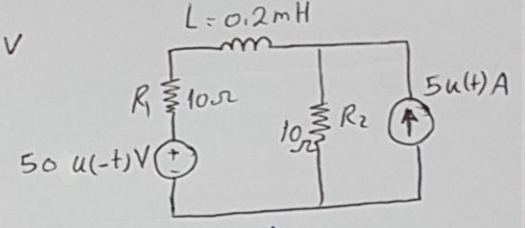


fig (31)

Now for $t > 0$

and $t \to \infty$ this current divides equally between the two 10 Ω resistor.

$i_L(\infty) = -2.5\ A$. the time constant of the circuit (τ)

$$\tau = \frac{0.2 * 10^{-3}}{R_1 + R_2} = \frac{0.2 * 10^{-3}}{20} = \frac{1}{100}\ mS.$$

and so, with t in ms and using $i_L(0^+) = i_L(0^-) = 2.5\ A$

$$i_L = i_L(\infty) + [i_L(0) - i_L(\infty)]\, e^{-t/\tau}$$

$$= 5\, e^{-100t} - 2.5\ (A)$$

and finally, using unit step functions to combine the expression for $t < 0$ and $t > 0$

$$i_L = 2.5\, u(-t) + (5\, e^{-100t} - 2.5\, u(t))\ A$$

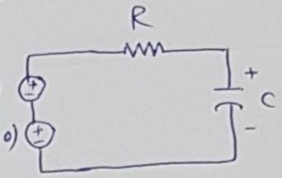
(24)

Ex 24- A series RC circuit with $R = 5\,K\Omega$ and $C = 20\,\mu f$, has two voltage sources in series: $V_1 = 25\,u(-t)\,V$ & $V_2 = 25\,u(t-t_0)\,V$ obtain the complete expression for the voltage across the capacitor and sketch if t_0 is positive?

Solution:-

$V_1 = 25\,u(-t)$

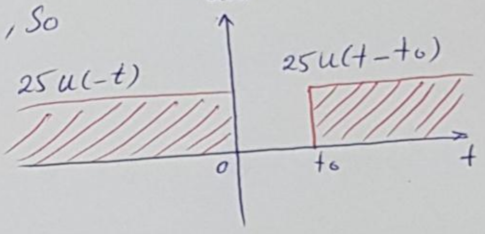
$V_2 = 25\,u(t-t_0)$



① for $t \leq 0$; $V_c = 25\,(V)$

② for $0 \leq t \leq t_0$ both sources are zero, so that V_c decays exponentially from 25 V towards zero $\tau = 5\times10^3 \times 20\times10^{-6} = 100\times10^{-3} = 0.1$

$$V_c = 25\,e^{-\frac{t}{\tau}} = 25\,e^{-10t}\ (V)$$



③ for $t \geq t_0$

$$V_c(t_0) = 25\,e^{-10t}$$

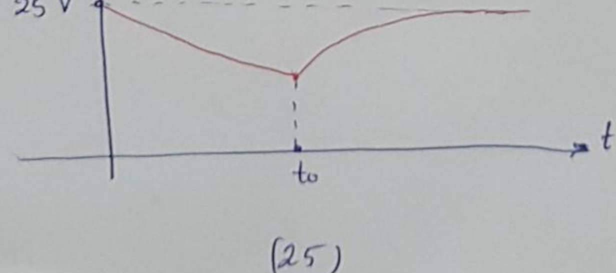
the V_c builds from $V_c(t_0)$ to the final value of 25 V established by V_2

$$V_c = [V_c(t_0) - V_c(\infty)]\,e^{-(t-t_0)/\tau} + V_c(\infty)$$

$$= 25\,[1 - (e^{+10t} - 1)\,e^{-10t}\ (V) \quad \text{for all } t \geq t_0$$

thus for t

$$V_c(t) = 25\,u(-t) + 25\,e^{-10t}\,[u(t) - u(t-t_0)] + 25\,[1 - (e^{10t} - 1)\,e^{-10t}]\,u(t-t_0)\ V$$



(25)

Ex5:- A series RLC circuit with $R = 50\Omega$; $L = 0.1$ H and $C = 50\,\mu F$, has a constant Voltage $V = 100$ V applied at $t = 0$ obtain the current transient, assuming Zero initial charge on the capacitor

Solution:-

$$\alpha = \frac{R}{2L} = \frac{50}{2 * 0.1} = 250\ s^{-1}$$

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{0.1 \times 50 * 10^{-6}} = 2 * 10^5\ s^{-2}$$

$$\beta = \sqrt{\alpha^2 - \omega_o^2} = \sqrt{(250)^2 - 2*10^5} = j\,370.8 \text{ rad/s.}$$

$\Rightarrow$ this oscillatory case ($\alpha < \omega_o$) and the general expression is

$$i = e^{-250t}(A_1 \cos 370.8t + A_2 \sin 370.8t)$$

initial condition, obtained is

$$i(0) = 0 \quad ;$$

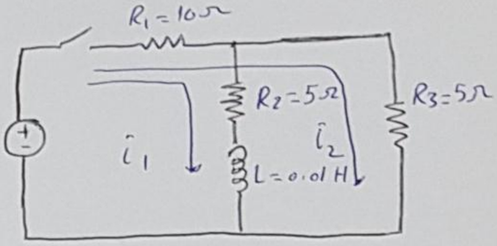
$$\left.\frac{di}{dt}\right|_0 = 1000 \text{ A/s.}$$

then $A_1 = 0$; $A_2 = 2.70$ A.

$$i = e^{-250t}(2.70 \sin 370.8t)$$

(26)

Ex6:- the switch in the two-mesh circuit shown in fig (32) is closed at $t=0$, obtain the currents i_1 and i_2 for $t > 0$



Solution:-

Method (1) two-mesh:

$$100 = 15 i_1 + 10 i_2 + 0.01 \frac{di_1}{dt} \quad \text{—— (1)}$$

$$100 = 15 i_2 + 10 i_1 \quad \text{—— (2)}$$

from eq. (2)

$$i_2 = \frac{100 - 10 i_1}{15} \quad \text{—— (3)}$$

sub eq. (3) in (1) we get

$$\frac{di_1}{dt} + 833 i_1 = 3333$$

$$i_1 = A e^{-833t} + 4 \quad A.$$

$$i_1(0) = 0 \Rightarrow A = -4$$

$$i_1 = 4(1 - e^{-833t}) \ A.$$

and

$$i_2 = 4 + 2.67 e^{-833t} \quad A.$$

Method(2)

$$Req = 5 + \frac{5 \times 10}{15} = 8.33 \ \Omega$$

$$\tau = \frac{L}{Req} = \frac{0.01}{8.33} = \frac{1}{833} \ ;$$

at $t = \infty$ $\quad RT = 10 + \frac{5 \times 5}{5+5} = 12.5 \ \Omega$

$$i_T = \frac{100}{12.5} = 8 \ A$$

$$i_L = i_1 = 4(1 - e^{-833t})$$

(27)