Derivatives

The derivative of y = f(x) is a function defined by:

$$y' = \frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdots\cdots\cdots(1)$$

The process of taking the derivative of a function is called differentiation and a function which possesses a derivative at x is said to be differentiable at x.

If the limit in (1) exists, we say that the function f(x) has a **tangent line** at the point $(x, f(x))$.

This is the line passing through the point $(x, f(x))$ with slope $f'(x)$.

The equation of line passing through the points (x_1, y_1) and (x_2, y_2) is given by:

$$y - y_1 = m(x - x_1) \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1} = slope$$

<u>Ex1:</u> Find $f'(x)$, if:

(1) $f(x) = x^2 + 4x - 10$

$$\because f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\therefore f'(x) = \lim_{\Delta x \to 0} \frac{\left[(x+\Delta x)^2 + 4(x+\Delta x) - 10\right] - \left[x^2 + 4x - 10\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(x^2 + 2x\Delta x + \Delta x^2 + 4x + 4\Delta x - 10\right) - \left(x^2 + 4x - 10\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 + 4\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (2x + 4 + \Delta x) = 2x + 4$$

$$\therefore f'(x) = 2x + 4$$

(2) $f(x) = \sqrt{x}$

$$\because f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \Rightarrow f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x+\Delta x} - \sqrt{x}\right)\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

Differentiation Formulas

(1) $f(x) = c, \; (c = \text{constant}) \Rightarrow \dfrac{df(x)}{dx} = f'(x) = 0$

(2) $f(x) = x \Rightarrow f'(x) = 1$

(3) $f(x) = x^n \Rightarrow f'(x) = n\,x^{n-1}$

(4) $\dfrac{d}{dx}\left[c\,f(x)\right] = c\; f'(x)$

(5) $\dfrac{d}{dx}\left[f_1(x) \pm f_2(x) \pm \cdots\cdots \pm f_n(x)\right] = f_1'(x) \pm f_2'(x) \pm \cdots\cdots \pm f_n'(x)$

(6) $\dfrac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\,g'(x) + g(x)\,f'(x) \Rightarrow$ Product Rule

(7) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{g^2(x)} \Rightarrow$ *Quotient Rule*

(8) $\dfrac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1} f'(x) \Rightarrow$ Power Rule

Ex2: Find the derivative of:

(1) $s(t) = 12 + 3t - 2t^2 + 6t^3 - 5\sqrt{t}$

$$\therefore s'(t) = 0 + 3(1) - 2(2t) + 6(3t^2) - 5\left(\frac{1}{2\sqrt{t}}\right) = 3 - 4t + 18t^2 - \frac{5}{2\sqrt{t}}$$

(2) $f(y) = (y^5 - 4y + 2)(7y + 3)$

$$\therefore f'(y) = (y^5 - 4y + 2)(7) + (7y + 3)(5y^4 - 4)$$

$$\therefore f'(y) = 7y^5 - 28y + 14 + 35y^5 + 15y^4 - 28y - 12 = 42y^5 + 15y^4 - 56y + 2$$

(3) $f(x) = \dfrac{5x^2 + 2x + 3}{\sqrt{x}}$

$$\therefore f'(x) = \frac{\sqrt{x}\,(10x + 2) - (5x^2 + 2x + 3)\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2} \times \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{2x\,(10x + 2) - (5x^2 + 2x + 3)}{2x\sqrt{x}} = \frac{20x^2 + 4x - 5x^2 - 2x - 3}{2x\sqrt{x}} = \frac{15x^2 + 2x - 3}{2x\sqrt{x}}$$

(4) $f(x)=\dfrac{6}{x^8} \Rightarrow f(x)=6x^{-8} \Rightarrow f'(x)=6\left(-8x^{-9}\right)$

$\therefore f'(x)=\dfrac{-48}{x^9}$

(5) $f(x)=\left(\dfrac{x^2+3}{3x+1}\right)^5$

$$\therefore f'(x)=5\left(\frac{x^2+3}{3x+1}\right)^{5-1}\left(\frac{(3x+1)(2x)-(x^2+3)(3)}{(3x+1)^2}\right)$$

$$=5\left(\frac{x^2+3}{3x+1}\right)^4\left(\frac{6x^2+2x-3x^2-9}{(3x+1)^2}\right)=\frac{5(x^2+3)^4(3x^2+2x-9)}{(3x+1)^6}$$

Derivative of Composite Functions- The Chain Rule

If $y=f(u)$ and $u=g(x)$, then:

$$\frac{dy}{dx}=\frac{d}{dx}[f(g(x))]=\frac{df}{du}\frac{du}{dx}$$

Ex3: Find the derivative of:

(1) $y=3u+2$ *and* $u=4x-1$

$$\therefore \frac{dy}{du}=3 \text{ and } \frac{du}{dx}=4 \Rightarrow \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=3(4)=12$$

Or $y=3(4x-1)+2=12x-3+2=12x-1 \Rightarrow \dfrac{dy}{dx}=12$

(2) $y=\sqrt{3x^4-5x^2+10}$

Let $u=3x^4-5x^2+10$ then $y=\sqrt{u}$

$$\therefore \frac{dy}{du}=\frac{1}{2\sqrt{u}} \text{ and } \frac{du}{dx}=12x^3-10x \Rightarrow \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{2\sqrt{u}}(12x^3-10x)$$

$$\therefore \frac{dy}{dx}=\frac{12x^3-10x}{2\sqrt{3x^4-5x^2+10}}=\frac{6x^3-5x}{\sqrt{3x^4-5x^2+10}}$$

Implicit Differentiation

Ex4: Find dy/dx, if:

(1) $x\,y + y^2 = x^2$

$$\frac{d}{dx}\left(x\,y+y^2\right)=\frac{d}{dx}\left(x^2\right) \Rightarrow x\frac{dy}{dx}+y+2y\frac{dy}{dx}=2x$$

$$\left(x+2y\right)\frac{dy}{dx}=2x-y \Rightarrow y'=\frac{dy}{dx}=\frac{2x-y}{x+2y}$$

Ex5: Find the equation of the line tangent to the curve $\left(x+y\right)^2+\sqrt{y}=2$ at the point (-2, 1).

Solution

$$\therefore 2\left(x+y\right)\left(1+y'\right)+\frac{1}{2\sqrt{y}}y'=0$$

$$\therefore 2\left(x+y\right)+2\left(x+y\right)y'+\frac{1}{2\sqrt{y}}y'=0 \Rightarrow \left(2x+2y+\frac{1}{2\sqrt{y}}\right)y'=-2\left(x+y\right)$$

$$\therefore \frac{dy}{dx}=\frac{-2\left(x+y\right)}{2\left(x+y\right)+\frac{1}{2\sqrt{y}}}$$

At the point (-2, 1)

$$\frac{dy}{dx}=\frac{-2\left(-2+1\right)}{2\left(-2+1\right)+\frac{1}{2\sqrt{1}}}=\frac{2}{-2+0.5}=\frac{2}{-3/2}=\frac{-4}{3}=m=\text{slope of tangent line}$$

$$\because y-y_1=m\left(x-x_1\right) \text{ Equation of line}$$

$$\therefore y-1=\frac{-4}{3}\left(x+2\right) \Rightarrow y=1-\frac{4}{3}x-\frac{8}{3} \Rightarrow y=\frac{-4}{3}x-\frac{5}{3}$$

Higher Order Derivatives

If $y=f(x)$, then:

$y'=\frac{dy}{dx}=f'(x)$ **First derivative of y with respect to x**

$y''=\frac{d^2y}{dx^2}=f''(x)$ **Second derivative of y with respect to x**

$$y''' = \frac{d^3y}{dx^3} = f'''(x)$$ **Third derivative of y with respect to x**

$$y^{(4)} = \frac{d^4}{dx^4} = f^{(4)}(x)$$ **Fourth derivative of y with respect to x**

$$y^{(n)} = \frac{d^n y}{dx^n} = f^{(n)}(x)$$ **nth derivative of y with respect to x**

<u>Ex6:</u> Let $y = f(x) = \dfrac{1}{x}$**, find** $y',\ y'',\ y''',$ and $y^{(4)}$**:**

$$y = x^{-1} \Rightarrow y' = -x^{-2}$$

$$y'' = -(-2x^{-3}) = 2x^{-3}$$

$$y''' = -6x^{-4} \Rightarrow y^{(4)} = 24x^{-5}$$

<u>Derivative of Trigonometric and Inverse Trigonometric Functions</u>

(1) $\dfrac{d}{dx}\sin[f(x)] = \cos[f(x)]\cdot f'(x)$

(2) $\dfrac{d}{dx}\cos[f(x)] = -\sin[f(x)]\cdot f'(x)$

(3) $\dfrac{d}{dx}\tan[f(x)] = \sec^2[f(x)]\cdot f'(x)$

(4) $\dfrac{d}{dx}\cot[f(x)] = -\csc^2[f(x)]\cdot f'(x)$

(5) $\dfrac{d}{dx}\sec[f(x)] = \sec[f(x)]\tan[f(x)]\cdot f'(x)$

(6) $\dfrac{d}{dx}\csc[f(x)] = -\csc[f(x)]\cot[f(x)]\cdot f'(x)$

(7) $\dfrac{d}{dx}\sin^{-1}[f(x)] = \dfrac{1}{\sqrt{1-f^2(x)}}\cdot f'(x)$

(8) $\dfrac{d}{dx}\cos^{-1}[f(x)] = \dfrac{-1}{\sqrt{1-f^2(x)}}\cdot f'(x)$

(9) $\dfrac{d}{dx}\tan^{-1}[f(x)] = \dfrac{1}{1+f^2(x)}\cdot f'(x)$

(10) $\frac{d}{dx}\cot^{-1}[f(x)]=\frac{-1}{1+f^2(x)}\cdot f'(x)$

(11) $\frac{d}{dx}\sec^{-1}[f(x)]=\frac{1}{|f(x)|\sqrt{f^2(x)-1}}\cdot f'(x)$

(12) $\frac{d}{dx}\csc^{-1}[f(x)]=\frac{-1}{|f(x)|\sqrt{f^2(x)-1}}\cdot f'(x)$

<u>Ex7:</u> Find y', if:

(1) $y=\sin\left(x^2+\sqrt{x}\right)$

$$\therefore\ y'=\cos\left(x^2+\sqrt{x}\right)\cdot\left(2x+\frac{1}{2\sqrt{x}}\right)$$

(2) $y=\sec^{-1}\left(x^2\right)$

$$\therefore\ y'=\frac{1}{|x^2|\sqrt{x^4-1}}\cdot 2x;\qquad \because |x^2|=x^2$$

$$\therefore\ y'=\frac{2}{x\sqrt{x^4-1}}$$

(3) $y=\sin(x+y)$

$$y'=\cos(x+y)\cdot(1+y') \Rightarrow y'=\cos(x+y)+y'\cos(x+y)$$

$$(1-\cos(x+y))y'=\cos(x+y) \Rightarrow y'=\frac{\cos(x+y)}{1-\cos(x+y)}$$

(4) $y=\tan^2\left(\frac{1-x}{1+x}\right)$

$$\therefore\ y'=2\left[\tan\left(\frac{1-x}{1+x}\right)\right]\cdot\sec^2\left(\frac{1-x}{1+x}\right)\cdot\left[\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}\right]$$

$$\therefore\ y'=\frac{-4}{(1+x)^2}\tan\left(\frac{1-x}{1+x}\right)\sec^2\left(\frac{1-x}{1+x}\right)$$

(5) $y=\sec\left(x^2+\tan^{-1}3x\right)$

$$\therefore\ y' = \sec\left(x^2 + \tan^{-1} 3x\right)\tan\left(x^2 + \tan^{-1} 3x\right)\cdot\left[2x + \frac{1}{1+(3x)^2}\cdot 3\right]$$

$$\therefore\ y' = \left[2x + \frac{3}{1+9x^2}\right]\cdot\sec\left(x^2 + \tan^{-1} 3x\right)\tan\left(x^2 + \tan^{-1} 3x\right)$$

(6) $y = \sqrt{3x^4 + \tan(3x + \sin x)}$

$$\therefore\ y = \left(3x^4 + \tan(3x + \sin x)\right)^{0.5}$$

$$\therefore\ y' = 0.5\left(3x^4 + \tan(3x + \sin x)\right)^{-0.5}\cdot\left(12x^3 + \sec^2(3x + \sin x)\cdot(3 + \cos x)\right)$$

Derivative of Exponential and Logarithmic Functions

(1) $\frac{d}{dx} e^x = e^x$

(2) $\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$

(3) $\frac{d}{dx} \ln(x) = \frac{1}{x}$

(4) $\frac{d}{dx} \ln[f(x)] = \frac{1}{f(x)}.f'(x)$

Ex8: Find y', if:

(1) $y = \ln(\sec\theta + \tan\theta)$

$$\therefore\ y' = \frac{1}{\sec\theta + \tan\theta}\cdot\left(\sec\theta\tan\theta + \sec^2\theta\right) = \frac{\sec\theta(\tan\theta + \sec\theta)}{\sec\theta + \tan\theta} \Rightarrow \therefore\ y' = \sec\theta$$

(2) $y = e^{\tan^{-1} x + \sin x}$

$$\therefore\ y' = e^{\tan^{-1} x + \sin x}\cdot\left(\frac{1}{1+x^2} + \cos x\right)$$

(3) $y = \ln\left(x + \sqrt{1+x^2}\right)$

$$\therefore\ y' = \frac{1}{x + \sqrt{1+x^2}}\cdot\left(1 + \frac{1}{2\sqrt{1+x^2}}\cdot 2x\right) = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}$$

$$\therefore y' = \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)} \Rightarrow \frac{dy}{dx} = y' = \frac{1}{\sqrt{1+x^2}}$$

(4) $y = \ln(\sec\theta)$

$$\therefore y' = \frac{1}{\sec\theta} \cdot \sec\theta \tan\theta \Rightarrow y' = \tan\theta$$

(5) $y = (\sin x)^x$

$$\therefore \ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{1}{y} \cdot y' = x\frac{1}{\sin x}\cos x + \ln(\sin x)$$

$$\therefore y' = y(x\cot x + \ln(\sin x)) = (\sin x)^x (x\cot x + \ln(\sin x))$$

(6) $y = \log_x(\sin x) \Rightarrow y = \frac{\ln\sin x}{\ln x}$

$$\therefore y' = \frac{\ln x \cdot \frac{1}{\sin x}\cos x - \ln\sin x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\cot x \ln x - \frac{\ln\sin x}{x}}{(\ln x)^2}$$

Ex9:

(1) Prove that $y = xe^x$ **is the solution of the differential equation:** $y'' + 4y' - 5y = 6e^x$ **.**

Solution

$$\because y = xe^x \Rightarrow y' = xe^x + e^x \Rightarrow y'' = xe^x + e^x + e^x = xe^x + 2e^x$$

$$\therefore y'' + 4y' - 5y = xe^x + 2e^x + 4\left(xe^x + e^x\right) - 5xe^x = 6e^x$$

$$\therefore y'' + 4y' - 5y = 6e^x$$

Ex10: Prove that the derivative of:

(1) $y = \tan[f(x)]$ *is given by* $y' = \sec^2[f(x)].\, f'(x)$

(2) $y = \sin^{-1}[f(x)]$ *is given by* $y' = \frac{1}{\sqrt{1-f^2(x)}}.\, f'(x)$

Solution

(1) $\because y = \tan[f(x)] \Rightarrow y = \frac{\sin[f(x)]}{\cos[f(x)]}$

$$\therefore y' = \frac{\cos[f(x)]\cos[f(x)].f'(x) - \sin[f(x)][-\sin[f(x)].f'(x)]}{\cos^2[f(x)]}$$

$$\therefore\ y' = \frac{f'(x)\left[\cos^2[f(x)] + \sin^2[f(x)]\right]}{\cos^2[f(x)]} = \frac{1}{\cos^2[f(x)]}.f'(x)$$

$$\therefore\ y' = \sec^2[f(x)].f'(x)$$

(2) $\because\ y = \sin^{-1}[f(x)] \Rightarrow \sin y = f(x)$

$$\therefore\ \cos y.y' = f'(x) \Rightarrow y' = \frac{1}{\cos y}\cdot f'(x)$$

For the right triangle shown in figure

$$\sin y = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\because\ \sin y = f(x) = \frac{f(x)}{1} \Rightarrow \text{opposite} = f(x) \text{ and hypotenuse} = 1$$

$$\therefore\ \text{adjacent} = \sqrt{1 - f^2(x)}$$

$$\because\ \cos y = \frac{\text{adjacent}}{\text{hypotenuse}} \Rightarrow \cos y = \frac{\sqrt{1 - f^2(x)}}{1}$$

$$\because\ y' = \frac{1}{\cos y}\cdot f'(x) \Rightarrow y' = \frac{1}{\sqrt{1 - f^2(x)}}\cdot f'(x)$$

Ex11: Find the derivative of: $y = 3^{\sin x + y^2}$

Solution

$$\therefore\ \ln y = \ln\left(3^{\sin x + y^2}\right) \Rightarrow \ln y = \left(\sin x + y^2\right)\ln 3$$

$$\therefore\ \frac{1}{y}\cdot y' = (\cos x + 2y.y')\ln 3 \Rightarrow \frac{1}{y}\cdot y' = \ln 3 \cos x + (2\ln 3)y\,y'$$

$$\therefore\ \frac{1}{y}\cdot y' - (2\ln 3)y\,y' = \ln 3\cos x \Rightarrow \left[\frac{1}{y} - (2\ln 3)y\right]y' = \ln 3\cos x$$

$$\therefore\ y' = \frac{\ln 3\cos x}{\frac{1}{y} - (2\ln 3)y}$$

Ex12: If $y = e^{ax}\sin(bx)$**, show that:** $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right)y = 0$

Solution

$$\because\ y = e^{ax}\sin(bx) \Rightarrow y' = e^{ax}[b\cos(bx)] + \sin(bx)[a\,e^{ax}]$$

$$\therefore\ y' = b\,e^{ax}\cos(bx) + a\,y \Rightarrow y' - a\,y = b\,e^{ax}\cos(bx)$$

$$\therefore\ y'' - a\,y' = b\,e^{ax}[-b\sin(bx)] + \cos(bx)[b\,a\,e^{ax}]$$

$$\therefore\ y'' - a\,y' = -b^2\,y + a\left[b\,e^{ax}\cos(bx)\right]$$

$$\because\ b\,e^{ax}\cos(bx) = y' - a\,y$$

$$\therefore\ y'' - a\,y' = -b^2\,y + a\,(y' - a\,y) \Rightarrow y'' - a\,y' = -b^2\,y + a\,y' - a^2\,y$$

$$\therefore\ y'' - 2a\,y' + (a^2 + b^2)y = 0$$

<u>Ex13:</u> Find the derivative of: $\cos y\ \ln x = \sin x\ \ln y$

<u>Solution</u>

$$\therefore\ \frac{d}{dx}(\cos y\ \ln x) = \frac{d}{dx}(\sin x\ \ln y)$$

$$\therefore\ \cos y\cdot\frac{1}{x} + \ln x\cdot[-\sin y\,.\,y'] = \sin x\cdot\frac{1}{y}\,y' + \ln y\cdot\cos x$$

$$\therefore\ \frac{\cos y}{x} - \ln y\cdot\cos x = \ln x\cdot\sin y\,.\,y' + \sin x\cdot\frac{1}{y}\ y'$$

$$\therefore\ \left(\ln x\sin y + \frac{\sin x}{y}\right)y' = \frac{\cos y}{x} - \ln y\cos x \Rightarrow y' = \frac{\dfrac{\cos y}{x} - \ln y\cos x}{\dfrac{\sin x}{y} + \ln x\sin y}$$

<u>Ex14:</u> If $y = (\sin^{-1}x)^2$**. Prove that:** $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$

<u>Solution</u>

$$\because\ y = (\sin^{-1}x)^2 \Rightarrow \sqrt{y} = \sin^{-1}x \Rightarrow \frac{1}{2\sqrt{y}}\,y' = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore\ \sqrt{1-x^2}\ y' = 2\sqrt{y} \Rightarrow \sqrt{1-x^2}\ y'' + y'\frac{1}{2\sqrt{1-x^2}}(-2x) = 2\frac{1}{2\sqrt{y}}\,y'$$

$$\therefore\ \sqrt{1-x^2}\ y'' - y'\frac{x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{y}}\,y'$$

$$\because \sqrt{1-x^2}\, y' = 2\sqrt{y} \;\Rightarrow\; y' = \frac{2\sqrt{y}}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2}\, y'' - y'\frac{x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{y}}\left(\frac{2\sqrt{y}}{\sqrt{1-x^2}}\right)$$

$$\therefore \sqrt{1-x^2}\, y'' - y'\frac{x}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}} \Rightarrow (1-x^2)y'' - x\,y' = 2$$

Application

Velocity is the derivative of the distance. Speed is the absolute value of velocity.

Speed = |velocity|
Speed = |v(t)| = |ds/dt|

Acceleration is the derivative of velocity with respect to time.

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Example

The distance of a ball falls freely from rest is proportional with time as $s=4.9t^2$

a- How long did it take the ball bearing to fall the first 14.7m?

b- What is the velocity, speed and acceleration after 2 second?

Solution

a- $s = 4.9t^2$

$14.7=4.9t^2$ so $t = \pm\sqrt{3}$ second

$t= \sqrt{3}$ (time increase from t=0 so we ignore the negative root)

b- velocity at any time

$$v(t) = \frac{ds}{dt} = 9.8t$$

Velocity after 2 second = v(2) = 19.6 m/s

Speed = |19.6| = 19.6 m/s

Acceleration at any time

$$a(t) = \frac{d^2 s}{dt^2} = 9.8 m/s^2$$

Acceleration after 2 second = 9.8 m/s^2

Ex16: **Let** $s(t)=2t^3-3t^2+8t+5$ **be the position function of a particle moving along an s-axis, where s is in meters and t is in seconds. Find (a) the initial position; (b) the initial velocity; (c) the initial acceleration.**

Solution

(a) The initial position is the position at t = 0.

$\therefore s(0)=2(0)^3-3(0)^2+8(0)+5=5[m]$

(b) The instantaneous velocity $=v(t)=s'(t)=6t^2-6t+8$

$v(0)=6(0)^2-6(0)+8=8\ [m/s]$

(c) The instantaneous acceleration $=a(t)=v'(t)=12t-6$

$\therefore a(0)=12(0)-6=-6\ [m/s^2]$

Ex17: **Let the position function of a particle moving along an s-axis is** $s(t)=\dfrac{-100}{t^2+12}\ [m]$**.**

Find the maximum speed of the particle for $t\geq 0$**.**

Solution

$$\because s(t)=\frac{-100}{t^2+12}\ \Rightarrow\ v(t)=s'(t)=\frac{(t^2+12)(0)+100(2t)}{(t^2+12)^2}=\frac{200t}{(t^2+12)^2}$$

$$\therefore v'(t)=\frac{(t^2+12)^2(200)-200t\left[2(t^2+12)(2t)\right]}{(t^2+12)^4}$$

$$\therefore v'(t)=\frac{200(t^2+12)^2-800t^2(t^2+12)}{(t^2+12)^4}=\frac{200(t^2+12)-800t^2}{(t^2+12)^3}$$

Maximum speed is occur when $v'(t)=0$

$$\therefore \frac{200(t^2+12)-800t^2}{(t^2+12)^3}=0\ \Rightarrow\ 200(t^2+12)-800t^2=0$$

$$\therefore 200t^2+2400-800t^2=0\Rightarrow 600t^2=2400\Rightarrow t^2=4\Rightarrow t=\pm 2$$

$$\therefore \text{for } t\geq 0\ \Rightarrow\ t=2\ [sec]$$

$$\therefore \text{Maximum speed}=v(2)=\frac{200(2)}{(2^2+12)^2}=\frac{400}{256}=1.5625\ [m/s]$$

Home Works 4

Q1: Find $f'(x)$**, if:**

1 $f(x)=\dfrac{1}{x}$

2 $f(x)=5x^2-3x+2$

Q2: Find the derivative of:

(1) $f(t)=\sqrt{x^3+3}\cdot\left(5x^3+2x-4\right)$

(2) $f(x)=\sqrt{x^3+2x^2-4x+3}$

(3) $f(x)=\dfrac{3x+4}{\sqrt{x^2+5}}$

(4) $f(x)=\dfrac{5}{x^2-7x+3}$

(5) $f(x)=5x^4+\dfrac{2}{x^3}+\dfrac{4}{\sqrt{x}}$

Q3: using chain rule Find the derivative of:

1 $y=3t^2+2t-5 \text{ and } t=x^2+2$

2 $y=\left(\dfrac{x^2}{3x+5}\right)^8$

Q4: using Implicit rule Find the derivative of

1 $\left(\sqrt{y}+xy\right)=y^2+x^2$

2 Find the equation of the line tangent to the curve $\sqrt{2+x^2y}+x^2=3$ **at the point (1, 2).**

Q5: Find the derivative of:

1 $y = \sin^3(2x+3)$

2 $y = \dfrac{x + \tan 4x}{\sec x}$

3 $y = \dfrac{\sin(x^2)}{\cos^2(x^2)}$

4 $y = \sqrt{\tan x + \sin^{-1} x}$

5 $y = \tan^{-1}\left[\dfrac{1+\sin x}{1-\sin x}\right]$

6 $y = \sec\left(\csc^{-1}\sqrt{x}\right)$

Q6: Find the derivative of:

1 $y = \ln\left(e^{\sin x} + \tan(x^2)\right)$

2 $y = \sqrt{\ln\sqrt{1+x} + \sin^2 x}$

3 $y = 4^{\sin x + Log_2(x)}$

4 $y = Log_{\ln x}\left(1 + e^{4x}\right)$

5 $y = (x + \tan x)^{\sin x}$

6 $y = \dfrac{x(1+\sin x)}{(x^2+7)(x+\ln x)}$

Q7 Prove that $y = e^{3x} + 2e^{x} + \sin x$ **is the solution of the differential equation:**

$y'' + 3y' + 2y = \sin x + 3\cos x$.

Q8: Prove that the derivative of:

1 $y = \sec[f(x)]$ *is given by* $y' = \sec[f(x)].\tan[f(x)].\, f'(x)$

2 $y = \sec^{-1}[f(x)]$ *is given by* $y' = \dfrac{1}{|f(x)|\sqrt{f^2(x)-1}}.\, f'(x)$

Q9: If $y = u^V$ and u = f(x) and V = g(x). Prove that: $\frac{dy}{dx} = v\,u^{v-1}\frac{du}{dx} + u^v(\ln u)\frac{dv}{dx}$

Q10: If $\tan^{-1}\left(\frac{y}{x}\right) = \ln(x^2 + y^2)$. Prove that $\frac{dy}{dx} = \frac{2x + y}{x - 2y}$.

Q11: Find the derivative of $x\,y = y^{\tan x}$.

Q12: Let $s(t) = t^3 - 6t^2 + 1$. (a) Find s and v when a = 0. (b) Find s and a when v = 0.

Q13:

A dynamite blast blows heavy rock straight up with a lunch velocity of 160 ft/sec. It reaches a height of $s=160t - 16t^2$ after time (sec)

a- How height does the rock ago?

b- what are thw velocity and speed of the rock when its 256 ft above the ground on the way up ? on the way down?

c- what is the acceleration of the rock at any time t during its flight (after the blast)?

d- when does the rock hit the ground again?